MAT327 - Introduction to Topology

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Contents

Ω1 Day 1: Introduction, Topological Spaces and Bases

This class is taught by Daniel Wilches Calderon during the Summer 2024 semester. We will mostly be following Topology by Munkres, with some suggested readings on the Quercus page

Ω1.1 Marking Scheme

1. Final Exam - 30%
2. Term Test - 20%
3. Quizzes - 50% (Best 8 of 10)

   Every week we will be given a problem set, and during tutorial one of those questions will be tested

Ω1.2 Open sets in the Real Numbers

Recall that $U \in \mathbb{R}$ is open if

$$\text{for all } x \in U, \text{ there exists } \varepsilon > 0 \text{ such that } x \in (x-\varepsilon, x+\varepsilon) \subseteq U$$

Open sets have some properties. That the arbitrary union and finite intersection of open sets is still open

1. Arbitrary Union

   Let $\{U_\alpha | \alpha \in \Lambda$[1]$\}$ be an arbitrary collection of open sets, if $x \in \bigcup_{\alpha\in\Lambda}$, then $x \in U_\alpha$ for some $\alpha \in \Lambda$, and since U_α is open, we have that there exists some $\varepsilon > 0$ such that

   $$x \in (x-\varepsilon, x+\varepsilon) \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$$

   This same result doesn't hold true for arbitrary intersection, but it does for finite.

2. Finite intersection

   Let $\{U_k | k \leqslant n\}$ be a finite collection of open sets. if $x \in \bigcap_{k\leqslant n} U_k$, then $x \in U_k$ for all $k \leqslant n$. Since each U_k is open, there are $\varepsilon_i > 0$ such that $x \in (x-\varepsilon_i, x+\varepsilon_i) \subset U_i$ for $i = 1, 2, \ldots, n$. Then pick $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, then it follows that

   $$x \in (x-\varepsilon, x+\varepsilon) \subseteq (x-\varepsilon_k, x+\varepsilon_k) \subseteq U_k$$

   for all $k \leqslant n$, so $(x-\varepsilon, x+\varepsilon) \subseteq \bigcap_{k\leqslant n} U_k$.

[1] We use Λ to denote an arbitrary indexing set

Ω1.3 Topological Spaces

Definition 1.1 (Topological Space)**.** A toplogy, τ, on X is a collection of subsets of X that satisfy

1. $X, \varnothing \in \tau$
2. τ is closed under arbitrary Union
3. τ is closed under finite intersection

The double (X, τ) is called a Topological Space. Furthermore, the elements of τ are called open sets, and the elements of X are called points.

We first notice that showing the intersection of 2 sets of τ is still in τ is equivalent to showing the finite intersection of sets of τ will still be in τ. This is true by an inductive argument.

The following are some examples of topologies on a set with 2 elements, the prof briefly went over the topologies on a set with 3 elements, but basically left it as an exercise.

Example 1.2 (Topologies on 2 Point Sets)

Let $X = \{x, y\}$, we have the following are possible topologies on X

1. $\{X, \varnothing\}$

   This is often called the indiscrete/trivial topology

2. $\{X, \varnothing, \{x\}, \{y\}\}$

   This is often called the discrete topology

3. Sierpinski Spaces

   The main characteristic of these spaces are that one point is open, while the other is not. More explicitly, the following are the Sierpinski Spaces

   a) $\{X, \{x\}, \varnothing\}$

   b) $\{X, \{y\}, \varnothing\}$

You might've noticed that we labelled some of those examples as the Discrete Topology and the Indiscrete/Trivial Topology.

Definition 1.3 (Discrete and Indiscrete Topology)**.** Given a set X, the discrete topology on X is equal to the Power Set of X, and the indiscrete topology is equal to $\{x, \varnothing\}$

Definition 1.4 (Cofinite Topology)**.** Let X be a set, the Cofinite Topology on X is the collection

$$\tau = \{U \subseteq X | X \backslash U \text{ is finite}\} \cup \{\varnothing\}$$

Now to prove that this is a topology, we must prove 3 things. The entire set and the empty set exist within τ. τ is closed under arbitrary union and finite intersection. The first of these is almost immediate as $\varnothing \in \tau$ by definition, and $X \backslash X = \varnothing$, which is a finite set.

1. Arbitrary Union
   Let $\{U_\alpha | \alpha \in \Lambda\}$ be open subsets of X, we have that $X \backslash U_\alpha$ is open. Then notice that
   $$X \backslash \left(\bigcup_{\alpha \in \Lambda} U_\alpha \right) = \bigcap_{\alpha \in \Lambda} (X \backslash U_\alpha)$$
   by DeMorgan's, which is an intersection of finite sets, so it must also be finite.

2. Finite Intersection
   Let $\{U_k | k \leqslant n\}$ be open, we have that $X \backslash U_k$ are finite, so then we notice that
   $$X \backslash \left(\bigcap_{k \leqslant n} U_k \right) = \bigcup_{k \leqslant n} (X \backslash U_k)$$
   By DeMorgan's, which is a finite union of finite sets, which implies that it is finite, implying it must be in τ

Definition 1.5 (Cocountable Topology)**.** The Cocountable Topology is defined synonymously to the cofinite topology. It is the collection
$$\tau = \{U \subseteq X | X \backslash U \text{ is countable } \} \cup \{\varnothing\}$$
Prove that this is a topology

Definition 1.6. Let τ, τ' be topologies on X, if $\tau \subseteq \tau'$, then we say

- τ is coarser than τ'
- τ' is finer than τ

As a more general statement, if $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$, then we say that τ and τ' are compareable.

Ω1.4 Bases

These are very similiar to the bases you're probably familiar with from linear algebra, where the linear combinations form the vector space, where as these bases are sets that "Unionize" to form the topological space.

Definition 1.7 (Bases)**.** A bases for a topology on a set X is a collection fo subsets of X, $\mathscr{B}$ such that

- for all $x \in X$, there exists $B \in \mathscr{B}$ such that $x \in B$
- if $x \in B_1 \cap B_2$, then there exists $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

Lemma 1.8 (Generated Topologies)

Let $\mathscr{B}$ be a basis of X, then the collection of all subsets U of X such that for all $x \in U$ there exists $B \in \mathscr{B}$ such that $x \in B \subseteq U$ is a topology on X. This is called the Generated Topology

Proof. We now need to prove that this is indeed a topology. We first have that $\varnothing \in \tau$ for vacuous reasons, then we also have that $X \in \tau$ because by letting $x \in X$ be arbitrary, we have that, by definition, there exists a $B \in \mathscr{B}$ such that $x \in B \subseteq X$.

1. Union

   Let $\{U_\alpha | \alpha \in \Lambda\}$ be some collection of open sets, let $x \in \bigcup_{\alpha\in\Lambda} U_\alpha$, then there is some $\alpha \in \Lambda$ such that $x \in U_\alpha$, then let $B \in \mathscr{B}$ be such that $x \in B \subseteq U_\alpha \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$

2. Intersection

   Let $U_1, U_2 \in \tau$, let $x \in U_1 \cap U_2$, since $U_1, U_2 \in \tau$, there exists B_1, B_2 such that

   - $x \in B_1 \subseteq U_1$
   - $x \in B_2 \subseteq U_2$

   so there must exist B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$

$\square$

Example 1.9 (Circles)

The bases of open disks in $\mathbb{R}^2$

$$\tau = \{\{x \in \mathbb{R}^2 | d(x, y) < \varepsilon\} | y \in \mathbb{R}^2, \varepsilon > 0\}$$

Example 1.10 (Rectangles)

$$\tau = \{(a_1, b_1) \times (a_2, b_2) | a_1, b_1, a_2, b_2 \in \mathbb{R}\}$$

Lemma 1.11

For subset $\mathscr{B}$ of τ on X, The following are equivalent

- $\mathscr{B}$ is a basis generating τ
- every non-empty element of τ is a union of elements of $\mathscr{B}$

Proof. Letting (X, τ) be a topological space, we begin with the forwards direction. Let $U \in \tau$ and let $x \in U$, since τ is generated by $\mathscr{B}$, there exists some $B_x \in \mathscr{B}$ such that $x \in B_x \subseteq U$, so it follows that

$$U = \bigcup_{x \in U} B_x$$ [2]

Conversely, let $x \in X$ be arbitrary. Since $X \in \tau$, $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ with $B_\alpha \in \mathscr{B}$ it follows that $x \in B_\alpha$ for some $\alpha \in \Lambda$. Now let $B_1, B_2 \in \mathscr{B}$ with x existing in their intersection. We first have that $B_1 \cap B_2$ is open, so we may represent it as a union,

$$B_1 \cap B_2 = \bigcup_{\alpha \in \Lambda} B_\alpha$$

so $x \in B_\alpha$ for some $\alpha \in \Lambda$. so $x \in B_\alpha \subseteq B_1 \cap B_2$ $\square$

[2] This equality wasn't immediately obvious to me why this is true, but you can prove it with double inclusion

Ω2 Day 2: Bases and Sub-bases

Proposition 2.1

Let $\mathscr{B}$ and $\mathscr{B}'$ be bases for topologies τ and τ' respectively on X. Then the following are equivalent.

- τ' is finer than τ
- for all $x \in X$ and for all $B \in \mathscr{B}$. if $x \in B$, then there exists $B' \in \mathscr{B}'$ such that $x \in B' \subseteq B$

Proof. Let $x \in X$ and $B \in \mathscr{B} \subseteq \tau$ so B is open with respect to τ, since τ' is finer than τ, we have that $B \in \tau'$, and therefor there is some $B' \in B'$ such that $x \in B' \subseteq B$.

Conversely let $U \in \tau$ be arbitrary. Since $U \in \tau$, we have that

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha$$

for some collection $B_\alpha \in \mathscr{B}$. We further have that each $B_\alpha = \bigcup_{\gamma \in \Lambda_\alpha} B'_\gamma$ with $B'_\gamma \in \mathscr{B}$. Then rewriting U yields

$$U = \bigcup_{\alpha \in \Lambda} \left(\bigcup_{\gamma \in \Lambda_\alpha} B'_\gamma \right)$$

Thus U is a union of elements from $\mathscr{B}'$, so $U \in \mathscr{B}'$ □

Recall from last lecture the topology generated by the bases of open disks and open rectangles of $\mathbb{R}^2$, using this proposition we may show that both topologies are equivalent, as we may always find sufficiently small circles and rectangles that fit within other circles and rectangles.

Example 2.2 (The Line)

The underlying set is $\mathbb{R}$ and τ is generated by the open intervals (x, y)

Example 2.3 (Sorgenfrey Line)

This is also commonly refered to as the Lower Limit Topology, the underlying set is $\mathbb{R}$ and τ is generated by clopen intervals $[x, y)$

Example 2.4 (K-Topology)
The K-Topology has an underlying set of $\mathbb{R}$ and τ is generated by

$$(x, y) \qquad\qquad (x, y)\backslash K$$

where $K = \{\frac{1}{n}|n \in \mathbb{N}\}$

An exercise left by the prof is to show that the basis generating the K-topology is indeed a basis.

Ω2.1 Sub-Bases

Definition 2.5 (Sub-Bases)**.** A Sub-Bases on X is a collection, $\mathscr{S}$ of subsets of X such that $\mathscr{S}$ covers X

Lemma 2.6
The finite intersection of elements of a sub-bases, $\mathscr{S}$, for a topology on X is a bases for a topology calle dthe basis generated by $\mathscr{S}$

Proof. Let $\mathscr{B}$ bet he collection of finite intersection of elements of $\mathscr{S}$, to prove $\mathscr{B}$ is a basis we must check that it covers X, which is true because $\mathscr{S} \subseteq \mathscr{B}$ and $\mathscr{S}$ covers X. and we may find a basis element inside the intersection of two elements. to do this we let $B_1, B_2 \in \mathscr{B}$ be arbitrary, let $x \in B_1 \cap B_2$, by definition, we have that B_1 and B_2 are a finite intersection of elements from $\mathscr{S}$, so $B_1 \cap B_2$ is also a finite inetrsection of elements of $\mathscr{S}$, so picking $B_3 = B_1 \cap B_2 \in \mathscr{B}$, it follows that

$$x \in B_3 = B_1 \cap B_2 \subseteq B_1 \cap B_2$$

Thus verifying that it is a bases □

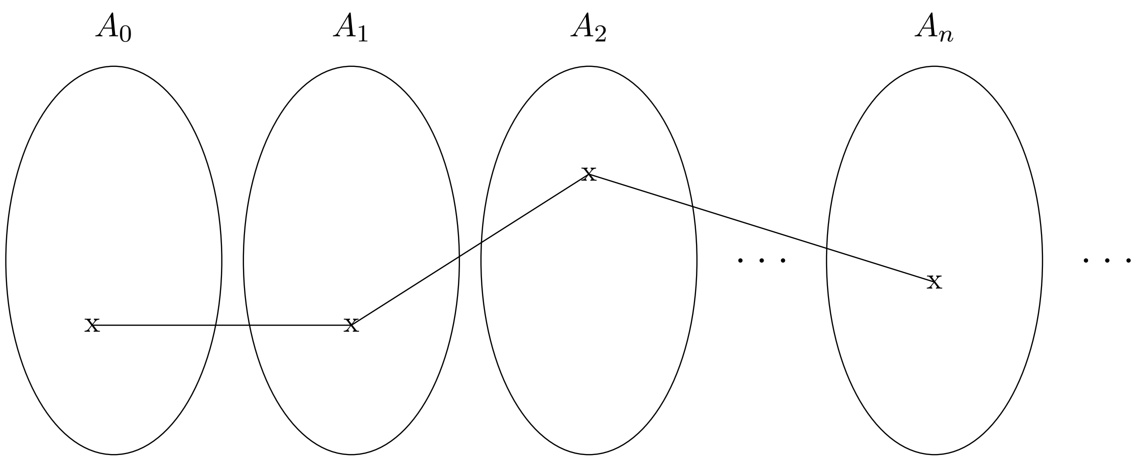
Ω3 Day 3: Axiom of Choice, Zorn's Lemma and Ordering

Ω3.1 Axiom of Choice

Theorem 3.1 (Axiom of Choice)
if $\{A_\alpha | \alpha \in \Lambda\}$ is a collection of non-empty sets, then their product

$$\prod_{\alpha \in \Lambda} A_\alpha$$

is non-empty as well



Remark 3.2. The Axiom of Choice is used to pick from a **non-finite** product of non-empty sets.

Picking from a finite product is pretty straight forward, take $\mathbb{R}^n$ as an example.

Ω3.2 Orderings of Sets

Definition 3.3 (Partial Order)**.** A Partial Order, $\leqslant$, on a set P, is a binary relation that satisfies

1. (Reflexitivity) for all $p \in P$, $p \leqslant p$
2. (Anti-Symmetry) for all $p, q \in P$ if $p \leqslant q$ and $q \leqslant p$, then $p = q$
3. (Transitivity) for all $p, q, r \in P$ if $p \leqslant q$ and $q \leqslant r$, then $p \leqslant r$

For any $p, q \in P$ if either $p \leqslant q$ or $q \leqslant p$ is always true, then we say that $\leqslant$ is total[3]

[3] or linear, but I will usually be using total

Let $(P, \leqslant)$ be a partially ordered set,

1. $p \in P$ is maximal if there is no q such that $p < q$
2. $q \in P$ is minimal if there is no p such that $p < q$
3. $Q \subset P$ is bounded, if there exists an $r \in P$ such that $q \leqslant r$ for all $q \in Q$
4. $C \subseteq P$ is a Chain if the restriction of $\leqslant$ to C is total

Further more we say that a totally ordered set P is Well-Ordered if every non-empty subset has a minimal element.

Theorem 3.4 (Well-Ordering Principle)
Every non-empty set is well-orderable

Proof. The proof of this requires the Axiom of Choice, the proof wasn't covered in class but you're interested heres a link □

Proposition 3.5
In a well ordered set S, every element, which is not maximal, has an immediate successor

> **Remark 3.6.** The intuition for this is that given a non maximal element, x, there exists nothing between x and it's successor. or more formally, for $x \in S$, there exists $y \in S$ such that $x < y$, and if $x < z$ it follows that $y \leqslant z$

Proof. let $x \in S$ be an arbitrary non-maximal element, then $\{z \in S | x < z\} = A$ is a non-empty set. By Well-Ordering, let $y = \min A$, then we have that $y > x$ and for all $z \in A$, $y \leqslant z$, by definition. □

Ω3.3 Zorn's Lemma

Lemma 3.7 (Zorn's Lemma)
Every non-empty partially ordered set in which every chain is bounded has a maximal element.

The proof for this is rather complicated and technical, much too far out of the scope for this course, but if you're interested you can look it up here. We're asked to prove a similiar result on the problem set, but we've been given the extra hypothesis of that every chain is finite. A rather surprising result is that in some axiomatic settings, Zorn's Lemma, the axiom of choice, and the well-ordering principle are all equivalent. You can again read about it here.

Ω3.4 Order Topologies

Definition 3.8 (Order Topology)**.** Given an ordered set $(X, <)$ with at least 2 elements, the order topology on X is the topology generated by the basis consisting of intervals

1. $(x, y) = \{z \in X | x < z < y\}$
2. $[x_{min}, y) = \{z \in X | x_{min} \leqslant z < y\}$
3. $(x, y_{max}] = \{z \in X | x < z \leqslant y_{max}\}$

We notice that the minimum and maximum might not exist. If either exists we add 2 and 3 to our basis accordingly

Before we delve into some examples, it is important to recall that a set X is countable if there exists a surjection

$$f : \mathbb{N} \to X$$

We require that it is surjective so that finite sets are also countable.[4]

Example 3.9 (The Line)

The underlying set is $\mathbb{R}$, with the usual order. The order topology is generated by the open intervals of $\mathbb{R}$, as we have that $\mathbb{R}$ does not have a maximum

Example 3.10 (The Natural Numbers)

The underlying set are the Natural Numbers. the order is as usual. We notice that $\mathbb{N}$ isn't generated by the open intervals, because 0 doesn't exist within any open interval. But we have that the basis for the order topology of $\mathbb{N}$ is indeed a basis, because we first may let $n \in \mathbb{N}$ be arbitrary, if $n = 0$ then $n \in [0, 1)$, otherwise we have that $n \in (n - 1, n + 2)$, which are both sets that are in the basis. We further have that the order and discrete topologies on $\mathbb{N}$ are equal.

Example 3.11 (Two Copies of the Natural Numbers)

The underlying set is $\{0, 1\} \times \mathbb{N}$, and the order is the lexicographic order. We actually have that every point in this topology is open, apart from $\langle 1, 0 \rangle$

[4] It is convention that the empty set is also countable

Example 3.12 (The Lexicographic Plane)

The underlying set is $\mathbb{R}^2$ and the order relation is given by

$$\langle a, b\rangle^{a} <_{lex} \langle c, d\rangle \text{ if either}$$

1. $a <_{\mathbb{R}} c$
2. $a = b$ and $b <_{\mathbb{R}} d$

[a] We use this notation to represent a tuple, to avoid confusion with intervals

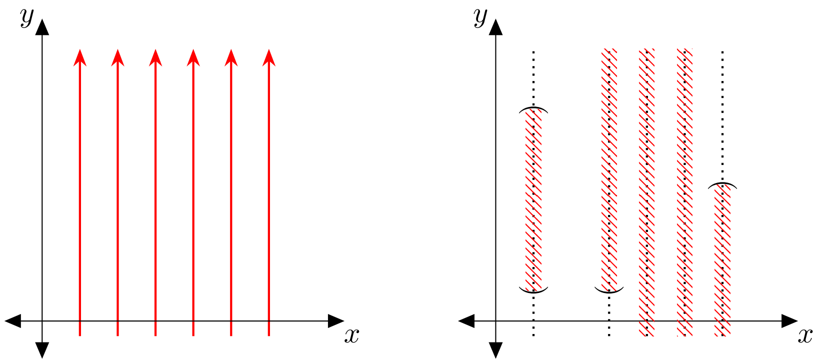


Figure 1: The arrows in the left diagram are to represent the order, and the shaded regions on the right are the open sets.

Example 3.13 (ω_1, The Smallest Uncountable Ordinal)

This is an uncountable, well ordered set such that $\forall \alpha \in \omega_1$, the set $\{\gamma | \gamma < \alpha\}$ is countable

Theorem 3.14

ω_1 exists

Proof. Let X be an uncountable set and let $\leqslant$ be a well ordering on X. Then consider the lexicographic order in $\{0, 1\} \times X$. We have the following two facts

- $\{x \in \{0, 1\} \times X | \{y \leqslant_{lex} x\} \text{ is uncountable}\}$ is non empty.
  This is true because $\langle 1, \min X\rangle$ exists within the set
- By Well-Ordering, there exists a minimum, x, with uncountably many predecessors, so the set of predecessors of x is uncountable.

□

Ω4 Day 4: Finite Products, Product Topology and Projections

Ω4.1 Finite Product and Product Topology

First recall that the Cartesian Product between two sets, X, Y is defined to be

$$X \times Y = \{\langle x, y\rangle | x \in X, y \in Y\}$$

Then we further define the Product topology as follow,

Definition 4.1 (Product Topology)**.** Let $(X, \tau), (Y, \mu)$ be topological spaces, the Product Topology on $X \times Y$ is the topology generated by the basis

$$\mathscr{B} = \{U \times V | U \in \tau \text{ and } V \in \mu\}$$

Proposition 4.2

$\mathscr{B}$ is a basis for a toplogy

Proof. We first begin by proving that $\mathscr{B}$ covers $X \times Y$, let $\langle x, y\rangle \in X \times Y$ be arbitrary, we notice that $X \in \tau$ and $Y \in \mu$, so we have that $\langle x, y\rangle \in X \times Y \in \mathscr{B}$.

Now let $B_1, B_2 \in \mathscr{B}$, by definition we have $B_1 = U_1 \times V_1$ and $B_2 = U_2 \times V_2$ for suitable sets. Let $\langle x, y\rangle \in (U_1 \times V_1) \cap (U_2 \times V_2)$. Then we notice that

$$\langle x, y\rangle \in (U_1 \cap U_2) \times (V_1 \cap V_2) \subseteq (U_1 \times V_1) \cap (U_2 \times V_2)$$

□

Theorem 4.3

If $\mathscr{B}$ and $\mathscr{S}$ are bases for X and Y, respectively. Then the set

$$\mathscr{D} = \{B \times C | B \in \mathscr{B}, C \in \mathscr{S}\}$$

is a basis for the product topology

Proof. We wish to show that it indeed is a basis, and that it does generate the product topology.

We begin by showing that it covers. Let $\langle x, y\rangle \in X \times Y$ be arbitrary, we then notice two things

1. Since $\mathscr{B}$ is a basis for X, there exists $B \in \mathscr{B}$ such that $x \in B$
2. Since $\mathscr{S}$ is a basis for Y, there exists $S \in \mathscr{S}$ such that $y \in S$

So we have $\langle x, y\rangle \in B \times S \in \mathscr{D}$

Further let $B_1 \times C_1, B_2 \times C_2 \in \mathscr{D}$, let $\langle x, y\rangle \in (B_1 \times C_1) \cap (B_2 \times C_2)$. Notice that since B_1 and B_2 belong to the same topological space, and that x exists in their intersection, we have the existence of some $B_3 \in \mathscr{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. We may similarly conclude the existence of a C_3. Thus we have

$$\langle x, y\rangle \in B_3 \times C_3 \subseteq (B_1 \times C_1) \cap (B_2 \times C_2)$$

Now that we have shown that $\mathscr{D}$ is indeed a bases, we are left to show that it generates the product topology. Let $U \in X$ and $V \in Y$ be open. We may rewrite them as

$$U = \bigcup_{\alpha\in\Lambda} B_\alpha, B_\alpha \in \mathscr{B} \text{ and } V = \bigcup_{\gamma\in\Pi} C_\gamma, C_\gamma \in \mathscr{S}$$

Thus it follows That

$$\begin{aligned} U \times V &= \bigcup_{\alpha\in\Lambda} B_\alpha \times \bigcup_{\gamma\in\Pi} C_\gamma \\ &= \bigcup_{\alpha\in\Lambda,\gamma\in\Pi} B_\alpha \times C_\gamma \end{aligned}$$

□

Proposition 4.2 and Theorem 4.3 basically say that we can mix and match topologies and bases. An example of this is the plane

Example 4.4 (The Plane)

Given $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ with each copy of the real line endowed with the usual topology, we have that the topology of $\mathbb{R}^2$ is generated by open rectangles.

Ω4.2 Projections

Definition 4.5 (Projection)**.** The functions $\pi_1 : X \times Y \twoheadrightarrow$ [5] X and $\pi_2 : X \times Y \twoheadrightarrow Y$ are given by

$$\pi_1\langle x, y\rangle = x \qquad\qquad \pi_2\langle x, y\rangle = y$$

are called the projections of $X \times Y$ onto X and Y respectively.

[5] This arrow is to symbolize that the function is surjective

Proposition 4.6

Given topological spaces, (X, τ) and (Y, μ), the collection

$$\mathscr{S} = \{\pi_1^{-1}(U), \pi_2^{-1}(V) | U \in X, V \in Y\}$$

form a subbases for the product topology on $X \times Y$

Remark 4.7. The pre-image of U and V are actually elements of $X \times Y$, this kinda confused me in lecture.

Proof. We begin by showing covering, we first notice that the preimage of X is necessarily $X \times Y$, thus we have covering.

Now to show that $\mathscr{S}$ does generate the product topology. Let $U \in \tau, V \in \mu$, we want to express $U \times V$ as a finite intersection of elements of $\mathscr{S}$. So we may notice that

$$U \times V = \pi_1^{-1}(U) \cap \pi_2^{-1}(V)$$

$\square$

Ω5 Day 5: Subspaces, Closed sets, Interiors and Closures

Ω5.1 Subspaces

Definition 5.1. Letting (X, τ) be a topological space, with $Y \subseteq X$, we define the Subspace Topology on Y to be

$$\tau_Y := \{U \cap Y | U \in \tau\}$$

We often refer to such subsets of X as a *Subspace*

Proposition 5.2

If $\mathscr{B}$ is a bases for the topology on X, and $Y \subseteq X$, then then collection

$$\mathscr{B}_Y := \{B \cap Y | B \in \mathscr{B}\}$$

is a basis for the subspace topology on Y

It's kind of like how in Linear Algebra you can take vectors from a basis of V and construct a basis for subspace U.

Proof. Let X be a topological space and $\mathscr{B}$ be a basis for τ. Let $U \cap Y$ be an open subset of Y, since we have that $\mathscr{B}$ is a basis for τ, it follows that there exists $\{B_\alpha | \alpha \in \Lambda\}$ such that

$$U = \bigcup_{\alpha \in \Lambda} B_\alpha$$

So it follows that

$$U \cap Y = \left(\bigcup_{\alpha \in \Lambda} B_\alpha \right) \cap Y = \bigcup_{\alpha \in \Lambda} (B_\alpha \cap Y)$$

□

Remark 5.3. As a side tangent, what does it mean to be open?

Example 5.4

We notice that in $[0, 1)$ is open in $[0, 1)$[a], but is isn't open in $\mathbb{R}$.

[a] The subspace topology

More generally, if Y is a subspace of X and $U \subset Y$ is open in Y, that doesn't imply that U is open in X. But with the stronger hypothesis that if Y is open in X and U is open in Y, we also have that U is open in X.

Remark 5.5. My intuition on why this is true is that the finite intersection between open sets in X is still in open set in X, and since bases elements are necessarily open, the claim immediately follows.

Theorem 5.6

If A and B are subspaces of X and Y respectively, then the product topology on $A \times B$ is eqaul to the topology on $A \times B$ as a subspace of $X \times Y$

Proof. Letting $U \times V$ be a bases element of $X \times Y$, it follows that $(X \times Y) \cap (A \times B)$ is a bases element of the subspace topology on $A \times B$, now we notice that

$$(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$$

And notice that $(U \cap A) \times (V \cap B)$ is a bases element for the product topology on $A \times B$, so we have that the bases for the product and subspace topology must be equal, implying that their topologies must also be the same. □

Remark 5.7. But orders and subspace topologies don't behave as nicely as this.

Example 5.8

I'm gonna need someone to send me the notes for this

Ω5.2 Closed Sets

Definition 5.9. Subset C of a topological space X is closed if its complement $X \backslash C$ is open.

Remark 5.10. Being closed and open aren't mutually exclusive properties, in any topological space, X, we always have that X and the empty set are always open and closed. And if you remember $Q10$ from the first problem set, it was shown that every open set is also actually closed.

Example 5.11 (Cofinite Topology)

Recall the cofinite topology on X, we have that the closed sets are X and all finite subsets of X

Example 5.12 (Discrete Topology)

In this case, the closed sets are all the open sets

Theorem 5.13

The collection of closed subsets of a topological space X satisfy

1. $\varnothing$ and X are in it
2. It is closed under finite unions
3. it is closed under arbitrary intersection

Proof. This is going to be a sketch of the proof, mostly because I'm kinda lazy. But 1. is immediate from definition, then you basically abuse DeMorgan's to prove 2. and 3. □

Proposition 5.14

Let Y be a subspace of X. Then a subset $A \subseteq Y$ is closed in Y if and only if, it is the intersection of closed subsets of X with Y

Proof. Assume that A is closed in Y, it follows that A^C is open in Y, so by definition, it must be equal to the intersection of an open set $U \subseteq X$ with Y, it follows that U^C is closed in X, and $A = Y \cap (U^C)$, so we have the desired result

Conversely, assume that $A = C \cap Y$, where C is closed in X, then C^C is open in X. We also have that $C^C \cap Y$ is open in Y, which is by definition of subspace topology. Then notice $C^C \cap Y = Y - A$, so it follows that $Y - A$ is open in Y, so A is closed in Y □

Proposition 5.15

Let Y be a subspace of X, if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. someone send me the proof of this plz □

Ω5.3 Interiors and Closures

Definition 5.16 (Interiors and Closures)**.** Let A be a subset of a topological space (X, τ)

- The interior of A is defined as

$$Int(A) = A^{\circ} = \bigcup \{U \subseteq X | U \text{ is open and } U \subseteq A\}$$

- The Closure of A is defined as

$$Cl(A) = \overline{A} = \bigcap \{C \subseteq X | C \text{ is closed and } A \subseteq C\}$$

Clearly we have that the interior is open and the closure is closed, and

$$A^o \subseteq A \subseteq \overline{A}$$

Proposition 5.17

Let Y be a subspace of X, and let $A \subseteq Y$. then

$$\mathrm{Cl}_Y(A) = \mathrm{Cl}_X(A) \cap Y$$

Proof.

$$\begin{aligned}
\mathrm{Cl}_Y(A) &= \bigcap\{C \subseteq Y | C \text{ is closed in } X \text{ and } A \subseteq C\} \\
&= \bigcap\{C \cap Y | C \text{ is closed in } X \text{ and } A \subseteq C\} \\
&= Y \cap \left(\bigcap\{C | C \text{ is closed in } X \text{ and } A \subseteq C\}\right) \\
&= Y \cap \mathrm{Cl}_X(A)
\end{aligned}$$

□

Definition 5.18. An open neighbourhood of a point X in a topological space X is an open set U such that $x \in U$

Proposition 5.19

Let X be a topological space and let $\mathscr{B}$ be a bases for the topology of X, if $A \subseteq X$, and $x \in X$, then the following are equivalent

1. $x \in \overline{A}$
2. Every open neightbourhood of x intersects A
3. Every basic open neighbourhood of x intersects A

Proof. Will do this later, I need to study for midterm □

Example 5.20 (Intervals)

Will do this later, I need to study for midterm

Example 5.21 (Points)
Will do this later, I need to study for midterm

Example 5.22 (The Integers)
Will do this later, I need to study for midterm

Example 5.23 (The Rationals)
Will do this later, I need to study for midterm