MAT327 - Introduction to Topology

Jimmy Xu

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Ω1 Day 1 : Introduction, Topological Spaces and Bases

This class is taught by Daniel Wilches Calderon during the Summer 2024 semester. We will mostly be following Topology by Munkres, with some suggested readings on the Quercus page

Ω1.1 Marking Scheme

1. Final Exam - 30%

2. Term Test - 20%

3. Quizzes - 50% (Best 8 of 10)

   Every week we will be given a problem set, and during tutorial one of those questions will be tested

Ω1.2 Open sets in the Real Numbers

Recall that $U \in \mathbb{R}$ is open if

$$\text{for all } x \in U, \text{ there exists } \varepsilon > 0 \text{ such that } x \in (x - \varepsilon, x + \varepsilon) \subseteq U$$

Open sets have some properties. That the arbitrary union and finite intersection of open sets is still open

1. Arbitrary Union

   Let $\{U_\alpha | \alpha \in \Lambda^1\}$ be an arbitrary collection of open sets, if $x \in \bigcup_{\alpha \in \Lambda}$, then $x \in U_\alpha$ for some $\alpha \in \Lambda$, and since U_α is open, we have that there exists some $\varepsilon > 0$ such that

   $$x \in (x - \varepsilon, x + \varepsilon) \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$$

   This same result doesn't hold true for arbitrary intersection, but it does for finite.

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[1]We use Λ to denote an arbitrary indexing set

2. Finite intersection

Let $\{U_k | k \leqslant n\}$ be a finite collection of open sets. if $x \in \bigcap_{k \leqslant n} U_k$, then $x \in U_k$ for all $k \leqslant n$. Since each U_k is open, there are $\varepsilon_i > 0$ such that $x \in (x - \varepsilon_i, x + \varepsilon_i) \subset U_i$ for $i = 1, 2, \ldots, n$. Then pick $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, then it follows that

$$x \in (x - \varepsilon, x + \varepsilon) \subseteq (x - \varepsilon_k, x + \varepsilon_k) \subseteq U_k$$

for all $k \leqslant n$, so $(x - \varepsilon, x + \varepsilon) \subseteq \bigcap_{k \leqslant n} U_k$.

Ω**1.3 Topological Spaces**

Definition 1.1 (Topological Space)**.** A toplogy, τ, on X is a collection of subsets of X that satisfy

1. $X, \varnothing \in \tau$

2. τ is closed under arbitrary Union

3. τ is closed under finite intersection

The double (X, τ) is called a Topological Space. Furthermore, the elements of τ are called open sets, and the elements of X are called points.

We first notice that showing the intersection of 2 sets of τ is still in τ is equivalent to showing the finite intersection of sets of τ will still be in τ. This is true by an inductive argument.

The following are some examples of topologies on a set with 2 elements, the prof briefly went over the topologies on a set with 3 elements, but basically left it as an exercise.

Example 1.2 (Topologies on 2 Point Sets)

Let $X = \{x, y\}$, we have the following are possible topologies on X

1. $\{X, \varnothing\}$

   This is often called the indiscrete/trivial topology

2. $\{X, \varnothing, \{x\}, \{y\}\}$

   This is often called the discrete topology

3. Sierpinski Spaces

   The main characteristic of these spaces are that one point is open, while the other is not. More explicitly, the following are the Sierpinski Spaces

   a) $\{X, \{x\}, \varnothing\}$
   b) $\{X, \{y\}, \varnothing\}$

You might've noticed that we labelled some of those examples as the Discrete Topology and the Indiscrete/Trivial Topology.

Definition 1.3 (Discrete and Indiscrete Topology)**.** Given a set X, the discrete topology on X is equal to the Power Set of X, and the indiscrete topology is equal to $\{x, \varnothing\}$

Definition 1.4 (Cofinite Topology)**.** Let X be a set, the Cofinite Topology on X is the collection

$$\tau = \{U \subseteq X | X \backslash U \text{ is finite}\} \cup \{\varnothing\}$$

Now to prove that this is a topology, we must prove 3 things. The entire set and the empty set exist within τ. τ is closed under arbitrary union and finite intersection. The first of these is almost immediate as $\varnothing \in \tau$ by definition, and $X \backslash X = \varnothing$, which is a finite set.

1. Arbitrary Union

   Let $\{U_\alpha | \alpha \in \Lambda\}$ be open subsets of X, we have that $X \backslash U_\alpha$ is open. Then notice that

   $$X \backslash \left(\bigcup_{\alpha \in \Lambda} U_\alpha \right) = \bigcap_{\alpha \in \Lambda} (X \backslash U_\alpha)$$

   , by DeMorgan's, which is an intersection of finite sets, so it must also be finite.

2. Finite Intersection

   Let $\{U_k | k \leqslant n\}$ be open, we have that $X \backslash U_k$ are finite, so then we notice that

   $$X \backslash \left(\bigcap_{k \leqslant n} U_k \right) = \bigcup_{k \leqslant n} (X \backslash U_k)$$

   By DeMorgan's, which is a finite union of finite sets, which implies that it is finite, implying it must be in τ

Definition 1.5 (Cocountable Topology)**.** The Cocountable Topology is defined synonymously to the cofinite topology. It is the collection

$$\tau = \{U \subseteq X | X \backslash U \text{ is countable }\} \cup \{\varnothing\}$$

Prove that this is a topology

Definition 1.6. Let τ, τ' be topologies on X, if $\tau \subseteq \tau'$, then we say

- τ is coarser than τ'

- τ' is finer than τ

As a more general statement, if $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$, then we say that τ and τ' are compareable.

Ω1.4 Bases

These are very similiar to the bases you're probably familiar with from linear algebra, where the linear combinations form the vector space, where as these bases are sets that "Unionize" to form the topological space.

Definition 1.7 (Bases). A bases for a topology on a set X is a collection fo subsets of X, $\mathscr{B}$ such that

- for all $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$

- if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Lemma 1.8 (Generated Topologies)

Let $\mathscr{B}$ be a basis of X, then the collection of all subsets U of X such that for all $x \in U$ there exists $B \in \mathscr{B}$ such that $x \in B \subseteq U$ is a topology on X. This is called the Generated Topology

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We now need to prove that this is indeed a topology. We first have that $\varnothing \in \tau$ for vacuous reasons, then we also have that $X \in \tau$ because by letting $x \in X$ be arbitrary, we have that, by definition, there exists a $B \in \mathscr{B}$ such that $x \in B \subseteq X$.

1. Union

   Let $\{U_\alpha | \alpha \in \Lambda\}$ be some collection of open sets, let $x \in \bigcup_{\alpha \in \Lambda} U_\alpha$, then there is some $\alpha \in \Lambda$ such that $x \in U_\alpha$, then let $B \in \mathscr{B}$ be such that $x \in B \subseteq U_\alpha \subseteq \bigcup_{\alpha \in \Lambda} U_\alpha$

2. Intersection

   Let $U_1, U_2 \in \tau$, let $x \in U_1 \cap U_2$, since $U_1, U_2 \in \tau$, there exists B_1, B_2 such that

   - $x \in B_1 \subseteq U_1$
   - $x \in B_2 \subseteq U_2$

   so there must exist B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

$\square$

Example 1.9 (Circles)

The bases of open disks in $\mathbb{R}^2$

$$\tau = \{\{x \in \mathbb{R}^2 | d(x,y) < \varepsilon\} | y \in \mathbb{R}^2, \varepsilon > 0\}$$

Example 1.10 (Rectangles)

$$\tau = \{(a_1, b_1) \times (a_2, b_2) | a_1, b_1, a_2, b_2 \in \mathbb{R}\}$$

> **Lemma 1.11**
>
> For subset $\mathscr{B}$ of τ on X, The following are equivalent
>
> - $\mathscr{B}$ is a basis generating τ
>
> - every non-empty element of τ is a union of elements of $\mathscr{B}$

Letting (X, τ) be a topological space, we begin with the forwards direction. Let $U \in \tau$ and let $x \in U$, since τ is generated by $\mathscr{B}$, there exists some $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$, so it follows that

$$U = \bigcup_{x \in U} B_x{}^2$$

Conversely, let $x \in X$ be arbitrary. Since $X \in \tau$, $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ with $B_\alpha \in \mathscr{B}$ it follows that $x \in B_\alpha$ for some $\alpha \in \Lambda$. Now let $B_1, B_2 \in \mathscr{B}$ with x existing in their intersection. We first have that $B_1 \cap B_2$ is open, so we may represent it as a union,

$$B_1 \cap B_2 = \bigcup_{\alpha \in \Lambda} B_\alpha$$

so $x \in B_\alpha$ for some $\alpha \in \Lambda$. so $x \in B_\alpha \subseteq B_1 \cap B_2$ $\square$

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[2]This equality wasn't immediately obvious to me why this is true, but you can prove it with double inclusion

$\Omega 2$ Day 2 : Bases and Sub-bases

> **Proposition 2.1**
>
> Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for topologies τ and τ' respectively on X. Then the following are equivalent.
>
> - τ' is finer than τ
>
> - for all $x \in X$ and for all $B \in \mathscr{B}$. if $x \in B$, then there exists $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$

Let $x \in X$ and $B \in \mathscr{B} \subseteq \tau$ so B is open with respect to τ, since τ' is finer than τ, we have that $B \in \tau'$, and therefor there is some $B' \in B'$ such that $x \in B' \subseteq B$.

Conversely let $U \in \tau$ be arbitrary. Since $U \in \tau$, we have that

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha$$

for some collection $B_\alpha \in \mathscr{B}$. We further have that each $B_\alpha = \bigcup_{\gamma \in \Lambda_\alpha} B'_\gamma$ with $B'_\gamma \in \mathscr{B}$. Then rewriting U yields

$$U = \bigcup_{\alpha \in \Lambda} \left(\bigcup_{\gamma \in \Lambda_\alpha} B'_\gamma \right)$$

Thus U is a union of elements from $\mathscr{B}'$, so $U \in \mathscr{B}'$ $\quad \square$

Recall from last lecture the topology generated by the bases of open disks and open rectangles of $\mathbb{R}^2$, using this proposition we may show that both topologies are equivalent, as we may always find sufficiently small circles and rectangles that fit within other circles and rectangles.

> **Example 2.2** (The Line)
>
> The underlying set is $\mathbb{R}$ and τ is generated by the open intervals (x, y)

> **Example 2.3** (Sorgenfrey Line)
>
> This is also commonly refered to as the Lower Limit Topology, the underlying set is $\mathbb{R}$ and τ is generated by clopen intervals $[x, y)$

> **Example 2.4** (K-Topology)
>
> The K-Topology has an underlying set of $\mathbb{R}$ and τ is generated by
>
> $$(x, y) \qquad\qquad\qquad\qquad (x, y) \backslash K$$
>
> where $K = \{\frac{1}{n} | n \in \mathbb{N}\}$

An exercise left by the prof is to show that the basis generating the K-topology is indeed a basis.

Definition 2.5 (Sub-Bases). A Sub-Bases on X is a collection, $\mathscr{S}$ of subsets of X such that $\mathscr{S}$ covers X

> **Lemma 2.6**
>
> The finite intersection of elements of a sub-bases, $\mathscr{S}$, for a topology on X is a bases for a topology calle dthe basis generated by $\mathscr{S}$

Let $\mathscr{B}$ bet he collection of finite intersection of elements of $\mathscr{S}$, to prove $\mathscr{B}$ is a basis we must check that it covers X, which is true because $\mathscr{S} \subseteq \mathscr{B}$ and $\mathscr{S}$ covers X. and we may find a basis element inside the intersection of two elements. to do this we let $B_1, B_2 \in \mathscr{B}$ be arbitrary, let $x \in B_1 \cap B_2$, by definition, we have that B_1 and B_2 are a finite intersection of elements from $\mathscr{S}$, so $B_1 \cap B_2$ is also a finite inetrsection of elements of $\mathscr{S}$, so picking $B_3 = B_1 \cap B_2 \in \mathscr{B}$, it follows that

$$x \in B_3 = B_1 \cap B_2 \subseteq B_1 \cap B_2$$

Thus verifying that it is a bases $\quad\square$