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An Introductory Course on Mathematical Game Theory

Julio González-Díaz
Ignacio García-Jurado
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10 9 8 7 6 5 4 3 2 1 15 14 13 12 11 10

to our families

Contents

Preface

This is a mathematical book on game theory and, thus, starts with a definition of game theory. This introduction also provides some preliminary considerations about this field, which mainly lies in the boundary between mathematics and economics. Remarkably, in recent years, the interest for game theory has grown in utterly disparate disciplines such as psychology, computer science, biology, and political science.

A definition of game theory. Game theory is the mathematical theory of interactive decision situations. These situations are characterized by the following elements: (a) there is a group of agents, (b) each agent has to make a decision, (c) an outcome results as a function of the decisions of all agents, and (d) each agent has his own preferences on the set of possible outcomes. Robert J. Aumann, one of the most active researchers in the field, said in an interview with Hart (2005): "game theory is optimal decision making in the presence of others with different objectives".

A particular collection of interactive decision situations are the so-called parlor games. Game theory borrows the terminology used in parlor games to designate the various elements that are involved in interactive decision situations: the situations themselves are called *games*, the agents are called *players*, their available decisions are called *strategies*, etc.

Classical game theory is an ideal normative theory, in the sense that it prescribes, for every particular game, how rational players should behave. By rational player we mean one who (a) knows what he wants, (b) has the only objective of getting what he wants, and (c) is able to identify the strategies that best fit his objective. More recently, a normative game theory for bounded-rational players and even a pure descriptive game theory have been developed. However, this book lies mostly inside the borders of classical game theory.

Noncooperative and cooperative models. Game theory is concerned with both noncooperative and cooperative models. The differences between these two types of models are explained, for instance, in van Damme and Furth (2002). They state that "noncooperative models assume that all the possibilities for cooperation have been included as formal moves in the game, while cooperative models are 'incomplete' and allow players to act outside of the detailed rules that have been specified". Another view comes from Serrano (2008): "One clear advantage of the (noncooperative) approach is that it is able to model how specific details of the interaction may impact the final outcome. One limitation, however, is that its predictions may be highly sensitive to those details. For this reason it is worth also analyzing more abstract approaches that attempt to obtain conclusions that are independent of such details. The cooperative approach is one such attempt". The necessity of cooperative models has been perceived by game theorists since the very beginning because in some situations the cooperation mechanisms are too complex as to be fully described by a mathematical model. Thus, cooperative game theory deals with coalitions and allocations, and considers groups of players willing to allocate the joint benefits derived from their cooperation (however it takes place). On the other hand, noncooperative game theory deals with strategies and payoffs, and considers players willing to use the strategies that maximize their individual payoffs.

Some stepping stones in the history of game theory. Most likely, the first important precursor of game theory was Antoine Augustin Cournot and his analysis of duopoly published in 1838. In Cournot (1838), the author studied a concept very close to Nash equilibrium for a duopoly model. At the beginning of the 20th century, some important mathematicians like Zermelo and Borel became interested in parlor games and in two-player zero-sum games.

However, the first important result of game theory was the minimax theorem proved by Hungarian mathematician John von Neumann in 1928. Some years later, von Neumann came back to interactive decision situations when he met the Austrian economist Oskar Morgenstern in Princeton. In 1944 they published their famous book, "The Theory of Games and Economic Behavior", which is considered to be the seminal work of game theory. Since then, game theory has advanced as a result of the cooperation between mathematicians and economists.

In 1950, John Nash published his first paper on the equilibrium concept. Nowadays, Nash equilibrium and, in general, equilibrium theory, which started from that concept, play a central role in the development of social sciences and, especially, of economic theory. In 1994, John Nash, John Harsanyi and Reinhard Selten were awarded the Nobel Prize in Economics "for their pioneering analysis of equilibria in the theory of noncooperative games". More recently, other game theorists have also received this award. In 2005 the prize was awarded to Robert Aumann and Thomas Schelling "for having enhanced our understanding of conflict and cooperation through game-theory analysis". In 2007 the prize went to L. Hurwicz, E. Maskin, and R. Myerson "for having laid the foundations of mechanism design theory" (through game theory). These awards illustrate the role of game theory as the main mathematical tool to analyze social interactions, where economics is the major field of application. This interaction between mathematics and social sciences continuously produces challenging problems for the scientific community.

In 1999, the Game Theory Society (GTS) was created to promote the research, teaching, and application of game theory. On its web site, the GTS provides resources related to game theory such as software tools, journals, and conferences. Periodically, the GTS organizes a world conference on game theory; the first one was held in Bilbao (Spain) in 2000.

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The authors maintain a web page that will be periodically updated to include the corrections of typos and other errors that we may be aware of. The web page is hosted by the AMS at: www.ams.org/bookpages/gsm-115/. In this respect, we appreciate any comments from our readers to improve this book. You may contact us via email at julio.gonzalez@usc.es.

Chapter 1

Introduction to Decision Theory

1.1. Preliminaries

In this chapter we provide a brief introduction to mathematical decision theory for problems with one *decision maker*, commonly known as *decision theory*. It lays the basis to develop a mathematical decision theory for situations in which several decision makers interact, *i.e.*, the basis to develop game theory. We would like to warn the reader that this is an instrumental chapter and, therefore, it is just a concise introduction. As a consequence, an unfamiliar reader may miss the kind of motivations, interpretations, and examples that appear in the rest of the book.[1]

Binary relations are fundamental elements of decision theory. Given a set A, every subset R of $A \times A$ is a binary relation over A. For each pair $a, b \in A$, we denote $(a, b) \in \mathrm{R}$ by $a \; \mathrm{R} \; b$ and, similarly, we denote $(a, b) \notin \mathrm{R}$ by $a \not\mathrm{R} \; b$.

Decision theory deals with *decision problems*. In a decision problem there is a *decision maker* who has to choose one or more *alternatives* out of a set A. The decision maker has *preferences* over A, which are usually modeled through a binary relation $\succeq \subset A \times A$, referred to as *preference relation* in this context. For each pair $a, b \in A$, $a \succeq b$ is interpreted as "the decision maker either prefers a over b or is indifferent between a and b". Two standard requirements are normally imposed on $\succeq$: i) $\succeq$ is *complete*, *i.e.*, for each pair $a, b \in A$, $a \succeq b$ or $b \succeq a$ (or both) and ii) $\succeq$ is *transitive*, *i.e.*, for each triple

[1]We refer the reader to Kreps (1988) and Mas-Colell et al. (1995) for deeper treatments of decision theory.

$a, b, c \in A$, if $a \succeq b$ and $b \succeq c$, then $a \succeq c$. A complete and transitive binary relation over a set A is referred to as a *weak preference* over A.

Definition 1.1.1. A *decision problem* is a pair $(A, \succeq)$, where A is a set of alternatives and $\succeq$ is a weak preference over A.

It is useful to introduce two more binary relations for any given decision problem $(A, \succeq)$: the *strict preference* $\succ$ and the *indifference* $\sim$. They are defined, for each pair $a, b \in A$, as follows:

- $a \succ b$ if and only if $b \not\succeq a$.
- $a \sim b$ if and only if $a \succeq b$ and $b \succeq a$.

There are other standard properties to look at when working with binary relations. A binary relation R on a set A is *reflexive* if, for each $a \in A$, $a \mathrel{\mathrm{R}} a$; it is *symmetric* if, for each pair $a, b \in A$, $a \mathrel{\mathrm{R}} b$ implies that $b \mathrel{\mathrm{R}} a$; it is *asymmetric* if, for each pair $a, b \in A$, $a \mathrel{\mathrm{R}} b$ implies that $b \not\mathrel{\mathrm{R}} a$; and it is *antisymmetric* if, for each pair $a, b \in A$, $a \mathrel{\mathrm{R}} b$ and $b \mathrel{\mathrm{R}} a$ together imply that $a = b$.

The following proposition presents some notable properties of the strict preference and the indifference relations associated with a decision problem. Its proof is immediate and it is left to the reader.

Proposition 1.1.1. *Let $(A, \succeq)$ be a decision problem. Then,*

i) *The strict preference, $\succ$, is asymmetric and transitive.*
ii) *The indifference, $\sim$, is an* equivalence relation, i.e.*, it is reflexive, symmetric, and transitive.*

Moreover, for each triple $a, b, c \in A$,

iii) *If $a \succ b$ and $b \sim c$, then $a \succ c$.*
iv) *If $a \sim b$ and $b \succ c$ then $a \succ c$.*
v) *Either $a \succ b$, or $b \succ a$, or $a \sim b$ (one and only one holds).*

We now show that, to each decision problem $(A, \succeq)$, we can associate a new one whose weak preference is antisymmetric. For each $a \in A$, let $I(a) := \{b \in A : b \sim a\}$. Take $(A/\sim, \succeq_a)$, where $A/\sim := \{I(a) : a \in A\}$ is the partition of A defined by $\sim$, and $\succeq_a$ is given, for each pair $I, J \in A/\sim$, by

- $I \succeq_a J$ if and only if, for each $a \in I$ and each $b \in J$, $a \succeq b$.

So defined, $(A/\sim, \succeq_a)$ is a well-defined decision problem (in the sense that $\succeq_a$ is complete and transitive) and, moreover, $\succeq_a$ is antisymmetric.

Dealing with preference relations in general can be difficult. Due to this, some alternative representations are commonly used. We devote the rest of this chapter to discussing the two main types of representations: ordinal utility and linear utility.

1.2. Ordinal Utility

Definition 1.2.1. Let $(A, \succeq)$ be a decision problem. A *utility function* representing $\succeq$ is a function $u\colon A \to \mathbb{R}$ satisfying, for each pair $a, b \in A$, that $a \succeq b$ if and only if $u(a) \geq u(b)$.

Note that an equivalent condition for u to be a utility function representing $\succeq$ is the following: for each pair $a, b \in A$, $a \succ b$ if and only if $u(a) > u(b)$. Besides, if u is a utility function representing $\succeq$, then, for each pair $a, b \in A$, $a \sim b$ if and only if $u(a) = u(b)$.

Next, we discuss two examples in which we analyze two particular decision problems from the point of view of ordinal utility.

Example 1.2.1. Let $(A, \succeq)$ be a decision problem where $A = \{a, b, c, d, e\}$ and $a \succeq a$, $a \succeq c$, $a \succeq d$, $b \succeq a$, $b \succeq b$, $b \succeq c$, $b \succeq d$, $b \succeq e$, $c \succeq a$, $c \succeq c$, $c \succeq d$, $d \succeq d$, $e \succeq a$, $e \succeq b$, $e \succeq c$, $e \succeq d$, $e \succeq e$. Notice that, in order to characterize a decision problem $(A, \succeq)$, it is sufficient to provide $\succ$, or $A/\!\sim$ and $\succeq_a$, or $A/\!\sim$ and $\succ_a$. In this case, for instance, it would be enough to say that $A/\!\sim = \{\{a, c\}, \{b, e\}, \{d\}\}$ and that $\{a, c\} \succ_a \{d\}$, $\{b, e\} \succ_a \{a, c\}$, $\{b, e\} \succ_a \{d\}$. There are infinitely many utility functions representing $\succeq$. For instance, $u\colon A \to \mathbb{R}$ given by $u(a) = 1$, $u(b) = 2$, $u(c) = 1$, $u(d) = 0$ and $u(e) = 2$ is one of these functions. Clearly, giving a utility function is a simple way of characterizing $(A, \succeq)$. $\diamond$

Example 1.2.2. Let $(\mathbb{R}^n, \succeq_L)$ be a decision problem where $\succeq_L$ is the *lexicographic order*, *i.e.*, for each pair $x, y \in \mathbb{R}^n$, $x \succeq_L y$ if and only if either $x = y$ or there is $i \in \{1, \ldots, n\}$ such that, for each $j < i$, $x_j = y_j$ and $x_i > y_i$. Being a relatively common order, whenever $n \geq 2$, there is no utility function representing the lexicographic order. To see this, suppose that u is a utility function representing $\succeq_L$. For each $x \in \mathbb{R}$, $(x, \ldots, x, 1) \succ_L (x, \ldots, x, 0)$. Hence, $u(x, \ldots, x, 1) > u(x, \ldots, x, 0)$ and we can find $f(x) \in \mathbb{Q}$ such that $u(x, \ldots, x, 1) > f(x) > u(x, \ldots, x, 0)$. Moreover, if $x, y \in \mathbb{R}^n$ are such that $x > y$, then $f(x) > u(x, \ldots, x, 0) > u(y, \ldots, y, 1) > f(y)$. Hence, any such function f would be an injective function from $\mathbb{R}$ to $\mathbb{Q}$, which is impossible. $\diamond$

A natural question arises from the examples above. Under what conditions can the weak preference of a decision problem be represented by a utility function? In the remainder of this section we tackle this question. The following result provides a partial answer.

Proposition 1.2.1. *Let A be a countable set and $(A, \succeq)$ a decision problem. Then, there is a utility function u representing $\succeq$.*

Proof. Since A is a countable set, we have $A = \{a_1, a_2, \dots\}$. For each pair $i, j \in \mathbb{N}$, define

$$h_{ij} := \begin{cases} 1 & a_i, a_j \in A \text{ and } a_i \succ a_j \\ 0 & \text{otherwise.} \end{cases}$$

For each $a_i \in A$, define $u(a_i) := \sum_{j=1}^{\infty} \frac{1}{2^j} h_{ij}$. Since $\sum_{j=1}^{\infty} \frac{1}{2^j} < \infty$, u is well-defined and, moreover, it represents $\succeq$. □

Next, we prepare the ground for a necessary and sufficient condition for the existence of a utility function representing a given preference relation.

Definition 1.2.2. Let $(A, \succeq)$ be a decision problem. A set $B \subset A$ is *order dense* in A if, for each pair $a_1, a_2 \in A$ with $a_2 \succ a_1$, there is $b \in B$ such that $a_2 \succeq b \succeq a_1$.

Definition 1.2.3. Let $(A, \succeq)$ be a decision problem. Let $a_1, a_2 \in A$, with $a_2 \succ a_1$. Then, (a_1, a_2) is a *gap* if, for each $b \in A$, either $b \succeq a_2$ or $a_1 \succeq b$. If (a_1, a_2) is a gap, then a_1 and a_2 are *gap extremes*. Let A^* be the set of gap extremes of A.

Lemma 1.2.2. *Let $(A, \succeq)$ be a decision problem and assume that $\succeq$ is antisymmetric.*

i) *If there is a countable set $B \subset A$ that is order dense in A, then A^* is countable.*

ii) *If there is a utility function representing $\succeq$, then A^* is countable.*

Proof. i) Let A_1^* and A_2^* be the sets of superior and inferior gap extremes, respectively, and let $B \subset A$ be a countable set order dense in A. If (a_1, a_2) is a gap, since $\succeq$ is antisymmetric, then there is $b \in B$ with $a_1 = b$ or $a_2 = b$. Hence, there is a bijection from $A_1^* \backslash B$ to a subset of B (matching each inferior gap extreme not being in B with its corresponding superior gap extreme, that must be in B). Hence, $A_1^* \backslash B$ is a countable set. Analogously, $A_2^* \backslash B$ is also a countable set. Therefore, $A^* = (A_1^* \backslash B) \cup (A_2^* \backslash B) \cup (A^* \cap B)$ is a countable set.

ii) Let u be a utility function representing $\succeq$. Note that, for each gap (a_1, a_2), there is $q \in \mathbb{Q}$ such that $u(a_2) > q > u(a_1)$. Hence, A^* is a countable set. □

Theorem 1.2.3. *Let $(A, \succeq)$ be a decision problem and assume that $\succeq$ is antisymmetric. Then, $\succeq$ can be represented by a utility function if and only if there is a countable set $B \subset A$ that is order dense in A.*

Proof. Let $B \subset A$ be a countable set order dense in A. We say that a is the first (last) element in A if there is not $\bar{a} \in A$, $\bar{a} \neq a$, such that $\bar{a} \succeq a$ ($a \succeq \bar{a}$). Note that the first or the last elements in A may not exist; however, if they exist, they are unique. Let $\bar{B}$ be the set containing B and the first and last elements of A (if they exist). By Lemma 1.2.2, A^* is countable and, hence, the set $B^* := \bar{B} \cup A^*$ is countable. By Proposition 1.2.1, there is a utility function $\bar{u}$ representing $\succeq$ in B^*. For each $a \in A$, define $u(a)$ in the following way:

$$u(a) := \sup\{\bar{u}(b) : b \in B^*, a \succeq b\}.$$

We now show that u is well defined. Let $a \in A$. If $a \in B^*$, then $u(a) = \bar{u}(a)$. If $a \notin B^*$, then there are $a_1, a_2 \in A$ such that $a_2 \succ a \succ a_1$. Since B is order dense in A and $a \notin B^*$, then there are $b_1, b_2 \in B$ such that $a_2 \succeq b_2 \succ a \succ b_1 \succeq a_1$. Hence, the set $\{\bar{u}(b) : b \in B^*, a \succeq b\}$ is nonempty ($\bar{u}(b_1)$ belongs to it) and bounded from above (by $\bar{u}(b_2)$), so it has a supremum. Therefore, for each $a \in A$, $u(a)$ is well defined.

We now check that u is a utility function representing $\succeq$ in A. Let $a_1, a_2 \in A$ be such that $a_2 \succ a_1$. We claim that there are $b_1, b_2 \in B^*$ such that $a_2 \succeq b_2 \succ b_1 \succeq a_1$. If $a_1, a_2 \in B^*$, then $b_1 = a_1$ and $b_2 = a_2$. If $a_1 \notin B^*$, since B is order dense in A, there is $b_2 \in B$ such that $a_2 \succeq b_2 \succ a_1$. Since (a_1, b_2) cannot be a gap, there is $\tilde{a} \in A$ such that $b_2 \succ \tilde{a} \succ a_1$. Hence, there is $b_1 \in B$ such that $\tilde{a} \succeq b_1 \succ a_1$. If $a_2 \notin B^*$, we can make an analogous reasoning. Since $u(a_2) \geq \bar{u}(b_2) > \bar{u}(b_1) \geq u(a_1)$, then $u(a_2) > u(a_1)$. The latter also implies that, for each pair $a_1, a_2 \in A$ such that $u(a_2) > u(a_1)$, we have $a_2 \succ a_1$ (recall that $\succeq$ is antisymmetric).

Conversely, assume that there is a utility function u representing $\succeq$ in A. Let $\bar{\mathbb{Q}}^2$ be the subset of $\mathbb{Q}^2$ given by:

$$\bar{\mathbb{Q}}^2 := \{(q_1, q_2) \in \mathbb{Q}^2 : \text{there is } a \in A \text{ such that } q_2 > u(a) > q_1\}.$$

Let $g\colon \bar{\mathbb{Q}}^2 \to A$ be a map such that, for each $(q_1, q_2) \in \bar{\mathbb{Q}}^2$, we have $q_2 > u(g(q_1, q_2)) > q_1$. The set $\bar{B} := g(\bar{\mathbb{Q}}^2)$ is countable and, by Lemma 1.2.2, so is the set $B := A^* \cup \bar{B}$. We claim that B is order dense in A. Let $a_1, a_2 \in A$ be such that $a_2 \succ a_1$ and assume that (a_1, a_2) is not a gap (otherwise we can take $b = a_1$ or $b = a_2$). Then, there are $q_1, q_2 \in \mathbb{Q}$ and $\bar{a} \in A$ such that $a_2 \succ \bar{a} \succ a_1$ and $u(a_2) > q_2 > u(\bar{a}) > q_1 > u(a_1)$. Hence, $(q_1, q_2) \in \bar{\mathbb{Q}}^2$ and there is $b \in B$ such that $q_2 > u(b) > q_1$, so $a_2 \succ b \succ a_1$. Therefore, B is order dense in A. $\square$

Corollary 1.2.4. *Let $(A, \succeq)$ be a decision problem. Then, $\succeq$ can be represented by a utility function if and only if there is a countable set $B \subset A$ that is order dense in A.*

Proof. Consider the decision problem $(A/\!\sim, \succeq_a)$, and recall that $\succeq_a$ is antisymmetric. Then, $\succeq$ can be represented by a utility function if and only if $\succeq_a$ can be represented by a utility function. It is easy to check that there is a countable subset of A order dense in A (with respect to $\succeq$) if and only if there is a countable subset of $A/\sim$ order dense in $A/\sim$ (with respect to $\succeq_a$). Hence, the result follows from Theorem 1.2.3. $\square$

To finish this section we provide a last result that states that the utility function representing a weak preference is unique up to strictly increasing transformations. The proof of the result is straightforward and we omit it. A function $f\colon \Omega \subset \mathbb{R} \to \mathbb{R}$ is *strictly increasing* if, for each pair $x, y \in \Omega$, $x > y$ if and only if $f(x) > f(y)$.

Proposition 1.2.5. *Let $(A, \succeq)$ be a decision problem and assume that u is a utility function representing $\succeq$. Then, $\bar{u}$ is another utility function representing $\succeq$ if and only if there is a strictly increasing function $f\colon u(A) \subset \mathbb{R} \to \mathbb{R}$ such that, for each $a \in A$, $\bar{u}(a) = f(u(a))$.*

The previous result illustrates why this section is entitled ordinal utility. Within the current framework, given a decision problem $(A, \succeq)$, a utility function u representing $\succeq$, and a pair of alternatives $a, b \in A$, the difference $|u(a) - u(b)|$ is meaningless; *i.e.*, $u(a) - u(b)$ tells, through its sign, the relative order of a and b and nothing else. In the following section we discuss a setting in which, under some extra assumptions, utility functions will also contain "cardinal" information about the intensities of the preferences.

1.3. Linear Utility

In this section we deal with *convex decision problems*. A convex decision problem is a decision problem $(X, \succeq)$ such that X is a convex subset of a finite dimensional real vector space.

Definition 1.3.1. Let $(X, \succeq)$ be a convex decision problem. A *linear utility function* representing $\succeq$ is a function $\bar{u}\colon X \to \mathbb{R}$ satisfying, for each pair $x, y \in X$, the following two conditions:

i) $x \succeq y$ if and only if $\bar{u}(x) \geq \bar{u}(y)$, *i.e.*, $\bar{u}$ represents $\succeq$.
ii) For each $t \in [0, 1]$, $\bar{u}(tx + (1-t)y) = t\bar{u}(x) + (1-t)\bar{u}(y)$.

Following the discussion at the end of the previous section, a linear utility function not only reveals the relative order of each pair of alternatives, but it also conveys information about how different they are. In this section we study the assumptions that a convex decision problem has to satisfy for its preference relation to be representable through a linear utility function. We begin by defining two properties that can be satisfied by the preference relation associated with a convex decision problem.

Definition 1.3.2. Let $(X, \succeq)$ be a convex decision problem. We say that $\succeq$ is *independent* if, for each triple $x, y, z \in X$ and each $t \in (0,1]$, $x \succeq y$ if and only if $tx + (1-t)z \succeq ty + (1-t)z$.

Definition 1.3.3. Let $(X, \succeq)$ be a convex decision problem. We say that $\succeq$ is *continuous* if, for each triple $x, y, z \in X$ such that $x \succ y \succ z$, there is $t \in (0,1)$ with $y \sim tx + (1-t)z$.

Both properties are self-explanatory and we do not make more comments on their interpretation; at the end of this section we include a brief discussion on their appropriateness. Note that, in the independence property, $t \neq 0$; otherwise the definition would be senseless.

Next, we show that independence and continuity are necessary and sufficient conditions for the existence of a linear utility function. In fact, it is straightforward to check that if in a convex decision problem $(X, \succeq)$ there is a linear utility function representing $\succeq$, then $\succeq$ is independent and continuous. The sufficiency, however, is not so immediate and we need some preliminary results. First note that if $\succeq$ is independent, then the statement in Definition 1.3.2 also holds if we replace $\succeq$ by $\succ$ or $\sim$.

Proposition 1.3.1. *Let $(X, \succeq)$ be a convex decision problem. Assume that $\succeq$ is independent and assume that there are $x, y \in X$ such that $y \succ x$. Let $s, t \in [0,1]$, $s > t$. Then, $sy + (1-s)x \succ ty + (1-t)x$.*

Proof. First, note that $t < 1$. Since $\succeq$ is independent,

$$\frac{s-t}{1-t}y + \frac{1-s}{1-t}x \succ \frac{s-t}{1-t}x + \frac{1-s}{1-t}x = x.$$

On the other hand,

$$sy + (1-s)x = ty + (1-t)\Big(\frac{s-t}{1-t}y + \frac{1-s}{1-t}x\Big).$$

Using the independence of $\succeq$ again,

$$ty + (1-t)\Big(\frac{s-t}{1-t}y + \frac{1-s}{1-t}x\Big) \succ ty + (1-t)x. \qquad \square$$

Corollary 1.3.2. *Let $(X, \succeq)$ be a convex decision problem. Assume that $\succeq$ is independent and continuous. Then, for each triple $x, y, z \in X$ such that $x \succ y \succ z$, there is* a unique *$t \in (0,1)$ such that $y \sim tx + (1-t)z$.*

We are ready to prove the main result concerning linear utility functions.

Theorem 1.3.3. *Let $(X, \succeq)$ be a convex decision problem. Assume that $\succeq$ is independent and continuous. Then, there is a linear utility function $\bar{u}$ representing $\succeq$. Moreover, the function $\bar{u}$ is unique up to positive affine transformations, that is,*

$\hat{u}$ is another linear utility function representing $\succeq$ if and only if there are $a, b \in \mathbb{R}$, with $a > 0$, such that, for each $x \in X$, $\hat{u}(x) = a\bar{u}(x) + b$.

Proof. If for each pair $x, y \in X$, $x \sim y$, then the result is trivially true (take $r \in \mathbb{R}$ and, for each $x \in X$, define $u(x) := r$). Hence, assume that there are $x_1, x_2 \in X$ such that $x_2 \succ x_1$. Let $[x_1, x_2] := \{x \in X : x_2 \succeq x \succeq x_1\}$ and define $u\colon [x_1, x_2] \to \mathbb{R}$ by:

$$u(x) := \begin{cases} 0 & x \sim x_1 \\ 1 & x \sim x_2 \\ \text{the unique } t \in (0,1) \text{ such that } x \sim tx_2 + (1-t)x_1 & \text{otherwise.} \end{cases}$$

In view of Corollary 1.3.2, u is well defined. By Proposition 1.3.1, u is a utility function representing $\succeq$ over $[x_1, x_2]$. We claim that u is linear. Let $x, y \in [x_1, x_2]$ and $t \in [0,1]$. Since $\succeq$ is independent, then

$$tx + (1-t)y \sim t(u(x)x_2 + (1-u(x))x_1) + (1-t)(u(y)x_2 + (1-u(y))x_1),$$

which is equal to

$$\big(tu(x) + (1-t)u(y)\big)x_2 + \big(t(1-u(x)) + (1-t)(1-u(y))\big)x_1.$$

Let $\bar{t} := tu(x) + (1-t)u(y)$. Then, we have $tx + (1-t)y \sim \bar{t}x_2 + (1-\bar{t})x_1$. Hence, $u(tx + (1-t)y) = \bar{t} = tu(x) + (1-t)u(y)$. We now show the uniqueness of u over $[x_1, x_2]$. Every positive affine transformation of u is a linear utility representing $\succeq$. Conversely, let $\bar{u}$ be a linear utility representing $\succeq$. We define another utility function, $\tilde{u}$, which is also a linear utility representing $\succeq$. For each $x \in [x_1, x_2]$, let

$$\tilde{u}(x) := \frac{\bar{u}(x)}{\bar{u}(x_2) - \bar{u}(x_1)} - \frac{\bar{u}(x_1)}{\bar{u}(x_2) - \bar{u}(x_1)}. \tag{1.3.1}$$

Since $\tilde{u}(x_2) = 1$ and $\tilde{u}(x_1) = 0$, we have that, for each $x \in [x_1, x_2]$,

$$\tilde{u}(x) = \tilde{u}(u(x)x_2 + (1-u(x))x_1) = u(x)\tilde{u}(x_2) + (1-u(x))\tilde{u}(x_1) = u(x).$$

If we separate $\bar{u}(x)$ in Eq. (1.3.1), we get $\bar{u}(x) = (\bar{u}(x_2) - \bar{u}(x_1))u(x) + \bar{u}(x_1)$, *i.e.*, $\bar{u}$ is a positive affine transformation of u.

To finish the proof we extend u to the whole set X. Let $x \in X \backslash [x_1, x_2]$. Let $[y_1, y_2] \subset X$ be such that $x \in [y_1, y_2]$ and $[x_1, x_2] \subset [y_1, y_2]$. Let u^* be a linear utility function representing $\succeq$ over $[y_1, y_2]$ defined as u above. Then, for each $y \in [y_1, y_2]$, define $\bar{u}(y)$ by:

$$\bar{u}(y) := \frac{u^*(y) - u^*(x_1)}{u^*(x_2) - u^*(x_1)}.$$

So defined, $\bar{u}$ is a linear utility representing $\succeq$ over $[y_1, y_2]$ and, hence, over $[x_1, x_2]$. Note that $\bar{u}(x_2) = u(x_2) = 1$ and $\bar{u}(x_1) = u(x_1) = 0$. Then, by the "uniqueness" of u, u and $\bar{u}$ coincide over $[x_1, x_2]$. We now show that the extension of u to X is independent of the chosen "interval", *i.e.*, independent of $[y_1, y_2]$. Let $[z_1, z_2]$ be different from $[y_1, y_2]$, such that $x \in [z_1, z_2]$ and $[x_1, x_2] \subset [z_1, z_2]$. Let $\hat{u}$ be the extension of u to $[z_1, z_2]$ defined as above. Assume, without loss of generality, that $x_1 \succ x$. Then, $\bar{u}$ and $\hat{u}$ are linear utility functions over $[x, x_2]$ with equal values on x_1 and x_2. Hence, they are equal over $[x, x_2]$. Therefore, we define $u(x) := \bar{u}(x)$.[2] To check that u is a linear utility function over X just note that, for each pair $x, y \in X$, u is a linear utility over $[\bar{x}, \bar{y}]$ with $x, y \in [\bar{x}, \bar{y}]$ and $[x_1, x_2] \subset [\bar{x}, \bar{y}]$. Finally, the uniqueness over X can be checked in the same way as the uniqueness over $[x_1, x_2]$ (when defining the new $\tilde{u}$, we also have that, for each $x \notin [x_1, x_2]$, $\tilde{u}(x) = u(x)$). $\square$

It is worth noting the difference between the uniqueness statement in Proposition 1.2.5 and the one included in Theorem 1.3.3. In the latter, given a linear utility function representing a decision problem, only a special type of increasing transformations can be performed on the utility function: those that do not affect the relative differences between any two alternatives.

Linear utility plays an important role in game theory because it is the theoretical basis that makes possible the definition of mixed strategies (Section 2.4). When a player chooses a mixed strategy, he chooses a lottery over his set of strategies and, hence, the actual result of his choice is random. This means that, to define a mixed strategy, we need to know how decision makers form their preferences in the presence of risk and, thus, linear utility will play an important role there. We now discuss why this is so.

Let A be a set of alternatives. Let ΔA be the set of lotteries over A. Formally,

$$\Delta A := \{x \in [0,1]^A : |\{a \in A : x(a) > 0\}| < \infty \text{ and } \sum_{a \in A} x(a) = 1\};$$

that is, ΔA is the set of probability distributions over A that have a finite support. For each $a \in A$, let $e_a \in \Delta A$ be the lottery defined by $e_a(a) := 1$ and, for each $\bar{a} \in A \backslash \{a\}$, $e_a(\bar{a}) := 0$. So defined, ΔA is a convex subset of the real vector space $\mathbb{R}^A$. Hereafter, we assume that A is a finite set, which implies that ΔA coincides with the convex hull of $\{e_a : a \in A\}$.[3] The concept of *von Neumann and Morgenstern utility function* is especially useful for convex decision problems whose set of alternatives has the form ΔA for some A. We introduce this concept in the next definition.

Definition 1.3.4. Let $(\Delta A, \succeq)$ be a convex decision problem. A function $u\colon A \to \mathbb{R}$ is a *von Neumann and Morgenstern utility function* representing $\succeq$

[2] The case $x \succ x_2$ is analogous but with $[x, x_2]$ being replaced by $[x_1, x]$.

[3] For a more general treatment, where the case with infinite alternatives is also discussed, refer, for instance, to Kreps (1988).

if, for each pair $x, y \in \Delta A$,

$$x \succeq y \Leftrightarrow \sum_{a \in A} u(a)x(a) \geq \sum_{a \in A} u(a)y(a).$$

The main advantage of von Neumann and Morgenstern utility functions is that they are defined on A, not on ΔA, and they represent the preferences of the decision maker on ΔA. Hence, they are a very convenient tool to represent the preferences of decision makers in a risk environment. However, what is the connection between von Neumann and Morgenstern utility functions and linear utility functions? The answer to this question is summarized in the next proposition.

Proposition 1.3.4. *Let $(\Delta A, \succeq)$ be a convex decision problem. Then, there is a von Neumann and Morgenstern utility function representing $\succeq$ if and only if there is a linear utility function representing $\succeq$.*

Proof. Suppose that u is a von Neumann and Morgenstern utility function representing $\succeq$ and define $\bar{u} \colon \Delta A \to \mathbb{R}$, for each $x \in \Delta A$, by $\bar{u}(x) := \sum_{a \in A} u(a)x(a)$. So defined, $\bar{u}$ is a linear utility function representing $\succeq$. Conversely, let $\bar{u}$ be a linear utility function representing $\succeq$ and define $u \colon A \to \mathbb{R}$, for each $a \in A$, by $u(a) := \bar{u}(e_a)$. Since each $x \in \Delta A$ can be written as $\sum_{a \in A} x(a)e_a$, u is a von Neumann and Morgenstern utility function representing $\succeq$. $\square$

Von Neumann and Morgenstern utility functions are also known as objective expected utility or cardinal utility functions, that is, the decision maker attaches some cardinal utility to the different alternatives and then derives his preferences over lotteries by calculating their expected values. This approach is more general than it might initially seem, as we illustrate in the following example.

Example 1.3.1. Suppose that a decision maker has to choose between the following two options: i) a sure 1 million dollars and ii) 2 million dollars with probability 0.5 and 0 dollars with probability 0.5. Expected utility does not imply that the decision maker will be indifferent toward the two lotteries, that is, the decision maker's utility when he gets 2 million dollars may not be twice his utility when he gets 1 million dollars. A decision maker whose utility for a monetary amount is exactly this monetary amount ($u(x) = x$) is said to be *risk neutral*; he would be indifferent regarding i) and ii). A *risk averse* decision maker ($u(x) = \sqrt{x}$) would go for option i) and a *risk prone* one ($u(x) = x^2$) would choose ii). Hence, when the set of alternatives are monetary payments, von Neumann and Morgenstern utility functions can account for very diverse attitudes toward risk. $\diamond$

As we will see in the following chapters, utility functions and, more particularly, von Neumann and Morgenstern utility functions play an important role in the development of game theory. In this chapter, we just intended to provide the corresponding definitions and to prove that, under general (and reasonable) conditions, there are utility functions and even von Neumann and Morgenstern utility functions representing the preferences of a decision maker. Fishburn (1970), French (1986), Kreps (1988), and Mas-Colell et al. (1995) provide discussions about whether these conditions are really natural, along with a deeper presentation of utility theory. Although we recommend the reader consult these books for a further study of decision theory, we finish this section with two examples that somewhat criticize the properties of independence and continuity, respectively. In any case, we believe that the criticisms raised by these examples, which are based on singular scenarios, are largely irrelevant for the purpose of this book and do not undermine the firm foundations of von Neumann and Morgenstern utility functions.

Example 1.3.2. (Allais paradox (Allais 1953)). The objective of this paradox was to show an inconsistency of actual observed behavior with the independence property. The paradox is as follows. Consider that a decision maker is asked to choose between option O_1 (a sure win of 1 million euro) and option P_1 (winning 5 million euro with probability 0.1, 1 million euro with probability 0.89, and nothing with probability 0.01). Then, the decision maker is asked to choose between option O_2 (winning 1 million euro with probability 0.11 and nothing with probability 0.89) and option P_2 (winning 5 million euro with probability 0.1 and nothing with probability 0.9). In practice, it has been observed that most people choose O_1 in the first case and P_2 in the second. However, this is inconsistent with the independence property. Assume that a decision maker has preferences over ΔA for $A = \{0, 1, 5\}$, where 0, 1, and 5 mean nothing, 1 million euro, and 5 million euro, respectively. Remember that, for each $a \in A$, e_a denotes the lottery which selects a with probability 1. Then, the four options can be identified with the lotteries depicted in Figure 1.3.1. If the weak preference of the decision maker over ΔA satisfies independence, then $O_1 \succeq P_1$ implies that $O_2 \succeq P_2$, which is inconsistent with the most frequent observed behavior. Allais paradox has been one of the motivations for the development of alternative utility theories for decision making under risk. However, von Neumann and Morgenstern utility theory is widely used nowadays, since Allais paradox can be interpreted as an example of the appearance of irrational behavior in singular situations when the decision makers are not particularly reflexive. $\diamond$

OPTION	LOTTERY
O_1	$0.11e_1 + 0.89e_1$
P_1	$0.11(\frac{0.01}{0.11}e_0 + \frac{0.1}{0.11}e_5) + 0.89e_1$
O_2	$0.11e_1 + 0.89e_0$
P_2	$0.11(\frac{0.01}{0.11}e_0 + \frac{0.1}{0.11}e_5) + 0.89e_0$

Figure 1.3.1. The lotteries corresponding with the four options in Allais paradox.

Example 1.3.3. Consider the decision problem $(\Delta A, \succeq)$, where A is the set $\{10, 0, D\}$, which stand for "10 euro for the decision maker", "0 euro for the decision maker", and "the death of the decision maker", respectively. For a standard decision maker it holds that $10 \succ 0 \succ D$. If $\succeq$ satisfies continuity, then there is $t \in (0,1)$ such that $0 \sim t10 + (1-t)D$. Yet, this means that the decision maker is willing to risk his life for just 10 euro; at first sight, even for t very close to 1, this does not seem credible. However, if one considers this example more carefully, the argument against continuity vanishes. Human beings are continuously taking minor risks in order to get small rewards (such as crossing the street to buy a newspaper or driving the car to go to the theater). $\diamond$

Chapter 2

Strategic Games

2.1. Introduction to Strategic Games

A *strategic game* is a static model that describes interactive situations among several players.[1] According to this model, all the players make their decisions simultaneously and independently. From the mathematical perspective, strategic games are very simple objects. They are characterized by the strategies available to the players along with their payoff functions. Even though one may think of the payoffs of the players as money, we have already seen in Chapter 1 that payoff functions may be representations of more general preferences of the players over the set of possible outcomes. These general preferences may account for other sources of utility such as unselfishness, solidarity, or personal affinities.

Throughout this book it is implicitly assumed that each player is rational in the sense that he tries to maximize his own payoff. Moreover, for a rational player, there is no bound in the complexity of the computations he can make or in the sophistication of his strategies.[2]

We start this chapter by formally introducing the concept of strategic game and then we move to the most widely studied solution concept in game theory: Nash equilibrium. We discuss some important classes of games, with special emphasis on zero-sum games. Later we study other solution concepts different from Nash equilibrium (Section 2.9) and, towards

[1] Strategic games are also known as *games in normal form*.

[2] This assumption is standard in classic game theory and in most of the fields in which it is applied, especially in economics. Rubinstein (1998) offers a deep treatment of different directions in which the rationality of the agents can be bounded along with the corresponding implications.

the end of the chapter, we try to establish a bridge between equilibrium behavior and rational behavior (Section 2.12).

We denote the set of players of a game by $N := \{1, \ldots, n\}$.

Definition 2.1.1. An n-player *strategic game* with set of players N is a pair $G := (A, u)$ whose elements are the following:

Sets of strategies: For each $i \in N$, A_i is the nonempty set of strategies of player i and $A := \prod_{i=1}^{n} A_i$ is the set of strategy profiles.

Payoff functions: For each $i \in N$, $u_i \colon A \to \mathbb{R}$ is the payoff function of player i and $u := \prod_{i=1}^{n} u_i$; u_i assigns, to each strategy profile $a \in A$, the payoff that player i gets if a is played.

Remark 2.1.1. In a play of G, each player $i \in N$ chooses, simultaneously and independently, a strategy $a_i \in A_i$.[3] Then, each player i gets payoff $u_i(a)$. One can imagine that, before playing the game, the players can communicate among themselves; if this is the case, during the course of this communication they can only make nonbinding agreements.

Remark 2.1.2. In the kind of interactive situations that strategic games model, the following elements are implicitly involved:

- $\{A_i\}_{i \in N}$, the strategy sets of the players.
- R, the set of possible outcomes.
- A function $f \colon A \to R$ that assigns, to each strategy profile a, its corresponding outcome.
- $\{\succeq_i\}_{i \in N}$, the preferences of the players over the outcomes in R. They are assumed to be complete, transitive, and representable through a utility function.[4]
- $\{U_i\}_{i \in N}$, the utility functions of the players, which represent their preferences on R.

Hence, a strategic game is a "simplification" in which, for each $i \in N$ and each $a \in A$, $u_i(a) = U_i(f(a))$.

Below, we show some examples of strategic games.

Example 2.1.1. (Prisoner's dilemma). Two suspects in a severe crime and a small robbery are put into separate cells. They are known to be guilty in the robbery but the police have no evidence for the crime. Both of them are given the chance to confess. If both confess the crime, each of them will spend 10 years in jail. If only one confesses, he will act as a witness against

[3] In Section 2.11 we introduce some structure to the situation in which the strategic game takes place and, in that setting, players can correlate their choices.

[4] That is, there is a countable subset of R that is order dense in R (see Theorem 1.2.3).

the other, who will spend 15 years in jail, and will receive no punishment. Finally, if no one confesses, they will be judged for the small robbery and each of them will spend 1 year in jail. Following the common terminology for this game, we refer to the confession as "defect" (D) and to no confession as "not defect" ND. Then, the prisoner's dilemma game is a strategic game (A, u) in which

- $A_1 = A_2 = \{ND, D\}$;
- $u_1(ND, ND) = -1$, $u_1(ND, D) = -15$, $u_1(D, ND) = 0$, and $u_1(D, D) = -10$; and
- $u_2(ND, ND) = -1$, $u_2(ND, D) = 0$, $u_2(D, ND) = -15$, and $u_2(D, D) = -10$.

Figure 2.1.1 shows a more convenient representation of this game, which is the standard way to represent strategic games with finite (and small) strategy sets.

	ND	D
ND	$-1, -1$	$-15, \ \ 0$
D	$0, -15$	$-10, -10$

Figure 2.1.1. The prisoner's dilemma.

The prisoner's dilemma game is a classic in game theory. It has been widely used not only within theoretical settings, but also for more applied purposes in sociology and behavioral economics. The "cooperative" outcome, $(-1, -1)$, is quite good for both players; it is almost all they can get in the game. Yet, for each player, the strategy D leads to a strictly higher payoff than ND, regardless of the strategy chosen by the other player. Hence, a rational decision maker should always play D. Thus, if both players behave rationally, they get payoffs $(-10, -10)$, which are much worse than the payoffs in the "cooperative" outcome. Many real life situations can be seen as prisoner's dilemma games. For instance, during the nuclear race between the US and the USSR in the Cold War, both countries could decide whether or not to produce nuclear weapons; in this situation, the payoffs would have the structure of those in Figure 2.1.1. $\diamond$

Example 2.1.2. (Cournot oligopoly (Cournot 1838)). The set N corresponds with the set of producers of a certain commodity. Each producer $i \in N$ has to choose a strategy $a_i \in [0, \infty)$ that denotes the number of units of the commodity that he will produce and bring to the market; $c_i(a_i)$ denotes the total cost player i has to face when choosing strategy a_i. The price of one unit of the commodity in the market depends on $\sum_{i \in N} a_i$ and is denoted by $\pi(\sum_{i\in N} a_i)$. This situation can be modeled by the strategic game $G = (A, u)$, where:

- $A_i = [0, \infty)$ and
- for each $i \in N$ and each $a \in A$, $u_i(a) = \pi(\sum_{j\in N} a_j)a_i - c_i(a_i)$. $\diamond$

Example 2.1.3. (A first-price auction). An object is to be auctioned. The set N corresponds with the potential bidders. Player i's valuation of the object is v_i. Suppose that $v_1 > v_2 > \ldots > v_n > 0$. The rules of the auction are as follows:

- The players submit bids simultaneously.
- The object is assigned to the player with the lowest index among those who have submitted the highest bid.[5]
- The player who gets the object pays his bid.

This auction can be modeled by the following strategic game G. For each $i \in N$, $A_i = [0, \infty)$ and, for each $a \in A$,

$$u_i(a) = \begin{cases} v_i - a_i & i = \min\{j \in N : a_j = \max_{l\in N} a_l\} \\ 0 & \text{otherwise.} \end{cases}$$ $\diamond$

Example 2.1.4. (A second-price auction). In a second-price auction, the rules are the same as in a first-price auction, except that the player who gets the object pays the highest of the bids of the other players. The strategic game that corresponds with this situation is the same as in Example 2.1.3, but now

$$u_i(a) = \begin{cases} v_i - \max\{a_j : j \in N, j \neq i\} & i = \min\{j \in N : a_j = \max_{l\in N} a_l\} \\ 0 & \text{otherwise.} \end{cases}$$ $\diamond$

Note that the model of strategic games is a static one, in the sense that it assumes that players choose their strategies *simultaneously*. Nevertheless, multistage interactive situations can also be represented by strategic games (although some information is lost in the process). Why is this reasonable? Suppose that several players are involved in a multistage conflict and that each of them wants to make a strategic analysis of the situation. Before the play starts, each player must consider all the possible circumstances in which he may have to make a decision and the corresponding consequences of every possible decision. This can be made constructing the strategic form of the game. We illustrate this in the following example.

Example 2.1.5. (A multistage interactive situation). Consider the following situation with two players and three stages:

[5] This is just an example. Many other tie-breaking rules and, indeed, many other auction formats can be defined.

Stage 1: Player 1 chooses between left (L_1) and right (R_1).

Stage 2: Player 2, after observing the choice of player 1, chooses between left (L_2) and right (R_2) when player 1 has chosen left and between l_2 and r_2 when player 1 has chosen right.

Stage 3: If left has been chosen in the first two stages, then player 1 chooses again between left (l_1) and right (r_1) and the game finishes. Otherwise, no more choices are made and the game finishes.

The payoffs for all possible outcomes are shown in the tree depicted in Figure 2.1.2 (the first and second components of the payoff vectors are the payoffs to players 1 and 2, respectively).

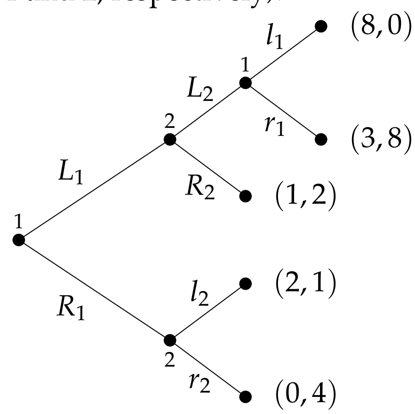


Figure 2.1.2. Tree representation of the game in Example 2.1.5.

Player 1 has to make a decision when the game starts; besides, he may have to make a decision at stage 3. A strategy of player 1 is a plan for all the circumstances in which he may have to make a decision. Thus the strategy set of player 1 is:

$$A_1 = \{L_1l_1, L_1r_1, R_1l_1, R_1r_1\},$$

where, for instance, L_1r_1 means "I plan to play L_1 at stage 1 and r_1 at stage 3". Player 2, after observing player 1's choice at stage 1, has to make a decision at stage 2. Hence, his strategy set is:

$$A_2 = \{L_2l_2, L_2r_2, R_2l_2, R_2r_2\},$$

where, for instance, R_2l_2 means "I plan to play R_2 if player 1 has played L_1, and to play l_2 if player 1 has played R_1". Figure 2.1.3 represents the strategic game associated with this multistage situation.[6] ◇

[6]It may seem strange to distinguish between R_1l_1 and R_1r_1. Indeed, both strategies are indistinguishable in the strategic form. However, as we will see in Chapter 3, the behavior in nodes that are not reached may make a difference when studying the equilibrium concepts of extensive games (through the tree/extensive representation).

	L_2l_2	L_2r_2	R_2l_2	R_2r_2
L_1l_1	8,0	8,0	1,2	1,2
L_1r_1	3,8	3,8	1,2	1,2
R_1l_1	2,1	0,4	2,1	0,4
R_1r_1	2,1	0,4	2,1	0,4

Figure 2.1.3. Representation of Example 2.1.5 as a strategic game.

2.2. Nash Equilibrium in Strategic Games

The next step is to propose solution concepts that aim to describe how rational agents should behave. The most important solution concept for strategic games is Nash equilibrium. It was introduced in Nash (1950b, 1951) and, because of its impact in both economic theory and game theory, John Nash was one of the laureates of the Nobel Prize in Economics in 1994. In Kohlberg (1990), the main idea underlying Nash equilibrium is said to be "to make a bold simplification and, instead of asking how the process of deductions might unfold, ask where its rest points may be". In fact, a Nash equilibrium of a strategic game is simply a strategy profile such that no player gains when unilaterally deviating from it; *i.e.*, the Nash equilibrium concept searches for *rest points* of the interactive situation described by the strategic game.

Given a game $G = (A, u)$ and a strategy profile $a \in A$, let $(a_{-i}, \hat{a}_i)$ denote the profile $(a_1, \ldots, a_{i-1}, \hat{a}_i, a_{i-1}, \ldots, a_n)$.

Definition 2.2.1. Let $G = (A, u)$ be a strategic game. A *Nash equilibrium* of G is a strategy profile $a^* \in A$ such that, for each $i \in N$ and each $\hat{a}_i \in A_i$,

$$u_i(a^*) \geq u_i(a^*_{-i}, \hat{a}_i).$$

Next, we study the Nash equilibria of the strategic games we presented in the previous section.

Example 2.2.1. The only Nash equilibrium of the prisoner's dilemma is $a^* = (D, D)$. Moreover, as we have already argued, this is the rational behavior in a noncooperative environment. $\diamond$

Example 2.2.2. We now study the Nash equilibria in a Cournot model like the one in Example 2.1.2. We do it under the following assumptions:

- We deal with a duopoly, *i.e.*, $n = 2$.
- For each $i \in \{1, 2\}$, $c_i(a_i) = ca_i$, where $c > 0$.
- Let d be a fixed number, $d > c$. The price function is given by:

$$\pi(a_1 + a_2) = \begin{cases} d - (a_1 + a_2) & a_1 + a_2 < d \\ 0 & \text{otherwise.} \end{cases}$$

For each $i \in \{1,2\}$ and each $a \in A$, the payoff functions of the associated strategic game are

$$u_i(a) = \begin{cases} a_i(d - a_1 - a_2 - c) & a_1 + a_2 < d \\ -a_i c & \text{otherwise.} \end{cases}$$

By definition, a Nash equilibrium of this game, sometimes called a Cournot equilibrium, is a pair $(a_1^*, a_2^*) \in A_1 \times A_2$ such that i) for each $\hat{a}_1 \in A_1$, $u_1(a_1^*, a_2^*) \geq u_1(\hat{a}_1, a_2^*)$ and ii) for each $\hat{a}_2 \in A_2$, $u_2(a_1^*, a_2^*) \geq u_2(a_1^*, \hat{a}_2)$. Now, we compute a Nash equilibrium of this game. For each $i \in \{1,2\}$ and each $a \in A$, let $f_i(a) := a_i(d - a_1 - a_2 - c)$. Then,

$$\frac{\partial f_1}{\partial a_1}(a) = -2a_1 + d - a_2 - c \quad \text{and} \quad \frac{\partial f_2}{\partial a_2}(a) = -2a_2 + d - a_1 - c.$$

Hence,

$$\frac{\partial f_1}{\partial a_1}(a) = 0 \Leftrightarrow a_1 = \frac{d - a_2 - c}{2} \quad \text{and} \quad \frac{\partial f_2}{\partial a_2}(a) = 0 \Leftrightarrow a_2 = \frac{d - a_1 - c}{2}.$$

Note that, for each $a \in A$, $\dfrac{\partial^2 f_1}{\partial a_1^2}(a) = \dfrac{\partial^2 f_2}{\partial a_2^2}(a) = -2$.

For each $i \in N$ and each $a_{-i} \in A_{-i}$, define the set $\mathrm{BR}_i(a_{-i}) := \{\hat{a}_i :$ for each $\tilde{a}_i \in A_i, u_i(a_{-i}, \hat{a}_i) \geq u_i(a_{-i}, \tilde{a}_i)\}$, where BR stands for "best reply". For each $a_2 \in A_2$, if $a_2 < d - c$, $\mathrm{BR}_1(a_2) = \frac{d-a_2-c}{2}$ and $\mathrm{BR}_1(a_2) = 0$ otherwise. Analogously, for each $a_1 \in A_1$, if $a_1 < d - c$, $\mathrm{BR}_2(a_1) = \frac{d-a_1-c}{2}$ and $\mathrm{BR}_2(a_1) = 0$ otherwise. Hence, since $a^* = (\frac{d-c}{3}, \frac{d-c}{3})$ is the unique solution of the system

$$\begin{cases} a_1 = \frac{d-a_2-c}{2} \\ a_2 = \frac{d-a_1-c}{2}, \end{cases}$$

a^* is the unique Nash equilibrium of this game. Note that, for each $i \in \{1,2\}$, $u_i(a^*) = \frac{(d-c)^2}{9}$.

Observe that, if instead of two duopolists, the market consists of a single monopolist, then his payoff function would be

$$u(a) = \begin{cases} a(d - a - c) & a < d \\ -ac & \text{otherwise.} \end{cases}$$

Let $f(a) := a(d - a - c)$. Now, $f'(a) = 0$ if and only if $a = (d - c)/2$. Since, for each $a \in \mathbb{R}$, $f''(a) = -2$, then the optimal production level and cost for a monopolist are $\bar{a} = \frac{d-c}{2}$ and $u(\bar{a}) = \frac{(d-c)^2}{4}$. Therefore, the profit of the monopolist is more than the sum of the profits of the two producers in the equilibrium a^*. Moreover, since $\bar{a} < a_1^* + a_2^*$, the market price is smaller in the duopoly case. ◇

Example 2.2.3. It is easy to check that in the first-price auction described in Example 2.1.3, the set of Nash equilibria consists of the strategy profiles $a^* \in [0, \infty)^n$ satisfying the following three conditions:

- $a_1^* \in [v_2, v_1]$,
- for each $j \in N \backslash \{1\}$, $a_j^* \leq a_1^*$, and
- there is $j \in N \backslash \{1\}$ such that $a_j^* = a_1^*$.

Note that, in a Nash equilibrium, player 1 always gets the object. ◇

Example 2.2.4. It is easy to check that in the second-price auction described in Example 2.1.3, the strategy profile $a^* = (v_1, \ldots, v_n)$ satisfies the following condition: for each $i \in N$, each $\hat{a}_i \in A_i$, and each $a_{-i} \in A_{-i}$, $u_i(a_{-i}, v_i) \geq u_i(a_{-i}, \hat{a}_i)$. This condition implies that a^* is a Nash equilibrium of this game. Note that, if a^* is played, player 1 gets the object. However, there are Nash equilibria of the second-price auction in which player 1 is not the winner. Take, for instance, $a = (0, v_1 + 1, 0, \ldots, 0)$. ◇

Example 2.2.5. Consider the strategic game in Example 2.1.5. The strategy profile $(L_1 l_1, R_2 r_2)$ is the unique Nash equilibrium of that game. Observe that, in order to find a Nash equilibrium in a game in which players have finite strategy sets, given the "tabular" representation, we only have to find a cell for which the payoff of player 1 is a maximum of its column and the payoff of player 2 is the maximum of its row. ◇

Note that, in all the strategic games that we have presented above, there is at least one Nash equilibrium. However, not every strategic game has a Nash equilibrium.

Example 2.2.6. (Matching pennies) Players 1 and 2 have to choose, simultaneously and independently, a natural number. If the sum of the chosen numbers is even, then player 1 wins; if the sum is odd, then player 2 wins. Strategically, all that matters in this game is whether a player chooses an even number (E) or an odd one (O). The strategic game which models this situation is given in Figure 2.2.1. This game does not have any Nash equilibria. ◇

	E	O
E	$1, -1$	$-1, 1$
O	$-1, 1$	$1, -1$

Figure 2.2.1. The matching pennies game.

Next, we present Nash theorem, which provides a sufficient condition for the existence of a Nash equilibrium in a strategic game. To state and prove Nash theorem we need some previous concepts and a classic result on correspondences: Kakutani fixed-point theorem.[7]

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$. A *correspondence* F from X to Y is a map $F\colon X \to 2^Y$. A correspondence F is *nonempty-valued*, *closed-valued*, or *convex-valued* if, for each $x \in X$, $F(x)$ is, respectively, a nonempty, closed, or convex subset of Y. Now, we move to the definition of continuity of a correspondence. First, recall the standard definition of continuity of a function. A function $f\colon X \to Y$ is *continuous* if, for each sequence $\{x^k\} \subset X$ converging to $\bar{x} \in X$ and each open set $Y^* \subset Y$ such that $f(\bar{x}) \in Y^*$, there is $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $f(x^k) \in Y^*$.[8] We present two generalizations of the above definition. A correspondence F is *upper hemicontinuous* if, for each sequence $\{x^k\} \subset X$ converging to $\bar{x} \in X$ and each open set $Y^* \subset Y$ such that $F(\bar{x}) \subset Y^*$, there is $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $F(x^k) \subset Y^*$. A correspondence F is *lower hemicontinuous* if, for each sequence $\{x^k\} \subset X$ converging to $\bar{x} \in X$ and each open set $Y^* \subset Y$ such that $F(\bar{x}) \cap Y^* \neq \emptyset$, there is $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $F(x^k) \cap Y^* \neq \emptyset$. Note that, if the correspondence F is a function (*i.e.*, only selects singletons in 2^Y), both upper and lower hemicontinuity properties reduce to the standard continuity of functions. Moreover, Figure 2.2.2 gives some intuition for the concepts of upper and lower hemicontinuity. Given a convergent sequence in the domain of the correspondence, upper hemicontinuity allows for "explosions" in the limit, whereas lower hemicontinuity allows for "implosions"; see $F(\bar{x})$ in Figures 2.2.2 (a) and 2.2.2 (b), respectively.

Theorem 2.2.1 (Kakutani fixed-point theorem)**.** *Let $X \subset \mathbb{R}^n$ be a nonempty, convex, and compact set. Let $F\colon X \to X$ be an upper hemicontinuous, nonempty-valued, closed-valued, and convex-valued correspondence. Then, there is $\bar{x} \in X$ such that $\bar{x} \in F(\bar{x})$,* i.e.*, F has a fixed-point.*

Proof. Refer to Section 2.13. □

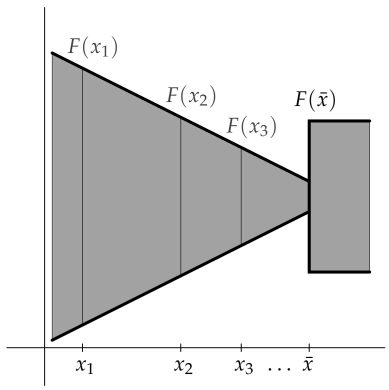
Definition 2.2.2. Let $G = (A, u)$ be a strategic game such that, for each $i \in N$, i) there is $m_i \in \mathbb{N}$ such that A_i is a nonempty and compact subset of $\mathbb{R}^{m_i}$ and ii) u_i is continuous. Then, for each $i \in N$, *i's best reply correspondence*, $\mathrm{BR}_i\colon A_{-i} \to A_i$, is defined, for each $a_{-i} \in A_{-i}$, by

$$\mathrm{BR}_i(a_{-i}) := \{a_i \in A_i : u_i(a_{-i}, a_i) = \max_{\tilde{a}_i \in A_i} u_i(a_{-i}, \tilde{a}_i)\}.^9$$

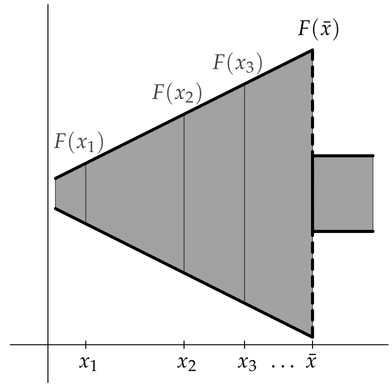
[7]The proof of the Kakutani fixed-point theorem is quite technical and requires some auxiliary notations and results. Since these auxiliary elements do not have much to do with game theory, we have relegated all of them, along with the proof of the Kakutani theorem, to an independent section (Section 2.13).

[8]We have defined sequential continuity instead of continuity but recall that, for metric spaces, they are equivalent.

[9]The assumptions on the strategy spaces and on the payoff functions guarantee that the best reply correspondences are well defined.



(a) A correspondence that is upper hemicontinuous but not lower hemicontinuous.



(b) A correspondence that is lower hemicontinuous but not upper hemicontinuous.

Figure 2.2.2. Two correspondences from $\mathbb{R}$ to $2^{\mathbb{R}}$.

Let $\mathrm{BR}\colon A \to A$ be defined, for each $a \in A$, as $\mathrm{BR}(a) := \prod_{i\in N} \mathrm{BR}_i(a_{-i})$.

Let $m \in \mathbb{N}$ and $A \subset \mathbb{R}^m$ be a convex set. A function $f\colon A \to \mathbb{R}$ is *quasi-concave* if, for each $r \in \mathbb{R}$, the set $\{a \in A : f(a) \geq r\}$ is convex or, equivalently, if, for each $a, \tilde{a} \in A$ and each $\alpha \in [0,1]$, $f(\alpha a + (1-\alpha)\tilde{a}) \geq \min\{f(a), f(\tilde{a})\}$. Quasi-concavity implies concavity, which requires $f(\alpha a + (1-\alpha)\tilde{a}) \geq \alpha f(a) + (1-\alpha) f(\tilde{a})$, but it is substantially more general. For instance, the convex function $f(x) = e^x$ is quasi-concave and so it is any monotone function from $\mathbb{R}$ to $\mathbb{R}$.

Proposition 2.2.2. *Let $G = (A, u)$ be a strategic game such that, for each $i \in N$,*

i) *A_i is a nonempty and compact subset of $\mathbb{R}^{m_i}$.*

ii) *u_i is continuous.*

iii) *For each a_{-i}, $u_i(a_{-i}, \cdot)$ is quasi-concave on A_i.*

Then, for each $i \in N$, BR_i is an upper hemicontinuous, nonempty-valued, closed-valued, and convex-valued correspondence. Therefore, BR *also satisfies the previous properties.*

Proof. Let $i \in N$.

Nonempty-valuedness: Obvious, since every continuous function defined on a compact set reaches a maximum.

Closed-valuedness: Also straightforward by the continuity of the payoff functions and the compactness of the sets of strategies.

Convex-valuedness: Let $a_{-i} \in A_{-i}$ and $\tilde{a}_i \in \mathrm{BR}_i(a_{-i})$. Let $r := u_i(a_{-i}, \tilde{a}_i)$. Then, $\mathrm{BR}_i(a_{-i}) = \{a_i \in A_i : u_i(a_{-i}, a_i) \geq r\}$. Convex-valuedness is implied by the quasi-concavity of u.

Upper hemicontinuity: Suppose BR_i is not upper hemicontinuous. Then, there are a sequence $\{a^k\} \subset A_{-i}$ converging to $\bar{a} \in A_{-i}$ and an open set $B^* \subset A_i$ with $\mathrm{BR}_i(\bar{a}) \subset B^*$, satisfying that, for each $k_0 \in \mathbb{N}$, there is $k \geq k_0$ such that $\mathrm{BR}_i(a^k) \not\subset B^*$. The latter implies that there is a sequence $\{\tilde{a}^m\} \subset A_i$ such that, for each $m \in \mathbb{N}$, $\tilde{a}^m \in \mathrm{BR}_i(a^m) \backslash B^*$. Since A_i is compact, $\{\tilde{a}^m\}$ has a convergent subsequence. Assume, without loss of generality, that $\{\tilde{a}^m\}$ itself converges and let $\hat{a} \in A_i$ be its limit. Since B^* is an open set, then $A_i \backslash B^*$ is closed. Hence, $\hat{a} \in A_i \backslash B^*$ and, therefore, $\hat{a} \notin \mathrm{BR}_i(\bar{a})$. For each $m \in \mathbb{N}$ and each $a \in A_i$, $u_i(a^m, \tilde{a}^m) \geq u_i(a^m, a)$. Letting $m \to \infty$ and using the continuity of u_i, we have that, for each $a \in A_i$, $u_i(\bar{a}, \hat{a}) \geq u_i(\bar{a}, a)$. Hence, $\hat{a} \in \mathrm{BR}_i(\bar{a})$ and we have a contradiction.

The result for BR is now straightforward. □

Exercise 2.6 asks you to show that, under the assumptions of Proposition 2.2.2, the BR_i functions need not be lower hemicontinuous.

We present below the main result of this section, whose proof is an immediate consequence of Kakutani theorem and Proposition 2.2.2.

Theorem 2.2.3 (Nash theorem[10]). *Let $G = (A, u)$ be a strategic game such that, for each $i \in N$,*

i) *A_i is a nonempty, convex, and compact subset of $\mathbb{R}^{m_i}$.*

ii) *u_i is continuous.*

iii) *For each a_{-i}, $u_i(a_{-i}, \cdot)$ is quasi-concave on A_i.*[11]

Then, the game G has, at least, one Nash equilibrium.

Proof. If a is a fixed point of the correspondence $\mathrm{BR}\colon A \to A$, then a is a Nash equilibrium of G. By Proposition 2.2.2, BR satisfies the conditions of Kakutani theorem (recall that now we have further assumed that the A_i sets are convex). Hence, BR has a fixed point. □

It is now an easy exercise to show that the set of Nash equilibria is closed (Exercise 2.5 asks the reader to prove this). Moreover, the role of

[10]In reality, what we present here is a generalization of the original Nash theorem. This version was proved in Rosen (1965). We present the Nash theorem in its original form in Theorem 2.4.1 (Section 2.4).

[11]In this context, the quasi-concavity of the function $u_i(a_{-i}, \cdot)$ can be phrased as follows: provided that all players different from player i are playing according to a_{-i}, we have that, for each $K \in \mathbb{R}$, if two strategies give at least payoff K to player i, then so does any convex combination of them.

quasi-concavity in the above theorem is to ensure that any convex combination of best responses is a best response, *i.e.*, BR is convex-valued.

The reader can easily verify that we cannot apply Nash theorem to any of the examples treated above. However, as we will see during this chapter, the sufficient condition provided by Nash theorem applies to a wide class of games.

2.3. Two-Player Zero-Sum Games

Many of the early works in game theory concentrated on a very special class of strategic games, namely *two-player zero-sum games*. They are characterized by the fact that the two players involved in the game have completely opposite interests.

Definition 2.3.1. A *two-player zero-sum game* is a strategic game given by $G = (\{A_1, A_2\}, \{u_1, u_2\})$, where, for each strategy profile $(a_1, a_2) \in A_1 \times A_2$, $u_1(a_1, a_2) + u_2(a_1, a_2) = 0$.

Note that in the previous definition we have imposed no restriction on the sets A_1 and A_2. We discuss the special case in which A_1 and A_2 are finite in Section 2.6; there we present and prove one of the classic results in game theory, the minimax theorem.

To characterize a two-player zero-sum game it is sufficient to give the payoff function of one of the players. Usually, player 1's payoff function is provided. Thus, when dealing with two-player zero-sum games we say that G is the triple (A_1, A_2, u_1). We assume that u_1 is bounded on $A_1 \times A_2$.

As mentioned above, the class of two-player zero-sum games was the first to be studied by game theorists and they represent situations in which players have totally opposite interests: whenever one player prefers (a_1, a_2) to $(\hat{a}_1, \hat{a}_2)$, then the other player prefers $(\hat{a}_1, \hat{a}_2)$ to (a_1, a_2). These games were initially analyzed by John von Neumann, who introduced the following concepts.

Definition 2.3.2. Let $G = (A_1, A_2, u_1)$ be a two-player zero-sum game.

Lower value: Let $\underline{\Lambda}\colon A_1 \to \mathbb{R}$ be defined, for each $a_1 \in A_1$, by $\underline{\Lambda}(a_1) := \inf_{a_2 \in A_2} u_1(a_1, a_2)$, *i.e.*, the worst payoff that player 1 can get if he plays a_1.

The *lower value* of G, denoted by $\underline{\lambda}$, is given by

$$\underline{\lambda} := \sup_{a_1 \in A_1} \underline{\Lambda}(a_1).$$

It provides the payoff that player 1 can guarantee for himself in G (or, at least, get arbitrarily close to it).

Upper value: Let $\bar{\Lambda}\colon A_2 \to \mathbb{R}$ be defined, for each $a_2 \in A_2$, by $\bar{\Lambda}(a_2) := \sup_{a_1 \in A_1} u_1(a_1, a_2)$, *i.e.*, the maximum loss that player 2 may suffer when he plays a_2.

The *upper value* of G, denoted by $\bar{\lambda}$, is given by

$$\bar{\lambda} := \inf_{a_2 \in A_2} \bar{\Lambda}(a_2).$$

It provides the supremum of losses that player 2 may suffer in G, *i.e.*, the maximum payoff that player 1 may get in G (or, at least, get arbitrarily close to it).

Note that $\underline{\lambda} \leq \bar{\lambda}$ since, for each $a_1 \in A_1$ and each $a_2 \in A_2$,

$$\underline{\Lambda}(a_1) = \inf_{\hat{a}_2 \in A_2} u_1(a_1, \hat{a}_2) \leq u_1(a_1, a_2) \leq \sup_{\hat{a}_1 \in A_1} u_1(\hat{a}_1, a_2) = \bar{\Lambda}(a_2).$$

Definition 2.3.3. A two-player zero-sum game $G = (A_1, A_2, u_1)$ is said to be *strictly determined* or to have a *value* if its lower value and its upper value coincide, *i.e.*, if $\underline{\lambda} = \bar{\lambda}$. In such a case, $V := \underline{\lambda} = \bar{\lambda}$ is the *value* of the game.

Definition 2.3.4. Let $G = (A_1, A_2, u_1)$ be a two-player zero-sum game with value V.

i) A strategy $a_1 \in A_1$ is *optimal for player* 1 if $V = \underline{\Lambda}(a_1)$.
ii) A strategy $a_2 \in A_2$ is *optimal for player* 2 if $V = \bar{\Lambda}(a_2)$.

Under the existence of optimal strategies, the value of a zero-sum game is the payoff that player 1 can guarantee for himself; similarly, it is the opposite of the payoff that player 2 can guarantee for himself. In those two-player zero-sum games that do not have a value, the situation is not strictly determined in the sense that it is not clear how the payoff $\bar{\lambda} - \underline{\lambda}$ is going to be allocated.

The following examples illustrate the different possibilities that may arise regarding the value and the optimal strategies of a two-player zero-sum game.

Example 2.3.1. (A finite two-player zero-sum game that is strictly determined). Consider the two-player zero-sum game in Figure 2.3.1. Clearly,

	L_2	R_2
L_1	2	2
R_1	1	3

Figure 2.3.1. A strictly determined game.

$\underline{\Lambda}(L_1) = 2$ and $\underline{\Lambda}(R_1) = 1$, so $\underline{\lambda} = 2$. Besides, $\bar{\Lambda}(L_2) = 2$ and $\bar{\Lambda}(R_2) = 3$, so

$\bar{\lambda} = 2$. Hence, the value of this game is 2 and L_1 and L_2 are optimal strategies for players 1 and 2, respectively. The latter is a general feature of finite two-player zero-sum games (those in which the strategy sets of the players are finite): if they have a value, then both players have optimal strategies. $\diamond$

Example 2.3.2. (The matching pennies revisited). In this game, introduced in Example 2.2.6, $\underline{\lambda} = -1$ and $\bar{\lambda} = 1$. Hence, it is not strictly determined. However, consider the following extension of the game. Each player, instead of selecting either E or O, can randomize over the two choices. Denote by a_1 and a_2 the probability that players 1 and 2 (respectively) choose E. Consider now the infinite two-player zero-sum game $([0,1],[0,1],u_1)$ where, for each pair $(a_1,a_2) \in [0,1] \times [0,1]$, $u_1(a_1,a_2) = 4a_1a_2 - 2a_1 - 2a_2 + 1$. This new game we have defined is just the *mixed extension* of the matching pennies game; we formally define this concept in the next section. We now show that the game $(\{[0,1],[0,1]\},u_1)$ defined above is strictly determined. For each $a_1 \in [0,1]$,

$$\underline{\Lambda}(a_1) = \inf_{a_2 \in [0,1]} \big((4a_1-2)a_2 - 2a_1 + 1\big) = \begin{cases} 2a_1 - 1 & a_1 < 1/2 \\ 0 & a_1 = 1/2 \\ 1 - 2a_1 & a_1 > 1/2 \end{cases}$$

and, for each $a_2 \in [0,1]$,

$$\bar{\Lambda}(a_2) = \sup_{a_1 \in [0,1]} \big((4a_2-2)a_1 - 2a_2 + 1\big) = \begin{cases} 1 - 2a_2 & a_2 < 1/2 \\ 0 & a_2 = 1/2 \\ 2a_2 - 1 & a_2 > 1/2. \end{cases}$$

Hence, $\underline{\lambda} = \bar{\lambda} = 0$, that is, the value of the game is 0. Moreover, $a_1 = a_2 = 1/2$ is the only optimal strategy of either player; the best thing a player can do in this game is to be completely unpredictable. $\diamond$

Example 2.3.3. (An infinite two-player zero-sum game that is not strictly determined). Take the two-player zero-sum game $([0,1],[0,1],u_1)$, where, for each $(a_1,a_2) \in [0,1] \times [0,1]$, $u_1(a_1,a_2) = \frac{1}{1+(a_1-a_2)^2}$. For each $a_1 \in [0,1]$,

$$\underline{\Lambda}(a_1) = \inf_{a_2 \in [0,1]} \frac{1}{1+(a_1-a_2)^2} = \begin{cases} \frac{1}{1+(a_1-1)^2} & a_1 \leq 1/2 \\ \frac{1}{1+a_1^2} & a_1 \geq 1/2 \end{cases}$$

and $\underline{\lambda} = \underline{\Lambda}(1/2) = 4/5$. For each $a_2 \in [0,1]$,

$$\bar{\Lambda}(a_2) = \sup_{a_1 \in [0,1]} \frac{1}{1+(a_1-a_2)^2} = 1.$$

Hence $\underline{\lambda} < \bar{\lambda}$ and the game is not strictly determined. $\diamond$

Example 2.3.4. (An infinite two-player zero-sum game with a value and with optimal strategies only for one player). Consider the two-player zero-sum game $((0,1),(0,1),u_1)$, where, for each pair $(a_1,a_2) \in (0,1)\times(0,1)$, $u_1(a_1,a_2) = a_1a_2$. It is easy to check that, for each $a_1 \in (0,1)$, $\underline{\Lambda}(a_1) = 0$. Hence, $\underline{\lambda} = 0$ and, for each $a_2 \in (0,1)$, $\bar{\Lambda}(a_2) = a_2$. Thus, $\bar{\lambda} = 0$. Therefore, the game is strictly determined, its value is zero, and the set of optimal strategies of player 1 is the whole interval $(0,1)$, but player 2 does not have optimal strategies. $\diamond$

Example 2.3.5. (An infinite two-player zero-sum game with a value but without optimal strategies for any of the players). Consider the two-player zero-sum game $((0,1),(1,2),u_1)$, where, for each pair $(a_1,a_2) \in (0,1)\times(1,2)$, $u_1(a_1,a_2) = a_1a_2$. It is easy to check that, for each $a_1 \in (0,1)$, $\underline{\Lambda}(a_1) = a_1$. Hence, $\underline{\lambda} = 1$ and, for each $a_2 \in (1,2)$, $\bar{\Lambda}(a_2) = a_2$. Thus, $\bar{\lambda} = 1$. Therefore, the game is strictly determined with value one, but players do not have optimal strategies. $\diamond$

So far, we have not discussed Nash equilibria of two-player zero-sum games, even though they are strategic games. We may wonder what is the connection between von Neumann's theory and Nash's theory, *i.e.*, what is the relation between the Nash equilibrium concept and the profiles of optimal strategies in two-player zero-sum games. We address this point below.

Let (A_1,A_2,u_1) be a two-player zero-sum game. A Nash equilibrium of G is a strategy profile $(a_1^*,a_2^*) \in A_1\times A_2$ such that, for each $\hat{a}_1 \in A_1$ and each $\hat{a}_2 \in A_2$, $u_1(a_1^*,a_2^*) \geq u_1(\hat{a}_1,a_2^*)$ and $-u_1(a_1^*,a_2^*) \geq -u_1(a_1^*,\hat{a}_2)$; equivalently,

$$u_1(\hat{a}_1,a_2^*) \leq u_1(a_1^*,a_2^*) \leq u_1(a_1^*,\hat{a}_2). \tag{2.3.1}$$

A strategy profile $(a_1^*,a_2^*) \in A_1\times A_2$ satisfying Eq. (2.3.1) is said to be a *saddle point* of u_1. Hence, in two-player zero-sum games, Nash equilibria and saddle points of the payoff function of player 1 represent the same concept. We now present two propositions illustrating that Nash's theory for strategic games is a generalization of von Neumann's theory for two-player zero-sum games.

Proposition 2.3.1. *Let $G = (A_1,A_2,u_1)$ be a two-player zero-sum game and let $(a_1^*,a_2^*) \in A_1\times A_2$ be a Nash equilibrium of G. Then:*

i) *G is strictly determined.*

ii) *a_1^* is optimal for player 1 and a_2^* is optimal for player 2.*

iii) *$V = u_1(a_1^*,a_2^*)$.*

Proof. By Eq. (2.3.1) we have:

- $\underline{\lambda} = \sup_{\hat{a}_1 \in A_1} \underline{\Lambda}(\hat{a}_1) \geq \underline{\Lambda}(a_1^*) = \inf_{\hat{a}_2 \in A_2} u_1(a_1^*, \hat{a}_2) \geq u_1(a_1^*, a_2^*)$,
- $u_1(a_1^*, a_2^*) \geq \sup_{\hat{a}_1 \in A_1} u_1(\hat{a}_1, a_2^*) = \bar{\Lambda}(a_2^*) \geq \inf_{\hat{a}_2 \in A_2} \bar{\Lambda}(\hat{a}_2) = \bar{\lambda}$.

Since we know that $\bar{\lambda} \geq \underline{\lambda}$, all the inequalities have to be equalities, which implies i), ii), and iii). □

Proposition 2.3.2. *Let $G = (A_1, A_2, u_1)$ be a two-player zero-sum game. Let G be strictly determined and let $a_1 \in A_1$ and $a_2 \in A_2$ be optimal strategies of players 1 and 2, respectively. Then (a_1, a_2) is a Nash equilibrium of G and $V = u_1(a_1, a_2)$.*

Proof. Since a_1 and a_2 are optimal strategies we have that, for each $\hat{a}_1 \in A_1$ and each $\hat{a}_2 \in A_2$,

$$u_1(\hat{a}_1, a_2) \leq \bar{\Lambda}(a_2) = V = \underline{\Lambda}(a_1) \leq u_1(a_1, \hat{a}_2).$$

Taking $\hat{a}_1 = a_1$ and $\hat{a}_2 = a_2$, we have that $V = u_1(a_1, a_2)$. □

Remark 2.3.1. In view of the propositions above, if (a_1^*, a_2^*) and (a_1, a_2) are Nash equilibria of a two-player zero-sum game G, then (a_1^*, a_2) and (a_1, a_2^*) are also Nash equilibria of G and, moreover,

$$u_1(a_1^*, a_2^*) = u_1(a_1, a_2) = u_1(a_1, a_2^*) = u_1(a_1^*, a_2).$$

The above observation is not true for every two-player game; see, for instance, the battle of the sexes in Example 2.5.1 (Section 2.5).

Remark 2.3.2. A strategic game $G = (\{A_1, A_2\}, u_1, u_2)$ is a two-player *constant-sum game* if there is $K \in \mathbb{R}$ such that, for each strategy profile $(a_1, a_2) \in A_1 \times A_2$, $u_1(a_1, a_2) + u_2(a_1, a_2) = K$. A zero-sum game is a constant-sum game ($K = 0$). However, from the strategic point of view, studying G is the same as studying the zero-sum game $\bar{G} := (A_1, A_2, u_1)$ because $(a_1, a_2) \in A_1 \times A_2$ is a Nash equilibrium of G if and only if it is a Nash equilibrium of $\bar{G}$. Therefore, the approach developed in this section can be readily extended to account for two-player constant-sum games.

2.4. Mixed Strategies in Finite Games

The main focus of the rest of the chapter is on finite games.

Definition 2.4.1. Let $G = (A, u)$ be a strategic game. We say that G is a *finite game* if, for each $i \in N$, $|A_i| < \infty$.

Since the sets of strategies in a finite game are not convex sets, Nash theorem cannot be applied to them. Moreover, we have already seen a finite game without Nash equilibria: the matching pennies (Example 2.2.6). However, there is a "theoretical trick" that allows us to extend the game and to guarantee the existence of Nash equilibria of the extended version of every finite game: this trick consists of enlarging the strategic possibilities of the players and allowing them to choose not only the strategies they initially had (henceforth called *pure strategies*), but also the lotteries over their (finite) sets of pure strategies. This extension of the original game is called its *mixed extension*, and the strategies of the players in the mixed extension are called *mixed strategies*. Actually, we have already studied the mixed extension of the matching pennies game (see Example 2.3.2).

Although we have referred above to the mixed extension of a game as a "theoretical trick", mixed strategies are natural in many practical situations. We briefly discuss this point in the following example (and also later in this book), in which we informally introduce the mixed extension of a strategic game. After the example, we provide the formal definition. For a much more detailed discussion on the importance, interpretations, and appropriateness of mixed strategies, Osborne and Rubinstein (1994, Section 3.2) can be consulted.

Example 2.4.1. Consider the matching pennies game (see Example 2.2.6). Suppose that the players, besides choosing E or O, can choose a lottery L that selects E with probability $1/2$ and O with probability $1/2$ (think, for instance, of a coin toss). The players have von Neumann and Morgenstern utility functions, *i.e.*, their payoff functions can be extended to the set of mixed strategy profiles computing the mathematical expectation (Definition 1.3.4). Figure 2.4.1 represents the new game we are considering. Note

	E	O	L
E	$1,-1$	$-1,\ 1$	$0,\ 0$
O	$-1,\ 1$	$1,-1$	$0,\ 0$
L	$0,\ 0$	$0,\ 0$	$0,\ 0$

Figure 2.4.1. Matching pennies allowing for a coin toss.

that the payoff functions have been extended, taking into account that players choose their lotteries independently (we are in a strategic game). For instance:

$$u_1(L,L) = \frac{1}{4}u_1(E,E) + \frac{1}{4}u_1(E,O) + \frac{1}{4}u_1(O,E) + \frac{1}{4}u_1(O,O) = 0.$$

Observe that this game has a Nash equilibrium: (L,L). The mixed extension of the matching pennies is a new strategic game in which players can choose not only L, but also any other lottery over $\{E,O\}$. It is easy to check that the only Nash equilibrium of the mixed extension of the matching pennies is (L,L). One interpretation of this can be the following. In a matching pennies situation it is very important for both players that each one does not have any information of what will be his final choice (E or O). In order to get this, it would be optimal for every player if the player himself does not know what his final choice will be: this reasoning justifies lottery L. $\diamond$

Definition 2.4.2. Let $G = (A, u)$ be a finite game. The *mixed extension* of G is the strategic game $E(G) := (S, u)$, whose elements are the following:

> **Sets of (mixed) strategies:** For each $i \in N$, $S_i := \Delta A_i$ and $S := \prod_{i \in N} S_i$. For each $s \in S$ and each $a \in A$, let $s(a) := s_1(a_1) \cdot \ldots \cdot s_n(a_n)$.[12]
>
> **Payoff functions:** For each $s \in S$, $u_i(s) := \sum_{a \in A} u_i(a) s(a)$ and $u := \prod_{i=1}^{n} u_i$.

Remark 2.4.1. The mixed extension of a finite game only makes sense if the players have preferences over the set of lotteries on R (the set of possible outcomes) and their utility functions are von Neumann and Morgenstern utility functions (see Remark 2.1.2 for more details).

Remark 2.4.2. $E(G)$ is indeed an extension of G, in the sense that, for each player $i \in N$, each element of A_i (pure strategy) can be uniquely identified with an element of S_i (mixed strategy). In this sense, we can write $A_i \subset S_i$. Also the payoff functions in $E(G)$ are extensions of the payoff functions in G.

Remark 2.4.3. Let $m_i := |A_i|$. Then, for each $i \in N$, S_i can be identified with the simplex of $\mathbb{R}^{m_i}$ given by:

$$\{s_i \in \mathbb{R}^{m_i} : \sum_{k=1}^{m_i} s_{i,k} = 1 \text{ and, for each } k \in \{1, \ldots, m_i\},\ s_{i,k} \geq 0\}.$$

Depending on the context, this vector notation for mixed strategies may be more convenient than the representation as functions and, because of this, both notations are used in the book.

Note that the mixed extension of a finite game satisfies the conditions of the Nash theorem. Hence, the mixed extension of a finite game always has, at least, one Nash equilibrium. Actually, this was the statement proved by Nash in his original paper. We formally write this result for the sake of completeness.

Theorem 2.4.1. *Let $G = (A, u)$ be a finite strategic game. Then, the mixed extension of G, $E(G)$, has, at least, one Nash equilibrium.*

Proof. Easily follows from Nash theorem. $\square$

[12] Since $s_i(a_i)$ is the probability that player i chooses strategy a_i, $s(a)$ is the probability that the profile a is played.

To finish this section, we give some definitions and basic results regarding finite games and their mixed extensions.

Definition 2.4.3. Let G be a finite game and $E(G)$ its mixed extension. Let $s_i \in S_i$ be a (mixed) strategy for player i and $s \in S$ a (mixed) strategy profile.

i) The *support* of s_i is the set $\mathscr{S}(s_i) := \{a_i \in A_i : s_i(a_i) > 0\}$. Analogously, the support of s is the set $\mathscr{S}(s) := \prod_{i \in N} \mathscr{S}(s_i) = \{a \in A : s(a) > 0\}$.

ii) We say that s_i is *completely mixed* if $\mathscr{S}(s_i) = A_i$. Analogously, we say that s is completely mixed if $\mathscr{S}(s) = A$ or, equivalently, if, for each $i \in N$, s_i is completely mixed.

iii) The set of *pure best replies* of player i to s_{-i} is given by $\mathrm{PBR}_i(s_{-i}) := \{a_i \in A_i : \text{for each } \hat{a}_i \in A_i,\ u_i(s_{-i}, a_i) \geq u_i(s_{-i}, \hat{a}_i)\}$. Let $\mathrm{PBR}(s) := \prod_{i \in N} \mathrm{PBR}_i(s_{-i})$.

Proposition 2.4.2. *Let G be a finite game and $E(G)$ its mixed extension. Then, for each $i \in N$, each $s_i \in S_i$, and each $s \in S$, the following properties hold:*

i) $s_i \in \mathrm{BR}_i(s_{-i})$ *if and only if* $\mathscr{S}(s_i) \subset \mathrm{PBR}_i(s_{-i})$.

ii) *s is a Nash equilibrium of $E(G)$ if and only if* $\mathscr{S}(s) \subset \mathrm{PBR}(s)$.

iii) *s is a Nash equilibrium of $E(G)$ if and only if for each $\hat{a}_i \in A_i$ and each $i \in N$, $u_i(s) \geq u_i(s_{-i}, \hat{a}_i)$.*

Proof. It is clear that $u_i(s) = \sum_{a_i \in A_i} u_i(s_{-i}, a_i) s_i(a_i)$. From this fact, the proposition immediately follows. □

2.5. Bimatrix Games

In this section we discuss finite two-player games, which can be easily represented using matrix notation. Because of this, the notations in this section and in the next two are somewhat independent from the notations used in the rest of the book. The finite sets of strategies of the players are given by $L := \{1, \ldots, l\}$ for player 1 and $M := \{1, \ldots, m\}$ for player 2. Since $N = \{1, 2\}$, there is no need to use letters i and j to index the elements of N and, hence, in these sections we use them to index the strategy sets of player 1 and player 2, respectively. We denote matrices by capital calligraphic letters such as $\mathcal{A}$ and $\mathcal{B}$ with entries a_{ij} and b_{ij}. By $\mathcal{M}_{l \times m}$ we denote the set of all $l \times m$ matrices with real entries. The transpose of a matrix $\mathcal{A}$ is denoted by $\mathcal{A}^t$. Finally, we use the notations $a_{i\cdot}$ and $a_{\cdot j}$ for the i-th row and the j-th column of $\mathcal{A}$, respectively.

Definition 2.5.1. A *bimatrix game* is the mixed extension of a finite two-player game $G = (\{L, M\}, u)$. Thus, a bimatrix game is a pair $(\{S_l, S_m\}, u)$ whose elements are the following:

Sets of Strategies: $S_l := \Delta L$, with generic element x. Then, for each $x \in \Delta L$ and each $i \in L$, x_i denotes the probability that player 1 plays strategy i. Analogously, $S_m := \Delta M$, with generic element y.

Payoff functions: For each $(x, y) \in S_l \times S_m$,

$$u_1(x, y) := \sum_{i \in L} \sum_{j \in M} u_1(i, j) x_i y_j = x\mathcal{A}y^t,$$

where $\mathcal{A} \in \mathcal{M}_{l \times m}$ has entries $(u_1(i, j))_{i \in L, j \in M}$ and, for each $(x, y) \in S_l \times S_m$,

$$u_2(x, y) := \sum_{i \in L} \sum_{j \in M} u_2(i, j) x_i y_j = x\mathcal{B}y^t,$$

where $\mathcal{B} \in \mathcal{M}_{l \times m}$ has entries $(u_2(i, j))_{i \in L, j \in M}$.

To characterize a bimatrix game it would suffice to give a couple of matrices $(\mathcal{A}, \mathcal{B})$. We have already seen some examples of bimatrix games: the prisoner's dilemma, matching pennies, and the game in Example 2.1.5. In this section we discuss some other examples.

The class of bimatrix games has been thoroughly studied. There are also many results concerning Nash equilibria of bimatrix games, as well as algorithms to compute them. The most popular algorithm was provided in Lemke and Howson (1964). Parthasarathy and Raghavan (1971) is a very good book on two-player games that, in spite of being relatively old, is still an important reference in this field.

Computing all the Nash equilibria of a 2×2 bimatrix game is especially simple. To show this, take an $l \times m$ bimatrix game given by $(\mathcal{A}, \mathcal{B})$, and consider the following sets:

$$\begin{aligned} B_1 &:= \{(x, y) \in S_l \times S_m : x \in \mathrm{BR}_1(y)\}, \text{ and} \\ B_2 &:= \{(x, y) \in S_l \times S_m : y \in \mathrm{BR}_2(x)\}. \end{aligned}$$

The set of Nash equilibria of $(\mathcal{A}, \mathcal{B})$ is $B_1 \cap B_2$. Note that, when $l = m = 2$, we can identify the strategies of the players with their first components; so the strategy set of each player can be represented by $[0, 1]$, with each of its elements denoting the probability of choosing the first pure strategy. Then, B_1 and B_2 are subsets of $\mathbb{R}^2$, and their intersection can be obtained geometrically. We illustrate this procedure with a pair of examples.

Example 2.5.1. (The battle of the sexes). A couple is deciding where to go for the evening. He (player 1) prefers to go to the cinema, but she (player 2) would like to go to the theater. Finally, both prefer to go together to the least preferred show than being alone at their first options. Figure 2.5.1 shows the strategic game corresponding with this situation. The battle of the sexes

	C	T
C	$2,\ \ 1$	$0,\ \ 0$
T	$-1,-1$	$1,\ \ 2$

Figure 2.5.1. The battle of the sexes.

is a coordination problem: there are two Nash equilibria in pure strategies, (C,C) and (T,T), but player 1 prefers (C,C) while player 2 prefers (T,T). Just to illustrate the procedure we described above, we compute all the Nash equilibria in mixed strategies of the battle of the sexes. Since the strategy sets of the players can be identified with $[0,1]$, a strategy of player 1 is an $x \in [0,1]$, and a strategy of player 2 is a $y \in [0,1]$ (x is actually a shortcut for the strategy $(x, 1-x)$ and y is a shortcut for $(y, 1-y)$). Then, for this game we have

$$B_1 = \{(0,y) : y \in [0,1/4]\} \cup \{(x,1/4) : x \in [0,1]\} \cup \{(1,y) : y \in [1/4,1]\},$$

$$B_2 = \{(x,0) : x \in [0,3/4]\} \cup \{(3/4,y) : y \in [0,1]\} \cup \{(x,1) : x \in [3/4,1]\}.$$

Figure 2.5.2 depicts the sets B_1 and B_2. Then, the set of Nash equilibria

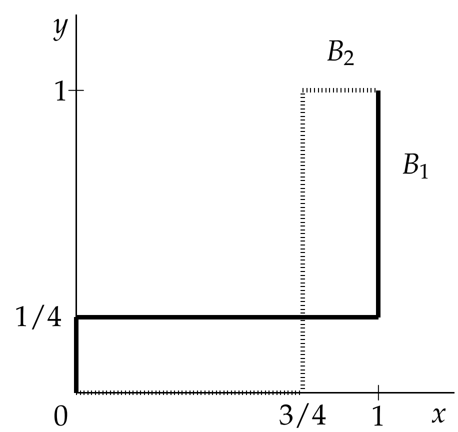


Figure 2.5.2. B_1 and B_2 in the battle of the sexes.

of the battle of the sexes is given by $B_1 \cap B_2 = \{(0,0),(3/4,1/4),(1,1)\}$. Note that, in the mixed strategy equilibrium, both players are indifferent between their two pure strategies, *i.e.*, both of them are best replies (this can be easily seen in Figure 2.5.2); although we already knew this from Proposition 2.4.2. The equilibrium payoff vectors are $(2,1)$, $(1/2,1/2)$, and

$(2,1)$, so the payoff in the mixed strategy equilibrium is dominated by any of the pure strategy ones. Moreover, the set of equilibrium payoff vectors is symmetric, as we should expect from a symmetric game (but not all the equilibrium payoff vectors are symmetric themselves). $\diamond$

Example 2.5.2. (The instigation game). A thief is deciding whether or not to steal from a warehouse tonight. A guard who works for the owner of the warehouse, guards it during the night. The guard is deciding whether to sleep or not tonight while he is supposed to be guarding the warehouse. If the thief steals and the guard sleeps, the thief will get a very expensive jewel (J) and the guard will be fired ($-F$). If the thief steals and the guard does not sleep, the thief will be sent to prison ($-P$) and the guard will get a medal (M). If the thief does not steal and the guard sleeps, the guard will get some rest (R). Suppose that J, F, P, M, and R are positive real numbers. Figure 2.5.3 illustrates the strategic game associated with this situation. We

	SL	DSL
ST	$J, -F$	$-P, \quad M$
DST	$0, \quad R$	$0, \quad 0$

Figure 2.5.3. The instigation game.

identify a strategy of player 1 (the thief) with the probability that he steals ($x \in [0,1]$), and a strategy of player 2 (the guard) with the probability that he sleeps ($y \in [0,1]$). Then, it is easy to check that:

$$\begin{aligned} B_1 &= \{(0,y) : y \in [0, \tfrac{P}{P+J}]\} \cup \{(x, \tfrac{P}{P+J}) : x \in [0,1]\} \\ &\quad \cup \{(1,y) : y \in [\tfrac{P}{P+J}, 1]\}, \text{ and} \\ B_2 &= \{(x,1) : x \in [0, \tfrac{R}{R+F+M}]\} \cup \{(\tfrac{R}{R+F+M}, y) : y \in [0,1]\} \\ &\quad \cup \{(x,0) : x \in [\tfrac{R}{R+F+M}, 1]\}. \end{aligned}$$

Figure 2.5.4 depicts B_1 and B_2. Then, $B_1 \cap B_2 = \{(\frac{R}{R+F+M}, \frac{P}{P+J})\}$, which determines the unique Nash equilibrium of this game. A remarkable feature concerning this example is that the equilibrium strategy of the thief does not depend on the prizes or penalties he may get (J and P), but on the prizes or penalties that the guard may achieve (M, R, and F). Hence, if an external regulator, who wants to reduce the probability that the thief steals, is to choose between increasing the punishment for the thief (P) or increasing the incentives and/or punishment for the guard (M and/or F), he must take into account the features summarized in Figure 2.5.5.

This example also provides a new interpretation of mixed strategies. In fact, the instigation game models an inspection problem: the "thief" is someone who is thinking of acting or not according to the law (for instance,

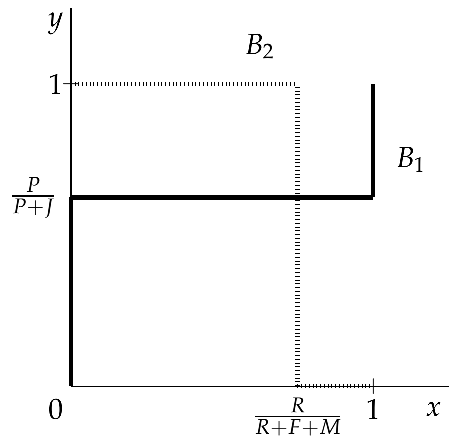


Figure 2.5.4. B_1 and B_2 in the instigation game.

INSTIGATION	CONSEQUENCE
Increasing P, the punishment for the thief.	The thief steals the same. The guard sleeps more.
Increasing M and/or F, the incentive and/or punishment for the guard.	The thief steals less. The guard sleeps the same.

Figure 2.5.5. Implications of the equilibria in the instigation game.

a tax payer), and the "guard" is an inspector. Further, we can interpret that player 1 is the population of tax payers, and player 2 is the population of tax inspectors. Now, a mixed strategy of player 1 may be thought of as the ratio of tax payers who are not acting according to the law, and a mixed strategy of player 2 can be interpreted as the ratio of tax inspectors who are not doing their job well. Then, if the regulator is the Government of the country, their target is that tax payers act according to the law. According to our analysis above, the Government must stress the measures concerning the incentives for the inspectors and not the measures concerning the punishment for the tax payers (which, by the way, are politically much more unpopular). $\diamond$

The class of 2×2 bimatrix games, in spite of being a simple and "small" one, provides many examples that can be useful to model and enlighten a large variety of real situations emerging from politics, economics, biology, literature, sociology, anthropology, etc. The prisoner's dilemma is especially emblematic and it has been widely used by social scientists. Next, we show one further example to illustrate this point.

Example 2.5.3. This example is taken from Deutsch (1958, 1960) and Straffin (1993). It illustrates an application of the prisoner's dilemma game in experimental psychology. In the 1950s, the F-scale was designed by a group of psychologists to test susceptibility to authoritarian ideologies. In the two works mentioned above, Deutsch investigated the connection between the F-scale and the standard concepts of trust, suspicion, and trustworthiness (suspicious and untrustworthy people may perhaps tend to prefer authoritarian ideologies). He asked 55 students from his introductory psychology course to play the following modification of the prisoner's dilemma game. Player 1 chooses between ND and D. Then, player 2 chooses between ND and D after observing the choice of player 1. The payoffs to the players are given in Figure 2.5.6. Yet, note that the actual game being played does not correspond with the one in Figure 2.5.6. Since player 2 observes the strategy of player 1, he has 4 strategies instead of 2 and the payoffs in each case are easily computed from those in Figure 2.5.6. Every student was asked to

	ND	D
ND	$9, \quad 9$	$-10, \quad 10$
D	$10, -10$	$-9, -9$

Figure 2.5.6. Description of the payoffs.

play this game twice: once as player 1, and once as player 2. The students never knew who the other player was in either case. In fact, every time a student played the role of player 2, player 1 was fictitious: the experimenter was player 1 and chose ND. Deutsch used the following definitions for his analysis:

- **Trust:** To choose ND when you are player 1.
- **Suspicion:** To choose D when you are player 1.
- **Trustworthiness:** To choose ND when you are player 2 and you know that player 1 has chosen ND.
- **Untrustworthiness** To choose D when you are player 2 and you know that player 1 has chosen ND.

To perform a statistical analysis of the results obtained, Deutsch divided the F-scale scores in three categories: low, medium and high. The corresponding contingency tables are depicted in Figure 2.5.7. Using a χ^2 test for independence, it can be checked that, for reasonable confidence levels, the hypothesis of independence of the qualitative variables considered should be rejected in both tables. Hence, in the experiment above, we can conclude that:

- Each student tends to think that the others will behave as he does.

	Trustworthy	Untrustworthy
Trusting	24	5
Suspicious	4	22

	Low	Medium	High
Trusting and trustworthy	12	10	2
Suspicious or untrustworthy	2	7	0
Suspicious and untrustworthy	0	13	9

Figure 2.5.7. Contingency tables of Deutsch's analysis.

- Suspicious and untrustworthy students tend to get high scores in the F-scale.

There are a pair of observations to be made with respect to this example. First, strictly speaking, since we restrict our attention to pure strategies, the experiment does not deal with a bimatrix game, but rather with a finite two-player game (recall that we defined a bimatrix game as the mixed extension of a finite two-player game). Moreover, with the payoffs given by Figure 2.5.6, it is clear that a rational player should be suspicious and untrustworthy. In fact, only if the numbers describe monetary gains, but not the true utility functions of the players (which should incorporate ethical and psychological considerations), do we get that the definitions of *trust*, *suspicion*, *trustworthiness*, and *untrustworthiness* are acceptable from a game theoretic point of view. $\diamond$

2.6. Matrix Games

Definition 2.6.1. A *matrix game* is the mixed extension of a finite two-player zero-sum game $G = (L, M, u_1)$, *i.e.*, a matrix game is a triple (S_l, S_m, u_1) such that:

Sets of Strategies: $S_l := \Delta L$, with generic element x, *i.e.*, for each $x \in \Delta L$ and each $i \in L$, x_i is the probability that player 1 plays strategy i. Analogously, $S_m := \Delta M$, with generic element y.

Payoff functions: For each $(x, y) \in S_l \times S_m$,

$$u_1(x, y) = \sum_{i \in L} \sum_{j \in M} u_1(i, j) x_i y_j = x\mathcal{A}y^t,$$

where $\mathcal{A} \in \mathcal{M}_{l \times m}$ has entries $(u_1(i, j))_{i \in L, j \in M}$ containing the payoffs to player 1.

Observe that the matrix $\mathcal{A}$ suffices to characterize a matrix game. Moreover, note that given $x \in S_l$, $xa_{\cdot j}$ corresponds to the payoff that player 1 gets when he plays mixed strategy x and player 2 plays pure strategy $j \in M$;

similarly, given $y \in S_m$, $a_{i\cdot}y^t$ corresponds to the payoff that player 1 gets when he plays pure strategy $i \in L$ and player 2 plays mixed strategy y. Note that every matrix game can be seen as a bimatrix game, though the converse is not true. Hence, Nash theorem can be applied to assure that every matrix game has a Nash equilibrium. Then, in view of Proposition 2.3.1, we know that every matrix game is strictly determined. This is the famous minimax theorem, which was proved in von Neumann (1928). Although we are presenting it as a corollary of Nash theorem, there are many direct proofs of the minimax theorem: one based on a separation theorem, one based on an alternative lemma for matrices, one based on a fixed point argument, and there is even one based on the induction principle. Below, we present the latter proof, which is due to Owen (1967); it is an elementary one, in the sense that it barely relies on any mathematical results. We first introduce an auxiliary result.

Proposition 2.6.1. *Take an $l \times m$ matrix game given by matrix $\mathcal{A}$. Then, for each $x \in S_l$ and each $y \in S_m$,*

$$\underline{\Lambda}(x) = \min_{j \in M} x a_{\cdot j} \quad \textit{and} \quad \bar{\Lambda}(y) = \max_{i \in L} a_{i\cdot} y^t.$$

Proof. We only prove the first equality, since the second one is analogous. Let $x \in S_l$. Recall that, for each $i \in M$, $e_i \in S_m$ denotes the mixed strategy that selects i with probability 1; hence, according to the notation in this section, $e_i \in \mathbb{R}^m$ and $(e_i)_j = 1$ if $j = i$ and 0 otherwise. Now, $\underline{\Lambda}(x) = \inf_{y \in S_m} x\mathcal{A}y^t \leq \min_{j \in M} x\mathcal{A}e_j^t = \min_{j \in M} xa_{\cdot j}$. For each $y \in S_m$,

$$x\mathcal{A}y^t = \sum_{k \in M} (xa_{\cdot k})y_k \geq \sum_{k \in M} (\min_{j \in M} xa_{\cdot j})y_k = (\min_{j \in M} xa_{\cdot j}) \sum_{k \in M} y_k = \min_{j \in M} xa_{\cdot j}.$$

Hence, $\underline{\Lambda}(x) = \inf_{y \in S_m} x\mathcal{A}y^t \geq \min_{j \in M} xa_{\cdot j}$. □

Note that this proposition immediately implies that the functions $\underline{\Lambda}$ and $\bar{\Lambda}$ are continuous, since they are, respectively, the minimum and the maximum of a finite number of continuous functions.

Theorem 2.6.2 (Minimax theorem)**.** *Every matrix game is strictly determined.*

Proof. Let $\mathcal{A}$ be an $l \times m$ matrix game. Then, we say that the *size* of $\mathcal{A}$ is just $l + m$. The proof is by induction on the size of $\mathcal{A}$. If the size is 2, then $\mathcal{A}$ is strictly determined. Assume that every matrix game is strictly determined if its size is smaller than t, and let $\mathcal{A}$ be a matrix game of size t. Since $\underline{\Lambda}$ and $\bar{\Lambda}$ are continuous functions and they are respectively defined on S_l and S_m, which are compact sets, there are $\bar{x} \in S_l$ and $\bar{y} \in S_m$ such that

$$\underline{\Lambda}(\bar{x}) = \min_{j \in M} \bar{x}a_{\cdot j} = \underline{\lambda} \quad \text{and} \quad \bar{\Lambda}(\bar{y}) = \max_{i \in L} a_{i\cdot}\bar{y}^t = \bar{\lambda}. \tag{2.6.1}$$

Hence, for each $j \in M$, $\bar{x}a_{\cdot j} \geq \underline{\lambda}$ and, for each $i \in L$, $a_{i\cdot}\bar{y}^t \leq \bar{\lambda}$. We distinguish three cases.

i) For each $j \in M$, $\bar{x}a_{\cdot j} = \underline{\lambda}$, and, for each $i \in L$, $a_{i\cdot}\bar{y}^t = \bar{\lambda}$.

ii) There is $k \in M$ such that $\bar{x}a_{\cdot k} > \underline{\lambda}$, which implies, in particular, that $m > 1$.

iii) There is $h \in L$ such that $a_{h\cdot}\bar{y}^t < \bar{\lambda}$, which implies, in particular, that $l > 1$.

In case i), $\mathcal{A}$ is strictly determined. We concentrate on case ii), since case iii) is analogous. Recall that, under case ii), $m > 1$. Let $\mathcal{A}^{-k}$ be the matrix resulting after deleting the k-th column of $\mathcal{A}$. The set of mixed strategies of player 2 in $\mathcal{A}^{-k}$ can be identified with $S_m^{-k} := \{y \in S_m : y_k = 0\}$. By the induction hypothesis, $\mathcal{A}^{-k}$ is strictly determined. Let $\underline{\lambda}^{-k}$ and $\bar{\lambda}^{-k}$ be the lower and upper values of the matrix game $\mathcal{A}^{-k}$, respectively. Then,

$$\underline{\lambda}^{-k} = \max_{x \in S_l} \min_{j \in M\backslash\{k\}} xa_{\cdot j} \geq \max_{x \in S_l} \min_{j \in M} xa_{\cdot j} = \underline{\lambda}$$

and

$$\bar{\lambda}^{-k} = \min_{y \in S_m^{-k}} \max_{i \in L} a_{i\cdot}y^t \geq \min_{y \in S_m} \max_{i \in L} a_{i\cdot}y^t = \bar{\lambda}.$$

We now show that $\underline{\lambda} = \underline{\lambda}^{-k}$, which implies that $\underline{\lambda} = \underline{\lambda}^{-k} = \bar{\lambda}^{-k} \geq \bar{\lambda} \geq \underline{\lambda}$ and, hence, $\mathcal{A}$ is strictly determined. Suppose, on the contrary, that $\underline{\lambda}^{-k} > \underline{\lambda}$ and let $\hat{x}$ be such that $\underline{\lambda}^{-k} = \min_{j \in M\backslash\{k\}} \hat{x}a_{\cdot j}$. Then, for each $j \in M\backslash\{k\}$, $\hat{x}a_{\cdot j} \geq \underline{\lambda}^{-k} > \underline{\lambda}$. The latter, together with Eq. (2.6.1), implies that, for each $\varepsilon \in (0,1)$ and each $j \in M\backslash\{k\}$, $(\varepsilon\hat{x} + (1-\varepsilon)\bar{x})a_{\cdot j} > \underline{\lambda}$. Moreover, since $\bar{x}a_{\cdot k} > \underline{\lambda}$, there is $\bar{\varepsilon} \in (0,1)$ small enough so that $(\bar{\varepsilon}\hat{x} + (1-\bar{\varepsilon})\bar{x})a_{\cdot k} > \underline{\lambda}$. Therefore,

$$\underline{\lambda} = \max_{x \in S_l} \underline{\Lambda}(x) \geq \underline{\Lambda}(\bar{\varepsilon}\hat{x} + (1-\bar{\varepsilon})\bar{x}) = \min_{j \in M}(\bar{\varepsilon}\hat{x} + (1-\bar{\varepsilon})\bar{x})a_{\cdot j} > \underline{\lambda},$$

which is impossible. □

In Section 2.7 we present a couple of algorithms to solve matrix games. However, before getting into the algorithms, we present a brief discussion on matrix games making use of two examples.

In parallel with the theoretical developments in game theory, as the celebrated minimax theorem we have just presented, one should expect to see tests of its implications in real life. However, this has proven to be very difficult in the literature, since it is often hard to determine the strategy sets of the players, measure individual choices and effort levels, and properly identify utility functions. Moreover, these real life situations typically involve complex problems and large strategy sets, which make it even harder to bridge theory and applications. An alternative approach has been to test theory in laboratory experiments, where one has control to design simple games and perfectly observe agents' behavior. However, there is some skepticism about how applicable behavioral insights obtained in a laboratory can be for understanding human behavior in real life. For instance, in Hart (2005), Robert J. Aumann argues that, in experiments, "the monetary payoff is usually very small. More importantly, the decisions that people face are not ones that they usually take, with which they are familiar. The whole setup is artificial. It is not a decision that really affects them and to which they are used." The example we present below, which is based on the paper Palacios-Huerta (2003), provides a natural setting that allows the author to overcome most of the aforementioned difficulties and test some of the implications of the minimax theorem in a real life setting.

Example 2.6.1. This example explores the use of mixed strategies in penalty kicks in soccer. A penalty kick involves two players, the kicker (player 1) and the goalkeeper (player 2). A goalkeeper, due to the speed of a typical kick and his own reaction time, cannot wait until the kicker has kicked the ball to start his movement. Thus, it seems reasonable to think of this situation as a simultaneous move game. The natural kick of a right-footed kicker is either to the right-hand side of the goalkeeper or to the center and the natural kick of a left-footed kicker is to the left-hand side of the goalkeeper or to the center. Based on the above considerations, we now present a simple matrix game that is a fair simplification of the strategic interaction involved in a penalty kick.[13] The strategy sets of the players are $A_1 = A_2 = \{N, U\}$, where N represents choosing the natural side of the kicker and U represents the unnatural side. This game is a kind of matching pennies game (Example 2.2.6), in the sense that the goalkeeper wants to match the action chosen by the kicker and the kicker wants to avoid it. That is, this game can be seen as a matrix game like the one in Figure 2.6.1, where $p_{a_1 a_2}$ denotes the percentage of scored penalty kicks if the kicker chooses a_1 and the goalkeeper chooses a_2. Since the goalkeeper wants to match the choice of the kicker, this game is such that $p_{UU} < p_{UN}$, $p_{UU} < p_{NU}$, $p_{NN} < p_{UN}$, and $p_{NN} < p_{NU}$. This payoff structure ensures, as in the matching pennies

	U	N
U	p_{UU}	p_{UN}
N	p_{NU}	p_{NN}

Figure 2.6.1. Matrix game associated with a penalty kick.

game, that equilibrium play requires the use of mixed strategies. In the

[13]We refer the reader to the original paper for further explanations concerning the necessity and implications of each of the assumed simplifications.

data set used in Palacios-Huerta (2003) the average probabilities are given in Figure 2.6.2.[14] In particular we see that scoring probabilities are very

	U	N
U	0.58	0.95
N	0.93	0.70

Figure 2.6.2. Payoffs from scoring probabilities in the data set used in Palacios-Huerta (2003).

high when the goalkeeper does not match the strategy of the kicker and, when he does match the strategy of the kicker, his odds are better if the ball was not kicked to the natural side of the kicker. Given these probabilities, the unique Nash equilibrium is given by the mixed strategy $(0.38, 0.62)$ for the kicker and $(0.42, 0.58)$ for the goalkeeper.[15] One of the main findings in Palacios-Huerta (2003) is that the observed frequencies chosen by the professional players in their data set are given by $(0.40, 0.60)$ for the kickers and $(0.42, 0.58)$ for the goalkeepers, that is, the aggregate behavior of the professional soccer players is extremely close to the Nash equilibrium of the associated matrix game. In the rest of the paper the author tests two important implications of the minimax theorem. First, in the equilibrium, both kickers and goalkeepers should be indifferent between their two actions. By running a series of statistical tests, the author shows that the null hypothesis that scoring probabilities are identical across strategies cannot be rejected at standard significance levels, neither for the aggregate population of players nor at the individual level. Second, a player's mixed strategy should be the same in each penalty kick and, therefore, one should observe that the realizations of a player's mixed strategy, *i.e.*, his choices, are serially independent; that is, he should switch his actions neither too often nor too rarely to be consistent with the randomness implied by equilibrium play. Part of the interest of this second aspect comes from the fact that usually, when subjects (typically college students) are asked to generate random sequences in a lab, for instance replicating repeated coin tosses, their sequences exhibit negative autocorrelation, *i.e.*, individuals tend to exhibit bias against repeating the same choice. Yet, this is not the case with the soccer players in this analysis, where one cannot reject the null hypothesis that their choices are independent draws from the probability distributions given by their mixed strategies.

[14] For the sake of exposition we round all the probabilities in this example to two precision digits.

[15] This equilibrium can be computed either by computing the Nash equilibria of the corresponding 2×2 bimatrix game or by using one of the algorithms in Section 2.7 below.

Interestingly, in the paper by Palacios-Huerta and Volij (2008), the authors go one step beyond and take professional soccer players and college students to a laboratory and ask them to play a matrix game that is formally identical to the penalty kick situation. They find that, consistent with their behavior on the field, soccer players play very close to equilibrium behavior, whereas college students do not. Moreover, soccer players generate serially independent sequences of choices whereas the sequences of the students exhibit negative autocorrelation. They interpret these results as "evidence that professionals transfer their skills across these vastly different environments and circumstances". ◇

We conclude this section with an application of matrix games to anthropology.

Example 2.6.2. This example is taken from Davenport (1960) and Straffin (1993). In this work, Davenport illustrates with an example that social behavior patterns can sometimes be functional responses to problems that the society must solve. Davenport studied a small village in Jamaica, where about two hundred inhabitants made their living by fishing. The fishing grounds can be divided into inside banks (5-15 miles offshore) and outside banks (15-22 miles offshore). Twenty-six fishing crews fish (in canoes) by setting pots which are drawn and reset three days per week. Because of special underwater contours, very strong currents move across the outside banks. These currents are unpredictable. The captains of the crews might take one of the following three strategies: fish in the inside banks (I), fish in the outside banks (O), or fish in both inside and outside banks (I-O). Fishing in the inside banks is safer, in the sense that when the current runs, the fish in the outside banks is lost; and easier because they are closer to the coast. However, in the outside banks much better fish can be obtained. Figure 2.6.3 displays the estimation made by Davenport of the average profits in English pounds per month for the captains of the canoes, depending on the fishing strategy chosen and on the fact if the current runs (R) or not (N). He points out that he made these estimations before planning a game theoretical analysis of the data.

	R	N
I	17.3	11.5
O	-4.4	20.6
I-O	5.2	17

Figure 2.6.3. Profit estimations.

Then, the captains of the canoes faced a decision problem that can be treated as a 3×2 matrix game. If we solve this game by any of the two methods described in Section 2.7 below, the result is that the first player has a unique optimal strategy $(0.67, 0, 0.33)$. Surprisingly enough, the real behavior of the captains was very close to this optimal strategy. Davenport observed that 69% of them chose strategy I, 31% followed strategy I-O, and no captains used strategy O. He also observed that the current ran 25% of the time (which is also very close to the unique optimal strategy of player 2, namely $(0.31, 0.69)$, although this is just a coincidence). A criticism that should be made to this analysis is that this situation is not really a game because player 2 is not choosing his "strategy". Hence, the rational behavior for player 1 should be observing the current and responding optimally. However, the strategy of player 1, which maximizes his expected payoff taking into account that the current runs 25% of the time, is $(0, 1, 0)$. Against this view it may be argued that the current is unpredictable and that the percentage of time it runs, viewed as a random variable, may have a large variance. This may explain why society prefers the more conservative maximin strategy, which, regardless of the true frequency of the currents, guarantees a reasonable payoff (actually, very close to the optimal one) and the continuity and survival of this society. $\diamond$

2.7. Algorithms for Matrix Games

In this section we present two algorithms to *solve* matrix games; that is, to find a matrix game's value and its sets of optimal strategies. Moreover, in Section 2.8 below, we show that solving a matrix game can be reduced to solving a pair of dual linear programming problems and, therefore, the algorithms to solve linear problems can be used to *solve* matrix games. Given a matrix game $\mathcal{A}$, we denote by $O_1(\mathcal{A})$ and $O_2(\mathcal{A})$, the sets of optimal strategies of players 1 and 2, respectively. We start with an algorithm for $2 \times m$ matrix games (actually, it is a geometric construction, not properly an algorithm). Take a $2 \times m$ matrix game given by a matrix $\mathcal{A}$. As we did in the previous section with 2×2 bimatrix games, we can identify S_2, the strategy set of player 1 with $[0,1]$, and $x \in [0,1]$ denotes the probability that player 1 chooses his first strategy. By Proposition 2.6.1, we have:

$$V = \max_{x \in [0,1]} \underline{\Lambda}(x) = \max_{x \in [0,1]} \min_{j \in M} (x, 1-x) a_{\cdot j} = \max_{x \in [0,1]} \min_{j \in M} (a_{2j} + (a_{1j} - a_{2j})x).$$

Given the above equation, one can compute the value of the game and the sets of optimal strategies of player 1 in a few simple steps. First, for each $j \in M$, draw the segment $\{(x, a_{2j} + (a_{1j} - a_{2j})x) : x \in [0,1]\}$. Then, obtain the lower envelope of all those segments. Finally, look for the strategies $\bar{x} \in S_l$ that maximize that envelope. Figure 2.7.1 illustrates this procedure.

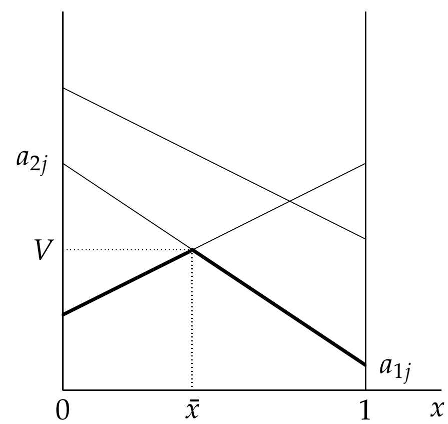


Figure 2.7.1. The value and optimal strategies of player 1.

To compute the set of optimal strategies of player 2, we must take into account that $y \in S_m$ is one of these strategies if and only if:

$$V = \bar{\Lambda}(y) = \max_{x\in[0,1]} \sum_{j\in M} (a_{2j} + (a_{1j} - a_{2j})x)y_j.$$

Hence, an optimal strategy of player 2 is any element of S_m providing a convex combination of the segments $\{(x, a_{2j} + (a_{1j} - a_{2j})x) : x \in [0,1]\}$ which lies under the segment $\{(x, V) : x \in [0,1]\}$. We illustrate this method in a couple of examples.

Example 2.7.1. Take the matrix game given by

$$\mathcal{A} = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 4 & 3 \end{pmatrix}.$$

In Figure 2.7.2, we show the segments corresponding to the columns of $\mathcal{A}$ and their lower envelope. We have that $V = 2$, $O_1(\mathcal{A}) = \{(1,0)\}$, and $O_2(\mathcal{A}) = \mathrm{conv}(\{(1,0,0),(2/3,1/3,0)\})$. $\diamond$

Example 2.7.2. Now we solve the matrix game given by

$$\mathcal{A} = \begin{pmatrix} 0 & 3 \\ 2 & 2 \\ 3 & 0 \end{pmatrix}.$$

Note first that, although this is not a $2 \times m$ matrix game, it can be solved with our method by interchanging the names of the players.[16] The resulting

[16]Another possibility would be to design the method for $l \times 2$ matrix games; and this design would be very similar to the one we have done for the $2 \times m$ method.

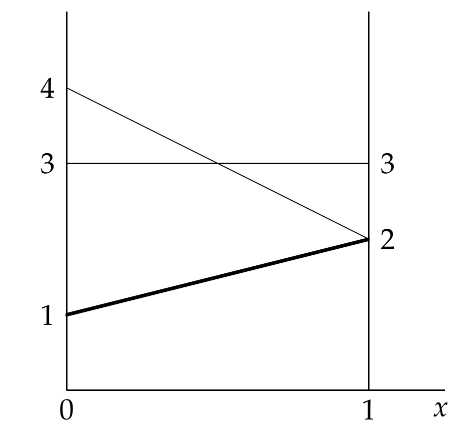


Figure 2.7.2. Solving the game in Example 2.7.1.

matrix game is characterized by

$$\mathcal{B} = \begin{pmatrix} 0 & -2 & -3 \\ -3 & -2 & 0 \end{pmatrix}.$$

We depict the segments corresponding to the columns of $\mathcal{B}$ and their lower envelope in Figure 2.7.3. We have $V_{\mathcal{B}} = -2$, $O_2(\mathcal{B}) = \{(0,1,0)\}$, and $O_1(\mathcal{B}) = \mathrm{conv}(\{(1/3,2/3),(2/3,1/3)\})$. Then, we have $V_{\mathcal{A}} = 2$, $O_1(\mathcal{A}) = \{(0,1,0)\}$, and $O_2(\mathcal{A}) = \mathrm{conv}(\{(1/3,2/3),(2/3,1/3)\})$. $\diamond$

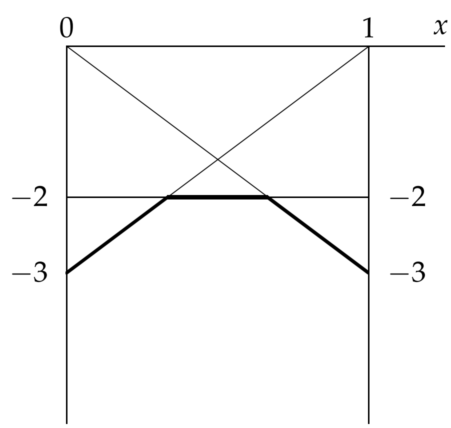


Figure 2.7.3. Solving the game in Example 2.7.2.

Next, we describe the *submatrices method*, an algorithm to solve any $l \times m$ matrix game. It is based on a characterization of the extreme points of the sets of optimal strategies of the players. We start with some preliminary results.

Proposition 2.7.1. *Let $\mathcal{A}$ be an $l \times m$ matrix game. Let $x \in S_l$ and $y \in S_m$. Then,*

i) $x \in O_1(\mathcal{A})$ *if and only if, for each* $j \in M$, $xa_{\cdot j} \geq V$.

ii) $y \in O_2(\mathcal{A})$ *if and only if, for each* $i \in L$, $a_{i\cdot}y^t \leq V$.

Proof. We only prove the first equivalence, the second one being analogous. Suppose that $x \in O_1(\mathcal{A})$. Then, by Proposition 2.6.1, $V = \underline{\Lambda}(x) = \min_{j \in M} xa_{\cdot j}$ and, hence, for each $j \in M$, $xa_{\cdot j} \geq V$. Conversely, if, for each $j \in M$, $xa_{\cdot j} \geq V$, then $V = \max_{\hat{x} \in S_l} \underline{\Lambda}(\hat{x}) \geq \underline{\Lambda}(x) = \min_{j \in M} xa_{\cdot j} \geq V$. □

Proposition 2.7.2. *Let $\mathcal{A}$ be an $l \times m$ matrix game. Then, $O_1(\mathcal{A})$ and $O_2(\mathcal{A})$ are convex and compact sets.*

Proof. In view of the last proposition, $O_1(\mathcal{A})$ and $O_2(\mathcal{A})$ are convex sets. Since the sets of strategies are bounded sets, $O_1(\mathcal{A})$ and $O_2(\mathcal{A})$ are also bounded sets. Since $\underline{\Lambda}$ and $\bar{\Lambda}$ are continuous functions, $O_1(\mathcal{A}) = \underline{\Lambda}^{-1}(\{V\})$ and $O_2(\mathcal{A}) = \bar{\Lambda}^{-1}(\{V\})$ are closed sets. □

Proposition 2.7.3. *Let $\mathcal{A}$ be an $l \times m$ matrix game. Let $x \in S_l$ and $y \in S_m$. Then, $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$ if and only if, for each $i \in L$ and each $j \in M$, $xa_{\cdot j} \geq a_{i\cdot}y^t$.*

Proof. The "only if" part is a consequence of Proposition 2.7.1. Conversely, if, for each $i \in L$ and each $j \in M$, $xa_{\cdot j} \geq a_{i\cdot}y^t$, then

$$V = \bar{\lambda} \leq \bar{\Lambda}(y) = \max_{i \in L} a_{i\cdot}y^t \leq \min_{j \in M} xa_{\cdot j} = \underline{\Lambda}(x) \leq \underline{\lambda} = V.$$

Hence, all the inequalities have to be equalities and, therefore, $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$. □

The following result implies that, if we add a constant to all the entries of a matrix game, the resulting matrix game is strategically equivalent to the original game.

Proposition 2.7.4. *Let $K \in \mathbb{R}$ and let $\mathcal{A}$ and $\mathcal{B}$ be two $l \times m$ matrix games such that, for each $i \in L$ and each $j \in M$, $b_{ij} = a_{ij} + K$. Then, $V_{\mathcal{B}} = V_{\mathcal{A}} + K$, $O_1(\mathcal{A}) = O_1(\mathcal{B})$, and $O_2(\mathcal{A}) = O_2(\mathcal{B})$.*

Proof. Since, for each $x \in S_l$ and each $y \in S_m$, $\underline{\Lambda}_{\mathcal{B}}(x) - \underline{\Lambda}_{\mathcal{A}}(x) = \bar{\Lambda}_{\mathcal{B}}(y) - \bar{\Lambda}_{\mathcal{A}}(y) = K$, the result is straightforward. □

Definition 2.7.1. Let $\mathcal{A}$ be an $l \times m$ matrix game. Let $x \in S_l$ and $y \in S_m$. The pair (x, y) is a *simple solution* of $\mathcal{A}$ if, for each $i \in L$ and each $j \in M$, $xa_{\cdot j} = a_{i\cdot}y^t$.

In a simple solution (x, y), the strategy of each player leaves the other indifferent among all his strategies. Hence, if (x, y) is a simple solution of $\mathcal{A}$, then it is also a Nash equilibrium of $\mathcal{A}$ (the reciprocal is not true).

We now present the three main theorems on which the submatrices method is based. The first of these theorems is a classical result on convex analysis known as the Krein-Milman theorem, whose proof is presented in Section 2.14 along with some other results on convex sets. The other two results are due to Shapley and Snow (1950). Also, recall that $|\mathcal{A}|$ denotes the determinant of $\mathcal{A}$. Let $\operatorname{ext}(S)$ denote the set of extreme points of the convex set S.

Theorem 2.7.5 (Krein-Milman theorem). *Let $S \subset \mathbb{R}^m$ be a nonempty, convex, and compact set. Then, i) $\operatorname{ext}(S) \neq \emptyset$ and ii) $\operatorname{conv}(\operatorname{ext}(S)) = S$.*

Proof. Refer to Section 2.14. $\square$

Before moving on, we need some extra matrix notations. Let $\mathcal{A}$ be a square matrix. Then, $|\mathcal{A}|$ denotes the determinant of $\mathcal{A}$ and a^*_{ij} denotes the ij-cofactor of $\mathcal{A}$.[17] Now, $\mathcal{A}^*$ denotes the matrix of cofactors, *i.e.*, the entries of $\mathcal{A}^*$ are the a^*_{ij} cofactors. If $\mathcal{A}$ is invertible, then $\mathcal{A}^{-1} = (\mathcal{A}^*)^t/|\mathcal{A}|$. Given a matrix $\mathcal{A}$ we define, for each $i \in M$, $\bar{r}_i := \sum_{j\in M} a^*_{ij}$ and, for each $j \in M$, $\bar{c}_j := \sum_{i\in M} a^*_{ij}$, *i.e.*, $\bar{r}_i$ ($\bar{c}_j$) is the sum of the entries in the i-th row (j-th column) of $\mathcal{A}^*$. There will be no need to make explicit the dependence of the $\bar{r}_i$ and $\bar{c}_j$ coefficients on the matrix $\mathcal{A}$. Let $\mathbb{1}_m := (1, \ldots, 1) \in \mathbb{R}^m$.

Theorem 2.7.6. *Let $\mathcal{A}$ be an $m \times m$ matrix game with $|\mathcal{A}| \neq 0$. Then, $\mathcal{A}$ has a simple solution if and only if the numbers $\bar{r}_1, \ldots, \bar{r}_m, \bar{c}_1, \ldots, \bar{c}_m$ are either all nonpositive or all nonnegative. Moreover, if $\mathcal{A}$ has a simple solution, then it has a unique simple solution given by:*

$$\Big(\frac{1}{\sum_{i\in M} \bar{r}_i}(\bar{r}_1, \ldots, \bar{r}_m), \frac{1}{\sum_{j\in M} \bar{c}_j}(\bar{c}_1, \ldots, \bar{c}_m)\Big).$$

Proof. Suppose that (x, y) is a simple solution of $\mathcal{A}$. By definition, for each pair $i, j \in M$, $xa_{\cdot j} = a_{i\cdot}y^t$ and, hence, $V = xa_{\cdot j} = a_{i\cdot}y^t$. Then, $x\mathcal{A} = V\mathbb{1}_m$ and $\mathcal{A}y^t = V\mathbb{1}^t_m$. Since $|\mathcal{A}| \neq 0$, $\mathcal{A}^{-1}$ is well defined and we have $x = V\mathbb{1}_m\mathcal{A}^{-1}$ and $y = V\mathbb{1}_m(\mathcal{A}^{-1})^t$. Hence, $1 = \sum_{i\in M} x_i = x\mathbb{1}^t_m = V\mathbb{1}_m\mathcal{A}^{-1}\mathbb{1}^t_m$ and, separating V,

$$V = \frac{1}{\mathbb{1}_m\mathcal{A}^{-1}\mathbb{1}^t_m} = \frac{|\mathcal{A}|}{\mathbb{1}_m(\mathcal{A}^*)^t\mathbb{1}^t_m} = \frac{|\mathcal{A}|}{\sum_{i\in M} \bar{r}_i} = \frac{|\mathcal{A}|}{\sum_{j\in M} \bar{c}_j}.$$

[17] If $\mathcal{M}_{ij}$ denotes the matrix obtained after removing the i-th row and the j-th column of $\mathcal{A}$, then $a^*_{ij} := (-1)^{i+j}|\mathcal{M}_{ij}|$.

Therefore,

$$x = V\mathbb{1}_m\mathcal{A}^{-1} = \frac{|\mathcal{A}|}{\sum_{i\in M}\bar{r}_i}\mathbb{1}_m\mathcal{A}^{-1} = \frac{\mathbb{1}_m(\mathcal{A}^*)^t}{\sum_{i\in M}\bar{r}_i},$$

and, hence, for each $i \in M$, $x_i = \bar{r}_i / \sum_{j\in M}\bar{r}_j$. Similarly we get that, for each $i \in M$, $y_i = \bar{c}_i / \sum_{j\in M}\bar{c}_j$. Since, for each $i \in M$, $x_i \geq 0$, all the numbers $\bar{r}_1,\ldots,\bar{r}_m$ and $\sum_{i\in M}\bar{r}_i$ have the same sign. Similarly, all the numbers $\bar{c}_1,\ldots,\bar{c}_m$ and $\sum_{j\in M}\bar{c}_j$ have the same sign. Since $\sum_{i\in M}\bar{r}_i = \sum_{j\in M}\bar{c}_j$, all the numbers $\bar{r}_1,\ldots,\bar{r}_m$, $\bar{c}_1,\ldots,\bar{c}_m$ have the same sign, *i.e.*, they are either all nonpositive or all nonnegative.

Conversely, suppose that the numbers $\bar{r}_1,\ldots,\bar{r}_m, \bar{c}_1,\ldots,\bar{c}_m$ are either all nonpositive or all nonnegative. Since $|\mathcal{A}| \neq 0$, all the rows of matrix $\mathcal{A}$ are linearly independent and all the columns are independent as well. Then, there are $i, j \in M$ such that $\bar{r}_i \neq 0$ and $\bar{c}_j \neq 0$. Since $|\mathcal{A}| \neq 0$, there are $i, j \in M$ such that $\bar{r}_i \neq 0$ and $\bar{c}_j \neq 0$. Let $V := |\mathcal{A}| / \sum_{i\in M}\bar{r}_i = |\mathcal{A}| / \sum_{j\in M}\bar{c}_j$. The system $x\mathcal{A} = V\mathbb{1}_m$ has a unique solution, namely $\bar{x}$. Then,

$$\bar{x} = V\mathbb{1}_m\mathcal{A}^{-1} = \frac{V}{|\mathcal{A}|}\mathbb{1}_m(\mathcal{A}^*)^t = \frac{V}{|\mathcal{A}|}(\bar{r}_1,\ldots,\bar{r}_m)$$

and, hence, for each $i \in M$, $\bar{x}_i = \bar{r}_i / \sum_{j\in M}\bar{r}_j$. Since all the numbers $\bar{r}_1,\ldots,\bar{r}_m$ have the same sign, for each $i \in M$, $\bar{x}_i \geq 0$ and $\sum_{j\in M}\bar{x}_j = 1$. Hence, $\bar{x} \in S_m$.

Similarly, by considering the system $\mathcal{A}y^t = V\mathbb{1}_m^t$ and the unique solution $\bar{y}$, we have $\bar{y} \in S_m$ and, for each $i \in M$, $\bar{y}_i = \bar{c}_i / \sum_{j\in M}\bar{c}_j$. For each pair $i, j \in M$, $\bar{x}a_{\cdot j} = V = a_{i\cdot}\bar{y}^t$ and, hence, $(\bar{x}, \bar{y})$ is a simple solution.

Finally, the uniqueness follows from the first part of the proof, where we uniquely determined the expression for the simple solution (x, y). □

Let $\mathcal{B}$ be a submatrix of the $l \times m$ matrix $\mathcal{A}$. Then, given a strategy of player 1 in game $\mathcal{A}$, namely $x \in S_l$, let $x_\mathcal{B}$ denote the strategy of player 1 in game $\mathcal{B}$ that is obtained after removing from x the components corresponding to rows of $\mathcal{A}$ that are not in $\mathcal{B}$. Similarly, given $y \in S_M$, we can define $y_\mathcal{B}$.

Theorem 2.7.7. *Let $\mathcal{A}$ be an $l \times m$ matrix game with a nonzero value. Then:*

i) *Let $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$. Then, $x \in \operatorname{ext}(O_1(\mathcal{A}))$ and $y \in \operatorname{ext}(O_2(\mathcal{A}))$ if and only if there is $\mathcal{B}$, a square and nonsingular submatrix of $\mathcal{A}$, such that $(x_\mathcal{B}, y_\mathcal{B})$ is a simple solution of $\mathcal{B}$.*
ii) *The sets $\operatorname{ext}(O_1(\mathcal{A}))$ and $\operatorname{ext}(O_2(\mathcal{A}))$ are finite.*

Proof. Let $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$. Suppose that $x \in \operatorname{ext}(O_1(\mathcal{A}))$ and $y \in \operatorname{ext}(O_2(\mathcal{A}))$. After reordering the rows and columns of $\mathcal{A}$ if necessary, we can assume, without loss of generality, that there are $h, k \in \mathbb{N}$, with $h \leq l$ and $k \leq m$, such that i) for each $i \in L$, $x_i > 0$ if $i \leq h$ and 0 otherwise and ii) for each $j \in M$, $y_j > 0$ if $j \leq k$ and 0 otherwise. Since $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$, there are $\bar{h}, \bar{k} \in \mathbb{N}$, with $h \leq \bar{h} \leq l$ and $k \leq \bar{k} \leq m$, such that i) for each $j \in \{1, \ldots, \bar{k}\}$, $xa_{\cdot j} = V$ and ii) for each $i \in \{1, \ldots, \bar{h}\}$, $a_{i\cdot}y^t = V$.

Let $\mathcal{C}$ be the $\bar{h} \times \bar{k}$ submatrix of $\mathcal{A}$ whose entries are defined, for each $i \in \{1, \ldots, \bar{h}\}$ and each $j \in \{1, \ldots, \bar{k}\}$, by $c_{ij} := a_{ij}$. We now show that the first h rows of $\mathcal{C}$ are linearly independent. Suppose they are not. Then, there is $z \in \mathbb{R}^{\bar{h}}$, $z \neq 0$ such that, for each $i \in \{h+1, \ldots, \bar{h}\}$, $z_i = 0$ and, for each $j \in \{1 \ldots, \bar{k}\}$, $zc_{\cdot j} = 0$. Moreover, note that, by the definition of $\mathcal{C}$, for each $i \in \{1, \ldots, \bar{h}\}$, $c_{i\cdot}y^t_{\mathcal{C}} = V$. Then,

$$0 = z\mathcal{C}y^t_{\mathcal{C}} = V \sum_{i \in \{1,\ldots,\bar{h}\}} z_i.$$

Since $V \neq 0$, we have $\sum_{i \in \{1,\ldots,\bar{h}\}} z_i = 0$. Let $\hat{z} \in \mathbb{R}^l$ be such that, for each $i \in \{1, \ldots, \bar{h}\}$, $\hat{z}_i = z_i$ and, for each $i \in \{\bar{h}+1, \ldots, l\}$, $\hat{z}_i = 0$. Clearly, $\sum_{i \in L} \hat{z}_i = 0$. Let $\varepsilon > 0$ and define $x^\varepsilon := x + \varepsilon\hat{z}$ and $x^{-\varepsilon} := x - \varepsilon\hat{z}$. Then, $x = \frac{1}{2}x^\varepsilon + \frac{1}{2}x^{-\varepsilon}$ and, if ε is small enough, $x^\varepsilon, x^{-\varepsilon} \in S_l$. Moreover, for ε small enough, we also have that,

$$\begin{array}{ll} \text{for each } j \in \{1, \ldots, \bar{k}\}, & x^\varepsilon a_{\cdot j} = x^{-\varepsilon} a_{\cdot j} = xa_{\cdot j} = V, \text{ and} \\ \text{for each } j \in \{\bar{k}+1, \ldots, l\}, & x^\varepsilon a_{\cdot j} > V \text{ and } x^{-\varepsilon} a_{\cdot j} > V. \end{array}$$

Hence, both x^ε and $x^{-\varepsilon}$ belong to $O_1(\mathcal{A})$. However, this contradicts that $x \in \operatorname{ext}(O_1(\mathcal{A}))$. Similarly, it can be shown that the first k columns of $\mathcal{C}$ are linearly independent. Let r be the rank of $\mathcal{C}$. Then, $r \leq \min\{\bar{h}, \bar{k}\}$. Let $\mathcal{B}$ be an $r \times r$ submatrix of $\mathcal{C}$ of rank r. By the construction of $\mathcal{B}$, for each $j \in \{1, \ldots, r\}$, $x_{\mathcal{B}}b_{\cdot j} = V$ and, for each $i \in \{1, \ldots, r\}$, $b_{i\cdot}y^t_{\mathcal{B}} = V$. Therefore, $(x_{\mathcal{B}}, y_{\mathcal{B}})$ is a simple solution of the matrix game $\mathcal{B}$.

Conversely, let $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$ be such that $(x_{\mathcal{B}}, y_{\mathcal{B}})$ is a simple solution of $\mathcal{B}$, an $r \times r$ submatrix of $\mathcal{A}$ with $|B| \neq 0$. After reordering the rows and columns of $\mathcal{A}$ if necessary, we can assume, without loss of generality, that, for each pair $i, j \in \{1, \ldots, r\}$, $b_{ij} = a_{ij}$. Since $x \in O_1(\mathcal{A})$, $y \in O_2(\mathcal{A})$, and $(x_{\mathcal{B}}, y_{\mathcal{B}})$ is a simple solution of $\mathcal{B}$, we have that, for each pair $i, j \in \{1, \ldots, r\}$,

$$V \leq xa_{\cdot j} = x_{\mathcal{B}}b_{\cdot j} = b_{i\cdot}y^t_{\mathcal{B}} = a_{i\cdot}y^t \leq V.$$

We now show that $x \in \operatorname{ext}(O_1(\mathcal{A}))$. Note that $O_1(\mathcal{A})$ is nonempty and, by Proposition 2.7.2, is a convex and compact subset of $\mathbb{R}^l$. Hence, by Theorem 2.7.5, there are $\hat{x}, \tilde{x} \in \operatorname{ext}(O_1(\mathcal{A}))$ such that $x = \alpha\hat{x} + (1-\alpha)\tilde{x}$ with $0 < \alpha < 1$. Given $i \in \{r+1, \ldots, l\}$, $x_i = 0$, and, hence, $\hat{x}_i = \tilde{x}_i = 0$ (recall that both are nonnegative). For each $j \in \{1, \ldots, r\}$, by optimality, $\hat{x}a_{\cdot j} \geq V$ and $\tilde{x}a_{\cdot j} \geq V$ and, moreover, $V = xa_{\cdot j} = \alpha\hat{x}a_{\cdot j} + (1-\alpha)\tilde{x}a_{\cdot j}$. Hence, for each $j \in \{1, \ldots, r\}$ it has to be the case that $\hat{x}a_{\cdot j} = \tilde{x}a_{\cdot j} = V$. Since the system $z\mathcal{B} = V\mathbb{1}_r$ has a unique solution, we have that $\hat{x}_{\mathcal{B}} = \tilde{x}_{\mathcal{B}} = x_{\mathcal{B}}$. Then,

$\hat{x} = \tilde{x} = x$, so x is indeed an extreme point of $(O_1(\mathcal{A}))$. A similar argument applies for y. This concludes the proof of i).

Finally, since the set of square and nonsingular submatrices of $\mathcal{A}$ that have a simple solution with value V is finite, and each one characterizes an extreme point of $O_1(\mathcal{A})$ and $O_2(\mathcal{A})$, both $\text{ext}(O_1(\mathcal{A}))$ and $\text{ext}(O_2(\mathcal{A}))$ are finite sets. $\square$

We are ready to present the submatrices method. In view of Proposition 2.7.2 and Theorem 2.7.5, the sets of optimal strategies of the players are the convex hulls of their extreme points. The submatrices method consists of finding all the extreme points of the sets of optimal strategies of the players (we already know that there is a finite number of such points). The way to do it is to search for simple solutions in all the square and nonsingular submatrices of the matrix of the game; Theorem 2.7.7 ensures that every extreme point is found with this approach.[18] Next, we provide the description of this method for a given $l \times m$ matrix game $\mathcal{A}$. We denote by $V_{\mathcal{A}}$ the value of matrix $\mathcal{A}$.

Step 1: Ensure that $V_{\mathcal{A}} \neq 0$. If necessary, apply the algorithm to solve the matrix game $\mathcal{B}$, where $\mathcal{B}$ is obtained by adding to all of the entries of $\mathcal{A}$ a positive constant such that all the entries of $\mathcal{B}$ are positive. Now, $V_{\mathcal{B}} > 0$ and, by Proposition 2.7.4, solving $\mathcal{B}$ is the same as solving $\mathcal{A}$. Hence, without loss of generality, assume that $V_{\mathcal{A}} \neq 0$.

Step 2: Look for pure Nash equilibria of $\mathcal{A}$ (recall that, in order to find the pure Nash equilibria of a matrix game $\mathcal{A}$, it suffices to find the entries of $\mathcal{A}$ that are a maximum of their column and a minimum of their row). If (x, y) is a pure Nash equilibrium of $\mathcal{A}$, then x is an extreme point of $O_1(\mathcal{A})$ and y is an extreme point of $O_2(\mathcal{A})$ (by the second statement in Theorem 2.7.7, with $\mathcal{B}$ being a 1×1 matrix).

Step 3: (To be repeated for each $i \in \{2, \ldots, \min\{l, m\}\}$) For each nonsingular $i \times i$ submatrix of $\mathcal{A}$, study if it has a simple solution (using Theorem 2.7.6). If it has such a solution, complete it with zeros[19] to have a pair $(x, y) \in S_l \times S_m$. If $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$ (which can be checked using Proposition 2.7.3), then x is an extreme point of $O_1(\mathcal{A})$ and y is an extreme point of $O_2(\mathcal{A})$ (Theorem 2.7.7).

Example 2.7.3. We use the submatrices method to solve the matrix game of Example 2.7.1.

[18] The main drawback of this method is that it is not very efficient from a computational point of view.

[19] In the rows and columns deleted from $\mathcal{A}$ to get the submatrix under consideration.

Step 1: All the entries of $\mathcal{A}$ are positive, so its value is positive.

Step 2: $((1,0),(1,0,0))$ is a pure Nash equilibrium of $\mathcal{A}$. Hence, $(1,0) \in \text{ext}(O_1(\mathcal{A}))$ and $(1,0,0) \in \text{ext}(O_2(\mathcal{A}))$.

Step 3.1:

$$\mathcal{A}_1 = \begin{pmatrix} 2 & 2 \\ 1 & 4 \end{pmatrix}.$$

Then, $|\mathcal{A}_1| \neq 0$ and

$$\mathcal{A}_1^* = \begin{pmatrix} 4 & -1 \\ -2 & 2 \end{pmatrix}.$$

Now, $\bar{r}_1 = 3$, $\bar{r}_2 = 0$, $\bar{c}_1 = 2$ and $\bar{c}_2 = 1$. Hence, $((1,0),(2/3,1/3))$ is a simple solution of $\mathcal{A}_1$. Take $((1,0),(2/3,1/3,0))$ and note that, for each $j \in \{1,2\}$ and each $i \in \{1,2,3\}$,

$$(1,0)a_{\cdot j} \geq 2 = a_{i\cdot}(2/3,1/3,0)^t.$$

Therefore, $((1,0),(2/3,1/3,0)) \in O_1(\mathcal{A}) \times O_2(\mathcal{A})$ and, moreover $(1,0) \in \text{ext}(O_1(\mathcal{A}))$ and $(2/3,1/3,0) \in \text{ext}(O_2(\mathcal{A}))$.

Step 3.2:

$$\mathcal{A}_2 = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}.$$

Then, $|\mathcal{A}_2| \neq 0$ and

$$\mathcal{A}_2^* = \begin{pmatrix} 3 & -1 \\ -3 & 2 \end{pmatrix}.$$

Now, $\bar{r}_1 = 2$, $\bar{r}_2 = -1$, $\bar{c}_1 = 0$ and $\bar{c}_2 = 1$. Hence, $\mathcal{A}_2$ does not have simple solutions.

Step 3.3:

$$\mathcal{A}_3 = \begin{pmatrix} 2 & 3 \\ 4 & 3 \end{pmatrix}.$$

Then, $|\mathcal{A}_3| \neq 0$ and

$$\mathcal{A}_3^* = \begin{pmatrix} 3 & -4 \\ -3 & 2 \end{pmatrix}.$$

Now, $\bar{r}_1 = -1$, $\bar{r}_2 = -1$, $\bar{c}_1 = 0$, and $\bar{c}_2 = -2$. Hence, $((1/2,1/2),(0,1))$ is a simple solution of $\mathcal{A}_3$. Take $((1/2,1/2),(0,0,1))$ and note that

$$(1/2,1/2)a_{\cdot 1} = 3/2 < 3 = a_{1\cdot}(0,0,1)^t.$$

Therefore, $((1,0),(2/3,1/3,0)) \notin O_1(\mathcal{A}) \times O_2(\mathcal{A})$.

Hence, $O_1(\mathcal{A}) = \{(1,0)\}$ and $O_2(\mathcal{A}) = \text{conv}(\{(1,0,0),(2/3,1/3,0)\})$. The value of the game is $V = u_1((1,0),(1,0,0)) = 2$. $\diamond$

2.8. Matrix Games and Linear Programming

In this section we present an interesting property of matrix games: solving a matrix game is, in some sense, equivalent to solving a pair of dual linear programming problems. Linear programming has been an important field within operations research since George B. Dantzig introduced the simplex method to solve linear programming problems in 1947. For those who are not familiar with linear programming, we now present a very brief introduction to this field. There are many books on linear programming; refer, for instance, to Bazaraa et al. (1990). Also Owen (1995), in spite of being a book on game theory, includes a chapter on linear programming in which the simplex method and the duality theory are discussed.

Definition 2.8.1. A *linear programming problem* is a constrained optimization problem in which both the objective function (the one that we want to maximize/minimize) and the constraints are linear. Every linear programming problem can be expressed as

$$\begin{array}{ll} \text{Minimize} & cx^t \\ \text{subject to} & x\mathcal{A} \geq b, \\ & x \geq 0. \end{array} \tag{2.8.1}$$

Given a linear programming problem such as the one in Eq. (2.8.1), we say that x is a *feasible solution* if $x\mathcal{A} \geq b$ and $x \geq 0$. We say that x is an *optimal solution* if it is feasible and optimizes the objective function (in this case, minimizes) within the set of feasible solutions.

Definition 2.8.2. The *dual problem* of the linear programming problem in Eq. (2.8.1) is the linear programming problem given by

$$\begin{array}{ll} \text{Maximize} & by^t \\ \text{subject to} & \mathcal{A}y^t \leq c^t, \\ & y \geq 0. \end{array} \tag{2.8.2}$$

If we have a pair of problems like the ones in Definitions 2.8.1 and 2.8.2, we refer to the first one as the primal problem (P) and to the second one as the dual problem (D). The dual of the dual problem is again the primal problem and, hence, the distinction between the two problems is really about what problem is defined first.[20] Below we present an important result

[20]It is worth noting that a linear problem as the one in Definition 2.8.1, but in which the restrictions $x \geq 0$ are not present, can be easily rewritten into the form of a primal linear problem and, hence, the analysis we develop below would also apply to any such problem. Just consider the first two problems below:

P1		P2		P3	
Minimize	cx^t	Minimize	$c(v-w)^t$	Maximize	by^t
subject to	$x\mathcal{A} \geq b$	subject to	$(v-w)\mathcal{A} \geq b$	subject to	$\mathcal{A}y^t = c^t$
	$x \in \mathbb{R}$		$v, w \geq 0$		$y \geq 0.$

of linear programming: the duality theorem. First, we prove two auxiliary results, the Farkas lemma and the Kuhn-Tucker optimality conditions.

Lemma 2.8.1 (Farkas lemma). *Let $\mathcal{A}$ be an $l \times m$ matrix and let c be an $1 \times m$ vector. Then, one and only one of the following systems of inequalities has a solution:*

i) $x\mathcal{A} \leq 0$ *and* $cx^t > 0$.

ii) $\mathcal{A}y^t = c^t$ *and* $y \geq 0$.

Proof. Suppose first that ii) has a solution $y \in \mathbb{R}^m$ and let $x \in \mathbb{R}^l$ be such that $x\mathcal{A} \leq 0$. Then, since $y \geq 0$ and $x\mathcal{A} \leq 0$, $cx^t = y\mathcal{A}^t x^t = x\mathcal{A}y^t \leq 0$. Thus, i) has no solution.

Conversely, suppose that ii) has no solution. Let $S := \{y\mathcal{A}^t : y \in \mathbb{R}^m,\ y \geq 0\}$. The set S is closed and convex. Since ii) has no solution, then $c \notin S$. Hence, there is a hyperplane separating c and S (see Theorem 2.14.1 in Section 2.14). In particular, there is $x \in \mathbb{R}^l$ such that, for each $y \geq 0$, $cx^t > y\mathcal{A}^t x^t = x\mathcal{A}y^t$. Taking $y = 0$, we get that $cx^t > 0$. Moreover, $x\mathcal{A} \leq 0$, since, otherwise, there would be $y \geq 0$, with the appropriate component being large enough, such that $cx^t \leq y\mathcal{A}^t x^t = x\mathcal{A}y^t$. Hence, i) has a solution. □

Theorem 2.8.2 (Kuhn-Tucker optimality conditions). *Consider the linear programming problem* (P)

$$\begin{array}{ll} \text{Minimize} & cx^t \\ \text{subject to} & x\mathcal{A} \geq b, \\ & x \geq 0, \end{array}$$

where $\mathcal{A}$ is an $l \times m$ matrix, c is a $1 \times l$ vector and b is a $1 \times m$ vector. Let $x \in \mathbb{R}^l$ be a feasible solution of (P)*. Then, x is an optimal solution of* (P) *if and only if there are $v \in \mathbb{R}^l$ and $w \in \mathbb{R}^m$ such that*

$$c - w\mathcal{A}^t - v = 0, \quad w \geq 0, \quad v \geq 0, \tag{2.8.3}$$

$$(x\mathcal{A} - b)w^t = 0, \quad xv^t = 0. \tag{2.8.4}$$

Proof. First, we prove the sufficiency part. Let $x \in \mathbb{R}^l$ be a feasible solution of (P) and let $v \in \mathbb{R}^l$ and $w \in \mathbb{R}^m$ satisfy Eqs. (2.8.3) and (2.8.4). Let $\hat{x} \in \mathbb{R}^l$ be a feasible solution of (P). By Eq. (2.8.3),

$$0 = (c - w\mathcal{A}^t - v)(x - \hat{x})^t = c(x - \hat{x})^t - w\mathcal{A}^t x^t - vx^t + w\mathcal{A}^t\hat{x}^t + v\hat{x}^t.$$

By transposing the terms in Eq. (2.8.4), we have $w\mathcal{A}^t x^t = wb^t$ and $vx^t = 0$. Then,

$$0 = c(x - \hat{x})^t - wb^t + w\mathcal{A}^t\hat{x}^t + v\hat{x}^t,$$

It is not hard to check that x is an optimal solution of P1 if and only if there is an optimal solution (v, w) of P2 such that $x = v - w$. Hence, solving P1 is equivalent to solving P2. Moreover, it can be readily verified that the dual problem of P2 is P3 and so, P3 can be seen as the dual of P1.

where $w \geq 0$, $v \geq 0$, $\hat{x}\mathcal{A} \geq b$, and $\hat{x} \geq 0$. Then, $c\hat{x}^t \geq cx^t$. Therefore, x is an optimal solution of (P).

Conversely, suppose that x is an optimal solution of (P). After reordering the rows of $\mathcal{A}$ and the columns of c if necessary,[21] we can assume, without loss of generality, that

- There is $p \in \{0, \ldots, l\}$ such that, for each $i \in \{1, \ldots, l\}$, $x_i > 0$ if $i \leq p$ and $x_i = 0$ otherwise.

After reordering the columns of $\mathcal{A}$ and the columns of b if necessary,[22] we can also assume, without loss of generality, that

- There is $k \in \{1, \ldots, m\}$ such that i) $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, with $\mathcal{A}_1 \in \mathcal{M}_{l\times k}$ and $\mathcal{A}_2 \in \mathcal{M}_{l\times(m-k)}$ ii) $b = (b_1, b_2)$, with $b_1 \in \mathcal{M}_{1\times k}$ and $b_2 \in \mathcal{M}_{1\times(m-k)}$, and iii) $x\mathcal{A}_1 = b_1$ and $x\mathcal{A}_2 > b_2$.

Consider the system

$$z\mathcal{A}_1 \geq 0, \quad z_{p+1}, \ldots, z_l \geq 0, \quad \text{and} \quad cz^t < 0. \tag{2.8.5}$$

This system has no solution. Otherwise, taking $\alpha > 0$ such that $x + \alpha z \geq 0$ and $(x + \alpha z)\mathcal{A} \geq b$, we would have that $c(x + \alpha z)^t < cx^t$, contradicting the fact that x is an optimal solution.

Note that the system (2.8.5) can be written as $z\mathcal{C} \leq 0$ and $-cz^t > 0$ where

$$\mathcal{C} := \left(\begin{array}{c|c} -\mathcal{A}_1 & \begin{array}{c} 0 \\ -I_{l-p} \end{array} \end{array} \right) \in \mathcal{M}_{l\times k+l-p}.$$

Since the system (2.8.5) has no solution, by Farkas lemma, there is $y \in \mathbb{R}^{k+l-p}$, $y \geq 0$, such that $\mathcal{C}y^t = -c^t$. Let $w \in \mathbb{R}^m$ be defined, for each $i \in \{1, \ldots, m\}$, by $w_i := y_i$ if $i \leq k$ and $w_i := 0$ otherwise. Let $v \in \mathbb{R}^l$ be defined, for each $i \in \{1, \ldots, l\}$, by $v_i := 0$ if $i \leq p$ and $v_i := y_i$ otherwise. By construction, $w \geq 0$, $v \geq 0$, and it is easy to check that, for each $i \in \{1, \ldots, l\}$, $-(w\mathcal{A}^t)_i - v_i = (y\mathcal{C}^t)_i = -c_i$ and, hence, $c - w\mathcal{A}^t - v = 0$. Thus, Eq. (2.8.3) holds. Moreover, for each $i \in \{p+1, \ldots, l\}$, $x_i = 0$, and, for each $i \in \{1, \ldots, p\}$, $v_i = 0$, and, thus, $xv^t = 0$. Now, since, for each $i \in \{k+1, \ldots, m\}$, $w_i = 0$, we have

$$(x\mathcal{A} - b)w^t = (x\mathcal{A}_1 - b_1, x\mathcal{A}_2 - b_2)w^t = (0, x\mathcal{A}_2 - b_2)w^t = 0,$$

and Eq. (2.8.4) also holds. $\square$

The Kuhn-Tucker conditions are a powerful tool to study the properties of a given solution of a linear programming problem. However, this kind of analysis is not in the scope of this book. We now present one of the most important results in linear programming: the duality theorem.

[21] The reordering of the rows of $\mathcal{A}$ and the coordinates of c being the same.

[22] Again, the reordering of the columns of $\mathcal{A}$ and the coordinates of b being the same.

Theorem 2.8.3 (Duality theorem)**.** *Let* (P) *and* (D) *be a pair of dual linear programming problems. Then,*

i) (P) *has an optimal solution if and only if* (D) *has an optimal solution.*

ii) *Let x and y be feasible solutions of* (P) *and* (D), *respectively. Then, x is an optimal solution of* (P) *and y is an optimal solution of* (D) *if and only if $cx^t = by^t$.*

Proof. We prove both statements together. Let $\hat{x}$ and $\hat{y}$ be feasible solutions of (P) and (D), respectively. Then,

$$c\hat{x}^t \geq \hat{y}\mathcal{A}^t\hat{x}^t = \hat{y}(\hat{x}\mathcal{A})^t \geq \hat{y}b^t, \tag{2.8.6}$$

which implies the sufficiency part of statement ii). We now present the rest of the proof in terms of the primal problem and, since the role of primal and dual problems is arbitrary, this accounts for both cases. Let x be an optimal solution of (P). By Theorem 2.8.2, there are $v \in \mathbb{R}^l$, $v \geq 0$, and $w \in \mathbb{R}^m$, $w \geq 0$, such that $c = w\mathcal{A}^t + v$, $xv^t = 0$, and $(x\mathcal{A} - b)w^t = 0$. Then,

$$cx^t = w\mathcal{A}^t x^t = bw^t,$$

and, since $v \geq 0$, $w\mathcal{A}^t \leq c$. Hence, w is a feasible solution of (D). Moreover, by Eq. (2.8.6), for each feasible solution $\hat{y}$ of (D), $b\hat{y}^t \leq cx^t = bw^t$. Therefore, w is an optimal solution of (D) and it is now easy to see that x and y are both optimal only if $cx^t = by^t$. $\square$

We are ready to show that, in order to solve a matrix game, it is enough to solve a certain pair of dual linear programming problems. Let $\mathcal{A}$ be a matrix game with a positive value (in view of Proposition 2.7.4 we can make this assumption without loss of generality). In the exposition below we slightly abuse notation and use $\underline{\Lambda}$ and $\bar{\Lambda}$ to denote variables of the optimization problems; we use this notation because these variables are related to the functions $\underline{\Lambda}$ and $\bar{\Lambda}$.

In order to find the value of the game and the set of optimal strategies of player 1, we can solve the linear programming problem below in which the variables are x and $\underline{\Lambda}$:

$$\begin{array}{ll} \text{Maximize} & \underline{\Lambda} \\ \text{subject to} & xa_{\cdot j} \geq \underline{\Lambda}, \quad \forall\, j \in M, \\ & x\mathbb{1}_l^t = 1, \\ & x \geq 0. \end{array} \tag{2.8.7}$$

Now consider the following two optimization problems:

$$\text{(2.8.8.a)}\qquad \begin{array}{ll} \text{Minimize} & 1/\underline{\Lambda} \\ \text{subject to} & x\mathcal{A} \geq \underline{\Lambda}\mathbb{1}_m, \\ & x\mathbb{1}_l^t = 1, \\ & x \geq 0, \\ & \underline{\Lambda} > 0. \end{array} \qquad \text{(2.8.8.b)}\qquad \begin{array}{ll} \text{Minimize} & v\mathbb{1}_l^t \\ \text{subject to} & v\mathcal{A} \geq \mathbb{1}_m, \\ & v \geq 0. \end{array}$$

Solving the (nonlinear) optimization problem (2.8.8.a) is equivalent to solving problem (2.8.7). On the other hand, problem (2.8.8.b) is a linear programming problem whose resolution is linked to the resolution of problem (2.8.8.a). The next proposition, whose (simple) proof is left as an exercise, formally states the last observation.

Proposition 2.8.4. *The following two statements hold:*

i) *Let* $(x, \underline{\Lambda})$ *be an optimal solution of* (2.8.8.a). *Let* $v \in \mathbb{R}^l$ *be defined, for each* $i \in \{1, \ldots, l\}$, *by* $v_i := x_i/\underline{\Lambda}$. *Then,* v *is an optimal solution of* (2.8.8.b).

ii) *Let* v *be an optimal solution of* (2.8.8.b). *Let* x *be defined, for each* $i \in \{1, \ldots, l\}$, *by* $x_i := v_i/(v\mathbb{1}_l^t)$, *and let* $\underline{\Lambda} := 1/(v\mathbb{1}_l^t)$. *Then,* $(x, \underline{\Lambda})$ *is an optimal solution of* (2.8.8.a).

Proof. Exercise 2.12. □

We can now proceed analogously from the point of view of player 2. In order to find the value of the game and the set of optimal strategies of player 2, we can solve the linear programming problem below in which the variables are y and $\bar{\Lambda}$:

$$\text{(2.8.9)}\qquad \begin{array}{ll} \text{Minimize} & \bar{\Lambda} \\ \text{subject to} & a_{i\cdot}y^t \leq \bar{\Lambda}, \quad \forall\, i \in L, \\ & y\mathbb{1}_m^t = 1, \\ & y \geq 0. \end{array}$$

Now consider the following two optimization problems:

$$\text{(2.8.10.a)}\qquad \begin{array}{ll} \text{Maximize} & 1/\bar{\Lambda} \\ \text{subject to} & \mathcal{A}y^t \leq \bar{\Lambda}\mathbb{1}_l^t, \\ & y\mathbb{1}_m^t = 1, \\ & y \geq 0. \end{array} \qquad \text{(2.8.10.b)}\qquad \begin{array}{ll} \text{Maximize} & w\mathbb{1}_m^t \\ \text{subject to} & \mathcal{A}w^t \leq \mathbb{1}_l^t, \\ & w \geq 0. \end{array}$$

Solving the (nonlinear) optimization problem (2.8.10.a) is equivalent to solving problem (2.8.9). Proposition 2.8.5 is the re-statement of Proposition 2.8.4 but from the point of view of player 2; again, the proof is left as an exercise.

Proposition 2.8.5. *The following two statements hold:*

i) *Let* $(y, \bar{\Lambda})$ *be an optimal solution of* (2.8.10.a). *Let* $w \in \mathbb{R}^m$ *be defined, for each* $j \in \{1, \ldots, m\}$, *by* $w_j := y_j/\bar{\Lambda}$. *Then,* w *is an optimal solution of* (2.8.10.b).

ii) *Let* w *be an optimal solution of* (2.8.10.b). *Let* y *be defined, for each* $j \in \{1, \ldots, m\}$, *by* $y_j := w_j/(w\mathbb{1}_m^t)$, *and let* $\bar{\Lambda} := 1/(w\mathbb{1}_m^t)$. *Then,* $(y, \bar{\Lambda})$ *is an optimal solution of* (2.8.10.a).

Proof. Exercise 2.12. □

Note that problems (2.8.8.b) and (2.8.10.b) are a pair of dual linear programming problems. Therefore, the two propositions above imply that, to solve the matrix game $\mathcal{A}$, it is enough to solve a pair of dual linear programming problems, which is what we wanted to show. In particular, this result implies that the simplex method can be used to solve a matrix game and, therefore, the information contained in the corresponding simplex tableau can be used to study certain properties of a given solution of a matrix game.[23]

We now prove that, conversely, in order to solve a pair of dual linear programming problems, it is enough to solve a certain matrix game. We need some auxiliary definitions and results.

Definition 2.8.3. An $m \times m$ matrix game $\mathcal{A}$ is *symmetric* if $\mathcal{A} = -\mathcal{A}^t$, *i.e.*, if the players are interchangeable.

Proposition 2.8.6. *Let* $\mathcal{A}$ *be an* $m \times m$ *symmetric matrix game. Then,* $O_1(\mathcal{A}) = O_2(\mathcal{A})$ *and* $V = 0$.

Proof. Let $x, y \in S_m$ be such that $x \in O_1(\mathcal{A})$ and $y \in O_2(\mathcal{A})$. Then, for each pair $\hat{x}, \hat{y} \in S_m$, $\hat{x}\mathcal{A}y^t \leq x\mathcal{A}y^t \leq x\mathcal{A}\hat{y}^t$. Equivalently, after taking transpose matrices and multiplying by -1, $-y\mathcal{A}^t\hat{x}^t \geq -y\mathcal{A}^t x^t \geq -\hat{y}\mathcal{A}^t x^t$. Since $\mathcal{A} = -\mathcal{A}^t$, we have $y\mathcal{A}\hat{x}^t \geq y\mathcal{A}x^t \geq \hat{y}\mathcal{A}x^t$ and, hence, $y \in O_1(\mathcal{A})$ and $x \in O_2(\mathcal{A})$. Therefore, $O_1(\mathcal{A}) = O_2(\mathcal{A})$. Moreover, $V = x\mathcal{A}y^t = x(-\mathcal{A}^t)y^t = y(-\mathcal{A})x^t = -V$ and, hence, $V = 0$. □

When dealing with a symmetric game, because of the above result, to refer to an optimal strategy of one of the players we simply refer to *an optimal strategy of the game*.

Definition 2.8.4. Let $\mathcal{A}$ be an $l \times m$ matrix game. A row $a_{i\cdot}$ is *relevant* if there is $x \in O_1(\mathcal{A})$ such that $x_i > 0$. A column $a_{\cdot j}$ is *relevant* if there is $y \in O_2(\mathcal{A})$ such that $y_j > 0$.

---

[23]The simplex method is the most popular algorithm for solving linear programming problems and it can be found in any text on linear programming. Refer, for instance, to Bazaraa et al. (1990).

It seems clear that a relevant strategy of a player has to be a best reply against every optimal strategy of the opponent. The next result shows that, not only this is true, but also the converse: a pure strategy that is a best reply against every optimal strategy of the opponent has to be part of some optimal strategy, *i.e.*, it is relevant.

Proposition 2.8.7. *Let $\mathcal{A}$ be an $l \times m$ matrix game.*

i) *$a_{\cdot j}$ is a relevant column if and only if, for each $x \in O_1(\mathcal{A})$, $xa_{\cdot j} = V$.*

ii) *$a_{i\cdot}$ is a relevant row if and only if, for each $y \in O_2(\mathcal{A})$, $a_{i\cdot}y^t = V$.*

Proof. We prove only the first statement; the second one being analogous. Let $a_{\cdot j}$ be a relevant column. Recall that, for each $x \in O_1(\mathcal{A})$ and each $k \in M$, $xa_{\cdot k} \geq V$. Suppose there is $x \in O_1(\mathcal{A})$ such that $xa_{\cdot j} > V$. Let $y \in O_2(\mathcal{A})$ be such that $y_j > 0$. Then,

$$V = x\mathcal{A}y^t = \sum_{k \in M}(xa_{\cdot k})y_k = (xa_{\cdot j})y_j + \sum_{k \in M\backslash\{j\}}(xa_{\cdot k})y_k > V,$$

which is a contradiction. To prove the converse, we show that if $a_{\cdot j}$ is a pure strategy that is a best reply against every optimal strategy of the opponent, then $a_{\cdot j}$ has to be part of some optimal strategy. We can assume, without loss of generality, that $V = 0$.[24] We claim that it suffices to prove that the following system of inequalities has a solution:

$$\mathcal{A}z^t \leq 0,\ z_j = 1,\ z \geq 0.$$

Once a solution z is obtained, we define, for each $k \in M$, $y_k := z_k / \sum_{r \in M} z_r$. Then, $y \in O_2(\mathcal{A})$ and $y_j > 0$. The above system can be rewritten as

$$\begin{pmatrix} \mathcal{A} & I_l \\ e_j & 0 \end{pmatrix} \begin{pmatrix} z^t \\ u^t \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},\ (z,u) \geq 0,$$

where $z \in \mathbb{R}^m$, $u \in \mathbb{R}^l$, and $e_j \in \mathbb{R}^m$ is the j-th vector of the canonical basis. Suppose that this new problem has no solution. Then, by Farkas lemma, there are $v \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$ such that

$$(v,\alpha)\begin{pmatrix} \mathcal{A} & I_l \\ e_j & 0 \end{pmatrix} \leq 0,\ \alpha > 0.$$

Hence, for each $i \in L$, $v_i \leq 0$, $va_{\cdot j} + \alpha \leq 0$, and, for each $k \in M$, $va_{\cdot k} \leq 0$. Note that $v \neq 0$ since, otherwise, $\alpha \leq 0$, which is a contradiction. For each $i \in L$, let $x_i := -v_i / \sum_{p \in L} v_p$. Then, for each $k \in M$, $xa_{\cdot k} \geq 0$ and, hence, $x \in O_1(\mathcal{A})$. However, $xa_{\cdot j} > 0$, which contradicts that $a_{\cdot j}$ is a best reply against every optimal strategy of player 1. $\square$

[24] Otherwise, we can sum $-V$ to all the payoffs in the game and, since this transformation does not affect the best reply correspondences, it suffices to prove the result for the new game, whose value is 0.

Take a pair of dual linear programming problems (P) and (D) as in Definitions 2.8.1 and 2.8.2. Consider the following matrix game

$$\mathcal{B} := \begin{pmatrix} 0 & \mathcal{A} & -c^t \\ -\mathcal{A}^t & 0 & b^t \\ c & -b & 0 \end{pmatrix}.$$

Since $\mathcal{B}$ is a symmetric matrix game, by Proposition 2.8.6, we have that $O_1(\mathcal{B}) = O_2(\mathcal{B})$ and its value is zero.

Theorem 2.8.8. *A pair of dual linear programming problems* (P) *and* (D) *have optimal solutions if and only if the associated matrix game $\mathcal{B}$ has an optimal strategy whose last component is positive.*

Proof. First, let $x \in \mathbb{R}^l$, $y \in \mathbb{R}^m$, and $\alpha > 0$ be such that (x, y, α) is an optimal strategy of $\mathcal{B}$. Then,

$$\begin{pmatrix} 0 & \mathcal{A} & -c^t \\ -\mathcal{A}^t & 0 & b^t \\ c & -b & 0 \end{pmatrix} \begin{pmatrix} x^t \\ y^t \\ \alpha \end{pmatrix} \leq 0.$$

Hence, $\mathcal{A}y^t - c^t\alpha \leq 0$, $-\mathcal{A}^t x^t + b^t\alpha \leq 0$, and $cx^t - by^t = 0$, where the last equality follows from the combination of Proposition 2.8.7 with the fact that $\alpha > 0$ and thus the last row of $\mathcal{B}$ is relevant. Let $\bar{x} := (1/\alpha)x$ and $\bar{y} := (1/\alpha)y$. So defined, $\bar{x}$ is a feasible solution of (P) and $\bar{y}$ is a feasible solution of (D). Since $cx^t = by^t$, by the duality theorem, $\bar{x}$ is an optimal solution of (P) and $\bar{y}$ is an optimal solution of (D).

Conversely, suppose that $\bar{x}$ is an optimal solution of (P) and $\bar{y}$ is an optimal solution of (D). Let $x := \alpha\bar{x}$ and $y := \alpha\bar{y}$, where

$$\alpha := \frac{1}{1 + \sum_{i \in L} \bar{x}_i + \sum_{j \in M} \bar{y}_j}.$$

Note that $\alpha > 0$ and $(x, y, \alpha) \in S_{l+m+1}$. The optimality of (x, y, α) as a strategy of $\mathcal{B}$ follows from the facts that the value of $\mathcal{B}$ is zero and that the following inequalities hold:

$$\begin{pmatrix} 0 & \mathcal{A} & -c^t \\ -\mathcal{A}^t & 0 & b^t \\ c & -b & 0 \end{pmatrix} \begin{pmatrix} x^t \\ y^t \\ \alpha \end{pmatrix} \leq 0. \qquad \square$$

2.9. Refinements of Nash Equilibrium in Finite Games

In this section we study refinements of Nash equilibrium. To do so, we restrict attention to finite games and their mixed extensions, as we discuss the different concepts, we briefly comment to what extent they can be extended to infinite games.

The main idea underlying the Nash equilibrium concept is to find strategy profiles that are *self-enforcing*, in the sense that, if players are asked (or they informally agree) to play according to one of such profiles, no player has incentives to unilaterally deviate from it. For a profile to be self-enforcing in the above sense, it must be a Nash equilibrium, but this is not a sufficient condition. We illustrate this statement with a couple of examples.

Example 2.9.1. Consider the bimatrix game in Figure 2.9.1. This game has

	L_2	R_2
L_1	1,1	0,0
R_1	0,0	0,0

Figure 2.9.1. The Nash equilibrium (R_1, R_2) is not self-enforcing.

two Nash equilibria in pure strategies: (L_1, L_2) and (R_1, R_2). However, (R_1, R_2) is not really self-enforcing. Suppose that the players have informally agreed to play (R_1, R_2) and take, for instance, player 1. He does not lose if he plays L_1 instead of R_1 and, in addition, he might gain by doing this (if player 2 also deviates from R_2). So, he will probably deviate to L_1, which means that (R_1, R_2) is not really self-enforcing. Note that player 2 can make an analogous reasoning so, even in the case that players have agreed to play (R_1, R_2), they will eventually play (L_1, L_2). Of course, (L_1, L_2) is a self-enforcing Nash equilibrium. ◇

Example 2.9.2. Consider the bimatrix game in Figure 2.9.2. This game also

	L_2	R_2
L_1	1,1	10,0
R_1	0,10	10,10

Figure 2.9.2. The socially desirable Nash equilibrium (R_1, R_2) is not self-enforcing.

has two pure Nash equilibria: (L_1, L_2) and (R_1, R_2). However, (R_1, R_2) is not really self-enforcing, and the reasoning is exactly the same as in Example 2.9.1. This example insists once more on the fact that noncooperative game theory does not look for socially optimal outcomes, but for "stable" outcomes: (R_1, R_2) is socially better than (L_1, L_2), but only (L_1, L_2) is eligible in a noncooperative environment. ◇

These examples can help us to understand that, if we wish to find self-enforcing profiles in a strategic game, we have to look for them within the set of Nash equilibrium points, but we must sometimes ask for some extra conditions. We should therefore *refine* the Nash equilibrium concept. In this section we present a brief introduction to this field of refinements of Nash equilibrium in the context of finite games (although some of the concepts introduced here for finite games can be extended in a direct way to any strategic game). For those who want to study this topic in depth, a good reference is van Damme (1991).

Take a finite strategic game $G = (A, u)$ and its mixed extension $E(G) = (S, u)$. We now define the *perfect equilibrium* concept, one of the most important refinements of Nash equilibrium. It was introduced in Selten (1975), and the idea underlying it is to select those Nash equilibria which are still in equilibrium even in the case that players might possibly make (small) mistakes when choosing their strategies: this is why perfect equilibrium is also known as *trembling hand perfect equilibrium*.

Definition 2.9.1. Let G be a strategic game. A *tremble* in G is a vector $\eta = (\eta_1, \ldots, \eta_n)$ such that, for each $i \in N$, η_i is a function from A_i to $\mathbb{R}$ satisfying that:

i) For each $a_i \in A_i$, $\eta_i(a_i) > 0$.

ii) $\sum_{a_i \in A_i} \eta_i(a_i) < 1$.

We denote by $T(G)$ the set of trembles in G.

Definition 2.9.2. Let $G = (A, u)$ be a finite game and let $\eta \in T(G)$. The η-*perturbation* of G is the strategic game $(G, \eta) := (S(\eta), u)$, whose elements are the following:

Sets of strategies: For each $i \in N$, $S_i(\eta_i) := \{s_i \in S_i : \text{for each } a_i \in A_i, s_i(a_i) \geq \eta_i(a_i)\}$ and $S(\eta) := \prod_{i \in N} S_i(\eta_i)$.

Payoff functions: We slightly abuse notation and use u_i to denote the restriction of player i's payoff function in $E(G)$ to $S(\eta)$ and $u := \prod_{i=1}^{n} u_i$.

Note that the η-perturbation of G is simply a modified version of $E(G)$ in which all players choose all their pure strategies with a positive probability (never smaller than the lower bound given by the tremble). Observe also that (G, η) fulfills the conditions of Nash theorem, so we can assure that it has, at least, one Nash equilibrium. In the next definition we formally introduce the perfect equilibrium concept.

Definition 2.9.3. Let G be a finite game and let $E(G)$ be its mixed extension. A strategy profile $s \in S$ is a *perfect equilibrium* of $E(G)$ if there are two sequences $\{\eta^k\} \subset T(G)$, with $\{\eta^k\} \to 0$, and $\{s^k\} \subset S$, with $\{s^k\} \to s$, such that, for each $k \in \mathbb{N}$, s^k is a Nash equilibrium of (G, η^k).[25]

It is an easy exercise to check that (L_1, L_2) is the unique perfect equilibrium in the games in Examples 2.9.1 and 2.9.2. The next two results show that every perfect equilibrium is a Nash equilibrium and that the mixed extension of a strategic game has, at least, one perfect equilibrium.

Theorem 2.9.1. *If $s \in S$ is a perfect equilibrium of $E(G)$, then it is a Nash equilibrium of $E(G)$.*

Proof. Let $s \in S$ be a perfect equilibrium of $E(G)$. Let $\{\eta^k\}$ and $\{s^k\}$ be in the conditions of Definition 2.9.3. Note that, for each $s^k \in S$ and each $i \in N$,

$$u_i(s^k) = \sum_{a_i \in A_i} u_i(s^k_{-i}, a_i) s^k_i(a_i).$$

Hence, s^k is a Nash equilibrium of (G, η^k) if, for each $i \in N$ and each $a_i \in A_i$:

$$a_i \notin \mathrm{PBR}_i(s^k_{-i}) \text{ implies that } s^k_i(a_i) = \eta^k_i(a_i).$$

Suppose that s is not a Nash equilibrium of $E(G)$. Hence, by statement ii) in Proposition 2.4.2, there are $i \in N$ and $a_i \in A_i$ such that $a_i \notin \mathrm{PBR}_i(s_{-i})$ and $s_i(a_i) > 0$. Then, since $\{\eta^k\} \to 0$ and $\{s^k\} \to s$, if k is large enough, $a_i \notin \mathrm{PBR}_i(s^k_{-i})$ and $s^k_i(a_i) > \eta^k_i(a_i)$, which contradicts the fact that s^k is a Nash equilibrium of (G, η^k). □

Theorem 2.9.2. *The mixed extension of a finite game G has, at least, one perfect equilibrium.*

Proof. Let $\{\eta^k\} \subset T(G)$, with $\{\eta^k\} \to 0$. For each $k \in \mathbb{N}$, let s^k be a Nash equilibrium of (G, η^k). Since S is a compact set, the sequence $\{s^k\} \subset S$ has a convergent subsequence converging to an $s \in S$, which is a perfect equilibrium of $E(G)$. □

The next proposition displays two alternative formulations of the perfect equilibrium concept that will be useful later on. One of them is based on the concept of *ε-perfect equilibrium*, which we introduce below.

Definition 2.9.4. Let G be a finite game and let $E(G)$ be its mixed extension. Let $\varepsilon > 0$. A strategy profile $s \in S$ is an *ε-perfect equilibrium* of $E(G)$ if it is completely mixed and satisfies that, for each $i \in N$ and each $a_i, \hat{a}_i \in A_i$,

(2.9.1) $$u_i(s_{-i}, a_i) < u_i(s_{-i}, \hat{a}_i) \Rightarrow s_i(a_i) \leq \varepsilon.$$

[25] Defining perfect equilibrium for infinite games is not straightforward. Indeed, different extensions have been discussed (Simon and Stinchcombe 1995, Méndez-Naya et al. 1995).

Therefore, in an ε-perfect equilibrium, those strategies that are not best replies are played with probability at most ε.

Proposition 2.9.3. *Let $E(G)$ be the mixed extension of a finite game G and let $s \in S$. The following three statements are equivalent.*

i) *s is a perfect equilibrium of $E(G)$.*

ii) *There are two sequences $\{\varepsilon^k\} \subset (0,\infty)$, with $\{\varepsilon^k\} \to 0$, and $\{s^k\} \subset S$, with $\{s^k\} \to s$, such that, for each $k \in \mathbb{N}$, s^k is an ε^k-perfect equilibrium of $E(G)$.*

iii) *There is a sequence of completely mixed strategy profiles $\{s^k\} \subset S$, with $\{s^k\} \to s$, such that, for each $k \in \mathbb{N}$ and each $i \in N$, $s_i \in \mathrm{BR}_i(s^k_{-i})$.*

Proof. i) $\Rightarrow$ ii). Since s is perfect, there are two sequences $\{\eta^k\}$ and $\{s^k\}$ in the conditions of Definition 2.9.3. Recall that if s^k is a Nash equilibrium of (G,η^k), then only best replies are chosen with probability higher than $\eta^k_i(a_i)$ in s^k. For each $k \in \mathbb{N}$, let $\varepsilon^k := \max_{i\in N} \max_{a_i \in A_i} \eta^k_i(a_i)$. Then, $\{\varepsilon^k\} \to 0$ and s^k is an ε^k-perfect equilibrium for each $k \in \mathbb{N}$.

ii) $\Rightarrow$ iii). Let $\{s^k\}$ and $\{\varepsilon^k\}$ be as in ii). By Eq. (2.9.1), if $a_i \in \mathscr{S}(s_i)$, then there is $\bar{k} \in \mathbb{N}$ such that, for each $k \geq \bar{k}$, $a_i \in \mathrm{PBR}_i(s^k_{-i})$. Hence, iii) holds.

iii) $\Rightarrow$ i). Let $\{s^k\}$ be a sequence as in iii). For each $k \in \mathbb{N}$, each $i \in N$, and each $a_i \in A_i$, let

$$\eta^k_i(a_i) := \begin{cases} s^k_i(a_i) & a_i \notin \mathscr{S}(s_i) \\ 1/k & \text{otherwise.} \end{cases}$$

Clearly, $\{\eta^k\} \to 0$. Moreover, there is $\bar{k} \in \mathbb{N}$ such that, for each $k \geq \bar{k}$, $\eta^k \in T(G)$, (G,η^k) is well defined, and $s^k \in S(\eta^k)$. Now, for each $k \geq \bar{k}$, each $i \in N$, and each $a_i \in A_i$, since $s_i \in \mathrm{BR}_i(s^k_{-i})$, we have that:

$$a_i \notin \mathrm{PBR}_i(s^k_{-i}) \;\Rightarrow\; s_i(a_i) = 0 \;\Rightarrow\; s^k_i(a_i) = \eta^k_i(a_i).$$

Hence, for each $k \geq \bar{k}$, s^k is a Nash equilibrium of (G,η^k). $\square$

Perfect equilibrium is closely related to the idea of dominance. We say that a strategy s_i of player i is *undominated* if there is no other strategy of player i that is never worse than s_i and sometimes is better.

Definition 2.9.5. Let G be a finite game and let $E(G)$ be its mixed extension. Let $s_i, \bar{s}_i \in S_i$ be two strategies of player i in $E(G)$. Then, $\bar{s}_i$ *dominates* s_i if:

i) For each $\hat{s}_{-i} \in S_{-i}$, $u_i(\hat{s}_{-i}, \bar{s}_i) \geq u_i(\hat{s}_{-i}, s_i)$, and

ii) There is $\tilde{s}_{-i} \in S_{-i}$ such that $u_i(\tilde{s}_{-i}, \bar{s}_i) > u_i(\tilde{s}_{-i}, s_i)$.

Equivalently, $\bar{s}_i$ dominates s_i if i) for each $\hat{a}_{-i} \in A_{-i}$, $u_i(\hat{a}_{-i}, \bar{s}_i) \geq u_i(\hat{a}_{-i}, s_i)$ and ii) there is $\tilde{a}_{-i} \in A_{-i}$ such that $u_i(\tilde{a}_{-i}, \bar{s}_i) > u_i(\tilde{a}_{-i}, s_i)$.[26]

Definition 2.9.6. Let $s_i \in S_i$ be a strategy of player i in $E(G)$. Then, s_i is an *undominated strategy* if there is no $\bar{s}_i \in S_i$ that dominates s_i. We say that $s \in S$ is an *undominated strategy profile* if, for each $i \in N$, s_i is undominated. Finally, an *undominated Nash equilibrium* of $E(G)$ is an undominated strategy profile that is a Nash equilibrium of $E(G)$.[27]

Note that, in the games in Examples 2.9.1 and 2.9.2, the unique perfect equilibrium, (L_1, L_2), is also the unique undominated Nash equilibrium. Theorem 2.9.5 below shows that the latter was not a coincidence because, in the mixed extension of a bimatrix game, the sets of undominated Nash equilibria and of perfect equilibria coincide.[28] First, we introduce one further property of undominated strategies in bimatrix games.

Let $G = (\{S_l, S_m\}, u)$ be a bimatrix game.[29] Given $\bar{s}_1 \in S_l$, define the matrix game $G_{\bar{s}_1} := (S_l, S_m, \bar{u}_1)$ where, for each $s_1 \in S_l$ and each $s_2 \in S_m$, $\bar{u}_1(s_1, s_2) := u_1(s_1, s_2) - u_1(\bar{s}_1, s_2)$; that is, the objective of player 1 in the game $G_{\bar{s}_1}$ is to choose a strategy s_1 that, given s_2, outperforms $\bar{s}_1$ as much as possible in the original game G. Since $G_{\bar{s}_1}$ is a (zero-sum) matrix game, the objective of player 2 is to prevent player 1 from doing so. The next result says that $\bar{s}_1$ being undominated in G is equivalent to the value of $G_{\bar{s}_1}$ being 0 and each strategy in M being chosen with positive probability at some optimal strategy of player 2 in $G_{\bar{s}_1}$. The intuition for one of the implications is quite simple. In $G_{\bar{s}_1}$, player 1 can ensure himself a payoff of 0 by playing $\bar{s}_1$, *i.e.*, $V(G_{\bar{s}_1}) = 0 \geq 0$. If he can guarantee for himself something strictly positive, then it has to be the case that $\bar{s}_1$ is dominated. Besides, suppose that the value of the game is 0 and, moreover, that there is a pure strategy j of player 2 that is not chosen with positive probability at any optimal strategy. This can only happen if, whenever a strategy of player 2 assigns positive probability to j, then player 1 can get a payoff greater than 0; that is, there is a strategy s_1 of player 1 that is never worse than $\bar{s}_1$ but that is strictly better against j. The next lemma formalizes the above discussion.

Lemma 2.9.4. *Let $G = (\{S_l, S_m\}, u)$ be a bimatrix game and let $\bar{s}_1 \in S_l$. Then, $\bar{s}_1$ is undominated if and only if $V(G_{\bar{s}_1}) = 0$ and each $j \in M$ belongs to the support of some optimal strategy of player 2 in game $G_{\bar{s}_1}$.*

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[26] This notion of dominance is often referred to as weak dominance.

[27] This concept can be extended in a direct way to infinite strategic games.

[28] To prove this result we follow the approach in van Damme (1991). More specifically, Lemma 2.9.4 and Theorem 2.9.5 essentially follow Lemma 3.2.1 and Theorem 3.2.2 in van Damme (1991).

[29] Recall the notation L and M for the sets of pure strategies in two-player games; $S_l = \Delta L$ and $S_m = \Delta M$ being the corresponding sets of mixed strategies.

Proof. First, note that $V(G_{\bar{s}_1}) \geq 0$ ($\bar{s}_1$ ensures payoff 0 to player 1). The strategy $\bar{s}_1$ is dominated by s_1 in G if and only if, for each $j \in M$, $u_1(s_1, j) \geq u_1(\bar{s}_1, j)$ and there is $k \in N$ such that $u_1(s_1, k) > u_1(\bar{s}_1, k)$. Hence, $\bar{s}_1$ is dominated by s_1 in G if and only if, for each $j \in M$, $\bar{u}_1(s_1, j) \geq 0$ and there is $k \in M$ such that $\bar{u}_1(s_1, k) > 0$. Hence, if $\bar{s}_1$ is dominated by s_1 in G either i) $V(G_{\bar{s}_1}) > 0$ or ii) $V(G_{\bar{s}_1}) = 0$ and k is not played at any optimal strategy of player 2. Conversely, i) immediately implies that $\bar{s}_1$ is dominated in G and, by Proposition 2.8.7, ii) implies that there is an optimal strategy s_1 of player 1 in $G_{\bar{s}_1}$ against which k is not a best reply and, since $V(G_{\bar{s}_1}) = 0$, s_1 dominates $\bar{s}_1$ in G. □

Theorem 2.9.5. *Let $E(G)$ be the mixed extension of a finite game G. Then:*

i) *Every perfect equilibrium of $E(G)$ is an undominated Nash equilibrium of $E(G)$.*

ii) *If G is a two-player game,* i.e., *if $E(G)$ is a bimatrix game, then a strategy profile is a perfect equilibrium of $E(G)$ if and only if it is an undominated Nash equilibrium of $E(G)$.*

Proof. Statement i) is quite straightforward by Theorem 2.9.1 and the fact that every strategy profile that satisfies statement iii) in Proposition 2.9.3 is undominated. Now, because of i), only one implication has to be proved to prove ii). Let $G = (\{S_l, S_m\}, u)$ be a bimatrix game and let $\bar{s} = (\bar{s}_1, \bar{s}_2)$ be an undominated Nash equilibrium of G. Now consider the game $G(\bar{s}_1)$. By Lemma 2.9.4, $V(G_{\bar{s}_1}) = 0$ and each $j \in M$ belongs to the support of some optimal strategy of player 2. By Proposition 2.7.2, the sets of optimal strategies are convex and, hence, there is a completely mixed optimal strategy of player 2 in $G_{\bar{s}_1}$. Let $\tilde{s}_2 \in S_n$ be one such strategy. Note that, since $V(G_{\bar{s}_1}) = 0$, $\bar{s}_1$ is optimal for player 1. Then, $\bar{s}_1$ is a best reply against $\tilde{s}_2$ in $G_{\bar{s}_1}$ and, hence, it is also a best reply against $\tilde{s}_2$ in G. Thus, $\bar{s}_1$ is also a best reply in G against any convex combination of $\bar{s}_2$ and $\tilde{s}_2$. For each $k \in \mathbb{N}$, let $s_2^k := (1 - \frac{1}{k})\bar{s}_2 + \frac{1}{k}\tilde{s}_2$. Hence, $\{s_2^k\} \to \bar{s}_2$, all the strategies in the sequence are completely mixed, and $\bar{s}_1$ is a best reply against all of them. Since $\bar{s}_2$ is also undominated, we can define the game $G_{\bar{s}_2}$ and use it to define strategies s_1^k such that $\{s_1^k\} \to \bar{s}_1$, all the strategies in the sequence are completely mixed and $\bar{s}_2$ is a best reply against all of them. Therefore, the sequence $\{(s_1^k, s_2^k)\}$ can be used to show that $\bar{s}$ satisfies iii) in Proposition 2.9.3. Hence, $\bar{s}$ is a perfect equilibrium of G. □

Corollary 2.9.6. *The mixed extension of a finite game G has, at least, one undominated Nash equilibrium.*

Proof. Follows from the combination of Theorem 2.9.2 and the first statement in Theorem 2.9.5. □

The second statement in Theorem 2.9.5 provides an easy way to look for perfect equilibria in bimatrix games. Unfortunately, the result does not hold with more than two players, *i.e.*, with three or more players there may be undominated Nash equilibria that are not perfect.

Example 2.9.3. Take the mixed extension of the finite three-player game given in Figure 2.9.3. Player 1 chooses a row, player 2 chooses a column,

	L_2	R_2
L_1	1,1,1	1,0,1
R_1	1,1,1	0,0,1

L_3

	L_2	R_2
L_1	1,1,0	0,0,0
R_1	0,1,0	1,0,0

R_3

Figure 2.9.3. Undominated Nash equilibria may not be perfect.

and player 3 chooses a matrix (the name of each matrix is indicated below it). Exercise 2.13 asks the reader to show that (R_1, L_2, L_3) is an undominated Nash equilibrium that is not perfect. $\diamond$

Although perfect equilibrium is an appealing concept it has some drawbacks. For instance, Myerson (1978) provides the following example to show that the addition of dominated strategies may enlarge the set of perfect equilibria.

Example 2.9.4. Consider the mixed extension of the finite two-player game in Figure 2.9.4. This game is a modification of the game in Example 2.9.1

	L_2	R_2	D_2
L_1	1, 1	0, 0	−9, −9
R_1	0, 0	0, 0	−7, −7
D_1	−9, −9	−7, −7	−7, −7

Figure 2.9.4. The addition of dominated strategies may enlarge the set of perfect equilibria.

after the addition of a dominated strategy for each player. By statement ii) in Theorem 2.9.5, since we have a bimatrix game and (R_1, R_2) is an undominated Nash equilibrium, it is also a perfect equilibrium. Alternatively, the perfection of (R_1, R_2) can be checked by taking the sequences $\{\varepsilon^k\}$ and $\{s^k\}$ given, for each $k \in \mathbb{N}$, by $\varepsilon^k := \frac{1}{k+10}$ and, for each $i \in \{1,2\}$, by $s_i^k := (\frac{1}{k+10}, 1 - \frac{2}{k+10}, \frac{1}{k+10})$. Now, $\{\varepsilon^k\} \to 0$, $\{s^k\} \to (R_1, R_2)$, and, for each $k \in \mathbb{N}$, s^k is an ε^k-perfect equilibrium. Hence, by statement iii) in Proposition 2.9.3, (R_1, R_2) is a perfect equilibrium of G. $\diamond$

To address this and other drawbacks of perfect equilibrium, Myerson (1978) introduces the *proper equilibrium* concept. Like perfect equilibrium, proper equilibrium selects those Nash equilibria which are still in equilibrium even in cases where players might make mistakes when choosing their strategies; however, the proper equilibrium concept is more demanding since it requires that the more costly a mistake is for a player, the less likely that mistake is. We now show the formal definition of proper equilibrium.

Definition 2.9.7. Let G be a finite game and let $E(G)$ be its mixed extension. Let $\varepsilon > 0$. A strategy profile $s \in S$ is an ε-proper equilibrium of $E(G)$ if it is completely mixed and satisfies that, for each $i \in N$ and each $a_i, \hat{a}_i \in A_i$,

$$u_i(s_{-i}, a_i) < u_i(s_{-i}, \hat{a}_i) \Rightarrow s_i(a_i) \leq \varepsilon s_i(\hat{a}_i). \tag{2.9.2}$$

Definition 2.9.8. Let G be a finite game and let $E(G)$ be its mixed extension. A strategy profile $s \in S$ is a proper equilibrium of $E(G)$ if there are two sequences $\{\varepsilon^k\} \subset (0, \infty)$, with $\{\varepsilon^k\} \to 0$, and $\{s^k\} \subset S$, with $\{s^k\} \to s$, such that, for each $k \in \mathbb{N}$, s^k is an ε^k-proper equilibrium of $E(G)$.[30]

Note that every ε-proper equilibrium of $E(G)$ is obviously ε-perfect. Hence, Proposition 2.9.3 implies that, in the mixed extension of a finite game, every proper equilibrium is also perfect. To show that the reciprocal is not true in general, consider the game in Example 2.9.4. Exercise 2.14 asks the reader to show that the perfect equilibrium (R_1, R_2) is not proper. So, properness is in general a strict refinement of perfectness. However, if G is a finite game in which all players have two strategies, the set of perfect equilibria of $E(G)$ coincides with the set of proper equilibria of $E(G)$. The next proposition provides a proof of this statement.

Proposition 2.9.7. *Let G be a finite game in which all players have two strategies. Then, s is a proper equilibrium of $E(G)$ if and only if it is perfect.*

Proof. It is sufficient to prove that, under the above conditions, perfect implies proper. Let $s \in S$ a perfect equilibrium of $E(G)$ and let $\{\varepsilon^k\}$ and $\{s^k\}$ be two sequences as in statement ii) in Proposition 2.9.3. For each $k \in \mathbb{N}$, let

$$c_i^k := \begin{cases} s_i^k(a_i) & \mathscr{S}(s_i) = \{a_i\} \\ 1 & \mathscr{S}(s_i) = A_i \end{cases} \qquad \text{and} \qquad \bar{\varepsilon}^k := \frac{\varepsilon^k}{\min_{i \in N} c_i^k}.$$

We now show that, for k large enough, s^k is an $\bar{\varepsilon}^k$ proper equilibrium. If k is large enough and $u_i(s_{-i}^k, a_i) < u_i(s_{-i}^k, \hat{a}_i)$, then it has to be the case that

[30] As with perfect equilibrium, extending the definition of proper equilibrium to infinite games is not straightforward (see Simon and Stinchcombe (1995) for a couple of different extensions).

$s_i(a_i) = 0$, which implies that $\mathscr{S}(s_i) = \{\hat{a}_i\}$ and $c_i^k = s_i^k(\hat{a}_i)$. Since s^k is ε^k-perfect, $s_i^k(a_i) \leq \varepsilon^k$ and, hence, $s_i^k(a_i) \leq \bar{\varepsilon}^k s_i^k(\hat{a}_i)$. Since $\{\bar{\varepsilon}^k\} \to 0$, the result follows. $\square$

Within his analysis of proper equilibrium, Myerson provided the following existence result.

Theorem 2.9.8. *The mixed extension of a finite game G has, at least, one proper equilibrium.*

Proof. We show first that, for each $k \in \mathbb{N}\backslash\{1\}$, there is a $\frac{1}{k}$-proper equilibrium of $E(G)$. Let $k \in \mathbb{N}\backslash\{1\}$. Let $m := \max_{i\in N}|A_i|$ and $\delta_k := \frac{1}{m}\frac{1}{k^m}$. Now, for each $i \in N$, let

$$S_i(\delta_k) := \{s_i \in S_i : \text{for each } a_i \in A_i, s_i(a_i) \geq \delta_k\}.$$

Each $S_i(\delta_k)$ is a nonempty compact subset of completely mixed strategies of S_i. Now, define a correspondence $F\colon \prod_{i\in N} S_i(\delta_k) \to \prod_{i\in N} S_i(\delta_k)$ given, for each $s \in \prod_{i\in N} S_i(\delta_k)$, by $F(s) := (F_1(s), \ldots, F_n(s))$, where, for each $i \in N$, $F_i(s) := \{\bar{s}_i \in S_i(\delta_k) : \text{for each pair } a_i, \hat{a}_i \in A_i, \text{ if } u_i(s_{-i}, a_i) < u_i(s_{-i}, \hat{a}_i) \text{ then } \bar{s}_i(a_i) \leq \frac{1}{k}\bar{s}_i(\hat{a}_i)\}$. We now show that each $F_i(s)$ is nonempty. For each $a_i \in A_i$, let $\tau(s, a_i) := |\{\hat{a}_i \in A_i : u_i(s_{-i}, a_i) < u_i(s_{-i}, \hat{a}_i)\}|$. Then, define $\bar{s}_i$, for each $a_i \in A_i$, by:

$$\bar{s}_i(a_i) := \frac{(1/k)^{\tau(s,a_i)}}{\sum_{\hat{a}_i\in A_i}(1/k)^{\tau(s,\hat{a}_i)}}.$$

It is easy to check that $\bar{s}_i \in F_i(s)$ (since, for each $a_i \in A_i$, $\bar{s}_i(a_i) \geq \delta_k$, we have that $\bar{s}_i \in S_i(\delta_k)$). Moreover, F satisfies the conditions of Kakutani theorem. Hence, there is $s^k \in \prod_{i\in N} S_i(\delta_k)$ such that $s^k \in F(s^k)$. So defined, s^k is a $\frac{1}{k}$-proper equilibrium. Since S is a compact set, $\{s^k\}$ has a convergent subsequence that converges to an $s \in S$, which is a proper equilibrium of $E(G)$. $\square$

One of the Myerson's reasons for introducing properness is that the perfectness concept has the drawback that adding dominated strategies can enlarge the set of perfect equilibria. The following example shows that the proper equilibrium concept has the same drawback.

Example 2.9.5. Take the mixed extension of the finite three-player game in Figure 2.9.5. This game can be seen as a modification of the game in Example 2.9.1 after adding a dominated strategy for player three. Note that (R_1, R_2, L_3) is now a proper equilibrium. To check it, simply take the sequences $\{\varepsilon^k\}$ and $\{s^k\}$ given, for each $k \in \mathbb{N}$, by $\varepsilon^k := \frac{1}{k+10}$, $s_1^k := (\frac{1}{k+100}, 1 - \frac{1}{k+100})$, $s_2^k := (\frac{1}{k+100}, 1 - \frac{1}{k+100})$ and $s_3^k := (1 - \frac{1}{k+20}, \frac{1}{k+20})$. Now,

L_3:

	L_2	R_2
L_1	1,1,1	0,0,1
R_1	0,0,1	0,0,1

R_3:

	L_2	R_2
L_1	0,0,0	0,0,0
R_1	0,0,0	1,1,0

Figure 2.9.5. The addition of strictly dominated strategies may enlarge the set of proper equilibria.

$\{\varepsilon^k\} \to 0$, $\{s^k\} \to (R_1, R_2, L_3)$ and, for each $k \in \mathbb{N}$, s^k is an ε^k-proper equilibrium. ◇

In spite of this drawback, proper equilibrium has proved to be an important equilibrium concept, with very interesting properties, especially in connection with extensive games. We discuss some of these properties later in this book.

Perfect and proper equilibrium are, perhaps, the most important refinements of the Nash equilibrium concept in strategic games. However, there are many other refinements in the literature, aiming to find an equilibrium concept that always selects self-enforcing profiles and that, at the same time, chooses at least one profile in the mixed extension of every finite game. The following example illustrates that there may be proper equilibria that are not self-enforcing.

Example 2.9.6. Consider the mixed extension of the finite two-player game in Figure 2.9.6. This game has two pure Nash equilibria that, moreover, are

	L_2	R_2
L_1	7,7	0,0
C_1	0,0	1,2
R_1	5,5	5,5

Figure 2.9.6. The proper equilibrium (R_1, R_2) is not self-enforcing.

proper: (L_1, L_2) and (R_1, R_2). However, the strategy profile (R_1, R_2) is not really self-enforcing. Suppose that the players have informally agreed to play (R_1, R_2), and take player 2. If player 1 sticks to the agreement, player 2 is indifferent between playing L_2 or R_2. Yet, if player 2 deviates, he might gain (if player 1 also deviates from R_1 and chooses L_1) or lose (if player 1 deviates from R_1 and chooses C_1). Nonetheless, player 1 will never choose C_1, which is always worse for him than R_1; if player 1 deviates, then he must select L_1. Hence, player 2, who can anticipate this, will deviate and choose L_2. Finally, player 1, who is able to guess that player 2 will make this reasoning, will also deviate and choose L_1. ◇

We now study two more refinements of Nash equilibrium: the first one, *strict equilibrium* (Harsanyi 1973b), is inspired in the definition of Nash equilibrium, and the second one, *strictly perfect equilibrium* (Okada 1981), is based on the definition of perfect equilibrium.

Definition 2.9.9. Let G be a finite game and $E(G)$ its mixed extension. A strategy profile $s \in S$ is a *strict equilibrium* of $E(G)$ if, for each $i \in N$ and each $\hat{s}_i \in S_i \backslash \{s_i\}$,

$$u_i(s) > u_i(s_{-i}, \hat{s}_i).^{31}$$

Remark 2.9.1. Harsanyi (1973b) called this concept strong equilibrium. The name strict equilibrium has prevailed to avoid confusion with the *strong equilibrium* of Aumann (1959), which is a strategy profile in which no coalition of players has a joint deviation after which all the players in the coalition obtain higher payoffs.

Remark 2.9.2. Every strict equilibrium is a Nash equilibrium. Note also that a strict equilibrium of $E(G)$ is always a pure strategy profile. Harsanyi (1973b) also defined the notion of *quasi-strict equilibrium* of $E(G)$ as a strategy profile s such that, for each player i, his set of pure best replies against s_{-i} coincides with the support of his own strategy.

Definition 2.9.10. Let G be a finite game and $E(G)$ its mixed extension. A strategy profile $s \in S$ is a *strictly perfect equilibrium* of $E(G)$ if, for each sequence $\{\eta^k\} \subset T(G)$ with $\{\eta^k\} \to 0$, there is a sequence $\{s^k\} \subset S$, with $\{s^k\} \to s$, such that, for each $k \in \mathbb{N}$, s^k is a Nash equilibrium of (G, η^k).

Remark 2.9.3. Every strictly perfect equilibrium is a perfect equilibrium (there is always a sequence $\{\eta^k\} \subset T(G)$ with $\{\eta^k\} \to 0$).

We now study the relationship between strict equilibria and strictly perfect equilibria, along with the existence of such equilibria in the mixed extensions of finite games.

Proposition 2.9.9. *Let G be a finite game and $E(G)$ its mixed extension. Let $s \in S$ be a strict equilibrium of $E(G)$. Then, s is a strictly perfect equilibrium.*

Proof. Let $s \in S$ be a strict equilibrium of $E(G)$. Recall that s has to be a pure strategy profile, *i.e.*, there is $a \in A$ such that $s = a$. Let $\{\eta^k\} \subset T(G)$ with $\{\eta^k\} \to 0$. For each $k \in \mathbb{N}$ and each $i \in N$, define s_i^k by:

- For each $\hat{a}_i \in A_i \backslash \{a_i\}$, $s_i^k(\hat{a}_i) = \eta_i^k(\hat{a}_i)$.
- $s_i^k(a_i) = 1 - \sum_{\hat{a}_i \in A_i \backslash \{a_i\}} \eta_i^k(\hat{a}_i)$.

[31] This concept can be extended in a direct way to infinite strategic games.

Now, $\{s^k\} \to a$. Moreover, if k is large enough, then s^k is a Nash equilibrium of (G, η^k). $\square$

The converse of this result is not true, as shown by the following example.

Example 2.9.7. Consider the mixed extension of the matching pennies (see Examples 2.2.6 and 2.3.2). We already know that the unique Nash equilibrium of this game is $s^* = ((1/2, 1/2), (1/2, 1/2))$. Since s^* is not a pure strategy profile, it is not a strict equilibrium. Moreover, a completely mixed Nash equilibrium is always strictly perfect. Hence, s^* is a strictly perfect equilibrium that is not strict. $\diamond$

The following example shows that the mixed extension of a finite game may not have any strictly perfect equilibrium.

Example 2.9.8. Consider the mixed extension of the finite two-player game in Figure 2.9.7. Exercise 2.15 asks the reader to show that this game has no strictly perfect equilibria. $\diamond$

	L_2	C_2	R_2
L_1	1,1	1,0	0,0
R_1	1,1	0,0	1,0

Figure 2.9.7. A game without strictly perfect equilibria.

Note that, by Theorem 2.9.8, we know that the above game has, at least, one proper equilibrium. Hence, it also illustrates that a proper equilibrium may not be strictly perfect. Figure 2.9.8 shows the relationship of the concepts that we have considered. The two concepts inside the shaded areas do not provide a strategy profile for the mixed extension of some finite games, *i.e.*, the existence of strict and strictly perfect equilibria is not ensured for every game. All the relations indicated in Figure 2.9.8 have been demonstrated above with two exceptions:

i) It is left up to the reader to prove that strict implies proper. It is an easy exercise whose solution can be found in van Damme (1991).
ii) The fact that strictly perfect does not imply proper was proved in Vermeulen and Jansen (1996), where a 3×4 bimatrix game with a strictly perfect equilibrium that is not proper is provided.

Remark 2.9.4. There are more refinements for the Nash equilibrium concept in strategic games than the ones we have defined here. We have tried to cover the most common ones. Some other refinements are the *essential*

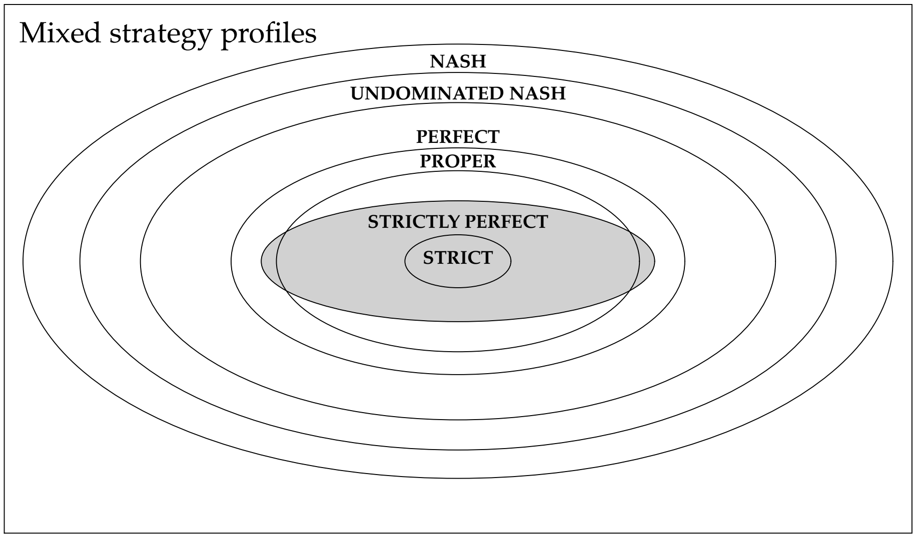


Figure 2.9.8. Relations between the different equilibrium concepts in extensive games that we have discussed in this chapter.

equilibrium (Wen-Tsun and Jia-He 1962), the *regular equilibrium* (Harsanyi 1973a), and the *persistent equilibrium* (Kalai and Samet 1984). Last, but not least, there is another equilibrium notion that takes a different approach and that is widely accepted among game theorists: the *stable sets* (Kohlberg and Mertens 1986). We refer the reader to Remark 3.6.3 for a brief discussion on this concept and for further references on it.

2.10. A Basic Model of Knowledge

In this section we introduce a model of knowledge that provides a complete description of the information each player has before the game is played; more precisely, the model describes the information that is relevant for the game at hand. We present this model with two objectives. First, it allows us to place the definition of a new equilibrium concept, *correlated equilibrium*, on a firm basis. Second, so far we have been implicitly assuming that all the players have complete (and symmetric) information about all the elements of the game. This model provides a convenient tool to model situations in which players may have different information when playing the game. This point will become clearer in the discussions below.

Definition 2.10.1. An *information model* for a set of players $N = \{1, \dots, n\}$ is a triple $I = (\Omega, \{\rho_i\}_{i \in N}, \{\mathcal{P}_i\}_{i \in N})$, whose elements are the following:

i) A finite set Ω, with generic element ω.

ii) For each player $i \in N$, a probability measure ρ_i on Ω.

iii) For each player $i \in N$, a partition $\mathcal{P}_i$ of Ω.

The set Ω represents the set of all possible states of the world. The probability measures represent the prior beliefs of the different players about the state of the world that is going to be realized. Finally, the partition $\mathcal{P}_i$ denotes the *information partition* of player i. Given $P \in \mathcal{P}_i$, if the true state of the world is $\omega \in P$, then player i just knows that the true state of the world is some *atom* of P, but does not know which one; that is, the different states of the world included in P are indistinguishable for player i. Let $P_i(\omega)$ denote the set of states that i cannot rule out when ω is the true state of the world. Note that, for each $\omega \in \Omega$, $\omega \in P_i(\omega)$. Finally, each $E \subset \Omega$ is an *event* of Ω.

Remark 2.10.1. Although the finiteness assumption for Ω is not completely without loss of generality, we have two good reasons to make it. First, the information model so defined suffices to rigorously discuss the issues for which we have defined it. Second, the use of an infinite set would require the use of more involved mathematical concepts; with the consequent loss of transparency in the results, proofs, and discussions.[32]

The following definition will be useful later on.

Definition 2.10.2. An information model I has a *common prior* if for each $i, j \in N$, $\rho_i = \rho_j$.

For the sake of notation, information models with a common prior are denoted by $(\Omega, \rho, \{\mathcal{P}_i\}_{i \in N})$ instead of $(\Omega, \{\rho_i\}_{i \in N}, \{\mathcal{P}_i\}_{i \in N})$. The common prior assumption is standard in most models in economic theory. Following the discussion in Aumann (1987) "…the common prior assumption expresses the view that probabilities should be based on information; that people with different information may legitimately entertain different probabilities, but there is no rational basis for people who have always been fed precisely with the same information to do so. (…) Under the common prior assumption differences in probabilities express differences in information *only*". The previous argument reflects Aumann's view and it may not be compelling to everybody. We conclude this discussion with an example that illustrates that, although the common prior assumption is quite natural, it has an impact on strategic situations that can be modeled.[33]

Example 2.10.1. Consider the information model without common prior given by $(\Omega, \{\rho_1, \rho_2\}, \{\mathcal{P}_1, \mathcal{P}_2\})$, where $\Omega = \{\omega_1, \omega_2\}$, $\rho_1(\omega_1) = \rho_2(\omega_2) = 0.75$, $\rho_1(\omega_2) = \rho_2(\omega_1) = 0.25$, and $P_1(\omega_1) = P_1(\omega_2) = P_2(\omega_1) = P_2(\omega_2) =$

[32] Refer to Mertens and Zamir (1985) for a paper in which the information model is defined with full generality.

[33] We refer the reader to Gul (1998) and Aumann (1998) for a deeper discussion on this assumption.

Ω; that is, there are only two states of the world and the two players have opposite beliefs about the likelihood of each of them. Player 1 believes that ω_1 is three times more likely than ω_2 and player 2 believes that ω_2 is three times more likely than ω_1. Moreover, once a state of the world is realized none of the players receives any extra information about it.

Suppose that state ω_1 is realized. Then, it is easy to see that, at state ω_1, it is common knowledge that player 1 assigns probability 0.75 at ω_1 being the true state of the world and it is also common knowledge that player 2 assigns probability 0.25 at ω_1 being the true state of the world. In words of Aumann (1976), both players "agree to disagree". Actually, Aumann (1976) shows that, under the common prior assumption, the players cannot agree to disagree. More precisely, under the common prior assumption, there can be no state of the world at which there is common knowledge that the players attach different (posterior) probabilities to a given event having being realized. Therefore, to model a situation in which differences in beliefs at some states of the world are common knowledge, the common prior assumption has to be relaxed. $\diamond$

Let $E \subset \Omega$ and $\omega \in \Omega$. We say that *i knows E* at ω if $P_i(\omega) \subset E$; that is, when ω is the true state of the world i knows that some atom of E has happened. Moreover, $\mathrm{K}_i\, E := \{\omega \in \Omega : P_i(\omega) \subset E\}$ denotes the states of the world at which i knows E. Similarly, $\mathrm{K}_*\, E := \bigcap_{i \in N} \mathrm{K}_i\, E$ denotes the states of the world at which all the players know E. Since $\mathrm{K}_i\, E \subset \Omega$, $K_i E$ is itself an event. Hence, $\mathrm{K}_j\, \mathrm{K}_i\, E$ is well defined and denotes the states at which j knows that i knows E. Finally, we can provide a formal definition of a very important concept: *common knowledge*.

Definition 2.10.3. An event $E \subset \Omega$ is *common knowledge* at $\omega \in \Omega$ if $\omega \in \mathrm{CK}\, E$, where $\mathrm{CK}\, E := \mathrm{K}_*\, E \cap \mathrm{K}_*\, \mathrm{K}_*\, E \cap \mathrm{K}_*\, \mathrm{K}_*\, \mathrm{K}_*\, E \cap \ldots$

In words, an event is common knowledge if all the players know it, all the players know that all other players know it,... and so on up to infinity. Note that common knowledge is much more than simply saying that something is known by all players. The following example illustrates this point.

Example 2.10.2. (Coordinated attack game). Two allied armies are situated on opposite hills waiting to attack their enemy. The commander in chief is in one of them and there is a captain in charge of the soldiers in the other. In order to ensure a successful battle, none of them will attack unless he is sure that the other will do so at the same time. The commander sends a messenger to the captain with the message "I plan to attack at night". The messenger's journey is dangerous and he may die on the way. Suppose that he gets to the other hill and informs the captain, who sends the messenger back with the confirmation. Unfortunately, the messenger dies on his way back. In this situation, both the commander and the captain know that *"the commander plans to attack at night"* but, can they be certain that both will attack at night? The problem is that the commander cannot be sure that the captain got the message and, similarly, the captain does not know whether the commander received the confirmation or not. Should each of them attack anyway? If the *event "the commander plans to attack at night"* were common knowledge, then there would be no doubt.[34] $\diamond$

2.11. Correlated Equilibrium

So far, we have been assuming that players choose their strategies independently. In this section we drop this assumption and consider noncooperative situations in which players are able to coordinate their strategies in a certain way; that is, they can *correlate* their strategies. This idea was first treated by Aumann (1974) when he introduced the concept of correlated equilibrium. We informally illustrate this concept in the following example.

Example 2.11.1. Consider the battle of the sexes, already discussed in Example 2.5.1. If players are able to coordinate when choosing their strategies, then they can decide, for instance, to toss a coin and go together to the cinema if heads is observed, or go together to the theater if the result is tails. This correlation cannot be achieved by means of a mixed strategy profile. Moreover, according to the definitions below, the previous correlation leads to a correlated equilibrium. Intuitively, this is an equilibrium, as no player gains when unilaterally deviating from this coordinated plan. $\diamond$

Definition 2.11.1. Let $G = (A, u)$ be a finite strategic game and let $I = (\Omega, \{\rho_i\}_{i\in N}, \{\mathcal{P}_i\}_{i\in N})$ be an information model. A *correlated strategy profile* is a map τ from Ω into A. Moreover, τ is *consistent with* I, shortly *I-consistent*, if for each $i \in N$, each $P \in \mathcal{P}_i$, and each pair $\omega, \bar{\omega} \in P$, we have that $\tau_i(\omega) = \tau_i(\bar{\omega})$.

So, in a correlated strategy profile, the players can condition their strategies upon the realization of a random event. The consistency requirement says that each player must know the strategy he chooses. In the game theoretical literature, to refer to this consistency condition, it is common to say

[34] This example was taken from Rubinstein (1989) (although originally introduced in Halpern (1986)). Refer to Rubinstein's paper for a deeper discussion on the strategic implications of (weakenings of) common knowledge.

that τ_i is measurable with respect to $\mathcal{P}_i$;[35] that is, if player i cannot distinguish between two states ω and $\bar{\omega}$, it is natural to assume that he is going to make the same choice in both cases.

Each mixed strategy profile can be seen as a correlated strategy profile, but the converse is not true. Although in both cases the players base their decisions on the realization of a random event, the difference is that, with mixed strategies, the random events used by the different players are independent of each other.

Next, we formally introduce the correlated equilibrium concept.

Definition 2.11.2. Let $G = (A, u)$ be a strategic game. A *correlated equilibrium* is a pair (I, τ^*), where I is an information model with common prior ρ and τ^* is an I-consistent correlated strategy profile such that, for each $i \in N$ and each I-consistent correlated strategy $\hat{\tau}_i$, we have

$$\sum_{\omega \in \Omega} \rho(\omega) u_i(\tau^*(\omega)) \geq \sum_{\omega \in \Omega} \rho(\omega) u_i(\tau^*_{-i}(\omega), \hat{\tau}_i(\omega)). \tag{2.11.1}$$

Remark 2.11.1. Let $\mathbb{E}[Y]$ denote the expectation of the random variable Y. Then, Eq. (2.11.1) is equivalent to $\mathbb{E}[u_i(\tau^*)] \geq \mathbb{E}[u_i(\tau^*_{-i}, \hat{\tau}_i)]$; that is, in a correlated equilibrium all the players maximize their expected utilities given their beliefs.

Remark 2.11.2. Moreover, the equilibrium condition also implies that for each $\omega \in \Omega$ such that $\rho(\omega) > 0$, we have that $\tau^*_i(\omega)$ is a player i's best reply to τ^*_{-i} given i's information at ω. In particular, $\tau^*(\omega)$ need not be a Nash equilibrium of G, so correlated equilibria go beyond alternating Nash equilibria as in Example 2.11.1. We illustrate this in Example 2.11.4.

Remark 2.11.3. Note that the information model is part of the correlated equilibrium; that is, given a correlated strategy profile τ, it will be part of a correlated equilibrium if there is an information model such that (i) all players are maximizing their expected utilities when playing τ, (ii) the information model has a common prior, and (iii) τ is consistent for the information model.

Remark 2.11.4. If we drop the common prior assumption for the information model in the definition of correlated equilibrium, then we have the definition of *subjective correlated equilibrium*, also discussed in Aumann (1974). This equilibrium concept is quite weak, since a very large number of correlated strategies will be part of a subjective correlated equilibrium. The reason for this is that players can have very different beliefs about what the

[35] Aumann (1987) wrote: "A random variable is *measurable* w.r.t. a partition if it is constant in each element of the partition. Intuitively, this means that the r.v. contains no more information than the partition…".

play of the game is going to be. Hence, using this disparity in the beliefs, we can support a wide variety of correlated strategies. Indeed, all the players will think that they are maximizing but, eventually, all of them might be far from doing so.[36]

Example 2.11.2. Every (pure) Nash equilibrium $a \in A$ of G induces a correlated equilibrium of G, (I, τ^*), given by:

- $I = (\Omega, \rho, \{\mathcal{P}_i\}_{i \in N})$, where $\Omega = \{a\}$, $\rho(a) = 1$, and, for each $i \in N$, $\mathcal{P}_i = \{\{a\}\}$.
- For each $i \in N$, $\tau_i^*(a) = a_i$. $\diamond$

Example 2.11.3. In the battle of the sexes (Example 2.5.1), the following specification of (I, τ^*) is a correlated equilibrium:

- $I = (\Omega, \rho, \{\mathcal{P}_i\}_{i \in N})$, where $\Omega = \{C, T\}$, $\rho(C) = \rho(T) = 1/2$, and $\mathcal{P}_1 = \mathcal{P}_2 = \{\{C\}, \{T\}\}$.
- $\tau_1^*(C) = \tau_2^*(C) = C$ and $\tau_1^*(T) = \tau_2^*(T) = T$. $\diamond$

The next theorem and its proof are the natural consequence of the relationship between mixed and correlated strategies discussed above. It says that Nash equilibrium can always be seen as a refinement of correlated equilibrium.

Theorem 2.11.1. *Let $G = (A, u)$ be a finite strategic game and let $s^* \in S$ be a Nash equilibrium of its mixed extension, $E(G)$. Then there is a correlated equilibrium of G, (I, τ^*), such that*

i) *for each $i \in N$, the probability distribution induced by τ_i^* on A_i is precisely s_i and*

ii) *the probability distribution induced by τ^* on A is precisely s.*

Proof. Let $I := (\Omega, \rho, \{\mathcal{P}_i\}_{i \in N})$ where i) $\Omega := A$, ii) for each $a \in A$, $\rho(A) := \prod_{i \in N} s_i^*(a_i)$, and iii) for each $i \in N$, $\mathcal{P}_i := \{P_i(a_i) : a_i \in A_i\}$, where, for each $a_i \in A_i$, $P_i(a_i) := \{\bar{a} \in A : \bar{a}_i = a_i\}$. For each $a \in \Omega$ and each $i \in N$, let $\tau_i^*(a) := a_i$.

Note that an I-consistent correlated strategy of player i is just a function from A to A_i such that, for each $a_i \in A_i$ and each pair $\hat{a}, \tilde{a} \in A$, $\hat{\tau}_i(\hat{a}_{-i}, a_i) = \hat{\tau}_i(\tilde{a}_{-i}, a_i)$. Hence, an I-consistent correlated strategy of player i can be simply represented by a function from A_i to A_i; thus, we use $\tau_i(\cdot, a_i)$ to denote the strategy chosen by player i when given information a_i. Clearly, τ^* is I-consistent. Note that, for each $i \in N$,

$$\sum_{a \in \Omega} \rho(a) u_i(\tau^*(a)) = u_i(s^*).$$

[36]Actually, to avoid this kind of situation, along with the common prior assumption it is also implicitly assumed that it coincides with the true distribution.

Let $i \in N$ and let $\hat{\tau}_i$ be an I-consistent correlated strategy of player i. For each $a_i \in A_i$, let $T_i(a_i) := \{\hat{a}_i \in A_i : \hat{\tau}_i(\cdot, \hat{a}_i) = a_i\}$. Now, define $\hat{s}_i \in S_i$, for each $a_i \in A_i$, by

$$\hat{s}_i(a_i) := \begin{cases} \sum_{\hat{a}_i \in T_i(a_i)} s_i^*(\hat{a}_i) & T_i(a_i) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

Hence, for each $i \in N$,

$$\sum_{a \in \Omega} \rho(a) u_i(\tau_{-i}^*(a), \hat{\tau}_i(a)) = u_i(s_{-i}^*, \hat{s}_i).$$

Therefore, since s^* is a Nash equilibrium of $E(G)$, (I, τ^*) is a correlated equilibrium of G. Finally, for each $i \in N$, the probability that a_i is chosen if τ_i^* is played is

$$\sum_{\hat{a} \in P_i(a_i)} \rho(\hat{a}) = s_i^*(a_i),$$

which proves i); ii) follows from the definition of the common prior, which implies that the choices of the players are independent of each other. $\square$

Corollary 2.11.2. *Every finite game has, at least, one correlated equilibrium.*

The result below says that the set of correlated equilibrium payoff vectors is convex. This is not true for any other of the equilibrium concepts we have discussed. Yet, recall that, in the case of matrix games, an even stronger result holds; namely, not only the set of Nash equilibrium payoff vectors is convex (it is pinned down by the value of the game, which is unique), but also the sets of optimal strategies are convex.

Theorem 2.11.3. *Let G be a finite strategic game. Then, the set of correlated equilibrium payoff vectors of G is convex.*

Proof. Let $v^1, v^2 \in \mathbb{R}^n$ be a pair of correlated equilibrium payoff vectors of G. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be such that $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, and $\alpha_1 + \alpha_2 = 1$. We now show that there is a correlated equilibrium of G with payoff vector $\alpha_1 v^1 + \alpha_2 v^2$.

Let (I^1, τ^1) and (I^2, τ^2) be the correlated equilibria associated with v^1 and v^2, respectively, where, for each $l \in \{1,2\}$, $I^l = (\Omega^l, \rho^l, \{\mathcal{P}_i^l\}_{i \in N})$. Assume, without loss of generality, that $\Omega^1 \cap \Omega^2 = \emptyset$. Let $I := (\Omega, \rho, \{\mathcal{P}_i\}_{i \in N})$ be the information model given by $\Omega := \Omega^1 \cup \Omega^2$, for each $l \in \{1,2\}$ and each $\omega \in \Omega^l$, $\rho(\omega) := \alpha_l \rho^l(\omega)$, and, for each $i \in N$, $\mathcal{P}_i = \mathcal{P}_i^1 \cup \mathcal{P}_i^2$. Let τ be the I-consistent strategy profile defined, for each $\omega \in \Omega$ and each $i \in N$, by $\tau_i(\omega) := \tau_i^l(\omega)$, where l is such that Ω^l is the unique set of states that contains ω. We now show that (I, τ) is the correlated equilibrium we are looking for. Since,

$$\begin{aligned}\sum_{\omega\in\Omega}\rho(\omega)u_i(\tau(\omega)) &= \textstyle\sum_{l=1}^{2}\sum_{\omega\in\Omega^l}\alpha_l\rho^l(\omega)u_i(\tau^l(\omega))\\ &= \textstyle\sum_{l=1}^{2}\alpha_l\sum_{\omega\in\Omega^l}\rho^l(\omega)u_i(\tau^l(\omega)) \quad = \quad \alpha_1 v^1+\alpha_2 v^2,\end{aligned}$$

we only need to check that (I,τ) is a correlated equilibrium. Suppose there is a player $i\in N$ and an I-consistent correlated strategy of player i, namely $\hat{\tau}_i$, such that i's expected payoff is higher with $\hat{\tau}_i$ than with τ_i. Then, there is $l\in\{1,2\}$ such that i's expected payoff in Ω^l is higher with $\hat{\tau}_i$ than with τ_i; which can be easily shown to contradict that (I^l,τ^l) is a correlated equilibrium of G. $\square$

After the previous result, we may wonder if, given a finite game G, its set of correlated equilibrium payoff vectors coincides with the convex hull of the set of Nash equilibrium payoff vectors of the mixed extension of G. The next example shows that the answer is, in general, negative.

Example 2.11.4. Consider the finite two-player game G in Figure 2.11.1. Different variations of this game are referred to as the chicken game or the hawk-dove game. The chicken game represents a situation in which two drivers drive towards each other; one must swerve (play L) to avoid the crash, but if only one swerves, he is called a chicken (coward). The hawk-dove game is mainly used in biology to represent two animals contesting an indivisible resource; they can play hawk (R) and fight for the resource or play dove (L) and withdraw if being attacked. The mixed extension of this game has three Nash equilibria: $((0,1),(1,0))$, $((1,0),(0,1))$, and $((2/3,1/3),(2/3,1/3))$, whose corresponding payoff vectors are $(7,2)$, $(2,7)$, and $(14/3,14/3)$, respectively.

	L_2	R_2
L_1	6, 6	2, 7
R_1	7, 2	0, 0

Figure 2.11.1. A game where the set of correlated equilibrium payoff vectors is larger than the convex hull of the Nash equilibrium payoff vectors.

Let (I,τ^*) be given by

- $I=(\Omega,\rho,\{\mathcal{P}_i\}_{i\in N})$, where $\Omega=\{\omega_1,\omega_2,\omega_3\}$, $\rho(\omega_1)=\rho(\omega_2)=\rho(\omega_3)=1/3$,
- $\mathcal{P}_1=\{\{\omega_1\},\{\omega_2,\omega_3\}\}$, $\mathcal{P}_2=\{\{\omega_1,\omega_2\},\{\omega_3\}\}$,
- $\tau_1^*(\omega_1)=L_1$, $\tau_1^*(\omega_2)=\tau_1^*(\omega_3)=R_1$, $\tau_2^*(\omega_1)=\tau_2^*(\omega_2)=L_2$, $\tau_2^*(\omega_3)=R_2$.

It is easy to check that (I, τ^*) is a correlated equilibrium of G. Note that the payoff vector associated to this correlated equilibrium is $(5,5)$, which does not belong to $\operatorname{conv}(\{(7,2),(2,7),(14/3,14/3)\})$ (see Figure 2.11.2). $\diamond$

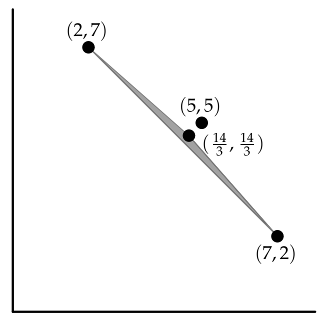


Figure 2.11.2. A correlated equilibrium payoff vector that is not in the convex hull of the Nash equilibrium payoff vectors.

2.12. On the Epistemic Foundations of the Different Solution Concepts for Strategic Games

In this section we use the model of knowledge introduced in Section 2.10 to briefly discuss the epistemic foundations of some of the solution concepts we have studied so far. One of the most active game theorists in this field has been Robert Aumann; due to his contributions to this and many other areas of game theory, he was awarded the Nobel Prize in Economics in 2005 (shared with Thomas C. Schelling).

To begin with, we want to uncover one implicit assumption that has been present from the very beginning of this chapter; that is, whenever we introduced a model, we implicitly assumed that it was *common knowledge*. In the basic model used to describe a strategic game, all the players had complete information about the payoff functions and the sets of strategies and, in addition, this information was assumed to be common knowledge. Throughout this section we also assume that the whole model is common knowledge.

Now, we explore what other assumptions are implicit in some of the equilibrium concepts we have studied. For instance, the assumption that players are rational supports any definition of the concept of best reply and, hence, it also underlies the motivations for the different equilibria. Indeed, behind many of the reasonings of this chapter lies the assumption of common knowledge of rationality. It is natural to wonder how close are our equilibrium concepts to the idea of rationality. To do so, we first need to know precisely what it means to be rational.

In the rest of this section we restrict the discussion to a finite setting: a finite number of states of the world and finite games. Let $G = (A, u)$ be a finite strategic game and let $I = (\Omega, \{\rho_i\}_{i \in N}, \{\mathcal{P}_i\}_{i \in N})$ be an information model for game G; that is, each state of the world contains the specification of all the variables that may be objects of uncertainty on the part of any player of G. In particular, each $\omega \in \Omega$ contains the specification of the strategy chosen by each player at ω. Moreover, following the motivation of the previous section, all the strategies have to be I-consistent. For each $\omega \in \Omega$, we denote by $a_i(\omega)$ the strategy chosen by player i when the true state of the world is ω. Henceforth, the game G is fixed and $\mathcal{I}^G$ denotes the set of (finite) consistent information models for G.

Remark 2.12.1. This framework might seem very restrictive, but it is not. The fact that, conditional on a given ω, all the players know everything, allows us to describe uncertainty by means of the information partitions of the different players (even the uncertainty about other players' decisions).

Remark 2.12.2. Note that, since we are assuming that Ω is finite, it might seem that mixed strategies are ruled out. Yet, this is not true. The idea is that all aspects of players' uncertainty are contained in I; that is, suppose that we already have an information model I for our game G. Suppose that player 1 comes up with the idea that, at a given element of his information partition, namely P, he will play a mixed strategy by tossing a coin and choosing his pure strategy depending on whether the result is heads or tails. Since each element of Ω has to contain a complete specification of all variables, to accommodate the coin toss into our model we have to change I. One way of doing so is to split each $\omega \in P$ into two new states of the world, each one containing one of the two possible realizations of the coin toss. Further, the probabilities have to change as well; in this case, the probability associated to ω is split equally into the two new states. Finally, for the information partitions, only the player tossing the coin can distinguish between the two elements in which ω has been split. Hence, whenever we discuss mixed strategies in this section, the reader may think of them as a shortcut for the above modification of the information model at hand.

Remark 2.12.3. Recall that the assumption of common knowledge for the model is not incompatible with the fact that at a given state of the world players have different information. Common knowledge of the model simply implies that, prior to the realization of the specific state of the world, all the players have the same information about the model (indeed, complete information).

Henceforth, we assume that players update their beliefs using Bayes rule. This is called *Bayesian updating*: at state $\omega \in \Omega$, the beliefs of player i

coincide with his prior beliefs, conditional on the fact that $P_i(\omega)$ has been realized (at ω, i just knows $P_i(\omega)$). We denote these beliefs by $\rho_i(\cdot\,|_{P_i(\omega)})$.

We say that player i is playing a best reply at $\omega \in \Omega$ if he is maximizing, given his beliefs about the (possibly correlated) strategy profiles of the other $n-1$ players.

Definition 2.12.1. Let $I = (\Omega, \{\rho_i\}_{i\in N}, \{\mathcal{P}_i\}_{i\in N}) \in \mathcal{I}^G$. Let $\omega \in \Omega$ and $i \in N$. Then, $a_i(\omega)$ is a best reply to i's beliefs at ω, *i.e.*, $a_i(\omega) \in \mathrm{BR}_i(\rho_i(\cdot\,|_{P_i(\omega)}))$, if, for each $\hat{a}_i \in A_i$,

$$\sum_{\hat{\omega}\in P_i(\omega)} \rho_i(\hat{\omega}\,|_{P_i(\omega)}) u_i(a(\hat{\omega})) \geq \sum_{\hat{\omega}\in P_i(\omega)} \rho_i(\hat{\omega}\,|_{P_i(\omega)}) u_i(a_{-i}(\hat{\omega}), \hat{a}_i).^{37}$$

Next, we present a formal definition of (Bayesian) rationality.

Definition 2.12.2. Let $I = (\Omega, \{\rho_i\}_{i\in N}, \{\mathcal{P}_i\}_{i\in N}) \in \mathcal{I}^G$ and let $\omega \in \Omega$. A player $i \in N$ is *rational* at ω if $a_i(\omega) \in \mathrm{BR}_i(\rho_i(\cdot\,|_{P_i(\omega)}))$.

Next, we present a result by Aumann (1987). This result is just a reinterpretation of the concept of correlated equilibrium and, with the structure we have already introduced, its proof is quite straightforward.

Theorem 2.12.1. *Let G be a finite game and $I \in \mathcal{I}^G$ an information model with common prior. If each player is rational at each state of the world, then the pair $(I, a(\cdot))$ is a correlated equilibrium.*

Proof. Let $I := (\Omega, \rho, \{\mathcal{P}_i\}_{i\in N}) \in \mathcal{I}^G$, where ρ is the common prior. First, note that $a(\cdot)$ is an I-consistent correlated strategy profile. Since all the players are rational at each state of the world, we know that, for each $i \in N$, each $\omega \in \Omega$, and each $\hat{a}_i \in A_i$,

$$\sum_{\hat{\omega}\in P_i(\omega)} \rho(\hat{\omega}\,|_{P_i(\omega)}) u_i(a(\hat{\omega})) \geq \sum_{\hat{\omega}\in P_i(\omega)} \rho(\hat{\omega}\,|_{P_i(\omega)}) u_i(a_{-i}(\hat{\omega}), \hat{a}_i).$$

Then, multiplying both sides by $\rho(P_i(\omega))$ we have,

$$\sum_{\hat{\omega}\in P_i(\omega)} \rho(\hat{\omega}) u_i(a(\hat{\omega})) \geq \sum_{\hat{\omega}\in P_i(\omega)} \rho(\hat{\omega}) u_i(a_{-i}(\hat{\omega}), \hat{a}_i).$$

Varying over all the elements of $\mathcal{P}_i$, we have that, for each I-consistent correlated strategy of player i, namely $\hat{\tau}_i$,

$$\sum_{\hat{\omega}\in \Omega} \rho(\hat{\omega}) u_i(a(\hat{\omega})) \geq \sum_{\hat{\omega}\in \Omega} \rho(\hat{\omega}) u_i(a_{-i}(\hat{\omega}), \hat{\tau}_i(\hat{\omega})),$$

which is the definition of correlated equilibrium. □

[37] Recall that for each $\hat{\omega} \in P_i(\omega)$, $a_i(\hat{\omega}) = a_i(\omega)$. Hence, although in the first term of the equation we wrote $a(\hat{\omega})$, we could have written $(a_{-i}(\hat{\omega}), a_i(\omega))$.

According to Theorem 2.12.1, whenever the players are rational and their knowledge can be derived from a common prior, then they must be playing a correlated equilibrium.

A special type of information model is one in which the information that each player receives is independent from that of the other players; that is, at each state $\omega \in \Omega$, player i knows that he is at the information partition $P_i(\omega)$, but this knowledge does not give him any new information about the probabilities of the information partitions at which the other players are. More formally, for each $j \neq i$ and each $P_j \in \mathcal{P}_j$, the events P_j and $P_i(\omega)$ are independent and, hence,

$$\rho_i(P_j \,|_{P_i(\omega)}) = \frac{\rho_i(P_j \cap P_i(\omega))}{\rho_i(P_i(\omega))} = \frac{\rho_i(P_j)\rho_i(P_i(\omega))}{\rho_i(P_i(\omega))} = \rho_i(P_j).$$

In any such information model there is no room for correlation and, hence, each correlated strategy corresponds with a mixed strategy. Therefore, the next result follows from Theorem 2.12.1.

Corollary 2.12.2. *Let G be a finite game and $I \in \mathcal{I}^G$ an information model with common prior. Assume that for each pair $i, j \in N$, each $P_i \in \mathcal{P}_i$, and each $P_j \in \mathcal{P}_j$ the events P_i and P_j are independent according to the common prior. If each player is rational at each state of the world, then the pair $(I, a(\cdot))$ corresponds with a Nash equilibrium.*

Proof. Follows from Theorem 2.12.1 and the fact that for any such information model a correlated strategy profile corresponds with a mixed strategy profile. $\square$

For a deeper discussion on the foundations of the Nash equilibrium concept refer to Aumann and Brandenburger (1995), Brandenburger and Dekel (1987), and Polak (1999).

After the above characterizations of correlated equilibrium and Nash equilibrium, it is natural to wonder what are the implications of rationality on its own, *i.e.*, in the absence of the common prior assumption. To address this issue, in the remainder of this section we assume that there is common knowledge of rationality and drop the common prior assumption.[38] First, we introduce a concept of dominance that is stronger than the one introduced in Section 2.9.

Definition 2.12.3. Let $\bar{s}_i, s_i \in S_i$ be two strategies of player i in $E(G)$. We say that $\bar{s}_i$ *strictly dominates* s_i if, for each $\hat{s}_{-i} \in S_{-i}$, $u_i(\hat{s}_{-i}, \bar{s}_i) > u_i(\hat{s}_{-i}, s_i)$. Equivalently, $\bar{s}_i$ strictly dominates s_i if, for each $a_{-i} \in A_{-i}$, $u_i(a_{-i}, \bar{s}_i) > u_i(a_{-i}, s_i)$.

[38] Recall that we are working with finite games and we are also assuming common knowledge of the model and Bayesian updating for the beliefs.

We say that a strategy $s_i \in S_i$ is *strictly dominated* if there is $\bar{s}_i \in S_i$ that strictly dominates s_i. Lemma 2.12.3 shows that having a strategy $s_i \in S_i$ that is never a best reply, regardless of the strategies of the other $n-1$ players, is equivalent to saying that such strategy is strictly dominated.

At each state of the world ω, each player i has a conjecture (belief) about the strategy profiles chosen by the other players. Indeed, if $P_i(\omega) = \{\omega\}$, this conjecture coincides with the chosen profile. This conjecture is just a probability distribution over the strategy profiles in A_{-i}. Hence, the conjectures of player i are defined over correlated profiles of the other players.

Definition 2.12.4. Let $i \in N$. The set of possible *conjectures* of player i about the strategy profiles in A_{-i} of the players in $N\backslash\{i\}$ is $\mathcal{C}_{-i}(A_{-i}) := \Delta A_{-i}$.

Given a player $i \in N$, a strategy $s_i \in S_i$, and a conjecture $\alpha \in \mathcal{C}_{-i}(A_{-i})$, then $u_i(\alpha, s_i) := \sum_{a_{-i} \in A_{-i}} \alpha(a_{-i}) u_i(a_{-i}, s_i)$.

Lemma 2.12.3. *Let G be a finite game and let $\bar{a}_i \in A_i$. Then, the following two statements are equivalent:*

i) *There is $\alpha \in \mathcal{C}_{-i}(A_{-i})$ such that $\bar{a}_i \in \mathrm{BR}_i(\alpha)$.*

ii) *The strategy $\bar{a}_i$ is not strictly dominated.*

Proof.[39]

i) $\Rightarrow$ ii) Suppose that there is $s_i \in S_i$ that strictly dominates $\bar{a}_i$. Then, for each $\beta \in \mathcal{C}_{-i}(A_{-i}), u_i(\beta, s_i) > u_i(\beta, \bar{a}_i)$. Hence, $\bar{a}_i$ is never a best reply.

ii) $\Rightarrow$ i) Assume, without loss of generality, that $i = 1$. Suppose that $\bar{a}_1$ is never a best reply to any element of $\mathcal{C}_{-1}(A_{-1})$. Then, there is a function $g\colon \mathcal{C}_{-1}(A_{-1}) \to S_1$ such that, for each $\alpha \in \mathcal{C}_{-1}(A_{-1})$, $u_1(g(\alpha), \alpha) > u_1(\bar{a}_1, \alpha)$. Now consider the two player zero-sum game $\bar{G} = (A_1, A_{-1}, \bar{u}_1)$, where, for each $a_1 \in A_1$ and each $a_{-1} \in A_{-1}$ (*i.e.*, for each $a \in A$), we have $\bar{u}_1(a_1, a_{-1}) := u_1(a_1, a_{-1}) - u_1(\bar{a}_1, a_{-1})$. By Theorem 2.2.3, the game $E(\bar{G})$ has a Nash equilibrium. Let (s_1^*, α^*) be a Nash equilibrium of $E(\bar{G})$. Note that a mixed strategy for player 2 in $\bar{G}$ is just an element of $\mathcal{C}_{-1}(A_{-1})$. Then, for each $\hat{\alpha} \in \mathcal{C}_{-1}(A_{-1})$,

$$\bar{u}_1(s_1^*, \hat{\alpha}) \geq \bar{u}_1(s_1^*, \alpha^*) \geq \bar{u}_1(g(\alpha^*), \alpha^*) > \bar{u}_1(\bar{a}_1, \alpha^*) = 0.$$

Since, for each $\hat{\alpha} \in \mathcal{C}_{-1}(A_{-1})$, $\bar{u}_1(s_1^*, \hat{\alpha}) > 0$, then, for each $a_{-1} \in A_{-1}$, $u_1(s_1^*, a_{-1}) > u_1(\bar{a}_1, a_{-1})$. Hence, s_1^* strictly dominates $\bar{a}_1$. □

Note that the previous lemma implies that, when working with finite games, the notion of strict dominance can be formally stated without using mixed strategies (but still assuming that players have von Neumann and Morgenstern utility functions). Another consequence is that common

[39] This proof is an adaptation of the proof of Lemma 3 in Pearce (1984).

knowledge of rationality (indeed, just rationality) implies that players do not choose strictly dominated strategies. Indeed, we show below that common knowledge of rationality has deeper implications.

We now introduce the idea of iterative elimination of strictly dominated strategies. We know that rational players never play strictly dominated strategies (Lemma 2.12.3). Since we are assuming common knowledge of rationality, all the players know that no player will assign a positive probability to any strictly dominated strategy in any state of the world. Let Λ_i^1 denote the set of pure strategies of player i that are not strictly dominated and $\Lambda^1 := \prod_{i \in N} \Lambda_i^1$. Now, consider the reduced game $G^1 := (\Lambda^1, u)$. Applying Lemma 2.12.3 to G^1, we have that players will assign probability 0 at the strictly dominated strategies of G^1. Moreover, this argument can be repeated up to infinitum. Hence, let Λ_i^k denote the set of strategies of player i that survive after the k-th step of the iterative elimination procedure. Finally, Λ_i^∞ denotes the set of strategies that survive infinitely many rounds. Indeed, since we are working with finite games, the procedure finishes in finitely many steps. This leads to the definition of *rationalizability*.

Definition 2.12.5. For each player $i \in N$ and each strategy $a_i \in A_i$, a_i is a *rationalizable strategy* if $a_i \in \Lambda_i^\infty$. Similarly, $\Lambda^\infty := \prod_{i \in N} \Lambda_i^\infty$ denotes the set of strategy profiles that are *rationalizable*.

Remark 2.12.4. The above definition of rationalizability stems from the combination of common knowledge of rationality with the idea that a rational player should never choose a strategy that is never a best reply against any conjecture over the strategy profiles chosen by the other players. Moreover, we have defined these conjectures over the set of correlated strategy profiles of the other players. However, in the seminal papers on rationalizability (Bernheim 1984, Pearce 1984), this notion was derived without allowing for correlation. In their setting, Lemma 2.12.3 would only hold when the number of players is 2 (since correlation is innocuous there). Therefore, when the number of players is greater than 2, depending on whether the conjectures are defined or not over correlated strategy profiles, one may get slightly different versions of rationalizability.

Theorem 2.12.4. *Let G be a finite game and $I \in \mathcal{I}^G$ be an information model. If there is common knowledge of rationality at each state of the world, then, for each $i \in N$ and each $\omega \in \Omega$, the strategy $a_i(\omega) \in A_i$ is rationalizable.*

Conversely, for each rationalizable strategy profile $a \in \Lambda^\infty$, there is an information model $I \in \mathcal{I}^G$ with common knowledge of rationality and a state $\omega \in \Omega$ such that $a(\omega) = a$.

Proof. The first part of the statement essentially follows from the discussion preceding the definition of rationalizable strategy (Definition 2.12.5).

Let $I \in \mathcal{I}^G$ be an information model with common knowledge of rationality. Let $i \in N$ and $\omega \in \Omega$. By common knowledge of rationality, the conjecture of i at ω assigns probability 0 to the strategy profiles outside Λ^∞_{-i}. Since i is rational, he has to choose a best reply to the conjecture. Hence, by definition of Λ^∞_i, his strategy will be in Λ^∞_i.

For the converse, consider the following construction of the information model $I := (\Omega, \{\rho_i\}_{i\in N}, \{\mathcal{P}_i\}_{i\in N}) \in \mathcal{I}^G$. Let $\Omega := \Lambda^\infty$, that is, each state of the world consists of the rationalizable strategy profile played at such state. Define the information partition of each $i \in N$ as follows: for each pair $a, \hat{a} \in \Omega = \Lambda^\infty$, $a \in P_i(\hat{a})$ if and only if $a_i = \hat{a}_i$. In order to define the priors of the players, we define first the posteriors at each state of the world. By Lemma 2.12.3, for each $i \in N$ and each $a_i \in \Lambda^\infty_i$, there is $\alpha \in \mathcal{C}_{-i}(\Lambda^\infty_{-i})$ such that $a_i \in \mathrm{BR}_i(\alpha)$. For each $\hat{a} \in P_i(a)$, let $\rho_i(\hat{a}\,|_{P_i(a)}) := \alpha(\hat{a}_{-i})$. Now, pick any weights $\{z(a)\}_{a\in\Lambda^\infty}$ such that for each $a \in \Lambda^\infty$, $z(a) > 0$ and $\sum_{a\in\Lambda^\infty} z(a) = 1$. Finally, define the priors of player i as follows. For each $a \in \Lambda^\infty$, $\rho_i(a) := z(a)\rho_i(a\,|_{P_i(a)})$. These priors generate the posteriors defined above and, moreover, for each $a \in \Lambda^\infty$ there is a state of the world (indeed, a itself) in which a is played.[40] Finally, note that at each state of the world all the players are rational and, moreover, this fact is common knowledge in I. $\square$

Remark 2.12.5. The previous theorem shows how far we can go with common knowledge of rationality. Under this assumption, all players choose rationalizable strategies, that is, strategies that cannot be removed by iterative elimination of strictly dominated strategies. Note that the first part of the statement in the theorem is very similar to that of Theorem 2.12.1, but without the common prior assumption and with the strengthening of rationality to common knowledge of rationality.

Remark 2.12.6. Refer to Rubinstein (1989) for an example in which a mild weakening of the common knowledge assumption for the rationality of the players has a severe impact on the strategic behavior of the players.

The next result shows that the set of strategies that can be played in some correlated equilibrium is a subset of the set of rationalizable strategies. In particular, combined with the existence result for correlated equilibrium (Corollary 2.11.2), it implies the existence of rationalizable strategy profiles.

Proposition 2.12.5. *Let (I^*, τ^*) be a correlated equilibrium of G, with $I^* \in \mathcal{I}^G$. Let $a_i \in A_i$ be a strategy that is played with positive probability according to the common prior. Then, a_i is rationalizable,* i.e., $a_i \in \Lambda^\infty_i$.

[40] We have just chosen one of the different priors that generate the posteriors we fixed before. Note that, in general, it is not possible to do it in such a way that the priors are common.

Proof. For each $i \in N$, let $\bar{A}_i$ be the set of strategies of player i that are played with positive probability according to (I^*, τ^*). Let $\bar{A} := \prod_{i \in N} \bar{A}_i$. To prove the result it suffices to show that, for each $i \in N$, $\bar{A}_i \cap (A_i \backslash \Lambda_i^\infty) = \emptyset$. Suppose, on the contrary, that there are $i \in N$ and $\hat{a}_i \in \bar{A}_i \cap (A_i \backslash \Lambda_i^\infty)$ such that $\hat{a}_i$ is one of the elements of $\cup_{i \in N} \big(\bar{A}_i \cap (A_i \backslash \Lambda_i^\infty)\big)$ that would be eliminated at an earlier stage when following the iterative elimination of dominated strategies procedure. Let $\omega \in \Omega$ be a state of the world in which $\hat{a}_i$ is played. Now, there is $s_i \in S_i$ that strictly dominates $\hat{a}_i$ in a game of the form (Λ^k, u), with $\bar{A} \subset \Lambda^k$. In particular, for each $a_{-i} \in \bar{A}_{-i}$, $u_i(a_{-i}, s_i) > u_i(a_{-i}, \hat{a}_i)$. Hence, $\hat{a}_i$ cannot be a best reply against any belief (conjecture) over the strategy profiles in $\bar{A}_{-i}$. Since it is common knowledge whether a strategy is played with positive probability or not according to (I^*, τ^*), the belief (conjecture) of player i at ω must put probability 0 outside $\bar{A}_{-i}$. Together, the last two observations contradict the fact that (I^*, τ^*) is a correlated equilibrium of G. □

Remark 2.12.7. In Brandenburger and Dekel (1987), it is shown that the notion of rationalizability (R) is equivalent to the concept of *a posteriori equilibrium* (PE). The latter was introduced in Aumann (1974) as a natural refinement of subjective correlated equilibrium (SCE); PE being itself refined by correlated equilibrium (CE). Hence, we have the following relations for the sets of strategies that can be played with positive probability in some of the solution concepts mentioned above

$$\text{NE} \subset \text{CE} \subset \text{PE} = \text{R} \subset \text{SCE}.$$

Besides, for each one of the above inclusion relations we can find a game in which such inclusion is strict. Note that these relations can be completed with those presented in Section 2.9 (Figure 2.9.8), where we studied different refinements of the Nash equilibrium concept.

2.13. Fixed-Point Theorems

This section and the following one contain the proofs of some technical results used in the preceding sections. The notations are completely independent from those in the rest of the book.

In this section we present a formal proof of Kakutani fixed-point theorem (Kakutani 1941). There are various ways in which this important theorem can be proved. Here, we follow the classic approach of deriving it from a series of results that rely on Sperner lemma.[41] Remarkably, McLennan and Tourky (2006) have proved Kakutani theorem by means of what

[41] We refer the reader to Ichiishi (1983) or Border (1985) for deeper discussions on fixed point theorems within game theoretical settings. Indeed, we follow here the proofs in the former book.

they called "a proof that is brief, elementary, and based on game theoretic concepts".

2.13.1. Some preliminary concepts. Let $k \in \mathbb{N}$ and let I be a set such that $|I| = k+1$. A finite set $\{x^i\}_{i\in I} \subset \mathbb{R}^n$ is *affinely independent* if, for each $\{r^i\}_{i\in I} \subset \mathbb{R}^n$ such that $\sum_{i\in I} r_i x^i = 0$ and $\sum_{i\in I} r_i = 0$, we have that, for each $i \in I$, $r_i = 0$. Let $\{x^i\}_{i\in I} \subset \mathbb{R}^n$ be an affinely independent set. Then, the set $S := \operatorname{conv}(\{x^i\}_{i\in I})$ is a *k-simplex*. The elements in $\{x^i\}_{i\in I}$ are the vertices of the simplex, *i.e.*, $\operatorname{ext}(S) = \{x^i\}_{i\in I}$. Given $\hat{I} \subset I$, the simplex $\operatorname{conv}(\{x^i\}_{i\in \hat{I}})$ is a $(|\hat{I}|-1)$*-face* of S. Given a set S, its *diameter* is the supremum of the distances between any two points in S.

We introduce some more concepts that allow us to state Sperner lemma in a simple way. Let S be a k-simplex. A collection $\mathcal{S}$ of k-simplices is a *dissection* of S if i) $\bigcup_{\hat{S}\in\mathcal{S}} \hat{S} = S$ and ii) for each pair $\hat{S}, \tilde{S} \in \mathcal{S}$, either $\hat{S} \cap \tilde{S} = \emptyset$ or $\hat{S} \cap \tilde{S}$ is a face of both $\hat{S}$ and $\tilde{S}$. For each collection of simplices $\mathcal{S}$, let $\mathcal{V}^{\mathcal{S}} := \bigcup_{S\in\mathcal{S}} \operatorname{ext}(S)$ be the set of "vertices" of $\mathcal{S}$ and let $d_{\mathcal{S}}$ denote the maximum of the diameters of the simplices in $\mathcal{S}$. Let $S = \operatorname{conv}(\{x^i\}_{i\in I})$ be a simplex and $\mathcal{S}$ a dissection of S. A *Sperner labelling associated with S and $\mathcal{S}$* is a map $f_l \colon \mathcal{V}^{\mathcal{S}} \to \operatorname{ext}(S)$ such that, for each $x \in \mathcal{V}^{\mathcal{S}}$ and each $\hat{I} \subset I$, if $x \in \operatorname{conv}(\{x^i\}_{i\in\hat{I}})$, then $f_l(x) \in \{x^i\}_{i\in\hat{I}}$. In words, the "vertices" of $\mathcal{V}^{\mathcal{S}}$ are labelled so that a point that is on a face of S must be given the label of one extreme point of that face. Each $\hat{S} \in \mathcal{S}$ such that $f_l(\operatorname{ext}(\hat{S})) = \operatorname{ext}(S)$ is said to be *completely labelled*.

2.13.2. Auxiliary results.

Lemma 2.13.1 (Sperner lemma)**.** *Let S be a simplex and $\mathcal{S}$ a dissection of S. Let f_l be a Sperner labelling associated with S and $\mathcal{S}$. Then, there is $\hat{S} \in \mathcal{S}$ such that $\hat{S}$ is completely labelled.*

Proof. Let $k \in \mathbb{N}$. Let I be such that $|I| = k+1$ and let $\{x^i\}_{i\in I}$ be an affinely independent set such that $S = \operatorname{conv}(\{x^i\}_{i\in I})$. Let $\mathcal{C} \subset \mathcal{S}$ be the set of completely labelled simplices of $\mathcal{S}$ under f_l. We show, by induction on k, that $|\mathcal{C}|$ is odd. The case $k=0$ is trivial. Assume that the result is true for $k-1$. Take $\hat{I} \subset I$ such that $|\hat{I}| = k$. For each $\hat{S} \in \mathcal{S}$, let $r_{\hat{S}}$ be the number of $(k-1)$-faces of $\hat{S}$ such that f_l maps its k vertices onto $\{x^i\}_{i\in\hat{I}}$. It can be readily checked that $\hat{S}$ is completely labelled if and only if $r_{\hat{S}} = 1$; otherwise, it is not hard to show that either $r_{\hat{S}} = 0$ or $r_{\hat{S}} = 2$.[42]

[42] Informally, the argument would run as follows. The case $r_{\hat{S}} = 0$ can arise, for instance, if all the vertices of $\hat{S}$ have the same label. Suppose that $r_{\hat{S}} \geq 1$ and take a $(k-1)$-face of $\hat{S}$ such that f_l maps its k vertices onto $\{x^i\}_{i\in\hat{I}}$. If the label of the remaining vertex of $\hat{S}$ belongs to $\{x^i\}_{i\in\hat{I}}$, then $r_{\hat{S}} = 2$; if not, then $r_{\hat{S}} = 1$.

Since $\mathcal{C} = \{\hat{S} \in \mathcal{S} : r_{\hat{S}} = 1\}$ and, for each $\hat{S} \in \mathcal{S}$, $r_{\hat{S}} \in \{0, 1, 2\}$, to show that $|\mathcal{C}|$ is odd it suffices to show that $\sum_{\hat{S} \in \mathcal{S}} r_{\hat{S}}$ is odd. Let $\mathcal{R} := \{R : R$ is a $(k-1)$-face of a simplex in $\mathcal{S}$ such that f_l maps its k vertices onto $\{x^i\}_{i \in \hat{I}}\}$. Given $R \in \mathcal{R}$, either R is not contained in a $(k-1)$-face of S or R is the intersection of two simplices in $\mathcal{S}$. In the latter case, R is counted twice in the sum $\sum_{\hat{S} \in \mathcal{S}} r_{\hat{S}}$. Hence, $\sum_{\hat{S} \in \mathcal{S}} r_{\hat{S}}$ is odd if and only if the number of elements of $\mathcal{R}$ that are contained in $(k-1)$-faces of S is odd. Yet, since f_l is a Sperner labelling, only the face $\operatorname{conv}(\{x^i\}_{i \in \hat{I}})$ can contain any such simplices. Now, we can apply the induction hypothesis to the face $\operatorname{conv}(\{x^i\}_{i \in \hat{I}})$ to obtain the desired result. □

The next result is a nice and useful application of Sperner lemma. Moreover, the result, along with its proof, already has some fixed-point flavor.

Theorem 2.13.2 (Knaster-Kuratowski-Mazurkiewicz theorem)**.** *Let S be the simplex in $\mathbb{R}^n$ defined by $S := \operatorname{conv}(\{x^i\}_{i \in I})$. Let $\{A^i\}_{i \in I}$ be such that i) for each $i \in I$, A^i is a closed subset of S and ii) for each $\hat{I} \subset I$, $\operatorname{conv}(\{x^i\}_{i \in \hat{I}}) \subset \bigcup_{i \in \hat{I}} A^i$. Then, $\bigcap_{i \in I} A^i \neq \emptyset$.*

Proof. Let $\{\mathcal{S}^m\}_{m \in \mathbb{N}}$ be a sequence of dissections of S such that $\{d_{\mathcal{S}^m}\} \to 0$. For each $m \in \mathbb{N}$, take the Sperner labelling f_l^m associated with S and $\mathcal{S}^m$ defined as follows. For each $x \in \mathcal{V}^{\mathcal{S}^m}$, let $\hat{I}$ be the smallest subset of I such that $x \in \operatorname{conv}(\{x^i\}_{i \in \hat{I}})$ (note that $\hat{I}$ is uniquely defined). Since $\operatorname{conv}(\{x^i\}_{i \in \hat{I}}) \subset \bigcup_{i \in \hat{I}} A^i$, there is $i \in \hat{I}$ such that $x \in A^i$. Take one such i and let $f_l^m(x) := x^i$. By Sperner lemma, for each $m \in \mathbb{N}$, there is $S^m \in \mathcal{S}^m$ that is completely labelled under f_l^m. Let $\operatorname{ext}(S^m) = \{x^{i,m}\}_{i \in I}$ and assume, without loss of generality, that $f_l^m(x^{i,m}) = x^i$. For each $i \in I$ and each $m \in \mathbb{N}$, $x^{i,m} \in A^i$. For each $i \in I$, the sequence $\{x^{i,m}\}_{m \in \mathbb{N}}$ has a convergent subsequence; let $\bar{x}^i$ be the limit of one such subsequence. Since the A^i sets are closed, for each $i \in I$, $\bar{x}^i \in A^i$. Moreover, since $\{d_{\mathcal{S}^m}\} \to 0$, all the $\bar{x}^i$ are the same point, which then belongs to $\bigcap_{i \in I} A^i$. □

2.13.3. Fixed-point theorems.

Theorem 2.13.3 (Brouwer fixed-point theorem)**.** *Let $A \subset \mathbb{R}^n$ be a nonempty, convex, and compact set. Let $f\colon A \to A$ be a continuous function. Then, there is $\bar{x} \in A$ such that $f(\bar{x}) = \bar{x}$,* i.e., *f has a fixed-point.*

Proof. We divide the proof in two cases.

Case 1: There are $k \in \mathbb{N}$, a set I such that $|I| = k + 1$, and affinely independent points $\{x^i\}_{i \in I} \subset \mathbb{R}^n$ such that A coincides with the simplex $\operatorname{conv}(\{x^i\}_{i \in I})$. For each $x \in A$, there is a unique way in which x can be written as a convex combination of the points in $ext(A)$, namely, $x = \sum_{i \in I} \alpha_i(x) x^i$, where $\sum_{i \in I} \alpha_i(x) = 1$ and, for each $i \in I, \alpha_i(x) \geq 0$. For each $i \in I$, let $A^i := \{x \in A : \alpha_i(f(x)) \leq \alpha_i(x)\}$. By the continuity of f, for each $i \in I$, A^i is a closed set. We now claim that, for each $\hat{I} \subset I$, $\operatorname{conv}(\{x^i\}_{i\in\hat{I}}) \subset \bigcup_{i\in\hat{I}} A^i$. Let $\hat{I} \subset I$ and $x \in \operatorname{conv}(\{x^i\}_{i\in\hat{I}})$. Then, $\sum_{i\in\hat{I}} \alpha_i(x) = 1 \geq \sum_{i\in\hat{I}} \alpha_i(f(x))$. Hence, there is $i \in \hat{I}$ such that $\alpha_i(x) \geq \alpha_i(f(x))$ and thus, $x \in A^i \subset \bigcup_{j\in\hat{I}} A^j$. By Theorem 2.13.2, there is $\bar{x} \in \bigcap_{i\in I} A^i$, *i.e.*, for each $i \in I$, $\alpha_i(f(\bar{x})) \leq \alpha_i(\bar{x})$. Finally, since $\sum_{i\in I} \alpha_i(f(\bar{x})) = \sum_{i\in I} \alpha_i(\bar{x}) = 1$, then, for each $i \in I$, $\alpha_i(f(\bar{x})) = \alpha_i(\bar{x})$. Therefore, $f(\bar{x}) = \bar{x}$.

Case 2: $A \subset \mathbb{R}^n$ is a nonempty, convex, and compact set. Then, there are $k \in \mathbb{N}$, a set I such that $|I| = k+1$, and affinely independent points $\{x^i\}_{i\in I} \subset \mathbb{R}^n$ such that $A \subset \operatorname{conv}(\{x^i\}_{i\in I})$. Among those such simplices, let S be one with the smallest k. Let $\tilde{x}$ be a point in the k-dimensional interior of S. Now, we define $\tilde{f}$, an extension of f to the whole simplex S, as follows. For each $x \in S$, let $\lambda(x) := \max\{\lambda \in [0,1] : (1-\lambda)\tilde{x} + \lambda x \in A\}$ and $\tilde{f}(x) := f((1-\lambda(x))\tilde{x} + \lambda(x)x)$. So defined, λ is a continuous function and, hence, $\tilde{f}$ is also continuous. Since the range of $\tilde{f}$ is contained in A, every fixed point of $\tilde{f}$ is also a fixed point of f. Now, by Case 1, $\tilde{f}$ has a fixed point and, hence, f also does. □

We are ready to prove Kakutani fixed-point theorem (Theorem 2.2.1 in Section 2.2).

Theorem (Kakutani fixed-point theorem)**.** *Let $A \subset \mathbb{R}^n$ be a nonempty, convex, and compact set. Let $F\colon A \to A$ be an upper hemicontinuous, nonempty-valued, closed-valued, and convex-valued correspondence. Then, there is $\bar{x} \in A$ such that $\bar{x} \in F(\bar{x})$,* i.e., *F has a fixed-point.*

Proof. Again, we divide the proof in two cases.

Case 1: There are $k \in \mathbb{N}$, a set I such that $|I| = k+1$, and affinely independent points $\{x^i\}_{i\in I} \subset \mathbb{R}^n$ such that A coincides with the simplex $\operatorname{conv}(\{x^i\}_{i\in I})$. Let $\{\mathcal{S}^m\}_{m\in\mathbb{N}}$ be a sequence of dissections of A such that $\{d_{\mathcal{S}^m}\} \to 0$. For each $m \in \mathbb{N}$, we define the function $g^m\colon A \to A$ as follows. If $x \in \mathcal{V}(\mathcal{S}^m)$, take $y \in F(x)$ and let $g^m(x) := y$. Then, for each $x \in A$, there is $S = \operatorname{conv}(\{x^i_S\}_{i\in I}) \in \mathcal{S}^m$ such that $x \in S$. Hence, $x = \sum_{i\in I} \alpha_i x^i_S$, where $\sum_{i\in I} \alpha_i = 1$ and, for each $i \in I, \alpha_i \geq 0$. Let $g^m(x) := \sum_{i\in I} \alpha_i g^m(x^i_S)$. All the g^m functions are piecewise linear and, hence, continuous. By Brouwer fixed-point theorem, for each $m \in \mathbb{N}$, there is x^m such that $g^m(x^m) = x^m$. Then, for each $m \in \mathbb{N}$, let $\{x^i_m\}_{i\in I}$ be the extreme points of a simplex of $\mathcal{S}^m$ that contains x^m. Then, $x^m = \sum_{i\in I} \alpha^m_i x^i_m$, where $\sum_{i\in I} \alpha^m_i = 1$ and, for each $i \in I, \alpha^m_i \geq 0$. Using subsequences if necessary, we can assume, without loss of generality, that, for each $i \in I$, $\{\alpha^m_i\}_{m\in\mathbb{N}} \to \bar{\alpha}_i$ and that $\{g^m(x^i_m)\}_{m\in\mathbb{N}} \to \bar{y}^i$. Now, $\sum_{i\in I} \bar{\alpha}_i = 1$ and, for each $i \in I, \bar{\alpha}_i \geq 0$. Moreover, the sequences $\{x^i_m\}_{m\in\mathbb{N}}$ and $\{x^m\}_{m\in\mathbb{N}}$ can also be assumed to be convergent and, since $\{d_{\mathcal{S}^m}\} \to 0$, all the limits coincide. Let $\bar{x} \in A$ be the common limit.

We claim that, for each $i \in I$, $\bar{y}^i \in F(\bar{x})$. Suppose there is $i \in I$ such that $\bar{y}^i \notin F(\bar{x})$. Then, since $F(\bar{x})$ is closed, there are open sets B_1 and B_2 in A such that $\bar{y}^i \in B_1$, $F(\bar{x}) \subset B_2$, and $B_1 \cap B_2 = \emptyset$ (recall that we are in a Hausdorff space). Since F is upper hemicontinuous and $\{x^i_m\}_{m\in\mathbb{N}} \to \bar{x}$, there is $k_0 \in \mathbb{N}$ such that, for each $k \geq k_0$, $g^k(x^i_k) \in B_2$. Similarly, since $\{g^m(x^i_m)\}_{m\in\mathbb{N}} \to \bar{y}^i$, there is $\hat{k}_0 \in \mathbb{N}$ such that, for each $k \geq \hat{k}_0$, $g^k(x^i_k) \in B_1$. The latter two properties of $\{g^m(x^i_m)\}_{m\in\mathbb{N}}$ contradict that $B_1 \cap B_2 = \emptyset$. Therefore, for each $i \in I$, $\bar{y}^i \in F(\bar{x})$.

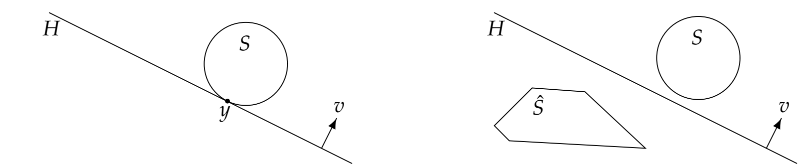
Finally, for each $m \in \mathbb{N}$, $x^m = g^m(x^m) = \sum_{i\in I} \alpha^m_i g^m(x^i_m)$. Passing to the limit we get $\bar{x} = \sum_{i\in I} \bar{\alpha}_i \bar{y}^i$. Hence, $\bar{x} \in \operatorname{conv}(F(\bar{x}))$. Therefore, since F is convex-valued, $\bar{x} \in F(\bar{x})$.

Case 2: $A \subset \mathbb{R}^n$ is a nonempty, convex, and compact set. This part is similar to Case 2 in the proof of Brouwer fixed-point theorem. The auxiliary function λ is defined analogously and the extension of F is defined as $\tilde{F}(x) := F((1-\lambda(x))\tilde{x} + \lambda(x)x)$. So defined, $\tilde{F}$ is nonempty-valued, closed-valued, and convex-valued. Moreover, since it is the composition of a continuous function and an upper hemicontinuous correspondence, it is also upper hemicontinuous. Therefore, we can apply Case 1 to $\tilde{F}$. □

2.14. On Extreme Points and Convex Sets: Krein-Milman Theorem

In this section we present a formal proof of Krein-Milman theorem (Krein and Milman 1940). To do so, we first need to introduce some notations along with three more theorems, which, remarkably, have shown to be quite useful in economic theory.

2.14.1. Some preliminary concepts. Throughout this section, given a set S contained in some euclidean space, let $\bar{S}$, $\mathring{S}$, and ∂S denote the closure, interior, and boundary of S, respectively. We say that a set is m-dimensional if it is contained in an m-dimensional affine space but no $(m-1)$-dimensional one contains it. Let $v, w \in \mathbb{R}^n$ and let $r \in \mathbb{R}$. Let vw denote the *scalar product* of v and w, *i.e.*, $vw := \sum_{i\in\{1,\ldots,n\}} v_i w_i$. Let $\|v\| := \sqrt{vv}$ denote the euclidean norm of v. Then, we define the *hyperplane* $H(v,r) := \{w \in \mathbb{R}^n : vw = r\}$ and the *half-spaces above* and *below* $H(v,r)$ as $H(v,r)^+ := \{w \in \mathbb{R}^n : vw \geq r\}$ and $H(v,r)^- := \{w \in \mathbb{R}^n : vw \leq r\}$, respectively. Let $S, \hat{S}$ be two sets in $\mathbb{R}^n$ and let $y \in \partial S$. A hyperplane $H(v,r)$, $v \in \mathbb{R}^n\backslash\{0\}$, is a *supporting hyperplane for S at y* if $vy = r$ and either $\inf_{x\in S} vx = r$ or $\sup_{x\in S} vx = r$, *i.e.*, if $y \in H(v,r)$ and either $S \subset H^+(v,r)$ or $S \subset H^-(v,r)$; note that, in such a case, $H(v,r) \cap \mathring{S} = \emptyset$. A hyperplane $H(v,r)$, $v \in \mathbb{R}^n \backslash \{0\}$, is a *separating hyperplane for S and $\hat{S}$* if they belong to different half-spaces; if, moreover, either $H \cap S = \emptyset$ or $H \cap \hat{S} = \emptyset$, we say that $H(v,r)$ *strictly separates* S and $\hat{S}$.



(a) A supporting hyperplane for S at y. (b) A separating hyperplane for S and $\hat{S}$.

Figure 2.14.1. Supporting and separating hyperplanes.

2.14.2. The theorems.

Theorem 2.14.1 (Separation of a point and a convex set)**.** *Let $S \subset \mathbb{R}^n$ be a nonempty and convex set and let $y \in \mathbb{R}^n \backslash S$. Then, there is a hyperplane $H(v,r)$ that separates $\{y\}$ from S. If S is also closed, the separation can be made strict.*

Proof. First of all, let $\bar{x} \in \mathbb{R}^n$ be such that $\|\bar{x} - y\| = \inf_{x \in S} \|x - y\|$.

We begin with the strict separation of a point from a nonempty, closed, and convex set S. Since S is closed, $\bar{x} \in S$ and, hence, $\bar{x} \neq y$. Since S is convex, for each $x \in S$ and each $\alpha \in (0,1]$, $\bar{x} + \alpha(x - \bar{x}) = \alpha x + (1-\alpha)\bar{x} \in S$. Hence, $\|\bar{x} - y\| \leq \|(\bar{x} + \alpha(x - \bar{x})) - y\| = \|(\bar{x} - y) + \alpha(x - \bar{x})\|$ Since, for each $v \in \mathbb{R}^n$, $\|v\|^2 = vv$, we have $(\bar{x} - y)(\bar{x} - y) \leq ((\bar{x} - y) + \alpha(x - \bar{x}))((\bar{x} - y) + \alpha(x - \bar{x}))$. Hence, $2\alpha(x - \bar{x})(\bar{x} - y) + \alpha^2(x - \bar{x})(x - \bar{x}) \geq 0$. Since the latter inequality holds for arbitrarily small α, then $(x - \bar{x})(\bar{x} - y) \geq 0$. Hence, $x(\bar{x} - y) \geq \bar{x}(\bar{x} - y)$. Similarly, since $\bar{x} \neq y$, $\|\bar{x} - y\| = (\bar{x} - y)(\bar{x} - y) > 0$ and, hence, $\bar{x}(\bar{x} - y) > y(\bar{x} - y)$. Let $v := (\bar{x} - y)$ and $r := \bar{x}(\bar{x} - y)$. By the inequalities above, for each $x \in S$, $xv \geq r > yv$. Hence, $H(v,r)$ strictly separates $\{y\}$ and S.

Suppose that S is just a nonempty and convex set. Hence, $\bar{x} \in \bar{S}$. Indeed, $\bar{x} \in \partial(\bar{S})$. We distinguish two cases.

Case 1: $\bar{x} \neq y$. In this case, $\|\bar{x} - y\| > 0$ and, hence, $y \notin \bar{S}$. Then, we can apply the above result to find a hyperplane $H(v,r)$ that strictly separates y and $\bar{S}$, *i.e.*, for each $x \in \bar{S}$, $xv \geq r > yv$ and $H(v,r)$ strictly separates y and S as well.

Case 2: $\bar{x} = y$. Since $\bar{x} \in \partial(\bar{S})$, there is $\{x^k\} \subset \mathbb{R}^n \backslash \bar{S}$ such that $\{x^k\} \to \bar{x}$. Since $\bar{S}$ is a nonempty, closed, and convex set, for each $k \in \mathbb{N}$ we can apply the above result to strictly separate $\{x^k\}$ and $\bar{S}$. Hence, there are $\{v^k\} \subset$

$\mathbb{R}^n \backslash 0$ and $\{r^k\} \subset \mathbb{R}$ such that, for each $k \in \mathbb{N}$, $H(v^k, r^k)$ strictly separates $\{x^k\}$ and $\bar{S}$. Without loss of generality, $\{v^k\} \subset \mathbb{R}^n \backslash 0$ can be chosen such that, for each $k \in \mathbb{N}$, $\|v^k\| = 1$. Hence, $\{(v^k, r^k)\} \subset \mathbb{R}^{n+1}$ has a convergent subsequence. Any such limit is a hyperplane that separates $\{\bar{x}\}$ and S and since $\bar{x} = y$, it also separates $\{y\}$ and S. □

The next two theorems are now easy to prove.

Theorem 2.14.2 (Supporting hyperplane theorem)**.** *Let $S \subset \mathbb{R}^n$ be a nonempty and convex n-dimensional set and let $y \in \partial S$. Then, there is a supporting hyperplane for S at y.*

Proof. The set $\mathring{S}$ is nonempty and convex and, moreover, $y \notin \mathring{S}$. Hence, by Theorem 2.14.1, there are $v \in \mathbb{R}^n \backslash \{0\}$ and $r \in \mathbb{R}$ such that $H(v, r)$ separates $\{y\}$ and $\mathring{S}$. Thus, $H(v, r)$ is a supporting hyperplane for S at y. □

Theorem 2.14.3 (Separating hyperplane theorem)**.** *Let S and $\hat{S}$ be two disjoint, nonempty, and convex subsets of $\mathbb{R}^n$. Then, there is a separating hyperplane for S and $\hat{S}$.*

Proof. Let $S - \hat{S} := \{z \in \mathbb{R}^n : z = x - y, x \in S \text{ and } y \in \hat{S}\}$. The convexity of S and $\hat{S}$ implies the convexity of $S - \hat{S}$. Since, $S \cap \hat{S} = \emptyset$, $0 \notin S - \hat{S}$. Hence, by Theorem 2.14.1, there are $v \in \mathbb{R}^n \backslash \{0\}$ and $r \in \mathbb{R}$ such that $H(v, r)$ separates 0 and $S - \hat{S}$, *i.e.*, for each $x \in S$ and each $y \in \hat{S}$, $v(x - y) \geq r \geq v0 = 0$. Hence, for each $x \in S$ and each $y \in \hat{S}$, $vx \geq vy$. Given $y \in \hat{S}$, $\{vx : x \in S\}$ is bounded from below by vy. Hence, $\hat{r} := \inf\{vx : x \in S\}$ is a real number. Thus, $H(v, \hat{r})$ separates S and $\hat{S}$. □

We are ready to prove Krein-Milman theorem (Theorem 2.7.5 in Section 2.6).

Theorem (Krein-Milman theorem)**.** *Let $S \subset \mathbb{R}^n$ be a nonempty, convex, and compact set. Then, i) $\operatorname{ext}(S) \neq \emptyset$ and ii) $\operatorname{conv}(\operatorname{ext}(S)) = S$.*

Proof. i) The proof is by induction on the dimension of S. First, if S is 0-dimensional, *i.e.*, a point, then $\operatorname{ext}(S) = S \neq \emptyset$. Assume that the assertion is true if S is $(k-1)$-dimensional. Let S be k-dimensional. We can assume, without loss of generality, that $0 \in S$ (otherwise, let $x \in S$ and take $S - x$)[43]. Hereafter we think of S as a subset of $\mathbb{R}^k$, where $\mathring{S} \neq \emptyset$. Let $y \in \partial S$ (in $\mathbb{R}^k$). By the compactness of S, $y \in S$. By Theorem 2.14.2, there is a supporting hyperplane $H \subset \mathbb{R}^k$ for S at y. Hence, $y \in H \cap S$ and $H \cap \mathring{S} = \emptyset$. The dimension of the nonempty, convex, and compact set $H \cap S$ is, at most, $k - 1$. By the induction hypothesis, $H \cap S$ has an extreme point, say z. We claim

[43] Let $z \in \mathbb{R}^n$ and let $S - z := \{x - z : x \in S\}$. Clearly, $y \in \operatorname{ext}(S - z)$ if and only if $y + z \in \operatorname{ext}(S)$; hence, $\operatorname{ext}(S - z) \neq \emptyset$ if and only if $\operatorname{ext}(S) \neq \emptyset$.

that $z \in \text{ext}(S)$. Let $x, \hat{x} \in S$ and $\alpha \in (0,1)$ be such that $\alpha x + (1-\alpha)\hat{x} = z$; we now show that $x = \hat{x}$. First, suppose that x and $\hat{x}$ are not both in H. Let $v \in \mathbb{R}^n \backslash \{0\}$ and $r \in \mathbb{R}$ be such that $H(v,r) \equiv H$ and suppose, without loss of generality, that $r \equiv \inf_{\tilde{x} \in S} v\tilde{x}$. Since $z \in H$, $vz = r$. Then, since H is a supporting hyperplane for S, $vx \geq r$ and $v\hat{x} \geq r$ and at least one of the inequalities is strict. Hence, $r = vz = \alpha x + (1-\alpha)\hat{x} > r$ and we get a contradiction. Then, both x and $\hat{x}$ belong to H. Since $z \in \text{ext}(H \cap S)$, we have $x = \hat{x} = z$. Therefore, $z \in \text{ext}(S)$.

ii) For the sake of exposition, denote $\text{conv}(\text{ext}(S))$ by C. We prove that $C = S$ in two steps. **Step 1:** We show that $\bar{C} = S$. Since $C \subset S$ and S is closed, $\bar{C} \subset S$. Suppose that there is $y \in S \backslash \bar{C}$. Then, by Theorem 2.14.1, there are $v \in \mathbb{R}^n \backslash \{0\}$ and $r \in \mathbb{R}$ such that $H(v,r)$ strictly separates $\{y\}$ and $\bar{C}$. Suppose, without loss of generality, that, for each $x \in \bar{C}$, $vy > vx$. Let $\hat{y} \in S$ be such that $v\hat{y} = \max_{x \in S} vx$, which is well defined by the compactness of S. Take the hyperplane $H(v, v\hat{y})$. By i), the nonempty, compact, and convex set $H(v, v\hat{y}) \cap S$ has an extreme point z. Repeating the reasonings in i), $z \in \text{ext}(S)$. Hence, since $z \in \bar{C}$, $vy > vz$. Therefore, $vy > vz = v\hat{y} = \max_{x \in S} vx \geq vy$ and we have a contradiction. **Step 2:** We show that $C = S$. By Step 1, we already know that $C \subset S$. Again, we prove the claim by induction on the dimension of S. If S is 0-dimensional the claim is trivial. Assume that it is true if S is $(k-1)$-dimensional. Let S be k-dimensional. Again, we can assume, without loss of generality, that $0 \in S$ and think of S and C as subsets of $\mathbb{R}^k$, where $\mathring{S} \neq \emptyset$. In $\mathbb{R}^k$, the interior of $\bar{C}$ coincides with $\mathring{C}$. Hence, by Step 1, if $y \in \mathring{S}$, then $y \in \mathring{C} \subset C$. Then, let $y \in S \backslash \mathring{S}$, *i.e.*, $y \in \partial S$. By Theorem 2.14.2, there is a supporting hyperplane $H \in \mathbb{R}^k$ for S at y. Hence, $y \in H \cap S$ and $H \cap \mathring{S} = \emptyset$. The dimension of the nonempty, compact, and convex set $H \cap S$ is, at most, $k-1$. Hence, by the induction hypothesis, $\text{conv}(\text{ext}(H \cap S)) = H \cap S$. Repeating the reasonings in i), $\text{ext}(H \cap S) \subset \text{ext}(S)$. Hence, $y \in \text{conv}(\text{ext}(H \cap S)) \subset \text{conv}(\text{ext}(S)) = C$. □

Remark 2.14.1. It is worth noting that Krein-Milman theorem is true for more general spaces. Indeed, it holds for arbitrary locally convex topological vector spaces (possibly infinite dimensional). Yet, the proofs for infinite dimensional spaces, although not being much more complicated, require the use of Zorn lemma.

Exercises of Chapter 2

2.1. Consider the following modification of the first-price and second-price auctions (Examples 2.1.3 and 2.1.4). In case of a tie at the highest bid, the object is awarded with equal probability to each one of the tied bidders:

(a) Write the payoff functions of the new strategic games.
(b) Find the set of equilibria in pure strategies of these new games (in the same lines of Examples 2.2.3 and 2.2.4).
(c) A correct resolution of the previous point would show that, in this new setting, we run into existence problems. Show that they can be overcome by discretizing the sets of strategies, *i.e.*, setting $\varepsilon > 0$ (small enough) such that only bids of the form $0 + k\varepsilon$, with $k \in \{0, 1, \ldots\}$, are possible.[44]

2.2. *Rock, paper, scissors game*. Two players participate in a game where they have to simultaneously and independently choose one of the following objects: rock, paper, and scissors. Rock defeats scissors, scissors defeats paper, and paper defeats rock. The player who chooses the winning object gets one unit; when both players choose the same object, the game is a draw. Describe the game and obtain the sets of optimal strategies.

2.3. *Bertrand duopoly* (Bertrand 1883). Consider a market of a homogeneous product with two companies. Assume that there is no fixed cost of production and that both marginal costs are identical and equal c. Each company decides on the selling price of its product. Given the lowest price p, the total quantity sold is $q = \max\{0, d - p\}$, for a given $d > 0$. If the companies charge different prices, then the company with the lower price sells everything. If both companies charge the same price, then they evenly split sales. Assume that each company has enough capacity to produce the total amount for the market at any price. Assume that $0 < c < d$.

(a) Write down the strategic game which describes this situation.
(b) Obtain the set of Nash equilibria of the game and the payoff that each company gets.

2.4. Consider the following variation of the Bertrand model above in which the two companies produce differentiated products. Assume that there is no fixed cost of production and that both marginal costs are identical and equal c. Each company decides on the selling price of its product. Once prices are chosen, say p_1 and p_2, the quantity that each company i sells is given by $q_i = \max\{0, d - p_i + rp_{-i}\}$, with $0 < r < 2$ and $0 < c < d$ (the parameter r indicates the degree of differentiation between products).

(a) Write down the strategic game which describes this situation.

[44] Another way out for the existence problem raised above is to use mixed strategies. Yet, since in both first-price and second-price auctions there is a continuum of strategies for each player, the definition of mixed strategies falls outside the scope of this book (they would be general distributions of probability defined over the sets of pure strategies).

(b) Obtain the set of Nash equilibria of the game and the payoff that each company gets.

2.5. Show that the set of Nash equilibria of a strategic game is closed.

2.6. Show that, under the assumptions of Proposition 2.2.2, the BR_i functions need not be lower hemicontinuous.

2.7. Consider the two-player zero-sum game (A_1, A_2, u_1) where $A_1 = A_2 = \mathbb{N}$ and, for each $a \in A$, $u_1(a) = 1/(a_1 + a_2)$. Show that this game has a value. Characterize the sets of optimal strategies.

2.8. Take a matrix game defined by an $l \times m$ matrix $\mathcal{A}$. Let $(x, y), (\bar{x}, \bar{y}) \in O_1(\mathcal{A}) \times O_2(\mathcal{A})$. Show that $(\bar{x}, y), (x, \bar{y}) \in O_1(\mathcal{A}) \times O_2(\mathcal{A})$.

2.9. Obtain the optimal strategies for the game $\mathcal{A} = \begin{pmatrix} 5 & -2 & -1 \\ -2 & 5 & -1 \\ 1 & 1 & 2 \end{pmatrix}$.

2.10. Take a matrix game defined by an $l \times m$ matrix $\mathcal{A}$ such that $V > 0$. Discuss the relationship between the optimal strategies of the matrix game $\mathcal{A}$ and the optimal strategies of the matrix game given by

$$\mathcal{B} := \begin{pmatrix} 0 & \mathcal{A} & -\mathbb{1}_m^t & \mathbb{1}_m^t \\ -\mathcal{A}^t & 0 & \mathbb{1}_n^t & -\mathbb{1}_n^t \\ \mathbb{1}_m & -\mathbb{1}_n & 0 & 0 \\ -\mathbb{1}_m & \mathbb{1}_n & 0 & 0 \end{pmatrix}.$$

2.11. Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be three $l \times m$ matrices such that $\mathcal{C} = \mathcal{A} + \mathcal{B}$. Find examples where $V_{\mathcal{C}} > V_{\mathcal{A}} + V_{\mathcal{B}}$ and $V_{\mathcal{C}} < V_{\mathcal{A}} + V_{\mathcal{B}}$.

2.12. Prove Propositions 2.8.4 and 2.8.5.

2.13. Show that the strategy profile (R_1, L_2, L_3) of the game in Example 2.9.3 is an undominated Nash equilibrium that is not perfect.

2.14. Show that the perfect equilibrium (R_1, R_2) of the game in Example 2.9.4 is not proper.

2.15. Show that the game in Example 2.9.8 does not have any strictly perfect equilibrium.

2.16. Consider the bimatrix game

	C_1	C_2	C_3	C_4	C_5
R_1	2,0	0,6	0,8	0,4	1,6
R_2	0,8	0,6	1,0	1,6	0,4

(a) Find the pure strategies that are strictly dominated for player 2.
(b) Find the sets of perfect and proper equilibria.

(c) Take the reduced game that results after eliminating the strictly dominated pure strategies for player 2. Compare the set of perfect equilibria of this bimatrix game with that of the original game. Compare as well the sets of proper equilibria.

2.17. Consider the matrix game $\mathcal{A} = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & -1 & 0 \end{pmatrix}$.

(a) Obtain the sets of optimal strategies for player 1 and for player 2.
(b) Obtain the sets of perfect and proper equilibria.
(c) Obtain the payoff player 2 gets when he uses any strategy involved in a perfect equilibrium and player 1 uses a pure strategy.

2.18. Take the matrix game $\mathcal{A} = \begin{pmatrix} 1 & 4 & 6 \\ 5 & 2 & 0 \end{pmatrix}$.

(a) Obtain the set of proper equilibria.
(b) Obtain the payoff player 2 gets when he uses any strategy involved in a proper equilibrium and player 1 uses a pure strategy.

2.19. Let $G = (A, u)$ be a finite strategic game. Show that every probability distribution over A that can be obtained in a correlated equilibrium of G can also be obtained in a correlated equilibrium $((\Omega, \rho, \{\mathcal{P}_i\}_{i\in N}), \tau)$, where $\Omega = A$ and, for each $i \in N$, $\mathcal{P}_i = \big\{\{a \in A : a_i = \hat{a}_i\} : \hat{a}_i \in A_i\big\}$.

2.20. Let $G = (A, u)$ be a finite strategic game. Show that the set of probability distributions over A that can be obtained in a correlated equilibrium of G is compact and convex.

2.21. Obtain the set of Nash equilibria and the set of probability distributions corresponding to correlated equilibria of the bimatrix game

	L_2	R_2
L_1	0,0	5,1
R_1	1,5	4,4

2.22. Consider the bimatrix game

	L_2	R_2
L_1	5,1	0,2
C_1	2,0	3,1
R_1	3,3	1,0

(a) Obtain the set of rationalizable strategies.
(b) Obtain the set of probability distributions corresponding to correlated equilibria.

2.23. Obtain the set of rationalizable strategies and the set of probability distributions corresponding to correlated equilibria of the bimatrix game

	L_2	C_2	R_2
L_1	0, 0	7, 2	1, −1
C_1	2, 7	6, 6	0, 5
R_1	1, 3	1, 3	2, 2

2.24. Consider the process of iterative elimination of dominated pure strategies. Show that the pure strategies that remain after this process depend on the order of elimination.

2.25. Show, by means of examples, that all of the assumptions in Kakutani theorem are needed for the result to go through.

Chapter 3

Extensive Games

3.1. Introduction to Extensive Games

In this chapter we move to a model that can capture nonstatic interactive situations. As we pointed out in Chapter 2 when discussing Example 2.1.5, every such situation can be represented by a strategic game. Yet, when doing this, we disregard some information that may be relevant in some circumstances. This is the reason why the more sophisticated model of extensive games (which aims to *extensively* describe the structure of the nonstatic interactive situations under study) is a valuable one. One of the game theorists that has been more influential in the development of the theory of extensive games has been Reinhard Selten. He was one of the laureates of the Nobel Prize in Economics in 1994, mainly because of his contributions to equilibrium theory in extensive games. In our exposition we basically follow Selten's notation for extensive games (Selten 1975). The notation in Osborne and Rubinstein (1994) for extensive games is simpler but, perhaps, less transparent. It is also important to remark that in this book we only formally cover the case of *finite* extensive games, although at the end of this chapter we also briefly discuss infinite extensive games (Sections 3.6.5 and 3.7). We start this chapter by formally introducing the model and other basic definitions, and then we present and discuss the basic equilibrium concepts along with some refinements. At the end of the chapter, we also present in some detail a special family of extensive games: repeated games. Although we briefly discuss the implications of rationality in extensive games in Example 3.4.3 (the centipede game), we do not cover the epistemic foundations of the different solution concepts for extensive games in this book.[1]

As in the previous chapter, we denote the set of players of a game by $N := \{1, \dots, n\}$. The general model of extensive games allows us to account for random events that may take place in the course of the dynamic interaction among the players. We refer to the random mechanism responsible for the random events in the game as *nature* and to the different realizations of the mechanism as *nature moves*; for notational convenience, although nature is not properly a player, we denote it as "player" 0. Note that $0 \notin N$.[2]

Definition 3.1.1. An n-player *extensive game* with set of players N is a 7-tuple $\Gamma := (X, E, P, W, C, p, U)$ whose elements are the following:

The game tree: (X, E) is a *finite tree*. By a finite tree we mean the following. X is a finite set of *nodes* and $E \subset X \times X$ is a finite set of *arcs* (or edges), satisfying that: i) there is a unique *root* node r, *i.e.*, a unique node r with no $x \in X$ such that $(x, r) \in E$, and ii) for each $x \in X \backslash \{r\}$ there is a unique *path* connecting r and x, where a path is a sequence of consecutive arcs; two arcs $(x, \hat{x})$ and $(y, \hat{y})$ are *consecutive* if $\hat{x} = y$. A node $x \in X$ is *terminal* if there is no arc starting at x, *i.e.*, if there is no $\hat{x} \in X$ such that $(x, \hat{x}) \in E$. Let Z be the set of terminal nodes.

The finite tree describes the structure of the dynamic situation: the nodes that are not terminal represent the decision points of the players; the arcs starting from each node represent the alternatives available in that particular decision point; the terminal nodes represent the possible endpoints of the game. Let $x, \hat{x} \in X$. We say that $\hat{x}$ *comes after* x if x belongs to one of the arcs on the path that connects the root and $\hat{x}$; in particular, any node comes after itself. Let $F(x)$ denote the set of all nodes that come after x (followers of x).

The player partition: $P := \{P_0, P_1, \dots, P_n\}$ is a partition[3] of $X \backslash Z$ that indicates, for each nonterminal node x, which player has to make a decision at x.

The information partition: $W := \{W_1, \dots, W_n\}$, where, for each $i \in N$, W_i is a partition of P_i. Each $w \in W_i$ contains the nodes of P_i

[1] The reader interested in this literature can refer, for instance, to Pearce (1984), Aumann (1995), Binmore (1996), Aumann (1996), Battigalli (1997), Battigalli and Siniscalchi (2002).

[2] Nature is not properly a player because its actions, *i.e.*, the distributions of probability over the random events, are fixed. Nature does not "play" strategically; indeed, nature's payoffs are not even defined.

[3] We allow for some elements of the partition to be empty sets.

in which player i has the same information about what has happened in the game up to that point. Hence, if a node in $w \in W_i$ is reached during the game, player i just knows that he has to make a decision at w, but he does not know what node in w has actually been reached. Thus, each $w \in W_i$ satisfies that:

- Each *play of the game* intersects w at most once, where a play of the game is a path from the root node to a terminal node.
- The number of arcs starting from x and $\hat{x}$ must be the same if $x, \hat{x} \in w$. Otherwise, i would have a different number of alternatives at x and at $\hat{x}$ and he would be able to distinguish between x and $\hat{x}$.

Each set $w \in W_i$ is an *information set* of player i.

The choice partition: Each player has to select an alternative at each decision node but, due to the information partition, he can only select a *choice* at each information set. Formally, a choice for player $i \in N$ at information set $w \in W_i$ is a set that contains, for each $x \in w$, one alternative at x, *i.e.*, an arc $(x, \hat{x}) \in E$. Moreover, each alternative belongs to one and only one choice. Let C be the set of all possible choices of the different players at the different information sets, which we call *choice partition*. Note that, so defined, C is a partition of the set of arcs starting at nodes outside P_0. For each $w \in W_i$, let C_w be the set of all choices at w. We say that node x *comes after* a choice c if one of the arcs of c is on the path that connects r and x.

The probability assignment: p is a map that assigns, to each $x \in P_0$, a probability distribution p_x, defined over the set of arcs starting at x. Hence, p provides a description of nature moves.

The utilities: $U := (U_1, \ldots, U_n)$ provides the utility functions of the players, defined over Z. We assume that each player $i \in N$ has preferences on Z (the set of possible endpoints of the game) and that his preferences are represented by a utility function U_i. In general, we assume that these utility functions are von Neumann and Morgenstern utility functions.

Example 2.1.5 in Section 2.1 was actually a well defined extensive game. Below, we present two more examples of extensive games.

Example 3.1.1. (A marketing game).[4] Two firms have developed a new product and plan to launch it. They do not know whether the market for such a product will be small (S), which would deliver aggregate profits of

[4]This example is taken from Straffin (1993).

80 million euros per year, or large (L), which would deliver aggregate profits of 200 million euros per year. No information is available on the nature of the market (the product has never been sold before), but the market analysis predicts that it will be small with probability $1/2$ and it will be large with probability $1/2$. Firms have to decide if they produce a high quality product (H), which will perform better if the market is small, or a medium quality product (M), which will perform better if the market is large. Firm 1 is an industry leader in the sector. Firm 2 is a smaller company which markets its products to a more sophisticated public. The analysts of both firms have estimated the market shares given in Figure 3.1.1 (these estimations are supposed to be based on well-known facts, so they can be considered common knowledge). Another relevant feature is that, since firm 2 is a

	M	H
M	$60, 20$	$20, 60$
H	$70, 10$	$50, 30$

S

	M	H
M	$160, 40$	$140, 60$
H	$70, 130$	$130, 70$

L

Figure 3.1.1. Market share estimates in the marketing game.

smaller and more flexible company, it needs less time to launch the product, so it can observe firm 1's decision before making its decision on the quality of the product. This nonstatic interactive situation can be modeled using the extensive game Γ in Figure 3.1.2 (we omit the explicit description of the elements of Γ). $\diamond$

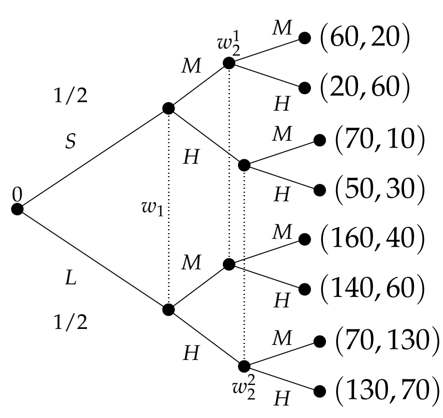


Figure 3.1.2. The extensive game Γ of Example 3.1.1. A marketing game.

Example 3.1.2. (A card game).[5] Consider the following interactive situation. Three persons A, B, and C are playing the card game described below (in which A and B play as a couple, sharing all their benefits and costs):

- Two cards, one marked ten and the other marked five, are dealt at random to A and C.
- The person with the highest card receives one euro from the other person and, in addition, decides either to stop (S) or to keep playing (K).
- If the play continues, B, not knowing who got the highest card in the stage before, decides if A and C exchange their cards (E) or not (N).
- Again, the person with the highest card receives one euro from the other person, and the game ends.

This interactive situation can be modeled as an extensive game in two different ways. We might consider that it is a two-player game (player 1 being A-B, and player 2 being C). In such a case, the extensive game corresponding to this situation is Γ_1 depicted in Figure 3.1.3. However, this situation

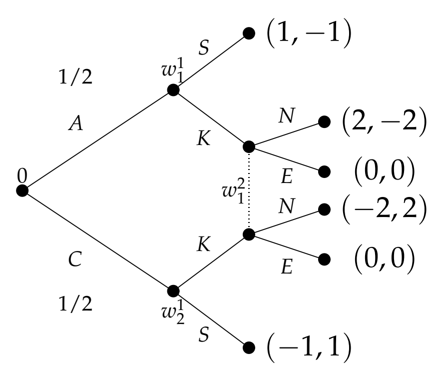


Figure 3.1.3. The extensive game Γ_1 of Example 3.1.2. A card game.

could also be considered as a three-player game (player 1 being A, player 2 being C, and player 3 being B).[6] In such a case, its corresponding extensive game is Γ_2 in Figure 3.1.4 below. $\diamond$

We now introduce some important classes of extensive games.

Definition 3.1.2. An extensive game Γ is a game with *perfect information* if for each $i \in N$, each $w \in W_i$ contains exactly one node of $X \backslash Z$. An extensive game without perfect information is said to have *imperfect information*.

[5]This example is taken from Kuhn (1953).

[6]This seems to be a more appropriate way of modeling this card game. We refer the reader to Selten (1975, page 27) for a discussion where Selten criticizes the two-player representation.

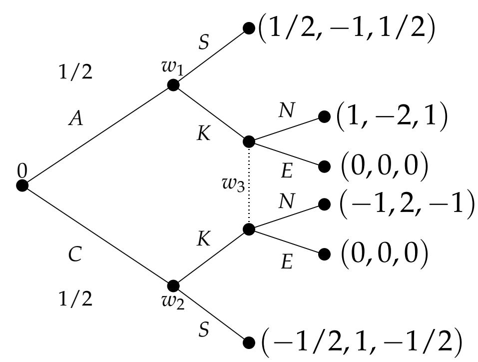


Figure 3.1.4. The extensive game Γ_2 of Example 3.1.2. A card game.

In a game with perfect information the players know the full history of the game whenever it is their time to move; that is, they know all the moves made by the other players and by nature.

Definition 3.1.3. An extensive game Γ is a game with *perfect recall* if, for each $i \in N$ and each pair $w, \hat{w} \in W_i$, the following condition holds: if one node $x \in \hat{w}$ comes after a choice $c \in C_w$, then every node $\hat{x} \in \hat{w}$ comes after c. An extensive game without perfect recall is said to have *imperfect recall*.

In words, a perfect recall game is one in which each player, in each of his information sets, remembers what he has known and done in all his previous information sets. We refer the reader to Selten (1975) for a discussion where the author argues that games with imperfect recall are not a natural way to model noncooperative situations with completely rational players.

By definition, every game with perfect information is a game with perfect recall. The game in Example 2.1.5 is a game with perfect information. The game in Example 3.1.1 is a game with imperfect information and perfect recall. Γ_1 in Example 3.1.2 is a game with imperfect recall; on the other hand, Γ_2 is a game with imperfect information and perfect recall.

3.2. Strategies in Extensive Games: Mixed Strategies vs. Behavior Strategies

We now define the strategies of the players in extensive games. As we did with strategic games, we want to introduce an equilibrium concept for which we can prove an existence result. We are dealing with *finite* extensive games so, as in Chapter 2, for this existence result to be possible, players must be able to take randomized decisions. We first define the pure strategies of the players in extensive games. Then we introduce the so-called *behavior strategies*.

Definition 3.2.1. Let Γ be an extensive game and let $i \in N$. A *pure strategy* a_i of player i is a map that assigns, to each $w \in W_i$, a choice $a_i(w) \in C_w$. A pure strategy profile of an extensive game is an n-tuple of pure strategies, one for each player. We denote by A_i the set of pure strategies of player i and by A the set of pure strategy profiles.

Definition 3.2.2. Let Γ be an extensive game and let $i \in N$. A *behavior strategy* b_i of player i is a map that assigns, to each $w \in W_i$, a lottery over C_w. For each $w \in W_i$ and each $c \in C_w$, $b_i(c)$ is the probability that player i assigns to choice c when he is at information set w. The lotteries for the different information sets of a player are independent of each other. A behavior strategy profile of an extensive game is an n-tuple of behavior strategies, one for each player. We denote by B_i the set of behavior strategies of player i and by B the set of behavior strategy profiles.

Note that every pure strategy of a player can be seen as a behavior strategy. In this sense we can say that, for each $i \in N$, $A_i \subset B_i$ and hence $A \subset B$.

Now take an extensive game Γ. For each $b \in B$ and each $x \in X$, let $p(x, b)$ be the probability that node x is reached if players play according to b. Note that $\{p(z, b) : z \in Z\}$ is a probability distribution over Z. Hence, if players have von Neumann and Morgenstern utility functions, for each $i \in N$, i's *payoff function* corresponds with the function $u_i\colon B \to \mathbb{R}$ given, for each $b \in B$, by

$$u_i(b) = \sum_{z \in Z} p(z, b) U_i(z).$$

Hence, since $A \subset B$, given an extensive game Γ we can formally define the *strategic game associated with* Γ by $G_\Gamma := (A, u)$. We had already pointed out that every nonstatic interactive situation can be represented as a strategic game and presented an informal illustration of this fact in Example 2.1.5. We can also consider the game $E(G_\Gamma) := (S, u)$, *i.e.*, the mixed extension of the strategic game associated with Γ. By definition, a mixed strategy in G_Γ is a lottery over the pure strategies of G_Γ and, hence, a lottery over the pure strategies of Γ; thus, we also call them mixed strategies of Γ.

After the previous definitions it is natural to wonder what is the relationship between the mixed strategies and the behavior strategies of an extensive game Γ. The rest of this section is devoted to addressing this question.

For each strategy profile $d \in \prod_{i\in N}(B_i \cup S_i)$ and each $x \in X$, $p(x,d)$ denotes the probability that node x is reached if players play according to d.[7] Similarly, let $\hat{x} \in X$ be a node such that x comes after $\hat{x}$; then, $p_{\hat{x}}(x,d)$ denotes the probability that, conditional on node $\hat{x}$ being reached, node x is reached if players play according to d (in the remainder of the game). If $p(\hat{x},d) > 0$, then $u_{i\hat{x}}(d)$ denotes the expected utility of player i when d is played but conditional on $\hat{x}$ being reached. Analogous (conditional) probabilities and utilities can be defined for information sets.

Definition 3.2.3. Let Γ be an extensive game. Let $i \in N$ and let $s_i \in S_i$ and $b_i \in B_i$. The strategies s_i and b_i are *equivalent* if they induce the same probabilities over A_i.

Every behavior strategy induces a lottery over pure strategies in the natural way. Let $b_i \in B_i$ and $a_i \in A_i$. The probability that a_i is played when i plays according to b_i is $\prod_{w\in W_i} b_i(a_i(w))$. Hence, for each behavior strategy, there is an equivalent mixed strategy. Example 3.2.1 below shows that the converse is not true.

Example 3.2.1. Consider the extensive game in Figure 3.2.1. Player 2 has four pure strategies: $A_2 = \{(L_2,l_2),(L_2,r_2),(R_2,l_2),(R_2,r_2)\}$. Let $s_2 \in S_2$ be the mixed strategy that selects both (L_2,l_2) and (R_2,r_2) with probability one half. There is no behavior strategy equivalent to s_2. When using behavior strategies, the players have to independently select the probabilities at the different information sets, whereas with mixed strategies the different choices can be correlated with one another. $\diamond$

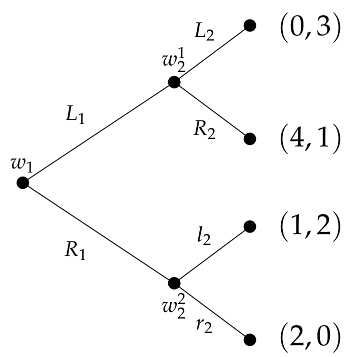


Figure 3.2.1.

Therefore, the sets of behavior and mixed strategy profiles are different even if we identify strategies that are equivalent in the sense we have

[7] That is, d is such that, for each $i \in N$, either $d_i \in B_i$ or $d_i \in S_i$.

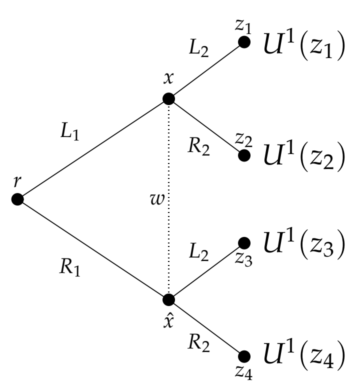
defined above. In fact, in this sense, we can say that $B \subset S$ although in general it is not true that $S \subset B$. Yet, Kuhn (1953) proved that these two sets are indeed equal if we take a weaker concept of equivalence and restrict attention to perfect recall games. Kuhn's equivalence result is at the basis of the equilibrium analysis literature of extensive games.

Definition 3.2.4. Let Γ be an extensive game. Let $i \in N$ and let $s_i \in S_i$ and $b_i \in B_i$. The strategies s_i and b_i are *realization equivalent* if, for each $\hat{s}_{-i} \in S_{-i}$ and each $x \in X$, $p(x, (\hat{s}_{-i}, s_i)) = p(x, (\hat{s}_{-i}, b_i))$; that is, s_i and b_i induce the same probabilities over X.

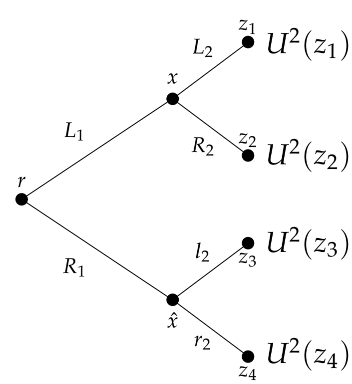
Definition 3.2.5. Let Γ be an extensive game. Let $i \in N$ and let $s_i \in S_i$ and $b_i \in B_i$. The strategies s_i and b_i are *payoff equivalent* if, for each $\hat{s}_{-i} \in S_{-i}$, $u(\hat{s}_{-i}, s_i) = u(\hat{s}_{-i}, b_i)$.

Note that being realization equivalent is stronger than being payoff equivalent. Moreover, since being equivalent is stronger than being realization equivalent, we already know that, for each behavior strategy, there is a realization equivalent mixed strategy. Kuhn theorem shows that the converse is also true for games with perfect recall. Before formally stating and proving Kuhn theorem, we informally discuss the role of perfect recall in this fundamental result.

Example 3.2.2. Since the difference between mixed and behavior strategies stems from the ability of a player to correlate his own elections when mixing, we can just look at 1-player games to illustrate the idea underlying Kuhn theorem. Consider the two 1-player games depicted in Figure 3.2.2.



(a) Game Γ^1 (without perfect recall).



(b) Game Γ^2 (with perfect recall).

Figure 3.2.2. The role of perfect recall in Kuhn theorem.

We begin by discussing the game Γ^1, which does not have perfect recall since, at w, the player does not know what he played at r. Let s be the mixed strategy that selects the pure strategy (L_1, L_2) with probability 1/2 and (R_1, R_2) with probability 1/2. We have that $p(x,s) = 1/2$, $p(\hat{x},s) = 1/2$, $p(z_1,s) = 1/2$, $p(z_2,s) = 0$, $p(z_3,s) = 0$, and $p(z_4,s) = 1/2$. We now try to construct a behavior strategy b that is realization equivalent to s. At r, b must choose L_1 with probability 1/2 and R_1 with probability 1/2. Now, what should be the choice at w? Since, according to s, the choice at w is correlated with the choice at r, b should choose L_2 if the choice at r was L_1 and R_2 if the choice at r was R_1; but this cannot be done with a behavior strategy. Indeed, if $b(L_1) = b(R_1) = 1/2$, for any distribution of probability over the choices at w, $p(z_1, b) > 0$ if and only if $p(z_3, b) > 0$ and $p(z_2, b) > 0$ if and only if $p(z_4, b) > 0$. Therefore, there is no behavior strategy that is realization equivalent to s. Now, consider the game Γ^2, which is analogous to Γ^1 except for the fact that it has perfect recall. Let $\hat{s}$ be the mixed strategy that selects the pure strategy (L_1, L_2, l_2) with probability 1/2 and (R_1, R_2, r_2) with probability 1/2. So defined, $\hat{s}$ is a kind of counterpart of the strategy s discussed above for Γ^1. Indeed, we have that $p(x,\hat{s}) = 1/2$, $p(\hat{x},\hat{s}) = 1/2$, $p(z_1,\hat{s}) = 1/2$, $p(z_2,\hat{s}) = 0$, $p(z_3,\hat{s}) = 0$, and $p(z_4,\hat{s}) = 1/2$. We now construct $\hat{b}$, a behavior strategy that is realization equivalent to $\hat{s}$. As before, at r, $\hat{b}$ must choose L_1 with probability 1/2 and R_1 with probability 1/2. At x, choice L_1 is made with probability 1 and, at $\hat{x}$, choice r_2 is made with probability 1. So defined, $\hat{b}$ is realization equivalent to $\hat{s}$.

In game Γ^1, when trying to define b, there is no way to define the choice of the player at w, since this choice should depend on the choice previously made by the player at r, and he does not have this information at w. On the other hand, under perfect recall there is no room for the latter problems.

$\diamond$

The idea of Kuhn theorem can be roughly summarized as follows. To construct a behavior strategy b that is realization equivalent to a mixed strategy s, we have to define the elections of the players at all the information sets. Let $i \in N$ and $w \in W_i$. As we have seen in Example 3.2.2, the main problem to define the election of player i at w is that this election might be correlated with the elections made by i at other information sets. Hence, to pin down the election of player i at w, he should use the information of the previous actions, which, under imperfect recall, might be unknown to him. Yet, if the game has perfect recall, i knows what his elections have been in all his information sets that have been reached before getting to w and, thus, the problem disappears.

Theorem 3.2.1 (Kuhn theorem)**.** *Let Γ be an extensive game with perfect recall. Let $i \in N$ and let $s_i \in S_i$. Then, there is $b_i \in B_i$ such that s_i and b_i are realization equivalent.*

Proof.[8] Let $s \in S$ be a mixed strategy profile. Let x be a node of the information set $w \in W_i$ such that $p(x,s) > 0$. Let $c \in C_w$ and let $\bar{x}$ be the node that is reached when choice c is taken at x. The conditional probability that c is chosen given that x has been reached is

$$p(c,x,s) := p_x(\bar{x},s) = \frac{p(\bar{x},s)}{p(x,s)}.$$

Now, we prove that $p(c,x,s)$ does not depend on s_{-i}. Moreover, we also show that, if $x, \hat{x} \in w \in W_i$, then $p(c,x,s) = p(c,\hat{x},s)$. Given $a_i \in A_i$, let $s_i(a_i\,|_x)$ be the probability that a_i is being played conditional on x having been reached. Given $\hat{x} \in X$, let $A_i(\hat{x}) \subset A_i$ be the set of pure strategies of player i that select the choices on the path to $\hat{x}$.[9] Then,

$$s_i(a_i\,|_x) = \begin{cases} \dfrac{s_i(a_i)}{\sum_{\hat{a}_i \in A_i(x)} s_i(\hat{a}_i)} & a_i \in A_i(x) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Hence, $p(c,x,s) = \sum_{\hat{a}_i \in A_i(\bar{x})} s_i(\hat{a}_i\,|_x)$, which is independent of s_{-i}. Let $x, \hat{x} \in w \in W_i$. The choices of i on the path to x are the same as the choices of i on the path to $\hat{x}$ because of the perfect recall assumption. Then, for each $a_i \in A_i$, $s_i(a_i\,|_x) = s_i(a_i\,|_{\hat{x}})$ and, hence, $p(c,x,s) = p(c,\hat{x},s)$.

Therefore, if an information set $w \in W_i$ is reached, then we can define the conditional choice probabilities of player i at w by $p(c,w,s_i) := p(c,x,s)$ where x is an arbitrary node of w_i. We use the p probabilities to construct the behavior strategy of player i we are looking for. Let $w \in W_i$. If there is $x \in w$ such that some strategy in $A_i(x)$ is selected by s_i with positive probability, then, for each $c \in C_w$, $b_i(c) := p(c,w,s_i)$. The construction of b_i is completed by assigning arbitrary choices at information sets that can never be reached when s_i is played by i (*i.e.*, those $w \in W_i$ such that, for each $x \in w$, s_i assigns 0 probability to the strategies in $A_i(x)$). Any such b_i is realization equivalent to s_i. □

In words, Kuhn theorem says that whatever a player can get with a mixed strategy can also be achieved by a realization equivalent behavior strategy. In the following sections we define equilibrium concepts for extensive games. Since the equilibrium analysis is based on the idea of best responses, and all that is needed for a player to best respond to a certain strategy is to know the probability of reaching each node, then, in view of Kuhn theorem, when dealing with perfect recall games, we can restrict attention to behavior strategies.

---

[8] We follow the same arguments of the proof in Selten (1975).

[9] More formally, $A_i(\hat{x}) = A_i \backslash \{\hat{a}_i \in A_i : \text{for each } \hat{s}_{-i} \in S_{-i}, p(\hat{x}, (\bar{s}_{-i}, \hat{a}_i)) = 0\}$.

3.3. Nash Equilibrium in Extensive Games

Before formally introducing the definition of Nash equilibrium we present the notion of best reply in extensive games.

Definition 3.3.1. Let Γ be an extensive game and let $b \in B$ and $\bar{b}_i \in B_i$. We say that $\bar{b}_i$ is a *best reply of player i against b* if

$$u_i(b_{-i}, \bar{b}_i) = \max_{\hat{b}_i \in B_i} u_i(b_{-i}, \hat{b}_i).$$

We now introduce the concept of Nash equilibrium in extensive games following the same idea of the definition for strategic games; that is, a Nash equilibrium is a strategy profile in which every player is maximizing his (expected) payoff when taking the strategies of the opponents as given. We define it for behavior strategies; the definitions for pure and mixed strategies are analogous.

Definition 3.3.2. Let Γ be an extensive game. A Nash equilibrium of Γ in behavior strategies is a strategy profile $b^* \in B$ such that, for each $i \in N$ and each $\hat{b}_i \in B_i$,

$$u_i(b^*) \geq u_i(b^*_{-i}, \hat{b}_i).$$

Proposition 3.3.1. *Let Γ be an extensive game and let $a^* \in A$. Then, a^* is a Nash equilibrium of Γ in behavior strategies if and only if, for each $i \in N$ and each $\hat{a}_i \in A_i$,*

$$u_i(a^*) \geq u_i(a^*_{-i}, \hat{a}_i).$$

Therefore, a^* is a Nash equilibrium of Γ in behavior strategies if and only if a^* is a Nash equilibrium of G_Γ.

Proof. Since $A \subset B$, if a^* is a Nash equilibrium of Γ in behavior strategies, then Definition 3.3.2 implies that, for each $i \in N$ and each $a_i \in A_i$, $u_i(a^*) \geq u_i(a^*_{-i}, a_i)$. Conversely, let $a^* \in A$ be such that, for each $i \in N$ and each $a_i \in A_i$, $u_i(a^*) \geq u_i(a^*_{-i}, a_i)$. Each behavior strategy b_i of player i defines a lottery s^{b_i} over his set of pure strategies A_i, *i.e.*, a strategy of player i in the mixed extension of G_Γ. Hence, for each $i \in N$ and each $b_i \in B_i$,

$$u_i(a^*_{-i}, b_i) = \sum_{a_i \in A_i} u_i(a^*_{-i}, a_i) s^{b_i}(a_i) \leq u_i(a^*). \qquad \square$$

Next we make use of Nash and Kuhn theorems to show that every extensive game has a Nash equilibrium. Indeed, in view of Nash theorem, we already know that the mixed extension of G_Γ, $E(G_\Gamma)$, has, at least, one Nash equilibrium; that is, the extensive game Γ always has an equilibrium in mixed strategies. However, using mixed strategies to study extensive games would be almost like going back to strategic games, *i.e.*, in this context behavior strategies seem to be more natural to capture the dynamic aspects of the games at hand. Hereafter, when we use the expression Nash equilibrium for an extensive game, we mean Nash equilibrium in behavior strategies, and we will mention it explicitly when we refer to equilibria in either pure or mixed strategies.

Theorem 3.3.2. *Let Γ be an extensive game with perfect recall. Then, Γ has, at least, one Nash equilibrium.*

Proof. By Nash theorem, the strategic game $E(G_\Gamma)$ has, at least, one Nash equilibrium. Let $s^* \in S$ be one of such equilibria. By Kuhn theorem, there is $b^* \in B$ such that s^* and b^* are realization equivalent and, hence, also payoff equivalent. The latter also implies that $u(s^*) = u(b^*)$. We now show that b^* is a Nash equilibrium of Γ. Suppose, on the contrary, that there are $i \in N$ and $\hat{b}_i \in B_i$ such that $u_i(b^*) < u_i(b^*_{-i}, \hat{b}_i)$. Hence, $u_i(s^*) = u_i(b^*) < u_i(b^*_{-i}, \hat{b}_i) = u_i(s^*_{-i}, \hat{b}_i)$. This implies that s^* is not a Nash equilibrium of $E(G_\Gamma)$ and we have a contradiction. $\square$

The above result implies that given an extensive game with perfect recall Γ and a Nash equilibrium s of $E(G_\Gamma)$, there is a Nash equilibrium b of Γ that is essentially identical to s. Moreover, the following converse result is also a consequence of Kuhn theorem.

Theorem 3.3.3. *Let Γ be an extensive game with perfect recall. If b^* is a Nash equilibrium of Γ, then b^* is also a Nash equilibrium of $E(G_\Gamma)$.*

Proof. First, recall that B can be considered as a subset of S. If b^* is not a Nash equilibrium of $E(G_\Gamma)$, then there are $i \in N$ and $\hat{s}_i \in S_i$ such that $u_i(b^*) < u_i(b^*_{-i}, \hat{s}_i)$. By Kuhn theorem, there is $\hat{b}_i \in B_i$ realization equivalent to $\hat{s}_i$ and, hence, also payoff equivalent. Therefore, $u_i(b^*) < u_i(b^*_{-i}, \hat{b}_i)$ and b^* is not a Nash equilibrium of Γ. $\square$

Yet, it is not true that every extensive game has a Nash equilibrium, *i.e.*, we cannot drop the perfect recall assumption in Theorem 3.3.2. We show this in the following example.

Example 3.3.1. Consider again the interactive situation described in Example 3.1.2 and the corresponding extensive game Γ_1 in Figure 3.1.3. The strategic game associated with Γ_1 is the two-player zero-sum game G_{Γ_1} depicted in Figure 3.3.1. It is easy to check that the unique Nash equilibrium of this matrix game is $((0, 1/2, 1/2, 0), (1/2, 1/2))$. Note that this mixed strategy of G_{Γ_1} is not a behavior strategy of Γ_1. The situation when the players are restricted to play behavior strategies can be represented by the two-player zero-sum game (B_1, B_2, u_1) given by:

- $B_1 = \{b_1 = (\alpha\beta, \alpha(1-\beta), (1-\alpha)\beta, (1-\alpha)(1-\beta)) : \alpha, \beta \in [0,1]\}$.

	S	K
SN	0	$-\frac{1}{2}$
SE	0	$\frac{1}{2}$
KN	$\frac{1}{2}$	0
KE	$-\frac{1}{2}$	0

Figure 3.3.1. The strategic game associated with the imperfect recall game Γ_1 when mixed strategies are not possible (Example 3.1.2).

- $B_2 = \{b_2 = (\delta, 1-\delta) : \delta \in [0,1]\}$.
- For each $b_1 \in B_1$ and each $b_2 \in B_2$,

$$u_1(b_1, b_2) = b_1 \begin{pmatrix} 0 & -1/2 \\ 0 & 1/2 \\ 1/2 & 0 \\ -1/2 & 0 \end{pmatrix} b_2^t = (\delta - \alpha)(\beta - 1/2).$$

In this game, for each $b_1 \in B_1$,

$$\underline{\Lambda}(b_1) = \inf_{b_2 \in Y} u_1(b_1, b_2) = \inf_{\delta \in [0,1]} (\delta - \alpha)(\beta - 1/2) \leq 0.$$

Since $\underline{\Lambda}((1/4, 1/4, 1/4, 1/4)) = 0$, we have that $\underline{\lambda} = 0$. Similarly, $\bar{\Lambda}(b_2) = \sup_{b_1 \in B_1} u_1(b_1, b_2)$ and then

$$\bar{\Lambda}(b_2) = \sup_{(\alpha,\beta) \in [0,1] \times [0,1]} (\delta - \alpha)(\beta - 1/2) = \begin{cases} \delta/2 & \delta \in [1/2, 1] \\ (1-\delta)/2 & \delta \in [0, 1/2]. \end{cases}$$

Hence, $\bar{\lambda} = \bar{\Lambda}((1/2, 1/2)) = 1/4$. Then, (B_1, B_2, u_1) is not strictly determined and, hence, (B_1, B_2, u_1) does not have any Nash equilibrium. Therefore, Γ_1 does not have any Nash equilibrium either. ◇

In the examples below we study the Nash equilibria of the extensive games introduced in Examples 3.1.2 and 3.1.1 along with a pair of variations of the latter.

Example 3.3.2. Consider again the interactive situation of Example 3.1.2 but, this time, modeled as a three-player game. The corresponding extensive game is Γ_2 (see Figure 3.1.4). Figure 3.3.2 shows its associated strategic game, G_{Γ_2}. It is easy to see that this game has two pure Nash equilibria: (K, K, N) and (S, S, E). ◇

Example 3.3.3. Consider the extensive game Γ in Example 3.1.1. Its corresponding strategic game G_Γ is shown in Figure 3.3.3. The strategies of player 2 mean the following:

$$\begin{array}{c|c|c|}
 & S & K \\ \hline
S & 0,\ 0,\ 0 & -\frac{1}{4},\ \frac{1}{2}, -\frac{1}{4} \\ \hline
K & \frac{1}{4}, -\frac{1}{2},\ \frac{1}{4} & 0,\ 0,\ 0 \\ \hline
\multicolumn{3}{c}{N}
\end{array}
\qquad
\begin{array}{c|c|c|}
 & S & K \\ \hline
S & 0,\ 0\ ,0 & \frac{1}{4}, -\frac{1}{2},\ \frac{1}{4} \\ \hline
K & -\frac{1}{4},\ \frac{1}{2}, -\frac{1}{4} & 0,\ 0,\ 0 \\ \hline
\multicolumn{3}{c}{E}
\end{array}$$

Figure 3.3.2. The three-player strategic game associated with game Γ_2 (Example 3.1.2),

- MM: Player 2 plays M regardless of the choice of player 1.
- MH: Player 2 plays the same as player 1.
- HM: Player 2 never plays the same as player 1.
- HH: Player 2 plays H regardless of the choice of player 1.

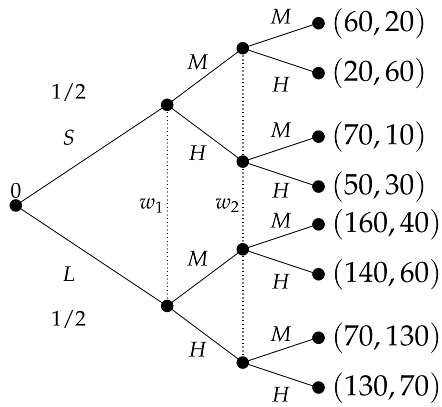
	MM	MH	HM	HH
M	$110, 30$	$110, 30$	$80, 60$	$80, 60$
H	$70, 70$	$90, 50$	$70, 70$	$90, 50$

Figure 3.3.3. The two-player strategic game associated with game Γ (Example 3.1.1).

Note that this is a constant-sum two-player game. Analyzing it is the same as analyzing the zero-sum game corresponding to player 1's payoff function. It is an easy exercise to check that, in constant-sum two-player games, the payoff for a player is the same in all the Nash equilibria of the game (as in the zero-sum case). Hence, we can speak of the value of the game as the payoff to player 1 in any of its Nash equilibria. Observe that (M, HM) is a pure Nash equilibrium of G_Γ, so it is a Nash equilibrium of Γ. Hence the value of the game is 80.

Suppose that player 2 cannot observe player 1's strategy before making his choice. Then the extensive game actually played, namely $\bar{\Gamma}$, is a different one. Figure 3.3.4 shows both $\bar{\Gamma}$ and its associated strategic game $G_{\bar{\Gamma}}$. In this case, $G_{\bar{\Gamma}}$ has no Nash equilibrium and $E(G_{\bar{\Gamma}})$ has a unique Nash equilibrium: $((2/5, 3/5), (1/5, 4/5))$. Since both players have a unique information set, the behavior strategies of $\bar{\Gamma}$ are the same as the strategies of the mixed extension of $G_{\bar{\Gamma}}$ and, hence, $((2/5, 3/5), (1/5, 4/5))$ is the unique Nash equilibrium of $\bar{\Gamma}$. The value of this game is 86. This means that, if player 2 can observe player 1's strategy before choosing, then his equilibrium payoff goes up, which is natural. ◇

Example 3.3.4. Consider again the situation described in Example 3.1.1, but now suppose that player 1 has conducted a marketing study and knows if the market will be small or large. Assume that player 2 knows that player



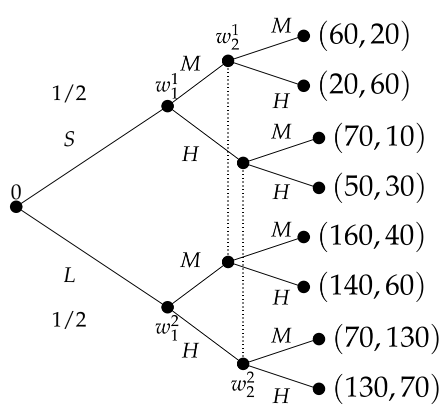
(a) Game $\bar{\Gamma}$.

	M	H
M	110,30	80,60
H	70,70	90,50

(b) Game $G_{\bar{\Gamma}}$.

Figure 3.3.4. $\bar{\Gamma}$ and $G_{\bar{\Gamma}}$ in Example 3.3.3.

1 has performed the study but that he does not know its result. Then, the extensive game that is played, namely $\hat{\Gamma}$, and its associated strategic game, $G_{\hat{\Gamma}}$, are given in Figure 3.3.5. The strategies of player 2 mean the same as in



(a) Game $\hat{\Gamma}$.

	MM	MH	HM	HH
MM	110,30	110,30	80,60	80,60
MH	75,65	95,45	45,95	75,65
HM	115,25	105,35	105,35	95,45
HH	70,70	90,50	70,70	90,50

(b) Game $G_{\hat{\Gamma}}$.

Figure 3.3.5. $\hat{\Gamma}$ and $G_{\hat{\Gamma}}$ in Example 3.3.4.

Example 3.3.3. The strategies of player 1 mean the following:

- *MM*: Player 1 plays *M* regardless of the size of the market.
- *MH*: Player 1 plays *M* if the market is small and *H* if the market is large.
- *HM*: Player 1 plays *H* if the market is small and *M* if the market is large.

- HH: Player 1 plays H regardless of the size of the market.

Now, (HM, HH) is a Nash equilibrium of $G_{\hat{\Gamma}}$ and, hence, it is also a Nash equilibrium of $\hat{\Gamma}$. The value of the game is 95 instead of 80 (see Example 3.3.3). Thus, player 1 should be willing to perform the marketing study if its cost is less than 15. Note that, if player 1 is able to keep secret the performance of the marketing study, then player 2 thinks that the game actually played is the one in Example 3.3.3, so he will choose HM and player 1 will have an extra benefit of 10. $\diamond$

We show below that, in extensive games, Nash equilibria need not be sensible. Apart from the reasons already discussed for strategic games, a new drawback arises because the "literal" translation of the definition of the Nash equilibrium concept to extensive games does not take into account the dynamic aspects of these games.

In Example 3.3.5 below, we illustrate the main weakness of the Nash equilibrium concept in extensive games; it does not disregard equilibria that are based on incredible threats.

Example 3.3.5. (Chain store game (Selten 1978)). Consider the following sequential game. There is a market with a potential *entrant* (player 1) and a *monopolist* (player 2). At the start of the game, the entrant decides whether to "enter the market" (E) or to "stay out" (O). The monopolist only has to play if the entrant has entered; if so, he has two options: to "yield" (Y) or to "fight" (F). If the entrant decides to stay out, the payoffs are 0 for the entrant and $\alpha > 1$ for the monopolist. Entering the market has a benefit β for the entrant, with $0 < \beta < 1$ and the duopoly payoff of the monopolist is 0. However, the action F entails a payoff loss of one unit for each player.[10] The corresponding extensive game is depicted in Figure 3.3.6. Note that (O, F) is a Nash equilibrium of this game. However, it is not self-enforcing. Namely, F is a best reply for player 2 because his information set is not reached if player 1 chooses O; nonetheless, if his information set is eventually reached, he will surely play Y. Then, if player 1 realizes this, he will also deviate and choose E. Hence, the only sensible Nash equilibrium in this game is (E, Y); that is, the equilibrium (O, F) is supported by the threat of player 2 of playing F if he is given the chance to play. Yet, this threat is not credible because, conditional on w_2 being reached, Y is a strictly dominant choice for him. $\diamond$

We now formally show where the weakness of the Nash equilibrium concept in extensive games lies. For each $b \in B$, each $i \in N$, and each $w \in$

[10] The chosen parametrization of the payoffs of the game follows Kreps and Wilson (1982a) and will be very convenient when we revisit this game in Section 4.3.

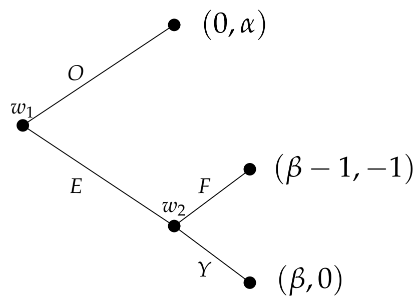


Figure 3.3.6. The chain store game. (O, F) is a Nash equilibrium supported by an incredible threat.

W_i such that $p(w, b) > 0$, let $u_{iw}(b)$ denote the conditional expected utility of player i once w has been reached; formally, $u_{iw}(b) := \sum_{z \in Z} p_w(z, b) u_i(z)$. Note that $p_w(z, b)$ needs not be well defined if $p(w, b) = 0$; since the relative probabilities of the nodes in w are unknown, $p_w(z, b)$ cannot be uniquely determined from b.

Definition 3.3.3. Let Γ be an extensive game and let $b \in B$ and $\bar{b}_i \in B_i$. Let $w \in W_i$ be such that $p(w, b) > 0$. We say that $\bar{b}_i$ is a *best reply of player i against b at w* if

$$u_{iw}(b_{-i}, \bar{b}_i) = \max_{\hat{b}_i \in B_i} u_{iw}(b_{-i}, \hat{b}_i).$$

Remark 3.3.1. The above definition is related to the definition of best reply given in Definition 3.3.1 in the following sense. Let Γ be an extensive game with perfect recall and let $i \in N$. Let $b \in B$ and $\bar{b}_i \in B_i$. Then, $\bar{b}_i$ is a best reply of player i against b if and only if $\bar{b}_i$ is a best reply of player i against b at all information sets $w \in W_i$ such that $p(w, b) > 0$.

According to the previous remark, for a behavior strategy profile b to be a Nash equilibrium, it suffices that it prescribes rational behavior at the information sets that are on the path it defines, *i.e.*, those that are reached with positive probability when b is played. Even though Nash equilibrium requires that every player maximizes his (expected) payoff when taking the strategies of the opponents as given, both Remark 3.3.1 and Example 3.3.5 show that this is not enough for extensive games because of the possibility of incredible threats outside the equilibrium path. We consider that a sensible equilibrium concept in extensive games should be a profile in which every player maximizes his (expected) payoff, *at all his information sets*, when taking the strategies of the opponents as given; that is, players maximize whenever it is their turn to play even if, eventually, such a decision never comes into effect because it is off the equilibrium path.

In the following sections we present a series of equilibrium concepts that follow the aforementioned idea and whose behavior is more sensible than the one exhibited by the Nash equilibrium. The two more important ones are the *subgame perfect equilibrium*, which is especially suitable for games with perfect information and, for general games with imperfect information and perfect recall, the *sequential equilibrium*. Yet, even the more restrictive of these equilibrium concepts has shortcomings and needs not be self-enforcing in general. Hence, in Section 3.6, we take an approach similar to the one in Section 2.9 and further refine the notion of equilibrium for extensive games.

3.4. Subgame Perfect Equilibrium

In Example 3.3.5, we used the chain store game to illustrate the main problem of the Nash equilibrium concept in extensive games: a Nash equilibrium might be based on irrational plans in some information sets that are not reachable if this equilibrium is played. Concerning this weakness of the Nash equilibrium concept, Selten (1975) argued that, if commitments are not possible, then the behavior in a subgame can depend only on the subgame itself. Hence, only those Nash equilibria of the original game that induce Nash equilibria in all the subgames are sensible. This leads to the definition of what Selten (1975) called subgame perfect equilibrium, which we develop below. Recall that by Definition 3.1.1, given an extensive game $\Gamma = (X, M, P, W, C, p, U)$ and a node $x \in X$, $F(x)$ denotes the nodes that come after x in the tree of Γ and $x \in F(x)$.

Definition 3.4.1. Let Γ be an extensive game and let $x \in X \backslash Z$. We say that Γ *can be decomposed at* x if there is no information set simultaneously containing nodes of the set $F(x)$ and nodes of the set $X \backslash F(x)$.

Note that, by the above definition, a game cannot be decomposed at a node that is in an information set that contains more than one node.

Definition 3.4.2. Let Γ be an extensive game that can be decomposed at $x \in X \backslash Z$. We denote by Γ_x the game that Γ induces in the tree whose root node is x. We say that Γ_x is a *subgame* of Γ.

If b is a behavior strategy profile in Γ, we denote by b_x the strategy profile in Γ_x induced by b. We are ready to formally introduce the definition of a subgame perfect equilibrium.

Definition 3.4.3. Let Γ be an extensive game. A *subgame perfect equilibrium* of Γ is a strategy profile $b \in B$ such that, for each subgame Γ_x of Γ, b_x is a Nash equilibrium of Γ_x.

Since every extensive game is a subgame of itself, every subgame perfect equilibrium is also a Nash equilibrium. We illustrate the above definitions in Example 3.4.1. This example also shows that there are Nash equilibria which are not subgame perfect.

Example 3.4.1. Let Γ be the chain store game introduced in Example 3.3.5. Apart from itself, the only subgame of Γ is the game given in Figure 3.4.1 (in which $P_1 = \emptyset$). The only Nash equilibrium of this game is A, so the unique subgame perfect equilibrium of Γ is (E, A). $\diamond$

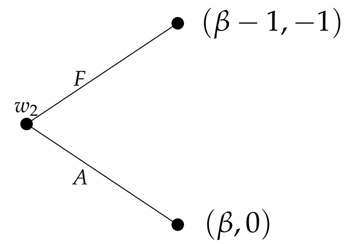


Figure 3.4.1. The unique proper subgame of the chain store game (Example 3.3.5).

Let Γ be an extensive game that can be decomposed at nodes $x_1, \ldots, x_m$ such that, for each pair $k, l \in \{1, \ldots, m\}$, x_k does not come after x_l. Let $b \in B$. Let $\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}$ be the extensive game resulting from Γ after i) deleting the subgames $\Gamma_{x_1}, \ldots, \Gamma_{x_m}$ and ii) defining, for each $i \in N$ and each $k \in \{1, \ldots, m\}$,

$$U_i^{\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}}(x_k) := u_{ix_k}(b).$$

Let $b_{-(x_1,\ldots,x_m)}$ denote the restriction of b to $\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}$. We now present a useful result whose proof is immediate (and left to the reader).

Proposition 3.4.1. *Let Γ be an extensive game. Suppose that Γ can be decomposed at nodes $x_1, \ldots, x_m$ such that, for each pair $k, l \in \{1, \ldots, m\}$, x_k does not come after x_l. If, for each $k \in \{1, \ldots, m\}$, b_{x_k} is a Nash equilibrium of Γ_{x_k} and $b_{-(x_1,\ldots,x_m)}$ is a Nash equilibrium of $\Gamma^{b_{x_1,\ldots,x_m}}_{-(x_1,\ldots,x_m)}$, then b is a Nash equilibrium of Γ.*

The following example illustrates Proposition 3.4.1.

Example 3.4.2. Consider the extensive game Γ given in Figure 3.2.1. Let x_1 and x_2 be the unique nodes of w_2^1 and w_2^2, respectively. L_2 is a Nash equilibrium of Γ_{x_1} and l_2 is a Nash equilibrium of Γ_{x_2}. The game $\Gamma^{(L_2 l_2)}_{-(x_1,x_2)}$ is given in Figure 3.4.2. Note that R_1 is a Nash equilibrium of this game. Hence, according to Proposition 3.4.1, $(R_1, L_2 l_2)$ is a Nash equilibrium of Γ. By construction, $(R_1, L_2 l_2)$ is not only a Nash equilibrium, but also a subgame perfect equilibrium. $\diamond$

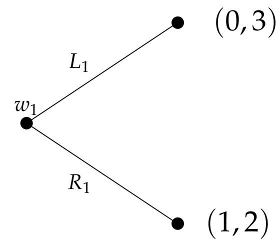


Figure 3.4.2. The game $\Gamma_{-(x_1,x_2)}^{(L_2,l_2)}$.

The previous example illustrates a backward induction procedure to identify subgame perfect equilibria in an extensive game: the game is decomposed in simpler games and, starting from one of the simplest (at the end of the original game), one proceeds backward identifying Nash equilibria and substituting subgames for the payoff vectors of the Nash equilibria in the subgames. The spirit of the backward induction procedure is also used to prove the next two existence results. Let $L(\Gamma)$ denote the *length* of the extensive game Γ, *i.e.*, the number of arcs of the longest play of Γ. Recall that we are working with finite games and, hence, $L(\Gamma) < \infty$.

Theorem 3.4.2. *Every extensive game with perfect recall has, at least, one subgame perfect equilibrium.*

Proof. We make the proof by induction on $L(\Gamma)$. If $L(\Gamma) = 1$, the result is straightforward. Assume that the result is true up to $t-1$ and let Γ be a game with length t. If Γ has no proper subgame, then all its Nash equilibria are subgame perfect and the result follows from the nonemptyness of the set of Nash equilibria. Otherwise, decompose Γ in a set of nodes $x_1, \ldots, x_m$ such that, for each pair $k, l \in \{1, \ldots, m\}$, x_k does not come after x_l and $\Gamma_{-(x_1,\ldots,x_m)}^{b_{x_1,\ldots,x_m}}$ does not have proper subgames (this is independent of the chosen b). For each $k \in \{1, \ldots, m\}$, $L(\Gamma_{x_k}) < t$ and, hence, by the induction assumption, there is a subgame perfect equilibrium $\bar{b}_{x_k}$ of Γ_{x_k}. Take a Nash equilibrium of $\Gamma_{-(x_1,\ldots,x_m)}^{\bar{b}_{x_1,\ldots,x_m}}$: $\bar{b}_{-(x_1,\ldots,x_m)}$. Now, $\bar{b} = (\bar{b}_{-(x_1,\ldots,x_m)}, \bar{b}_{x_1}, \ldots, \bar{b}_{x_m})$ is a subgame perfect equilibrium of Γ. $\square$

Theorem 3.4.3. *Every extensive game with perfect information Γ has, at least, one pure strategy profile that is a subgame perfect equilibrium.*

Proof. Again, we make the proof by induction on $L(\Gamma)$. If $L(\Gamma) = 1$, the result is straightforward. Assume that the result is true up to $t-1$ and let Γ be a game with length t. Decompose Γ in a set of nodes $x_1, \ldots, x_m$ such that, for each pair $k, l \in \{1, \ldots, m\}$, x_k does not come after x_j and $\Gamma_{-(x_1,\ldots,x_m)}^{b_{x_1,\ldots,x_m}}$ has length 1 (this is independent of the chosen b); the latter can be done because

Γ has perfect information. For each $k \in \{1, \dots, m\}$, $L(\Gamma_{x_k}) < t$ and, hence, by the induction assumption, there is a pure subgame perfect equilibrium a_{x_j} of Γ_{x_j}. Since the length of $\Gamma^{a_{x_1,\dots,x_m}}_{-(x_1,\dots,x_m)}$ is 1, it has a pure Nash equilibrium, namely, $a_{-(x_1,\dots,x_m)}$. Clearly, $a = (a_{-(x_1,\dots,x_m)}, a_{x_1}, \dots, a_{x_m})$ is a pure subgame perfect equilibrium of Γ. $\square$

Below, we present the centipede game, one of the most widely studied extensive games, where backward induction leads to the unique subgame perfect equilibrium of the game.

Example 3.4.3. (*The centipede game* (Rosenthal 1981)).[11] Two players are engaged in the following game. There are two pots with money: a small pot and a big pot. At the start of the game, the small pot contains one coin and the big pot contains four coins. When given the turn to play, a player has two options: i) stop the game (S), in which case he gets the big pot and the small one goes to the other player, and ii) pass (P), in which case the number of coins in each pot gets doubled and the other player is given the turn to play. Player 1 is the one playing at the start of the game and the total number of periods is 6. If no player has stopped after period 6, the number of coins in the pots doubles once again and the player who passed last gets the small pot, with the other player getting the big one. This version of the centipede game is depicted in Figure 3.4.3 below. This game

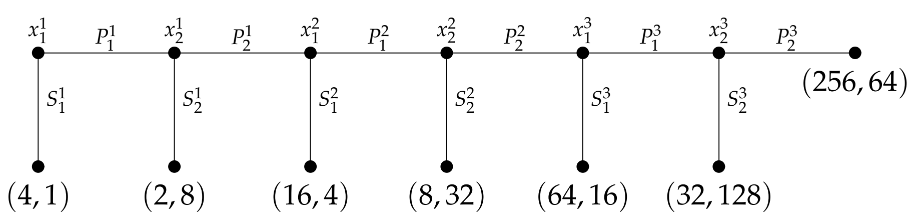


Figure 3.4.3. The centipede game.

can be easily solved by backward induction. If node x_2^3 is reached, then player 2 will surely stop, which gives him payoff 128, instead of the 64 he would get by passing. Now, if node x_1^3 is reached, player 1, anticipating that player 2 will stop at x_2^3, will surely stop as well, getting 64 instead of the 32 he would get by passing and having player 2 stopping afterwards. This reasoning unravels all the way to period 1 and, therefore, the unique subgame perfect equilibrium of this game prescribes that both players should stop whenever they are given the chance to play. Thus, the subgame perfect equilibrium outcome of this game coincides with the most inefficient

[11]Our exposition is partially based on Palacios-Huerta and Volij (2009) and, therefore, we present here the version of the centipede game used in their analysis.

one from the social point of view and, moreover, it gives player 2 his worst payoff in the game and player 1 his second worst payoff. Actually, it is easy to see that, in any Nash equilibrium of this game, player 1 stops in period 1, that is, despite the large gains both players might make if they pass in early rounds, they get payoff $(4, 1)$ in any Nash equilibrium of the game. This unambiguous prediction has been regarded by many game theorists as very counterintuitive and a good number of studies have been devoted to get a better understanding of this game. Several papers, starting with McKelvey and Palfrey (1992), have shown that human behavior in laboratory experiments is far from the game theoretical prediction.[12] In these experiments, the proportion of games in which player 1 stops in period 1 is typically below 5 percent. Indeed, this is the case even if each player plays the centipede game several times, which might allow for some learning.

Thus, because of this conflict between the theoretical prediction and intuitively reasonable behavior in the centipede game, theorists wondered to what extent rationality alone would lead to the backward induction outcome. Aumann (1992) shows in an example that it is possible for rationality to be mutually known to a high degree and still observe that both players pass for several rounds.[13] Soon after that, Aumann himself showed that it is not rationality, not even mutual knowledge of rationality, but common knowledge of rationality that pins down the backward induction outcome (Aumann 1995, 1998).[14]

After all these studies on the centipede game, two explanations to conciliate theory and observed behavior stood over the rest. According to the first explanation, the observed behavior would be the result of some form of social preferences, such as altruism, which might induce players to pursue socially preferable outcomes, even at the risk of lowering one's own monetary payoff. In such a case, since the above game theoretic analysis implicitly assumes that the utility of a player is represented by his own monetary payoff, one should not be surprised by the divergence between equilibrium play and actual play. The second explanation comes from the insights obtained through the epistemic approach, whose results we have just outlined. In this case, the observed behavior would be a consequence of the lack of common knowledge of rationality. Palacios-Huerta and Volij (2009) develop an experimental analysis that allows us to neatly discriminate between the two explanations, providing strong evidence supporting

[12] Refer to Palacios-Huerta and Volij (2009) for a deep review of the experimental literature on the centipede game.

[13] A similar analysis can be found in Reny (1992) and Ben-Porath (1997).

[14] Using a different formalization of the idea of rationality in extensive games, Reny (1993) showed that the backward induction outcome may not arise even if there is common knowledge of rationality. See also Ben-Porath (1997) and Asheim and Dufwenberg (2003) for related studies.

the second one; we briefly summarize the findings in their paper. Their main departure from earlier experiments on the centipede game is that the subject pool they consider is partially formed by chess players. In their own words, the main reason is that "one can safely say that it is common knowledge among most humans that chess players are highly familiar with backward induction reasoning" and then they "use these subjects to study the extent to which knowledge of an opponents rationality is a key determinant of the predictive power of subgame-perfect equilibrium in this game". They ran two separate experiments. First they went to the "field", in this case to chess tournaments, and asked chess players to play, only once, the centipede game. Then, they took chess players to the laboratory and asked them to play the centipede game ten times among themselves and also against college students, which were the typical subjects of study in the preceding works. The key idea was that a game involving chess players should be perceived as "closer" to common knowledge of rationality and therefore one might expect actual play to get closer to equilibrium play. In a nutshell, they observed that the presence of chess players qualitatively changes the observed behavior in the centipede game, which should not happen under the interpretations based on social preferences. More specifically, their findings can be summarized as follows:

Chess players *vs.* chess players: Both in the field and in the laboratory, the outcome was quite close to the game theoretic prescription. In the field experiment, with chess players playing the centipede game once, stop was played at the initial node in 69 percent of the games. In the laboratory experiment, where each chess player played the centipede game ten times against different opponents, stop was played at the initial node in more than 70 percent of the games. Moreover, the play of every chess player fully converged to equilibrium after the fifth time they played the game.

Students *vs.* students: The results were consistent with earlier literature, with stop being played at the initial node only 3 percent of the games and, moreover, there was no sign of convergence to equilibrium play as the repetitions progressed.

Chess players *vs.* students: The outcome was much closer to equilibrium play than in the case above. When students played in the role of player 1, stop was played at the initial node in 30 percent of the games. When chess players played in the role of player 1, 37.5 percent of the games ended in the first node.

In view of these findings, the authors argued that "these results offer strong support for standard approaches to economic modeling based on the principles of self-interested rational economic agents and on their assessments of the behavior of their opponents in a game". $\diamond$

In the following subsection we present the one shot deviation principle, a result that facilitates the verification of whether a given strategy profile is a subgame perfect equilibrium or not.

3.4.1. The one shot deviation principle. Let Γ be an extensive game with perfect information and let $b \in B$. To check that b is a subgame perfect equilibrium of Γ we have to check that there is no player i that can gain by unilaterally deviating from b at any subgame. Since we are assuming perfect information, all the information sets are singletons and, hence, each of them can be identified with the unique node it contains. Given $x \in P_i$, a *one shot deviation by player i from b at x* is a behavior strategy that coincides with b_i in the choices made at the nodes different from x but that makes a different choice at node x. The one shot deviation principle says that, for finite extensive games with perfect information, to check that b is a subgame perfect equilibrium it suffices to check that there is no profitable one shot deviation from b (by any player at any node). Hence, the one shot deviation principle provides a very useful tool when studying subgame perfection for both theoretical and applied purposes.

Proposition 3.4.4 (One shot deviation principle)**.** *Let Γ be an extensive game with perfect information and let $b \in B$. Then, the two following statements are equivalent,*

i) *b is a subgame perfect equilibrium of Γ,*
ii) *no one shot deviation from b is profitable.*

Proof. [15] Clearly, i) $\Rightarrow$ ii). Next we prove that ii) $\Rightarrow$ i) also holds. Suppose that b satisfies ii) but it is not a subgame perfect equilibrium. Let Γ_x be a subgame where a player, namely i, can profitably deviate. Among all such deviations, let $\hat{b}_i$ be the one that differs from b_i in the minimum number of nodes. Among the nodes that come after x, let $\hat{x}$ be one such that 1) the number of nodes in the path connecting x and $\hat{x}$ is maximal and 2) $\hat{b}_i$ and b_i differ at $\hat{x}$. Now, we claim that $\hat{b}_i$ is also a profitable deviation at $\Gamma_{\hat{x}}$; otherwise, taking $\tilde{b}_i$ equal to $\hat{b}_i$ in all the nodes different from $\hat{x}$ and equal to b_i at x, we would have a deviation in Γ_x that differs from b_i in less nodes than $\hat{b}_i$ but, by construction, this is not possible. Thus, the behavior strategy that coincides with b_i at all the nodes different from $\hat{x}$ and with $\hat{b}_i$ at $\hat{x}$ is a profitable one shot deviation from b. $\square$

[15] Our proof follows the one in Osborne and Rubinstein (1994, Lemma 98.2).

The idea of the one shot deviation principle comes from dynamic programming, where it was introduced by Blackwell (1965). Here we have presented it as a tool to study subgame perfect equilibria in finite extensive games with perfect information, but it can also be applied to other equilibrium concepts and other classes of extensive games. For instance, refer to Mailath and Samuelson (2006, Section 2.2) for an application to subgame perfect equilibrium in infinitely repeated games with perfect monitoring and to Osborne and Rubinstein (1994, Exercise 227.1) for an application to sequential equilibrium in finite games with perfect recall.

3.4.2. Weaknesses of the subgame perfect equilibrium concept in extensive games. In general, the subgame perfect equilibrium concept leads to sensible strategy profiles when working with games with perfect information. Unfortunately, this is not so when imperfect information appears. The reason for this is that, in a game with perfect information, whenever a player has to make a choice, he is at a single-node information set and there is a subgame beginning at that node. Yet, this is not the case for games with imperfect information where subgame perfection imposes no restriction to the behavior of the players at information sets that are not in the equilibrium path of any subgame. Example 3.4.4 illustrates why this can lead to equilibria that are not self-enforcing.

Example 3.4.4. Consider the extensive game Γ of Figure 3.4.4. Since Γ does not have a subgame different from itself, all its Nash equilibria are subgame perfect. Thus (R_1, R_2) is a subgame perfect equilibrium of Γ. However, it is not really self-enforcing because if player 2's information set is reached, he will play L_2; since player 1 is able to realize this, he will play L_1. Note that the problem with (R_1, R_2) in this example is the same one we had raised for Nash equilibrium in the chain store game (Example 3.3.5); namely, (R_1, R_2) is sustained by an incredible threat of player 2 at w_2. Hence, (L_1, L_2) is the unique sensible equilibrium of this game. $\diamond$

Thus, for games with imperfect information the concept of subgame perfection does not ensure that players act rationally at all their information sets. In Section 3.5 we define the concept of *sequential equilibrium*, which addresses this problem.

3.5. Sequential Equilibrium

We have shown by means of Example 3.4.4 above that, in general, subgame perfection does not perform well in games with imperfect information. This is because in games with imperfect information there might be

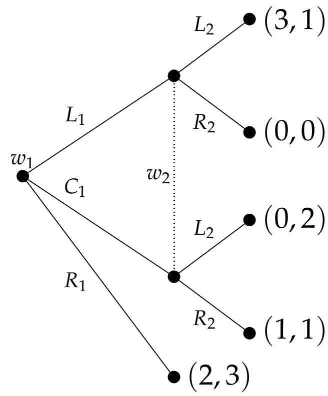


Figure 3.4.4. (R_1, R_2) is a subgame perfect equilibrium that is not self-enforcing.

few proper subgames (if any) and, hence, in such games, subgame perfection imposes almost no additional restriction as compared to Nash equilibrium. Therefore, if we are keen on finding an equilibrium concept in which every player maximizes his (expected) payoff, *at all his information sets*, when taking the strategies of the opponents as given, then we need to impose restrictions on the information sets that are not singletons. The latter point is the main idea underlying the definition of the sequential equilibrium (Kreps and Wilson 1982b). A player, when taking a decision at a given information set, should first form some beliefs about the probabilities of being at each one of the nodes contained in the information set and play so as to maximize his expected utility. Hence, each player should, at each information set, take a choice that is a best reply given his beliefs and the strategies of the opponents. Then, a natural question is whether or not we should impose restrictions on the beliefs of the players in the different information sets. Since the equilibrium approach assumes that players take the strategies of the opponents as given, then, whenever possible, they should use these strategies to form their beliefs; that is, players should use Bayes rule to calculate the probabilities of the different nodes in each information set. This has been called Bayesian updating and it underlies most of the equilibrium concepts that have been defined for extensive games with imperfect information; recall that we already talked briefly about Bayesian updating in Section 2.12.

Following Kreps and Wilson (1982b), we define a *system of beliefs over* $X \backslash Z$ as a function $\mu : X\backslash Z \to [0,1]$ such that, for each information set w, $\sum_{x \in w} \mu(x) = 1$. For each information set, the beliefs represent the probabilities of being at each one of its different nodes (conditional on the information set being reached). To formally define what a sequential equilibrium is, we need to complement a strategy profile with a system of beliefs over $X\backslash Z$.

Definition 3.5.1. Let Γ be an extensive game. An *assessment* is a pair (b,μ), where b is a behavior strategy profile and μ is a system of beliefs.

Given an assessment (b,μ), for each information set w and each $z \in Z$, let $\mu_w(z,b)$ denote the probability that z is reached conditional on: i) w being reached, ii) being at each of the nodes of w with probabilities given by μ, and iii) b to be played thereafter. Formally, $\mu_w(z,b) := \sum_{x\in w}\mu(x)p_x(z,b)$. Now, let $u^{\mu}_{iw}(b) := \sum_{z\in Z}\mu_w(z,b)u_i(z)$ denote the conditional expected utility of player i once w has been reached if his beliefs are given by μ.[16]

Definition 3.5.2. Let Γ be an extensive game, (b,μ) an assessment, and $\bar{b}_i \in B_i$. Let $w \in W_i$. We say that $\bar{b}_i$ is a *best reply of player i against* (b,μ) *at* w if

$$u^{\mu}_{iw}(b_{-i},\bar{b}_i) = \max_{\hat{b}_i\in B_i} u^{\mu}_{iw}(b_{-i},\hat{b}_i).$$

Definition 3.5.3. Let Γ be an extensive game. An assessment (b,μ) is *sequentially rational* if, for each $i \in N$ and each $w \in W_i$, b_i is a best reply of player i against (b,μ) at w.

Note that sequential rationality does not even imply Nash equilibrium. Since there is complete freedom for the beliefs, the players might be best replying against their beliefs but not against the strategy profile that is actually being played. As indicated above, a first natural requirement for the beliefs associated with a given strategy profile is that the players form them using Bayes rule at those information sets that are in the path. This idea leads to the concept of *weak perfect Bayesian equilibrium*.

Definition 3.5.4. Let Γ be an extensive game. An assessment (b,μ) is *weakly consistent with Bayes rule* if μ is derived using Bayesian updating in the path of b, *i.e.*, for each information set w such that $p(w,b) > 0$ we have that, for each $x \in w$, $\mu(x) = p_w(x,b)$.

Definition 3.5.5. Let Γ be an extensive game. A *weak perfect Bayesian equilibrium* is an assessment that is sequentially rational and weakly consistent with Bayes rule.

Remark 3.5.1. Note that, if the system of beliefs is weakly consistent with Bayes rule, then, for each information set w such that $p(w,b) > 0$ we have that, for each $x \in w$, $p_w(x,b) = \mu(x)$; hence, $u_{iw}(b) = u^{\mu}_{iw}(b)$. Therefore, given an assessment (b,μ) and an information set w in the path of b

[16] Note that the utilities $u^{\mu}_{iw}(b)$ are defined for every $w \in W$ whereas, on page 106, the utilities $u_{iw}(b)$ could only be defined for $w \in W$ such that $p(w,b) > 0$ (where Bayes rule applies and, hence, conditional probabilities are well defined).

($p(w,b) > 0$), if μ is consistent with Bayes rule, then best replying against (b,μ) at w (Definition 3.5.2) implies best replying against b at w (Definition 3.3.3). Hence, weak perfect Bayesian equilibrium implies Nash equilibrium. Exercise 3.2 asks to formally show that weak perfect Bayesian equilibrium is indeed a strict refinement of Nash equilibrium.

Therefore, as compared to sequential rationality alone, weak perfect Bayesian equilibrium is a step forward since, at least, it refines Nash equilibrium. Yet, since it imposes no restriction on how off-path beliefs have to be formed, it does not even imply subgame perfection. Example 3.5.1 below illustrates the latter point and Exercise 3.3 asks the reader to show that subgame perfection does not imply weak perfect Bayesian equilibrium either.

Example 3.5.1. Consider the extensive game Γ of Figure 3.5.1 and the strategy profile $b = (R_1, L_2, R_3)$. To associate a system of beliefs with b, it suffices to say what are the beliefs at the information set of player 3, *i.e.*, at w_3. Since w_3 is not in the path of b, any beliefs at w_3 will be weakly consistent with Bayes rule. In particular, let μ denote the beliefs in which player 3 puts probability 1 at the node in which player 2 has chosen R_2. By doing so, we have that (b,μ) is a weakly perfect Bayesian equilibrium. However, b is not subgame perfect. $\diamond$

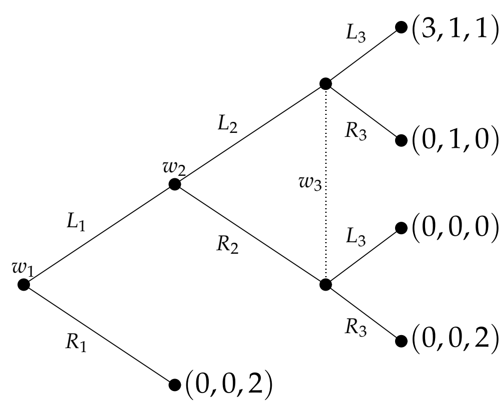


Figure 3.5.1. A weak perfect bayesian equilibrium may not be subgame perfect.

Therefore, being weakly consistent with Bayes rule is not demanding enough because of this important drawback of weak perfect Bayesian equilibrium.[17] For specific classes of games, such as multistage games, the concept of weak perfect Bayesian equilibrium has been strengthened by imposing some restrictions on the way off-path beliefs are formed, leading to the definition of *perfect Bayesian equilibrium* (discussed in Section 4.6). Essentially, the idea is to use Bayes rule whenever possible, not only inside the path, *i.e.*, once some arbitrary beliefs have been chosen for some off-path information set, then the beliefs thereafter should be formed again according to Bayes rule. Within this approach, equilibria like the one in Example 3.5.1 would be ruled out.

Yet, there are deeper considerations, different from those imposed by the above "perfect Bayesian" approach, that may lead to some restrictions in the way beliefs are formed even in those information sets that are reached with probability 0. Roughly speaking, the structure of the game by itself might affect the beliefs to be formed. This was first argued by Kreps and Wilson (1982b) by means of an example similar to the situation described in Figure 3.5.2 below. Nature moves first and players receive no informa-

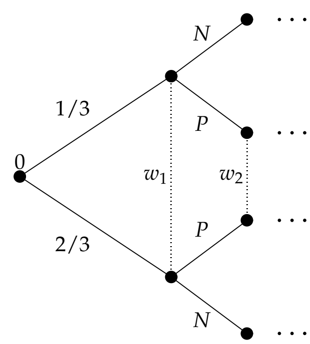


Figure 3.5.2.

tion about its move. Then, player 1 has to choose whether to pass the move to player 2 (P) or not (N). Suppose that player 1 chooses action (N) at w_1. Bayes rule imposes no restriction on the beliefs of player 2 at w_2; should he be free to form any beliefs? Kreps and Wilson (1982b) argue that the answer to this question should be no. If w_2 is reached it means that player 1's strategy was not N. Yet, regardless of the lottery player 1 might have chosen at w_1, since he has to take the same action at the two nodes in w_1, player 2 should attach probabilities 1/3 and 2/3 respectively at the nodes in w_2. Thus, there seems to be some minimal consistency requirements that we

[17] The interested reader may refer to Mas-Colell et al. (1995) for a deeper discussion of weak perfect Bayesian equilibrium.

should impose on the beliefs at information sets that are off the path of the given strategy profile. We now present the consistency requirement imposed by Kreps and Wilson when they introduced the sequential equilibrium concept.[18]

In Definition 3.5.7 below we define sequential equilibrium as an assessment (b, μ) satisfying two properties. First, each player, at each one of his information sets, is best replying given b and μ. Second, the beliefs have to satisfy a consistency requirement that aims to account for the issues raised above: i) players should use Bayes rule whenever possible and ii) even at the information sets where Bayes rule does not apply, there should be some deviation from b that originates the given beliefs.

We say that a behavior strategy $b \in B$ is *completely mixed* if at each information set all the choices are taken with positive probability. Let B^0 denote the set of all completely mixed behavior strategy profiles. Note that, if a strategy $b \in B^0$ is played, then all the nodes of the game are reached with positive probability. In such a case, there is a unique system of beliefs associated with b that is consistent with Bayes rule. For each $b \in B^0$, let μ^b denote the (unique) system of beliefs that is derived from b using Bayes rule.

Definition 3.5.6. Let Γ be an extensive game. An assessment (b, μ) is *consistent* if there is a sequence $\{b^n\}_{n\in\mathbb{N}} \subset B^0$, such that $(b, \mu) = \lim_{n\to\infty}(b^n, \mu^{b^n})$.

The system of beliefs of a consistent assessment is weakly consistent with Bayes rule. Indeed, it is consistent with Bayes rule in a much stronger sense, since the limit property of the beliefs ensures that Bayes rule is, in some sense, also being applied off-path.

Definition 3.5.7. Let Γ be an extensive game. A *sequential equilibrium* is an assessment that is sequentially rational and consistent.

Hereafter, if no confusion arises, we abuse the language and say that a strategy profile $b \in B$ is a sequential equilibrium if it is *part of* a sequential equilibrium, *i.e.*, if there is a system of beliefs μ such that the assessment (b, μ) is a sequential equilibrium.

Proposition 3.5.1. *Let Γ be an extensive game. If (b, μ) is a sequential equilibrium of Γ, then b is a subgame perfect equilibrium. Moreover, if Γ is a game with perfect information, then every subgame perfect equilibrium is also a sequential equilibrium.*

[18] There are other consistency requirements. Refer to Kreps and Wilson (1982b) and Fudenberg and Tirole (1991b) and (Fudenberg and Tirole 1991a, Section 8.3.2) for a discussion on those.

Proof. Let Γ be an extensive game and let (b, μ) be a sequential equilibrium of Γ. Let $x \in X$ be such that Γ_x is a proper subgame and let w be the information set containing x. Recall that, for Γ_x to be a subgame, w has to be a singleton. Hence, $\mu(x) = 1$ and best replying against (b, μ) at w is the same as best replying against b at w. Since (b, μ) is sequentially rational, all players are best replying at w. Hence, b induces a Nash equilibrium in Γ_x. The converse for games with perfect information is now straightforward. □

Remark 3.5.2. There are also some classes of games with imperfect information in which the sets of subgame perfect equilibria and sequential equilibria coincide. We briefly discuss this issue in Section 3.7.

We use Example 3.4.4 to show that the converse of Proposition 3.5.1 does not hold; that is, sequential equilibrium is more demanding than subgame perfection when information is not perfect. Take the subgame perfect equilibrium (R_1, R_2). Clearly, it cannot be a sequential equilibrium. Conditional on the information set w_2 being reached, and regardless of the beliefs player 2 might form there, it is a strictly dominant action for him to play L_2. Although sequential equilibrium is more demanding than subgame perfection, its existence is ensured for every extensive game with perfect recall.

Theorem 3.5.2. *Let Γ be an extensive game with perfect recall. Then, Γ has, at least, one sequential equilibrium.*

Proof. We do not provide a direct proof of this result. It is an immediate consequence of Proposition 3.6.1 and Corollary 3.6.4 in Section 3.6.1; there we define perfect equilibrium for extensive games and show that every perfect equilibrium is a sequential equilibrium and that every extensive game with perfect recall has, at least, one perfect equilibrium. □

We consider that, from the point of view of the discussion at the beginning of this section, the sequential equilibrium concept for extensive games with perfect recall is a natural counterpart of the Nash equilibrium concept for strategic games since, in a sequential equilibrium, every player maximizes his (expected) payoff, *at all his information sets*, when taking the strategies of the opponents as given. As it happened with the refinements for strategic games, a sequential equilibrium may not be self-enforcing in general. We illustrate this fact in Example 3.5.2.

Example 3.5.2. Consider the extensive game Γ of Figure 3.5.3. We claim that the pure strategy profile $a = (R_1, R_2)$ is a sequential equilibrium of Γ that is not sensible. We now construct the sequential equilibrium assessment based on (R_1, R_2). For player 2 to be a best reply to play R_2 he must believe that it is more likely to be at $\hat{x}_2$ than at x_2. Hence, a natural candidate for

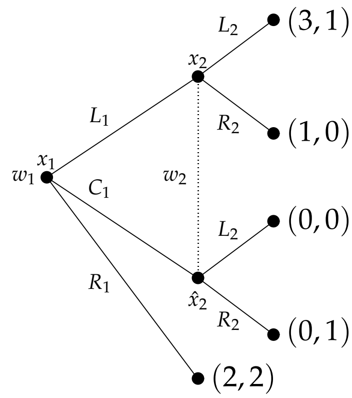


Figure 3.5.3. (R_1, R_2) is a sequential equilibrium that is not self-enforcing.

μ is defined as $\mu(x_1) = 1$, $\mu(x_2) = 0$, and $\mu(\hat{x}_2) = 1$. So defined, (a, μ) is sequentially rational. To see that it is also consistent, let b^k be such that $b_1^k(L_1) = \frac{1}{k^2}$, $b_1^k(C_1) = \frac{1}{k}$, $b_1^k(R_1) = 1 - \frac{1}{k} - \frac{1}{k^2}$, $b_2^k(L_2) = \frac{1}{k}$, and $b_2^k(R_2) = 1 - \frac{1}{k}$. Whereas the induced beliefs are

$$\mu^{b^k}(x_2) = \frac{\frac{1}{k^2}}{\frac{1}{k^2} + \frac{1}{k}} \quad \text{and} \quad \mu^{b^k}(\hat{x}_2) = \frac{\frac{1}{k}}{\frac{1}{k^2} + \frac{1}{k}}.$$

Now, $\lim_{k \to \infty}(b^k, \mu^{b^k}) = (a, \mu)$. Hence, (a, μ) is a sequential equilibrium. However, it is not really self-enforcing. Namely, if player 2's information set is reached, he should deduce that player 1 played L_1; otherwise, he would get a worse payoff than what he would get with R_1. Indeed, C_1 is strictly dominated by both R_1 and L_1. Hence, player 2, if given the turn to move would play L_2. Since player 1 must be able to realize this, he should play L_1. $\diamond$

Remark 3.5.3. The above reasoning illustrates an important idea in the theory of refinements in extensive games: *forward induction*. So far we have said that players should use backward induction to find the equilibrium profiles. Yet, upon reaching an information set that is not in the equilibrium path, the players should also try to think about what might have happened for that information set to be reached, in order to form the beliefs there. By doing so in the game of Example 3.5.2, player 2 would never attach positive probability at node $\hat{x}_2$ once w_2 is reached. Indeed, the above forward induction reasoning suggests that eliminating the strictly dominated choice C_1 should not affect the equilibrium analysis. However, in this book we do not cover the theory of refinements that exploits the idea of forward induction, though it became quite popular after the seminal paper by Kohlberg and Mertens (1986). Refer to Remark 3.6.3 for some references on the topic.

In Section 2.9 we showed that many refinements of the Nash equilibrium concept for strategic games have the problem that the set of equilibria might change if we add to the game strictly dominated strategies. Example 3.5.3 shows that the situation is similar in extensive games and illustrates this drawback along with a related one.[19]

Example 3.5.3. Consider the extensive game Γ_1 in Figure 3.5.4 (a). The strategy U can be supported as part of a sequential equilibrium. Namely, let b be such that $b_1(U) = 1$ and $b_2(L_2) = 1$; and define the corresponding beliefs such that $\mu(x_2) = 1$. So defined, the assessment (b, μ) is sequentially rational. Moreover, it is also consistent. Just take b^k defined as follows: $b_1^k(L_1) = \frac{1}{k}$, $b_1^k(R_1) = \frac{1}{k^2}$, $b_1^k(U) = 1 - \frac{1}{k} - \frac{1}{k^2}$, $b_2^k(R_2) = \frac{1}{k}$, and $b_2^k(L_2) = 1 - \frac{1}{k}$. Now, look at the extensive game Γ_2 in Figure 3.5.4 (b). The only difference with respect to Γ_1 is that we have added the apparently irrelevant move NU for player 1. First he has to decide whether to play up or not (node x_1^0) and then, at node x_1, he has to choose between L_1 and R_1. It seems that this change should be innocuous for the strategic analysis of the game. Yet, U is not part of a sequential equilibrium anymore. To see this, note that the subgame Γ_{x_1} has a unique Nash equilibrium: (R_1, R_2), in which player 1 gets 4. Hence, U cannot be part of any sequential equilibrium (not even subgame perfect). Moreover, in the game Γ_1 the strategy L_1 is strictly dominated and, if deleting it, U would not be part of a sequential equilibrium anymore. Hence, we have also seen that the deletion of strictly dominated actions affects the set of sequential equilibria. ◇

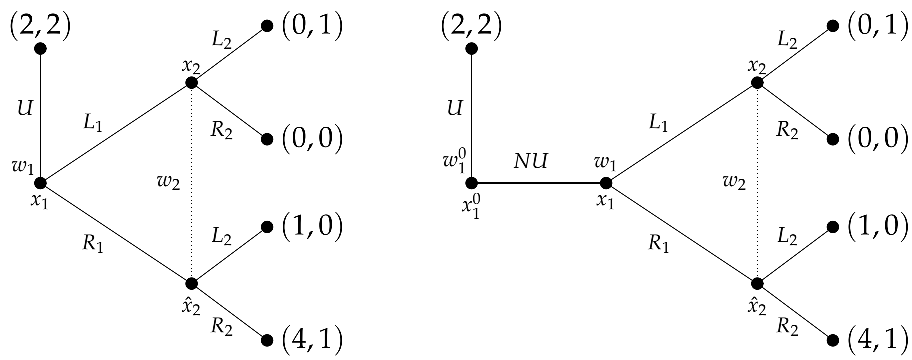
Based on the above considerations, it is also natural to think of refinements of the sequential equilibrium concept. We tackle this problem in Section 3.6.

3.6. Further Refinements

This section is mainly devoted to study two refinements of sequential equilibrium for extensive games: the (trembling hand) perfect equilibrium and the proper equilibrium. We have already introduced them for strategic games (Section 2.9) but, since the underlying ideas also apply to extensive games, they have also been studied in this setting.

3.6.1. Perfect equilibrium. Actually, perfect equilibrium was originally introduced by Selten (1975) as a refinement of subgame perfection in extensive games. There, Selten incidentally defined the concept of perfect equilibrium for strategic games (Definition 2.9.3) as an auxiliary tool to get an

[19] We have taken this example from Fudenberg and Tirole (1991a).



(a) Game Γ_1.

(b) Game Γ_2.

Figure 3.5.4. Adding "irrelevant" actions.

existence result for perfect equilibrium in extensive games with perfect recall. Finally, perfect equilibrium has become one of the major refinement concepts for both strategic and extensive games.

The idea of the perfect equilibrium concept in extensive games is the same as in strategic games: look for equilibrium strategies that remain optimal when introducing trembles in the strategies of the players. Even though the definitions are analogous to the ones in Section 2.9, we present them again for the sake of mathematical completeness. Given an extensive game Γ, for each $i \in N$, we define the set of all the choices of player i as $C_i := \bigcup_{w\in W_i} \bigcup_{c\in C_w} \{c\}$.

Definition 3.6.1. Let Γ be an extensive game. A tremble in Γ is a vector $\eta := (\eta_1, \ldots, \eta_n)$ such that, for each $i \in N$, η_i is a function from C_i to $\mathbb{R}$ satisfying:

i) For each $c \in C_i$, $\eta_i(c) > 0$.
ii) For each $w \in W_i$, $\sum_{c\in C_w} \eta_i(c) < 1$.

We denote by $T(\Gamma)$ the set of trembles in Γ.

Definition 3.6.2. Let Γ be an extensive game and let $\eta \in T(\Gamma)$. The *η-perturbation of* Γ is the extensive game (Γ, η), whose only difference with Γ is that, for each $i \in N$, his set of behavior strategies, B_i, has been reduced to $B_i^\eta := \{b_i \in B_i : \text{for each } w \in W_i \text{ and each } c \in C_w, b_i(c) \geq \eta_i(c)\}$.

The interpretation of the η-perturbation of a game Γ is the same as in the case of strategic games. The players have to take all their choices with a positive probability bounded from below by the trembles. The definition of the Nash equilibrium concept for perturbed games is done in the obvious way.

Definition 3.6.3. Let Γ be an extensive game and let $\eta \in T(\Gamma)$. Let (Γ, η) be the η-perturbation of Γ. A behavior strategy profile $b^* \in B^\eta$ is a Nash equilibrium of (Γ, η) if, for each $i \in N$ and each $\hat{b}_i \in B_i^\eta$,

$$u_i(b^*) \geq u_i(b^*_{-i}, \hat{b}_i).$$

Definition 3.6.4. Let Γ be an extensive game. A behavior strategy $b \in B$ is a perfect equilibrium of Γ if there are two sequences $\{\eta^k\} \subset T(\Gamma)$ and $\{b^k\} \subset B^{\eta^k}$ such that:

i) $\{\eta^k\} \to 0$ and $\{b^k\} \to b$.

ii) For each $k \in \mathbb{N}$, b^k is a Nash equilibrium of (Γ, η^k).

Proposition 3.6.1. *Let Γ be an extensive game. Then, every perfect equilibrium of Γ is (part of) a sequential equilibrium.*

Proof. Let Γ be an extensive game. Let b be a perfect equilibrium of Γ and let $\{\eta^k\} \subset T(\Gamma)$ and $\{b^k\} \subset B^{\eta^k}$ be two sequences taken according to Definition 3.6.4. For each $k \in \mathbb{N}$, $b^k \in B^0$ and, hence, μ^{b^k} is well defined. The sequence $\{\mu^{b^k}\}$ is defined on a compact set so it has a convergent subsequence. Assume, without loss of generality, that the sequence itself is convergent and let μ be its limit. Now it is easy to check that the assessment (b, μ) is a sequential equilibrium. $\square$

We show in Example 3.6.1 below that the converse of Proposition 3.6.1 does not hold, *i.e.*, perfect equilibrium is more demanding than sequential equilibrium.

Example 3.6.1. Consider the extensive game depicted in Figure 3.6.1. The assessment (b, μ) where b is such that players 1 and 2 play L_1 and R_2, respectively and μ is uniquely defined using Bayes rule is a sequential equilibrium. Yet, since R_2 is dominated by L_2, it is easy to check that (L_1, R_2) cannot be a perfect equilibrium. $\diamond$

Remark 3.6.1. The perfect equilibrium concept was already known when Kreps and Wilson introduced the concept of sequential equilibrium. Indeed, they note in their paper that every perfect equilibrium is also sequential and use Selten's existence result for perfect equilibrium to show that every extensive game with perfect recall has a sequential equilibrium. Thus, one might wonder why the sequential equilibrium has become one of the most widely used refinements in applied game theory, even ahead of perfect equilibrium. Kreps and Wilson argue that sequential equilibrium mainly has two advantages: first, in most situations it is much easier to verify that an equilibrium is sequential than to verify that it is perfect and second, the formulation in terms of beliefs gives a tool to discriminate

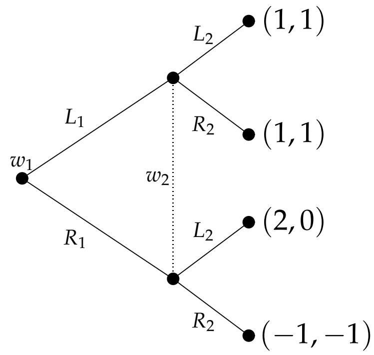


Figure 3.6.1. A sequential equilibrium may not be a perfect equilibrium.

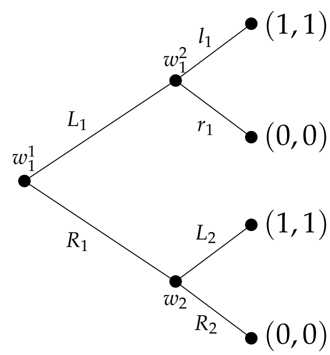
among different equilibria depending on which can be supported by more plausible beliefs.[20]

We now elaborate on the behavior of the perfect equilibrium concept in extensive games. For the sake of exposition, we postpone the existence result until the end of the discussion. We have already seen that perfect equilibrium refines sequential equilibrium. Nonetheless, it does not necessarily lead to self-enforcing equilibria in all extensive games. To see this we just need to reexamine the game of Example 3.5.2 (Figure 3.5.3). There we showed that the equilibrium (R_1, R_2) was not self-enforcing and, still, it was part of a sequential equilibrium assessment. It is easy to check that it is also a perfect equilibrium. Just define $\eta^k \in T(\Gamma)$ such that, for each $i \in \{1,2\}$ and each $c \in w_i$, $\eta_i^k(c) = \frac{1}{k}$. Also, define $b_1^k(L_1) = b_1^k(C_1) = \frac{1}{k}$, $b_1^k(R_1) = 1 - \frac{2}{k}$ and $b_2^k(L_2) = \frac{1}{k}$ and $b_2^k(R_2) = 1 - \frac{1}{k}$. Then, for $k > 3$, each profile b^k is a Nash equilibrium of (Γ, η^k). Since $\{\eta^k\} \to 0$ and $\{b^k\} \to (R_1, R_2)$, (R_1, R_2) is a perfect equilibrium. Indeed, the perfect equilibrium not only fails to eliminate the kind of undesirable equilibria presented above, but also the problems raised in Example 3.5.3 for sequential equilibrium concerning the addition of irrelevant actions persist for perfect equilibrium; Exercise 3.5 asks the reader to formally show that this is so.

3.6.2. Perfect equilibrium in strategic games *vs.* perfect equilibrium in extensive games. Next we provide two examples that show that no inclusion relation holds between the set of perfect equilibria of an extensive game Γ and the set of perfect equilibria of the mixed extension of its corresponding strategic game G_Γ.

[20]Furthermore, though the converse of Proposition 3.6.1 is not true, Kreps and Wilson (1982b) also show that for generic payoffs the two equilibrium concepts coincide; they paraphrase their result as "for almost every game, almost every sequential equilibrium is perfect". Regarding Example 3.6.1, the equilibrium (L_1, R_2) is sequential, but not perfect, because of the indifference of player 2 after player 1 plays L_1; the result in Kreps and Wilson (1982b) says that these situations are nongeneric.

Example 3.6.2. Consider the extensive game Γ depicted in Figure 3.6.2 (a). This game has two subgame perfect equilibria in pure strategies: $(L_1 l_1, L_2)$ and $(R_1 l_1, l_2)$. Moreover, they are also perfect equilibria. Figure 3.6.2 (b) shows G_Γ, the strategic game associated with Γ. The game $E(G_\Gamma)$ has a unique perfect equilibrium: $(L_1 l_1, L_2)$. Hence, $(R_1 l_1, L_2)$ is a perfect equilibrium of Γ that is not a perfect equilibrium of $E(G_\Gamma)$.



(a) The game Γ.

	L_2	R_2
$L_1 l_1$	1,1	1,1
$L_1 r_1$	0,0	0,0
$R_1 l_1$	1,1	0,0
$R_1 r_1$	1,1	0,0

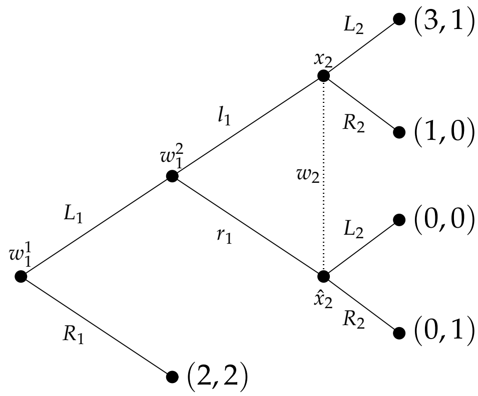
(b) The game G_Γ.

Figure 3.6.2. Perfect in Γ does not imply perfect in $E(G_\Gamma)$.

Note that the strategy $R_1 l_1$ is dominated by $L_1 l_1$. Hence, this example also shows that a perfect equilibrium in extensive games can involve dominated strategies. This might be surprising since one of the advantages of the perfect equilibrium for strategic games was precisely to avoid this kind of equilibria (Theorem 2.9.5). An argument supporting the equilibrium $(R_1 l_1, L_2)$ in the extensive game might be that player 1 may consider it more likely that he makes a mistake at w_1^2 than player 2 does at w_2; there is no room for such reasoning in the game $E(G_\Gamma)$. On the other hand, an implication of Proposition 3.6.3 below is that, for extensive games, a perfect equilibrium cannot contain dominated choices. $\diamond$

Example 3.6.3. Consider the extensive game Γ of Figure 3.6.3 (a). Since the unique Nash equilibrium of the subgame that begins at w_1^2 is (l_1, L_2), $(L_1 l_1, L_2)$ is the unique subgame perfect equilibrium of Γ and, hence, the unique perfect equilibrium as well. Nonetheless, the game $E(G_\Gamma)$ (see Figure 3.6.3 (b)) has three pure perfect equilibria: $(L_1 l_1, L_2)$, $(R_1 l_1, R_2)$, and $(R_1 r_1, R_2)$. Hence, $(R_1 l_1, R_2)$ and $(R_1 r_1, R_2)$ are perfect equilibria of $E(G_\Gamma)$ that are not even subgame perfect equilibria of Γ. $\diamond$

If we compare the extensive games in Figures 3.5.3 and 3.6.3 (a) (discussed in Examples 3.5.2 and 3.6.3, respectively), we can see that the situation is very similar to that illustrated in Example 3.5.3. The two extensive



(a) The game Γ.

	L_2	R_2
L_1l_1	3,1	1,0
L_1r_1	0,0	0,1
R_1l_1	2,2	2,2
R_1r_1	2,2	2,2

(b) The game G_Γ.

Figure 3.6.3. Perfect in $E(G_\Gamma)$ does not imply perfect in Γ.

games are essentially identical but in the game in Figure 3.6.3 (a) the outcome $(3,1)$ is the only one that can be achieved as a perfect equilibrium outcome, whereas in the game in Figure 3.5.3 the outcome $(2,2)$ can also be obtained as the outcome of (unreasonable) perfect equilibria.

3.6.3. The agent strategic game and the existence result. We have seen in Example 3.6.3 that, given an extensive game Γ, a perfect equilibrium of the corresponding strategic game needs not even be subgame perfect in Γ. Thus, we cannot think of perfection in the strategic game as a good refinement for extensive games. Although the idea of the perfect equilibrium concept is the same in the two settings, namely, players best reply against trembles of the others, the source of the problem hinges on the meaning of best reply in each scenario. In a strategic game a best reply is just a strategy that maximizes the expected utility, whereas in the extensive game, since the trembles ensure that all the nodes are reached with positive probability, a best reply is a strategy that maximizes the expected utility at each information set. If we go back to Example 3.6.3, in the strategic game, as far as player 2 chooses R_2 with high enough probability, it is a best reply of player 1 to choose R_1, regardless of whether it is accompanied by l_1 or r_1. Then, for R_2 to be a best reply of player 2, it suffices that player 1 makes the mistake L_1r_1 with higher probability than L_1l_1.

To overcome the aforementioned problems of the perfect equilibrium concept in $E(G_\Gamma)$, Selten (1975) introduced the agent strategic game and showed the relation between perfect equilibrium in the extensive game and in the agent strategic game. Let Γ be an extensive game. Now, we define the *extensive game with agents* that we denote by $A\Gamma$, as follows: $A\Gamma$ has the same game tree as Γ but, for each player $i \in N$, at each information set $w_i \in W_i$, there is an agent playing on i's behalf. Each agent has the same utility function as the player he represents. The strategic game associated with the extensive game with agents, that we denote by $A\Gamma$ is what Selten called the *agent strategic game*, *i.e.*, $G_{A\Gamma}$. By definition, a mixed strategy of an agent of player i in $G_{A\Gamma}$ is just a lottery over the choices at the information set in which the agent represents i. Thus, a one to one mapping can be established between behavior strategies in Γ and mixed strategies in $G_{A\Gamma}$, *i.e.*, every strategy profile in $E(G_{A\Gamma})$ induces a *unique* behavior strategy profile in Γ and *vice versa*. Similarly, given a tremble $\eta \in T(\Gamma)$, it induces a *unique* tremble in $G_{A\Gamma}$ in the natural way. We slightly abuse notation and also denote by η the induced tremble.

Lemma 3.6.2. *Let Γ be an extensive game with perfect recall and let $\eta \in T(\Gamma)$. Then, each Nash equilibrium of (Γ, η) induces a Nash equilibrium of $(G_{A\Gamma}, \eta)$ and* vice versa*.*

Proof. Each strategy profile (Γ, η) has the property that all the information sets are reached with positive probability. The result in Remark 3.3.1 is also true when we restrict attention to strategies in B^{η} instead of B. Therefore, a Nash equilibrium of (Γ, η) is a strategy profile at which each player best replies (within B^{η}) at each one of his information sets. This is equivalent to saying that each player (agent) in $(G_{A\Gamma}, \eta)$ best replies in the induced strategy profile. The converse implication is analogous. □

Proposition 3.6.3. *Let Γ be an extensive game with perfect recall. Then, each perfect equilibrium of Γ induces a perfect equilibrium of $G_{A\Gamma}$ and* vice versa*.*

Proof. Let $b \in B$ be a perfect equilibrium of Γ. Let $\{\eta^k\} \subset T(\Gamma)$ and let $\{b^k\}$ be such that each $b^k \in B^{\eta^k}$ is an equilibrium of the perturbed game (Γ, η^k). By Lemma 3.6.2, the sequence of strategy profiles of $E(G_{A\Gamma})$ induced by the $\{b^k\}$ are equilibria of the corresponding perturbations of $E(G_{A\Gamma})$. Moreover, the sequence of induced strategy profiles converges to the one induced in $E(G_{A\Gamma})$ by b. The converse implication is analogous. □

Corollary 3.6.4. *Let Γ be an extensive game with perfect recall. Then, Γ has, at least, one perfect equilibrium.*

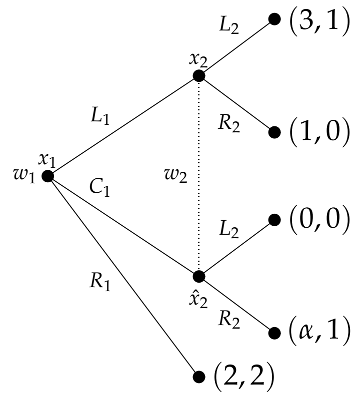
Proof. It follows from the combination of the existence result of perfect equilibrium for strategic games (Theorem 2.9.2) and the proposition above. □

3.6.4. Proper equilibrium. We now discuss to what extent the proper equilibrium concept might be used to disregard the kind of unreasonable equilibria of Example 3.5.2 and also to avoid "inconsistencies" as the ones raised in Example 3.5.3. Our analysis is mainly based on the contributions of van Damme (1984, 1991).

The exposition above for perfect equilibrium suggests that there are two natural definitions of the proper equilibrium for extensive games: either as the proper equilibrium of $E(G_\Gamma)$ or as the proper equilibrium of $E(G_{A\Gamma})$. Now, it is the definition with respect to $E(G_\Gamma)$, the one that, in general, leads to more reasonable equilibria. Note that Theorem 2.9.8 guarantees the existence of proper equilibrium in both cases. Recall that, for strategic games, proper equilibrium is a refinement of perfect equilibrium. Thus, by Proposition 3.6.3, if we look at proper equilibrium through $E(G_{A\Gamma})$, then we have a refinement of the perfect equilibrium for extensive games. On the other hand, we show below that a proper equilibrium of $E(G_{A\Gamma})$ may not be proper in $E(G_\Gamma)$.

Example 3.6.4. As a first approach to properness in extensive games, we begin by studying the game in Figure 3.5.3, in which the agent strategic game and the strategic game coincide. Thus, if we say proper equilibrium we mean proper equilibrium in both $E(G_\Gamma)$ and $E(G_{A\Gamma})$. We claim that the undesirable perfect equilibrium (R_1, R_2) is not proper. We argue informally why this is so (Exercise 3.7 asks to formalize our arguments). Suppose that both players think that (R_1, R_2) is the equilibrium to be played, but there is some probability that player 2 makes the mistake L_2. Then, between the two errors of player 1, L_1 is the less costly (it strictly dominates C_1). Knowing that player 1 will play L_1 more often than C_1, player 2 will prefer L_2 to R_2. As soon as this happens, and player 2 plays L_2 often enough, player 1 will prefer L_1 to R_1. Thus, (R_1, R_2) is not an equilibrium point in a model of mistakes if the likelihood of the mistakes depend on how costly they are, *i.e.*, (R_1, R_2) is not a proper equilibrium. Moreover, the previous reasoning also illustrates why (L_1, L_2), the reasonable equilibrium of Γ, is a proper equilibrium. Yet, a small modification of the game in Figure 3.5.3 shows that there are also proper equilibria which are not self-enforcing.

Consider the extensive game in Figure 3.6.4 (a). If $\alpha = 0$, we are back to the game in Figure 3.5.3. Suppose that $1 < \alpha < 2$. In this case, (R_1, R_2) is still a perfect equilibrium and, again, we think it is not self-enforcing. The arguments are the same as in Example 3.5.2: if player 2's information set is reached, he should guess that player 1 has played L_1 and, hence, play L_2. Unfortunately, (R_1, R_2) is also a proper equilibrium; also in this new game $G_{A\Gamma} = G_\Gamma$, therefore, there is no need to distinguish between the two notions of proper equilibrium. Again, we argue informally why this is so (and, as above, Exercise 3.7 asks to formalize our arguments). Suppose that both players think that (R_1, R_2) is the equilibrium to be played, but there is some probability that player 2 makes the mistake L_2. If L_2 is played with



(a) The game Γ.

	L_2	R_2
L_1	3,1	1,0
C_1	0,0	α,1
R_1	2,2	2,2

(b) The game G_Γ.

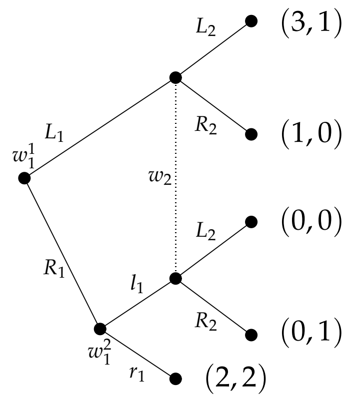
Figure 3.6.4. (R_1, R_2) is not a proper equilibrium

low enough probability, then, between the two errors of player 1, C_1 is now the less costly. Moreover, once player 1 plays C_1 with higher probability than L_1, player 2 prefers R_2 to L_2. Hence, R_1 is a best reply for player 1. Therefore, (R_1, R_2) is a proper equilibrium of both $E(G_{A\Gamma})$ and $E(G_\Gamma)$. $\diamond$

One criticism of perfect equilibrium in extensive games is that apparently equivalent representations of the same game lead to different sets of perfect equilibria. We illustrated this through the games in Example 3.5.3 and also through the games in Figures 3.5.3 and 3.6.3 (a). It is easy to see that the definition of proper equilibrium with respect to $E(G_{A\Gamma})$ exhibits a good behavior in the previous situations; that is, when studying the agent strategic game, the strategy U cannot be part of any proper equilibrium of any of the games in Example 3.5.3 and $(3,1)$ is the only proper equilibrium outcome in the games of Figures 3.5.3 and 3.6.3 (a). Example 3.6.5 presents a further variation of the previous games to show that the set of proper equilibria in the agent strategic game is also sensitive to the chosen representation for the game.

Example 3.6.5. Consider the extensive game Γ of Figure 3.6.5 (a). Note that the strategy profile $(R_1 r_1, R_2)$ is a perfect equilibrium of $E(G_{A\Gamma})$ (we only need that the agent playing at w_1^1 makes the mistake L_1 with a smaller probability than the other agent makes his mistake l_1 at w_1^2). Again, we consider that this equilibrium is unreasonable. Since the choice l_1 is strictly dominated by r_1 at w_1^2, player 2 would prefer to play L_2 instead of R_2 at w_2. In the game $E(G_{A\Gamma})$, every player has only two pure strategies and, hence, by Proposition 2.9.7, $(R_1 r_1, R_2)$ is also a proper equilibrium of $E(G_{A\Gamma})$. This game is essentially identical to the games of Figures 3.5.3 and 3.6.3 (a). We now argue why the only reasonable outcome in the three games is $(3,1)$.

Player 1 has to decide whether to force the outcome $(2,2)$ or pass the move to player 2. In the latter case, player 2 does not know in which of his two nodes he is at. Yet, provided that player 1 is rational, only node x_2 can be reached. Hence, player 2 should play L_2. Player 1 should anticipate this and play so that x_2 is reached. ◇



(a) The game Γ.

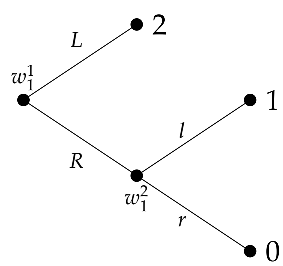
	L_2	R_2
$L_1 l_1$	3,1	1,0
$L_1 r_1$	3,1	1,0
$R_1 l_1$	0,0	0,1
$R_1 r_1$	2,2	2,2

(b) The game G_Γ.

Figure 3.6.5. The set of proper equilibria is sensitive to the chosen representation for the game.

We have noted that, although the three games of Figures 3.5.3, 3.6.3 (a) and 3.6.5 (a) describe essentially the same strategic situation, neither the perfect nor the proper equilibrium of $E(G_{A\Gamma})$ select the same set of profiles in the three games. On the other hand, since the strategic forms of the three games only differ in the addition of duplicated strategies, both the perfect and the proper equilibrium of $E(G_\Gamma)$ select the same set of profiles in the three games (up to duplications). Yet, in Example 3.6.3, when discussing the game in Figure 3.6.3 (a), we showed that a perfect equilibrium of $E(G_\Gamma)$ does not even need to be subgame perfect in Γ. Indeed, both $(3,1)$ and $(2,2)$ are perfect equilibrium outcomes of $E(G_\Gamma)$. Fortunately, the proper equilibrium of the strategic game suffices to get rid of the undesirable outcome $(2,2)$ regardless of the representation of the game, which shows, in particular, that a proper equilibrium of $E(G_{A\Gamma})$ may not be a proper equilibrium of $E(G_\Gamma)$. To see that $(3,1)$ is the only proper equilibrium outcome of the strategic game associated with any of the three extensive games at hand, it suffices to repeat the arguments we made above to show that (R_1, R_2) is not a proper equilibrium of the game in Figure 3.5.3. Hence, we can restrict attention to the proper equilibria of $E(G_\Gamma)$ to eliminate the unreasonable equilibria in these games. However, for this refinement there are also some problems that have to be dealt with carefully. If we consider the game $E(G_\Gamma)$ in Figure 3.6.5 (b), we see that both $(L_1 l_1, L_2)$ and $(L_1 r_1, L_2)$ are proper equilibria. The strategy l_1 is a strictly dominated choice for player 1 at w_1^2 but, when looking at the strategic game, all that matters is that L_1 has already been played and whether to play l_1 or r_1 is irrelevant, *i.e.*, strategies $L_1 l_1$ and $L_1 r_1$ are identical. Thus, a proper equilibrium of $E(G_\Gamma)$ may prescribe irrational behavior outside the equilibrium path. We use Example 3.6.6 below to isolate the problem and then we explain how van Damme (1984) addressed it.

Example 3.6.6. Consider the one-player game in Figure 3.6.6, taken from van Damme (1984). Both Ll and Lr are proper equilibria of $E(G_\Gamma)$. Yet, since the choice r is strictly dominated at w_1^2, Lr seems quite unreasonable. Indeed, the behavior strategy induced by Lr in Γ is not even subgame perfect. However, below we show that there is a compelling way to eliminate these unreasonable behavior strategy profiles that can be induced by the proper equilibria of $E(G_\Gamma)$. $\diamond$



(a) The game Γ.

Ll	2
Lr	2
Rl	1
Rr	0

(b) The game G_Γ.

Figure 3.6.6. A proper equilibrium of $E(G_\Gamma)$ may prescribe irrational behavior outside the equilibrium path

By Kuhn theorem we know that, for each mixed strategy profile s of $E(G_\Gamma)$, there is a behavior strategy profile of Γ that is realization equivalent to s. Moreover, if s is completely mixed, then the construction in the proof of Kuhn theorem leads to a unique behavior strategy profile, which we call *behavior strategy profile induced by* s.

Definition 3.6.5. Let Γ be an extensive game. Let the strategy profile s be a proper equilibrium of $E(G_\Gamma)$. Let $\{\varepsilon^k\} \to 0$ and let $\{s^k\}$ be a sequence of ε^k-proper equilibria that converges to s and let $\{b^k\}$ be the sequence of behavior strategy profiles induced in Γ by $\{s^k\}$. Let $b \in B$ be such that there is a subsequence of $\{b^k\}$ converging to b. Then, we say that b is a *limit behavior strategy profile induced by* s.

Every sequence of behavior strategy profiles as the one defined above has a convergent subsequence. Hence, every extensive game Γ has, at least, one limit behavior strategy profile induced by a proper equilibrium of $E(G_\Gamma)$.

We use the one-player game in Example 3.6.6 to illustrate the idea of the previous definition. We now show that if a behavior strategy profile of Γ chooses L at w_1^1 and assigns positive probability at the choice r at w_1^2, then it is not a limit behavior strategy profile induced by a proper equilibrium of $E(G_\Gamma)$. Let s be a proper equilibrium of $E(G_\Gamma)$, *i.e.*, a mixed strategy profile with support $\{Ll, Lr\}$. Let $\{s^{\varepsilon^k}\}$ be a sequence of ε^k-proper equilibria converging to s. Since the mistake Rr is always more costly than Rl, then, by definition of ε^k-proper equilibrium, we have $s^{\varepsilon^k}(Rr) \leq \varepsilon^k s^{\varepsilon^k}(Rl)$. Therefore, in any limit behavior strategy profile induced by s, player 1 will choose l at w_1^2 with probability one. Moreover, a completely analogous reasoning can be used to show that, in the game of Example 3.6.5, a limit behavior strategy profile induced by some proper equilibrium must choose r_1 at w_1^2 with probability one.

Remark 3.6.2. With respect to the above discussion about properness in the strategic game and the concept of limit behavior strategy profile, the following relationships hold (van Damme 1984, Kohlberg and Mertens 1986): Let Γ be an extensive game; then i) for each proper equilibrium of $E(G_\Gamma)$, there is a sequential equilibrium of Γ that leads to the same outcome (the same probability distribution over terminal nodes) and ii) if $b \in B$ is a limit behavior strategy profile induced by a proper equilibrium of $E(G_\Gamma)$, then b is a sequential equilibrium of Γ.[21]

Nonetheless, even this refined version of properness fails to produce self-enforcing equilibria in every extensive game. Consider again the game in Figure 3.6.4 with $1 < \alpha < 2$. We already argued in Example 3.6.4 that the strategy profile (R_1, R_2) is a proper equilibrium of both $E(G_\Gamma)$ and $E(G_{A\Gamma})$ (the two games coincide). Our discussion also gives the idea of how to show that the behavior strategy profile that consists of choosing R_1 and R_2 at w_1 and w_2, respectively, is a limit behavior strategy profile induced by (R_1, R_2). In addition, we also argued, by means of forward induction reasonings, that such an equilibrium is not reasonable (see Remark 3.5.3 and the discussions of Examples 3.5.2 and 3.6.5).

Figure 3.6.7 shows the relationship of the different refinements we have discussed in this chapter. Recall that we have proved that the existence of all of them is guaranteed for every extensive game with perfect recall.

[21] Note that if we complement statement ii) with the generic coincidence of sequential and perfect equilibrium outlined in footnote 20, then we could say that, for almost every game, b is a perfect equilibrium of $E(G_{A\Gamma})$. Indeed, van Damme (1991) goes further and argues that, for almost every game, b is a proper equilibrium of $E(G_{A\Gamma})$.

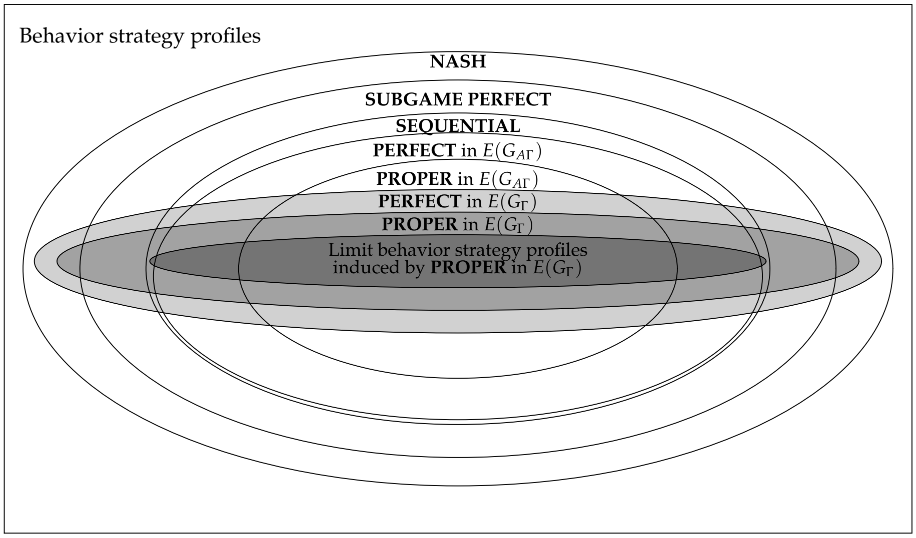


Figure 3.6.7. Relations between the different equilibrium concepts in extensive games that we have discussed in this chapter.

Remark 3.6.3. The scope of the theory of refinements in extensive games covers much more concepts than the ones discussed here. We might define, for instance, the strictly perfect equilibrium for extensive games, though we would again run into existence problems. It is worth mentioning that the equilibrium refinements we have presented capture neither the idea of forward induction nor the related one of elimination of dominated strategies. Within this line of attack, maybe the most compelling approach is the one initiated by Kohlberg and Mertens (1986). They showed that there cannot be a single-valued solution that leads to self-enforcing strategy profiles for every game. Thus, they took a set-valued approach and, based on some stability requirements, they first defined the concept of *stable sets*. According to Van Damme, "[this stability concept], loosely speaking, is the set-valued analogue of strict perfectness". Some classic references of the above approach are Kohlberg and Mertens (1986), Cho (1987), Cho and Sobel (1990), Mertens (1989, 1991), van Damme (1989, 1991), and Hillas (1990).

Remark 3.6.4. As we have seen in this section, there are some relations between the extensive game refinements of Nash equilibrium and the strategic game ones. These connections motivated some research to understand to what extent the strategic game associated with an extensive game can capture all of the strategically relevant information. This issue was already discussed in Kohlberg and Mertens (1986) and, later on, a deep analysis of the potential of strategic games to account for extensive game reasoning was developed in a series of papers by Mailath et al. (1993, 1994, 1997).

3.6.5. Infinite games: An example. So far in this chapter, attention has been restricted to finite extensive games. However, there are two directions in which an extensive game can be infinite. We say that a game has *infinite length* if the length of the game tree is infinite or, more formally, there is at least one play of the game that contains an infinite number of arcs. On the other hand, the game has *infinite width* if the players may have a continuum of actions at some of their information sets. From the technical point of view, games of infinite length and games of infinite width have different implications for the analysis we carried out in this chapter. Although virtually all the analysis can be naturally extended to extensive games of infinite length, one has to be more careful when dealing with games of infinite width. We already mentioned this issue in Chapter 2, where we referred the reader to Simon and Stinchcombe (1995) and Méndez-Naya et al. (1995) for extensions of perfect and proper equilibrium to infinite strategic games. Moreover, also some of the refinements that are specific to extensive games are difficult to extend to games of infinite width. The definition of sequential equilibrium heavily relied on sequences of (completely mixed) strategies in which, at each information set, all the choices were made with positive probability. So far there has been no successful extension of sequential equilibrium to games of infinite width.

We now present the model of *Stackelberg duopoly*. This example provides an application of subgame perfection to a game of infinite width. Moreover, Section 5.9 also contains other applications of subgame perfection to games of infinite width, including the classic *alternating offers game* defined in Rubinstein (1982).

Example 3.6.7. (Stackelberg duopoly (von Stackelberg 1934)). A Stackelberg duopoly only differs from a Cournot duopoly by the fact that the two producers do not choose *simultaneously* the number of units they produce and bring to the market. On the contrary, one of the producers (the leader) makes his choice first. Then, the second producer (the follower), after observing the choice of the leader, makes his choice. Note that this situation can be modeled as an infinite extensive game with perfect information. Hence, subgame perfection leads to sensible outcomes. We follow the same notation we used in Examples 2.1.2 and 2.2.2 when discussing the Cournot model. A subgame perfect equilibrium of this game is a pair (a_1^*, g^*), with $a_1^* \in A_1$ and $g^* \in A_2^{A_1}$, satisfying that:

- for each $a_1 \in A_1$, $g^*(a_1) \in \mathrm{BR}_2(a_1)$ and
- for each $a_1 \in A_1$, $u_1(a_1^*, g^*(a_1^*)) \geq u_1(a_1, g^*(a_1))$.

A *Stackelberg equilibrium* is a pair (a_1^*, a_2^*) such that $a_2^* = g^*(a_1^*)$ for a subgame perfect equilibrium (a_1^*, g^*). In Example 2.2.2 we obtained that, for

	a_1^*	a_2^*	$a_1^* + a_2^*$	$u_1(a_1^*, a_2^*)$	$u_2(a_1^*, a_2^*)$	$u_1 + u_2$
Monopoly			$\frac{d-c}{2}$			$\frac{(d-c)^2}{4}$
Cournot	$\frac{d-c}{3}$	$\frac{d-c}{3}$	$\frac{2(d-c)}{3}$	$\frac{(d-c)^2}{9}$	$\frac{(d-c)^2}{9}$	$\frac{2(d-c)^2}{9}$
Stackelberg	$\frac{d-c}{2}$	$\frac{d-c}{4}$	$\frac{3(d-c)}{4}$	$\frac{(d-c)^2}{8}$	$\frac{(d-c)^2}{16}$	$\frac{3(d-c)^2}{16}$

Figure 3.6.8. Summary of the results.

each $a_1 \in A_1$,

$$\mathrm{BR}_2(a_1) = \begin{cases} \frac{d-a_1-c}{2} & a_1 < d-c \\ 0 & \text{otherwise.} \end{cases}$$

Hence, in order to obtain a Stackelberg equilibrium it is enough to find a maximum of the following function:

$$f(a_1) = a_1\Big(d - a_1 - \frac{d-a_1-c}{2} - c\Big) = a_1\frac{d-a_1-c}{2}.$$

Note that

$$f'(a_1) = \frac{d-a_1-c}{2} - \frac{a_1}{2}$$

and, hence, the maximum of f is attained at $(d-c)/2$. Thus, the unique Stackelberg equilibrium is $((d-c)/2, (d-c)/4)$ and the corresponding payoffs to the players are $(d-c)^2/8$ and $(d-c)^2/16$, respectively. Figure 3.6.8 summarizes the results for the firms in a monopoly, a Cournot duopoly, and a Stackelberg duopoly. Note that, in this example, it is better for a firm to be the leader and to reveal his strategy to the other player. This is precisely the contrary of what happened in the marketing game treated in Example 3.3.3. Thus, to have information about the other's behavior in a two-player game does not necessarily increase the payoff of the player that gets the extra information. The changes in the production in the equilibria of the different settings imply that the prices decrease from monopoly to Cournot duopoly and from Cournot duopoly to Stackelberg duopoly; hence, the consumers are better off in the latter case. ◇

3.7. Repeated Games

In this section, differently from the example discussed in Section 3.6.5, we study a class of games in which we let the length of the game be infinite. We also allow for the width of the game to be infinite, though we restrict the sets of choices available to each player at each one of his information sets to be compact.

It is very common in real life to face models of repeated interaction among strategic agents. Thus, because of their wide range of applications, repeated games have been extensively studied by game theorists. Quoting Osborne and Rubinstein (1994), "the model of a repeated game is designed to examine the logic of longterm interaction. It captures the idea that a player will take into account the effect of his current behavior on the other players' future behavior, and aims to explain phenomena like cooperation, revenge, and threats." Seminal contributions to this literature are Aumann and Shapley (1976) and Rubinstein (1978, 1979). Since then, the literature on repeated games has been mainly concerned with the so-called *folk theorems*, which aim to characterize the set of payoff vectors that can be achieved by Nash or subgame perfect equilibria in the repeated game. Refer to Mailath and Samuelson (2006) for a comprehensive analysis of repeated games and their applications.

The situations we study in this section are easy to describe. There is a strategic game, which we refer to as the *stage game*, that is repeated a possibly infinite number of times. Repeated games can be easily modeled through extensive games. Yet, because of their special structure, they allow for some other representations that are easier to handle. Hereafter, to avoid confusion, we use the word actions for the strategies of the stage game and reserve the word strategies for the repeated game.

3.7.1. The assumptions. Before we introduce specifics of the model, we briefly discuss the assumptions that we make in this section. Though all of them have been widely used, they are not equally important for the results we present here. The reader that is interested in the impact of each assumption is referred to the specialized literature.

Compact sets of actions: The sets of actions of the different players in the stage game are assumed to be compact. Note that this technical assumption accounts for both finite strategic games and the corresponding mixed extensions.

Continuous payoff functions: To assume that the payoff functions of the stage game are continuous is a natural complement of the compactness assumption above. Again, finite strategic games and the corresponding mixed extensions meet this requirement.

Average discounted payoffs: The payoffs in the repeated game are essentially computed as the average of the payoffs in the different periods. Nonetheless, in order to have a common framework for finitely and infinitely repeated games, the payoffs in the repeated game are discounted so that the payoffs of the stage game are worth less as the repeated game evolves. Alternative definitions for the payoffs in the infinite horizon case are discussed in Aumann and Shapley (1976) and Rubinstein (1978, 1979).

Perfect monitoring: Each player, at the end of each period, observes the actions chosen by the others. This is the closest we can get to perfect information in a repeated game, *i.e.*, the only source of imperfect information comes from the fact that, at the beginning of each period, the players choose their actions simultaneously. Kandori (2002) presents an overview of the literature on repeated games with imperfect monitoring.

Public randomization: The players can condition their actions on the realization of an exogenous random variable, that is, they can use correlated strategies. For most of the results we present here, this assumption is without loss of generality. We briefly discuss public randomization in Remark 3.7.3.

Observability of mixed actions: During this section, our analysis is more natural if we think of the stage game as a game with pure actions. If the stage game corresponds with the mixed extension of some strategic game, and a player chooses a mixed action at some period, then perfect monitoring implies that the players can observe not only the pure action resulting from the randomization but the random process itself. In the latter setting this assumption is not natural but, for the folk theorems below, it can be dispensed with. Yet, the corresponding proofs become more involved and do not fall within the scope of this brief introduction to the topic. This assumption is discussed (and dropped), for instance, in Fudenberg and Maskin (1986) and Gossner (1995).

3.7.2. The formal model. The *stage game* is just a strategic game $G = (A, u)$ where , for each $i \in N$, the sets A_i are compact and the functions u_i are continuous. We need some notations and concepts related to the stage game. For each $i \in N$ and each $a \in A$, let $M_i(a_{-i}) := \max_{\hat{a}_i \in A_i}\{u_i(a_{-i}, \hat{a}_i)\}$ and let $\bar{M}_i := \max_{a \in A}\{u_i(a)\}$. For each $i \in N$, let $\underline{v}_i := \min_{a \in A}\{M_i(a_{-i})\}$ and let $a^{\underline{v}_i} \in \operatorname{argmin}_{a \in A}\{M_i(a_{-i})\}$. Note that $u_i(a^{\underline{v}_i}) = \underline{v}_i$. The vector $\underline{v} := \{\underline{v}_1, \ldots, \underline{v}_n\}$ is the *minimax payoff vector*; $\underline{v}_i$ is the maximum payoff that player i can get in the stage game provided that all the other players are trying to minimize his profits.[22] Let $F := \operatorname{conv}\{u(a) : a \in A\}$ be the set of *feasible* payoffs. We say that F is *full dimensional* if it has a nonempty interior in $\mathbb{R}^n$.

Next, given a game $G = (A, u)$ and $T \in \mathbb{N} \cup \{\infty\}$, we define the *repeated game* $G(\delta, T)$: the T-fold repetition of G with discount factor $\delta \in (0, 1]$. If $T = \infty$, we slightly abuse notation and, given $k \in \mathbb{N}$, use $\{k, \ldots, T\}$ to

[22]The minimax payoff of a player in an n-player game is related to the concepts of lower and upper value of a two-player zero-sum game (Sections 2.3 and 2.6).

denote $\mathbb{N} \cap [k, \infty)$. A *history* at period $t \in \{1, \ldots, T\}$, namely h, is defined as follows:

i) for $t = 1$, an element of $A^0 = \{*\}$, where $*$ is any element not belonging to $\bigcup_{k \in \mathbb{N}} A^k$.

ii) for $t \in \{2, \ldots, T\}$, an element of A^{t-1}.

The set of all histories is $H := \bigcup_{t=1}^{T} A^{t-1}$. Then, the strategy profiles and payoff functions of the game $G(\delta, T)$ are defined as follows.

- A strategy for player $i \in N$ is a mapping from H to A_i. Let Y_i be the set of strategies of player $i \in N$ and $Y := \prod_{i \in N} Y_i$ the set of strategy profiles. Hence, a strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ prescribes, for each player, what action to take after each possible history. Let σ be a strategy profile and let $h \in H$; then, we denote the action profile $(\sigma_1(h), \ldots, \sigma_n(h))$ by $\sigma(h)$. A strategy profile σ recursively determines the sequence of action profiles $\pi(\sigma) = (\pi^1(\sigma), \ldots, \pi^T(\sigma)) \in A^T$ as follows: $\pi^1(\sigma) := \sigma(*)$ and, for each $t \in \{2, \ldots, T\}$, $\pi^t(\sigma) := \sigma(\pi^1(\sigma), \ldots, \pi^{t-1}(\sigma))$. We refer to $\pi(\sigma)$ as the *path* determined by σ.
- The payoff function u^δ is defined as follows. Let σ be a strategy profile. Then, when $\delta < 1$, player i's payoff in $G(\delta, T)$ is his average discounted period payoff:

$$u_i^\delta(\sigma) := \frac{1-\delta}{1-\delta^T} \sum_{t=1}^{T} \delta^{t-1} u_i(\pi^t(\sigma)).$$

  If $T < \infty$ and $\delta = 1$, *i.e.*, if we have finite horizon and undiscounted payoffs, then we have the standard average: $u_i^1(\sigma) := (1/T) \sum_{t=1}^{T} u_i(\pi^t(\sigma)) = \lim_{\delta \to 1} u_i^\delta(\sigma)$. When working with infinitely repeated games, there are some problems to defining the average of the payoffs. For instance, we cannot simply define the payoffs when $T = \infty$ as the limit of the average payoffs when $T \to \infty$, since $\lim_{T\to\infty} (1/T) \sum_{t=1}^{T} u_i(\pi^t(\sigma))$ does not always exist. Hence, whenever $T = \infty$, we assume that $\delta < 1$.

Remark 3.7.1. The use of discounted payoffs in a repeated game can be motivated in two different ways. First, the usual motivation consists of thinking of the players as being impatient and, hence, they are more concerned with their payoffs in the early stages of the game. The idea of the second motivation is to think of $1 - \delta$ as the probability that the game stops at each period $t < T$. Although in our context both approaches are strategically equivalent, they may lead to different interpretations. For instance, when $T = \infty$, under the impatience approach, the game is played for an infinite number of periods, whereas under the second motivation, the game stops after a finite number of periods with probability one.

Remark 3.7.2. When dealing with finitely repeated games, Kuhn theorem ensures that there is no loss of generality in restricting attention to behavior strategies. The same equivalence result holds for infinite games but, although the proof is conceptually similar, many technical issues arise. We refer the reader to Aumann (1964) and Mertens et al. (1994, Chapter 2) for detailed discussions and proofs.

Remark 3.7.3. We briefly explain what the role of the public randomization assumption is. Suppose that we want to get some $v \in F$ as the payoff of the players in the repeated game. Ideally, if there is $a \in A$ such that $u(a) = v$, then we can play a in every period and we are done. In general, v might be a convex combination of the payoffs of several profiles in A and things become more complicated, mainly because of the discount factor. Once public randomization is assumed, the players can correlate their pure strategies and choose, at each period, each of the different profiles that lead to v with the required probability so that the expected payoff at each period equals v. Nonetheless, even without public randomization, if δ is big enough and the game is repeated long enough, then the players can get as close to v as desired just by alternating the different profiles that lead to v with the appropriate frequencies.[23] Due to this approximation procedure, the assumption of public randomization is almost without loss of generality in many folk theorems. We refer the reader to Fudenberg and Maskin (1991) and Olszewski (1997) for a formal treatment on the dispensability of public randomization.

Although the definition of Nash equilibrium for a repeated game is straightforward, one can make use of the special structure of a repeated game to define subgame perfect equilibrium. We present both definitions below.

Definition 3.7.1. Let $G(\delta, T)$ be a repeated game. A Nash equilibrium of $G(\delta, T)$ is a strategy profile σ^* such that, for each $i \in N$ and each $\hat{\sigma}_i \in Y_i$,

$$u_i^\delta(\sigma^*) \geq u_i^\delta(\sigma^*_{-i}, \hat{\sigma}_i).$$

[23]Consider, for instance, the following construction. Let $v \in F$. Since v is feasible, for each $\varepsilon > 0$ there is a vector of action profiles $(a^1, \ldots, a^k)$ and a vector of weights $(p^1, \ldots, p^k) \in \mathbb{Q}^k$ such that $\|\sum_{l=1}^k p^l u(a^l) - v\| < \varepsilon$. Let $Q^1, \ldots, Q^k \in \mathbb{N}$ be such that, for each par $l, \hat{l} \in \{1, \ldots, k\}$, if $p^{\hat{l}} = 0$, then $Q^{\hat{l}} = 0$ and, otherwise, $Q^l / Q^{\hat{l}} = p^l / p^{\hat{l}}$. Let $T = \sum_{l=1}^k Q^l$. Then, let σ be the strategy profile of $G(1, T)$ that consists of playing a^1 during the first Q^1 periods, a^2 during the following Q^2, and so on. Then, $\|u^1(\sigma) - v\| < \varepsilon$. Finally, since u^δ is continuous in δ, there is $\delta_0 \in (0,1)$ such that, for each $\delta \in (\delta_0, 1]$, $\|u^\delta(\sigma) - v\| < \varepsilon$.

Next, we define subgame perfection. In a repeated game, a subgame begins after each period. Let $G(\delta, T)$ be a repeated game and let σ be a strategy profile. Given $h \in H$, a new subgame that begins after history h, which, moreover, is a new repeated game that we denote by $G(\delta, T\,|h)$. Given a strategy profile σ, its restriction to the subgame $G(\delta, T\,|h)$, $\sigma_{|h}$, is defined in the natural way.

Definition 3.7.2. Let $G(\delta, T)$ be a repeated game. A subgame perfect equilibrium of $G(\delta, T)$ is a strategy profile σ such that, for each $h \in H$, $\sigma_{|h}$ is a Nash equilibrium of $G(\delta, T\,|h)$.

We might try to further refine subgame perfection for repeated games. Nonetheless, under perfect monitoring, we do not gain much by looking for more restrictive refinements. When we discussed the weaknesses of subgame perfection in Section 3.4.2, we argued that a subgame perfect equilibrium might fail to make sensible choices at information sets that are not in the equilibrium path of any subgame. Yet, in a repeated game, for each information set, there is a subgame in which that information set is reached (regardless of the strategy profile being played) and, hence, the above weakness of subgame perfection disappears. Furthermore, suppose that the stage game is the mixed extension of some finite game so that sequential equilibrium is well defined. Then, at each information set, each player knows all the past history of the game and his only uncertainty concerns the strategies chosen by his opponents at the present period. Hence, the players can just use the strategy at hand to calculate the probabilities of the different nodes in their information sets. Therefore, systems of beliefs have no bite and sequential equilibrium does not add anything to subgame perfection in repeated games with perfect monitoring. Proposition 3.7.1 formalizes the above arguments.[24]

Proposition 3.7.1. *Let $G(\delta, T)$ be a repeated game. Assume that the stage game is the mixed extension of a finite game. Then, the strategy profile σ is a subgame perfect equilibrium if and only if it is (part of) a sequential equilibrium.*

Proof.[25] By Proposition 3.5.1, every sequential equilibrium induces a subgame perfect equilibrium. Now, we prove that the converse is true in this context. Let σ be a subgame perfect equilibrium of $G(\delta, T)$. There is some system of beliefs μ such that (σ, μ) is a consistent assessment. We claim that every such assessment is also sequentially rational, *i.e.*, for each $i \in N$

[24] Refer to González-Pimienta and Litan (2008) for general conditions under which sequential equilibrium and subgame perfection are equivalent.

[25] Though we are aware that it is not a good practice, in this proof we mix the standard notation for extensive games with the specific notation for repeated games. We think that the loss in rigor is compensated with the extra clarity.

and each $w \in W_i$, σ_i is a best reply against (σ,μ) at w. Let $i \in N$ and $w \in W_i$. There is $h \in H$ such that w is in the path of $\sigma_{|h}$ in the subgame $G(\delta, T\,|h)$. Since σ is a subgame perfect equilibrium, $\sigma_{|h}$ is a Nash equilibrium of $G(\delta, T\,|h)$. Then, by Remark 3.3.1, $(\sigma_{|h})_i$ is a best reply against $\sigma_{|h}$ at w. Hence, σ_i is a best reply against (σ,μ) at w. □

We want to study the set of Nash and subgame perfect equilibrium payoff vectors of a repeated game. Each player i knows that, no matter what the others do, he can get at least $\underline{v}_i$ if he responds optimally; thus, in any equilibrium, each player must receive, at least, his minimax payoff. Owing to this, the payoff vectors $v \in F$ such that $v > \underline{v}$ are called (strictly) *individually rational*. Let $\bar{F} := \{v \in F : v > \underline{v}\}$ denote the set of feasible and (strictly) individually rational payoff vectors. A strategy profile of the repeated game that consists of playing equilibria of the stage game in every period is a subgame perfect equilibrium of the repeated game. One of the objectives of the literature on repeated games is to explore to what extent the set of equilibrium payoff vectors of the stage game can be enlarged in the repeated setting. The folk theorems look for conditions on the stage game that ensure that every payoff in $\bar{F}$ can be achieved by Nash or subgame perfect equilibria in the repeated game. These conditions depend not only on the equilibrium concept at hand, but also on whether the horizon is finite or infinite. Example 3.7.1 shows that the results might change drastically when we move from finite to infinite horizon.

Example 3.7.1. (The repeated prisoner's dilemma). We already know that the only Nash equilibrium of the prisoner's dilemma game (Example 2.1.1) is the action profile in which both players defect. Now, consider the repeated prisoner's dilemma, $G(\delta, T)$. In the infinite-horizon case, *i.e.*, $T = \infty$, consider the strategy profile in which no player defects as far as no one has defected in the past and, as soon as one of the players defects at one stage, then both players will defect forever. Formally, let σ be a strategy profile such that, for each $i \in N$ and each $h \in H$, $\sigma_i(h) = D$ if there is some period in which (ND, ND) has not been played and, otherwise, $\sigma_i(h) = ND$. So defined, the strategy profile σ is a subgame perfect equilibrium in which both players get payoff $(-1, -1)$. Yet, if $T \in \mathbb{N}$ there is no way in which an outcome different from $(-10, -10)$ can be achieved in equilibrium. The idea follows a simple backward induction reasoning. For a strategy profile to be a Nash equilibrium of a finitely repeated game, the actions chosen in the last period have to be a Nash equilibrium of the stage game. Hence, in the last period both players have to defect, which is the only equilibrium of the stage game. Provided that both players anticipate that the opponent is going to defect in the last period, it is strictly dominant for them to defect in the next to last period. Thus, both players defect in the last two periods. If we repeat this reasoning backward to the beginning, we get that in any Nash equilibrium of the finitely repeated prisoner's dilemma, on the equilibrium path, both players defect in every period; no matter how big T might be. Actually, it is easy to see that this repeated game has a unique subgame perfect equilibrium in which both players defect at the start of every subgame. $\diamond$

The key difference between the finite and infinite horizon, which drives the gap between the results in the two settings, is the existence of a last period. Regardless of the interpretation given to the discounted payoffs in the game, either as modeling the impatience of the players or as a probability of stopping the game at each period, under finite horizon, if period T is reached, both players know that this is the last time they will have to choose an action and there is no room for future punishments. Therefore, in any equilibrium of a finitely repeated game, a Nash equilibrium of the stage game has to be played at period T.

Remark 3.7.4. It is worth noting that many experiments have shown that, quite often, both players "cooperate" in the earlier stages of the repeated prisoner's dilemma.[26] These kind of departures from the unique Nash equilibrium of the game will typically lead to higher payoffs for both players in the repeated game, which might raise some doubts on the adequacy of game theory to model these repeated interactions. This situation is similar to the one described below: the chain store paradox. In Section 4.3 we revisit these situations and, by slightly enriching the model, we argue why game theory can actually explain the observed behavior.

Example 3.7.2. (The chain store paradox (Selten 1978, Kreps and Wilson 1982a)) Consider the following variation of the chain store game (Example 3.3.5). Suppose that the chain store game is to be played a finite number of times (periods). Now, the monopolist faces one different entrant in each period and each entrant observes what has been played in earlier stages. The payoff of each entrant is his stage game payoff, whereas the payoff of the monopolist is the undiscounted sum of the payoffs obtained in the different periods. Note that this repeated interaction is not exactly a repeated game in the sense defined in this chapter, since we are repeating an extensive game and, more importantly, the set of players changes in each period.

It is natural to think that, in this repeated environment, the monopolist can decide to fight early entrants in order to build a reputation and deter later entries. In doing so, the short run loss experienced by fighting early entrants might be compensated by the lack of competitors in the long run.

[26]See Axelrod (1981) and Smale (1980) for references.

Yet, although the strategy in which all entrants stay out of the competition and the monopolist fights whenever he is given the chance to play is a Nash equilibrium, it fails to be subgame perfect because of the same kind of backward induction reasoning presented above for the repeated prisoner's dilemma. In a subgame perfect equilibrium, upon entry in the last period, the monopolist will choose his best reply yield. Hence, the last entrant would definitely enter the market. Thus, the monopolist has no incentive to build a reputation in the next to last period and, in case of entrance, he will also yield there. The penultimate entrant, realizing this, will enter the market as well. This reasoning can be repeated all the way to period 1. Therefore, in the unique subgame perfect equilibrium of this game, all the entrants will enter and the monopolist will yield every single time.

This was named "the chain store paradox" because the model fails to account for the possibility of the monopolist building a reputation by fighting early entrants, which seems to be a reasonable way to play. In Section 4.3 we show how, by slightly enriching the model and introducing a small amount of incomplete information, the reputation effect arises in equilibrium. $\diamond$

3.7.3. The folk theorems. Our first result is the classic folk theorem, which deals with infinitely repeated games and Nash equilibrium. For many years this result was well known by game theorists but no formal statement or proof had been published, *i.e.*, this result belonged to the game theory folklore and, thus, it received the name "folk theorem", which was also passed on to its sequels and extensions.

The proofs of most of the results in this section are not as detailed as those in the rest of the book. We have tried to develop these proofs in order that the intuitions behind them become clear, but we have omitted most of the algebra.

Theorem 3.7.2 (Infinite horizon Nash folk theorem). *Let $G(\delta,\infty)$ be an infinitely repeated game. Then, for each $v \in \bar{F}$, there is $\delta^0 \in (0,1)$ such that, for each $\delta \in [\delta^0, 1)$, there is a Nash equilibrium of $G(\delta,\infty)$ with payoff v.*

Proof. Let a be a (possibly correlated) action profile such that $u(a) = v$. Let σ be a strategy profile defined, for each $i \in N$, as follows:

A) Main path: Play a_i. If one and only one player j does not play a_j at some period, go to B).

B) Minimax phase: Play according to a^{v_j} thereafter.

Upon the observation of a multilateral deviation, the players ignore it, *i.e.*, they keep playing according to a_i. Note that $\pi(\sigma) = (a, a, \ldots)$. If δ is close enough to 1, then any short run gain a player may get by deviating at some period would be completely wiped out because of the minimaxing periods thereafter. Hence, it is straightforward to check that σ is a Nash equilibrium of $G(\delta, \infty)$. $\square$

From the discussion of Example 3.7.1 above, we already know that we cannot expect the same result to be true for finite horizon, *i.e.*, that some assumptions on the stage game are needed for the Nash folk theorem with finite horizon.

Theorem 3.7.3 (Finite horizon Nash folk theorem; adapted from Benoît and Krishna (1987)[27]). *Let $G(\delta, T)$ be a finitely repeated game. Suppose that, for each player $i \in N$, there is a Nash equilibrium e^i of G such that $u_i(e^i) > \underline{v}_i$. Then, for each $v \in \bar{F}$ and each $\varepsilon > 0$, there are $\delta^0 \in (0,1)$ and $T^0 \in \mathbb{N}$ such that, for each $\delta \in [\delta_0, 1]$ and each $T \geq T^0$, there is a Nash equilibrium of $G(\delta, T)$ whose payoff vector $\hat{v}$ satisfies $\|\hat{v} - v\| < \varepsilon$.*[28]

Proof. Let a be a (possibly correlated) action profile such that $u(a) = v$. We claim that there are $\delta^0 \in (0,1)$ and natural numbers $T^0, T^1, \ldots, T^n$ such that there is a Nash equilibrium of $G(\delta^0, T^0)$ with path $\bar{\pi} \in A^{T^0}$, where

$$\bar{\pi} := (\underbrace{a, \ldots, a}_{T^0 - \sum_{i=1}^n T^i}, \underbrace{e^1, \ldots, e^1}_{T^1}, \underbrace{e^2, \ldots, e^2}_{T^2}, \ldots, \underbrace{e^n, \ldots, e^n}_{T^n} \}.$$

Let σ be a strategy profile defined, for each $i \in N$, as follows:

A) Main path: Play according to $\bar{\pi}$. If one and only one player j deviates from $\bar{\pi}$ at some period, go to B).

B) Minimax phase: Play according to $a^{\underline{v}_j}$ thereafter.

Upon the observation of a multilateral deviation, all the players ignore it. Note that $\pi(\sigma) = \bar{\pi}$. For each $i \in N$, since $u_i(e^i) > \underline{v}_i$, we can define T^i so that $\bar{M}_i + T^i \underline{v}_i < v + T^i u_i(e^i)$. For each $\varepsilon > 0$, we can define T^0 so that $\|u^1(\sigma) - v\| < \varepsilon$. Now, let $\delta^0 \in (0,1)$ be such that, for each $\delta \in [\delta^0, 1]$, the previous $n+1$ inequalities also hold (with u^δ instead of u^1 in the last one). So defined, it is straightforward to check that, for each $\delta \in [\delta^0, 1]$, σ is a Nash equilibrium of $G(\delta, T^0)$ whose payoff vector is within ε of v. Finally, for each $T > T^0$ we just need to modify σ so that a is played during $T - T^0$ more periods at the beginning of the game. The new profile is a Nash equilibrium of $G(\delta, T)$ within ε of v. $\square$

[27]The version of this result in Benoît and Krishna (1987) is for average undiscounted payoffs ($\delta = 1$) and assumes neither public randomization nor observability of mixed actions.

[28]Note that a game with a unique Nash equilibrium whose payoff vector is strictly greater than $(\underline{v}_1, \ldots, \underline{v}_n)$ would meet the requirements of this folk theorem.

Strategy profiles like the ones used in the proofs of Theorems 3.7.2 and 3.7.3 are known as *grim trigger strategies*; as soon as one player deviates, he is minimaxed forever. Although strategies of this kind lead to Nash equilibria, it might well be the case that minimaxing a deviating player is very costly for some players; thus, the credibility of these punishments is very limited. To ensure that we get more reasonable equilibria, we move to subgame perfection, where the above trigger strategies are not so useful. Instead, for the punishments to be credible threats, we use strategies in which, after an initial phase of periods in which the deviating player is minimaxed, all the other players are rewarded for having conformed during the minimaxing phase.

The proofs of the next two theorems are notably more involved than the previous ones, so more effort is needed for the understanding of the underlying intuitions.

Theorem 3.7.4 (Infinite horizon perfect folk theorem; adapted from Fudenberg and Maskin (1986)[29])**.** *Let $G(\delta,\infty)$ be an infinitely repeated game. If F is full dimensional, then, for each $v \in \bar{F}$, there is $\delta^0 \in (0,1)$ such that, for each $\delta \in [\delta^0,1)$, there is a subgame perfect equilibrium of $G(\delta,\infty)$ with payoff v.*

Proof. Let a be a (possibly correlated) action profile such that $u(a) = v$. The construction we present below relies heavily on the full dimensionality assumption. Let $\tilde{v}$ be a payoff in the interior of $\bar{F}$ such that $\tilde{v} < v$. For each $i \in N$, let $\lambda > 0$ be such that

$$(\tilde{v}_1+\lambda,\ldots,\tilde{v}_{i-1}+\lambda,\tilde{v}_i,\tilde{v}_{i+1}+\lambda,\ldots,\tilde{v}_n+\lambda)$$

lies in $\bar{F}$ and let $\tilde{a}^i \in A$ be a (possibly correlated) action profile that realizes the above payoffs. Let $t_i \in \mathbb{N}$ be such that

$$\bar{M}_i + t_i \underline{v}_i < (t_i+1)\tilde{v}_i.$$

Let σ be a strategy profile defined, for each $i \in N$, as follows:

A) Main path: Play a_i. If one and only one player j does not play a_j at some period, go to B).

B) Minimax phase: Play according to $a^{\underline{v}_j}$ during t_j periods. Then, go to C). If one and only one player k, with $k \neq j$, deviates, restart B) with k instead of j.

C) Reward phase: Play according to $\tilde{a}^j$ thereafter. If one and only one player k deviates, go to B) with k instead of j.

[29] The version of this result in Fudenberg and Maskin (1986) does not require observability of mixed actions and the role of public randomization is minimal. Actually, public randomization is removed altogether in Fudenberg and Maskin (1991).

Moreover, multilateral deviations are ignored. Note that $\pi(\sigma) = (a, a, \ldots)$ and $u^\delta(\sigma) = v$. To check that there is $\delta^0 \in (0,1)$ such that, for each $\delta \in [\delta^0, 1)$, σ is a subgame perfect equilibrium of $G(\delta, \infty)$ is now a simple exercise that requires some algebra. $\square$

Remark 3.7.5. One may wonder why the reward phase is needed in the above proof. Suppose that we remove it. Suppose as well that, during the minimax phase of player j, a player $k \neq j$ deviates. Note that the payoff player k gets when minimaxing player j may be below his own minimax payoff. Hence, to punish k's deviation, it does not suffice to play $a^{\underline{v}_k}$ during t_k periods, since we not only have to ensure that the one period benefits of k's deviation are wiped out, but also the gain he gets during some periods by receiving the minimax payoff instead of a lower one must be accounted for. Further deviations may require longer and longer punishments, getting unboundedly long minimaxing phases as nested deviations take place. Therefore, regardless of the discount factor, there will always be a subgame in which the punishment phase is so long that the incentives to punish cannot be sustained.

Theorem 3.7.5 (Finite horizon perfect folk theorem; adapted from Benoît and Krishna (1985)[30]). *Let $G(\delta, T)$ be a finitely repeated game. Suppose that F is full dimensional and that, for each player $i \in N$, there are two Nash equilibria e^i and $\tilde{e}^i$ of G such that $u_i(e^i) > u_i(\tilde{e}^i)$. Then, for each $v \in \bar{F}$ and each $\varepsilon > 0$, there are $\delta^0 \in (0,1)$ and $T^0 \in \mathbb{N}$ such that, for each $\delta \in [\delta_0, 1]$ and each $T \geq T^0$, there is a subgame perfect equilibrium of $G(\delta, T)$ whose payoff vector $\hat{v}$ satisfies $\|\hat{v} - v\| < \varepsilon$.*[31]

Proof.[32] Let a be a (possibly correlated) action profile such that $u(a) = v$. As in Theorem 3.7.4, let $\tilde{v}$ be a payoff vector in the interior of $\bar{F}$ such that $\tilde{v} < v$. For each $i \in N$, let $\lambda > 0$ be such that

$$(\tilde{v}_1 + \lambda, \ldots, \tilde{v}_{i-1} + \lambda, \tilde{v}_i, \tilde{v}_{i+1} + \lambda, \ldots, \tilde{v}_n + \lambda)$$

lies in $\bar{F}$ and let $\tilde{a}^i \in A$ be an action profile that realizes the above payoffs. Let $t \in \mathbb{N}$ be such that, for each $i \in \mathbb{N}$,

$$\bar{M}_i + t\underline{v}_i < (t+1)\tilde{v}_i.$$

[30]The version of this result in Benoît and Krishna (1985) does not require public randomization. Although they assumed observability of mixed actions, this assumption was later dropped in Gossner (1995).

[31]Note that a game with two Nash equilibria in which the payoff vector of one is strictly greater than the payoff vector of the other would meet the requirements of this folk theorem.

[32]We follow the proof of the folk theorem in Smith (1995) rather than the one in Benoît and Krishna (1985) where public randomization was not assumed.

Moreover, let $r \in \mathbb{N}$ be such that, for each $i \in \mathbb{N}$, each $j \neq i$, and each $\hat{t} \in \{1, \dots, t\}$,

$$\bar{M}_i + r\tilde{v}_i + (t - \hat{t})v_i < (t - \hat{t} + 1)u_i(a^{v_j}) + r\tilde{v}_j.$$

Now, we claim that there is $\delta^0 \in (0,1)$ and there are natural numbers $T^0, T^1, \dots, T^n$ such that there is a subgame perfect equilibrium of $G(\delta^0, T^0)$ with path $\bar{\pi} \in A^{T^0}$, where

$$\bar{\pi} := (\underbrace{a, \dots, a}_{T^0 - \sum_{i=1}^n T^i}, \underbrace{e^1, \dots, e^1}_{T^1}, \underbrace{e^2, \dots, e^2}_{T^2}, \dots, \underbrace{e^n, \dots, e^n}_{T^n} \}.$$

Let σ be a strategy profile defined, for each $i \in N$, as follows. A deviation that takes place during the first $T^0 - \sum_{i=1}^n T^i - (t+r)$ periods is referred to as an *early* deviation. All others are *late* deviations.

A) Main path: Play according to $\bar{\pi}$. If one and only one player j deviates early from $\bar{\pi}$, go to B); if he deviates late, go to D).

B) Minimax phase: Play according to a^{v_j} for t periods. Then, go to C). If one and only one player k, with $k \neq j$, deviates, start C) with k instead of j.

C) Reward phase: Play according to $\tilde{a}^j$ for r periods.[33] If one and only one player k deviates early, go to B) with k instead of j; if he deviates late, go to D) with k instead of j. If no one deviates, return to A).

D) Bad Nash phase: Play $\tilde{e}^j$ until there are only $\sum_{i=1}^n T^i$ periods left. Then, play as in A) but with the $\tilde{e}^l$ equilibria instead of the e^l ones.

Multilateral deviations are ignored. Note that $\pi(\sigma) = \bar{\pi}$. For each $i \in N$, since $u_i(e^i) > u_i(\tilde{e}^i)$, we can define T^i so that late deviations are not profitable. The definitions of T^0 and δ^0 follow similar lines to the corresponding ones in the proof of Theorem 3.7.3. The rest of the proof is just a matter of basic algebra. □

Remark 3.7.6. The folk theorems we have stated in this section present sufficient conditions for the payoff vectors in $\bar{F}$ to be achievable in equilibrium. Actually, there are more refined versions of these folk theorems that present conditions that are both necessary and sufficient. They are developed in Abreu et al. (1994), Wen (1994), Smith (1995), and González-Díaz (2006).

[33] The natural number r has been defined to ensure that no player has incentives to deviate when a given player j is being minimaxed, *i.e.*, if some player $k \neq j$ deviates, then the reward phase is long enough so that the gains k achieved by deviating are overcome by the losses because $\tilde{a}_k$ is played in the reward phase instead of $\tilde{a}_j$.

Exercises of Chapter 3

3.1. Consider the extensive game depicted in Figure 3.5.4 (b).

(a) Consider the mixed strategy profile (s_1, s_2), where $s_1(UL_1) = 1/4$, $s_1(UR_1) = 1/2$, $s_1(NUL_1) = 1/4$, and $s_2(L_2) = 1$. Calculate $u(s)$. Obtain a behavior strategy profile $b = (b_1, b_2)$ such that for each $i \in \{1,2\}$, s_i and b_i are realization equivalent. Obtain a system of beliefs μ such that the assessment (b, μ) is weakly consistent with Bayes rule. Is this assessment sequentially rational?

(b) Consider the behavior strategy profile (b_1, b_2), where $b_1(U) = 3/4$, $b_1(NU) = 1/4$, $b_1(L_1) = 1/2$, $b_1(R_1) = 1/2$, $b_2(L_2) = 2/3$, and $b_2(R_2) = 1/3$. Obtain $p(\hat{x}_2, b)$, $p(w_2, b)$, and a system of beliefs μ such that (b, μ) is weakly consistent with Bayes rule. Is this assessment sequentially rational?

3.2. Show formally that weak perfect Bayesian equilibrium implies Nash equilibrium and that the converse is not true.

3.3. We showed in Example 3.5.1 that weak perfect Bayesian equilibrium does not imply subgame perfect equilibrium. Show that subgame perfect equilibrium does not imply weak perfect Bayesian equilibrium either.

3.4. Show that the imperfect recall game depicted in Figure 3.7.1 does not have any Nash equilibrium (this game has been taken from Myerson (1991)).

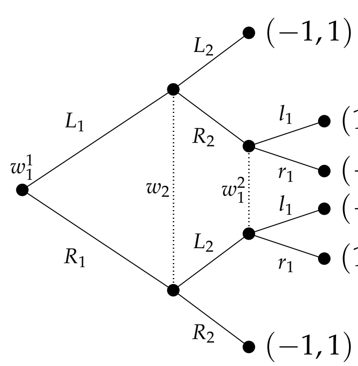


Figure 3.7.1. The game of Exercise 3.4.

3.5. Show that the shortcomings of the sequential equilibrium illustrated in Example 3.5.3 cannot be overcome by using the perfect equilibrium concept.

3.6. Find the set of perfect equilibria of the game of Figure 3.7.2 (this extensive game has been taken from Selten (1975)).

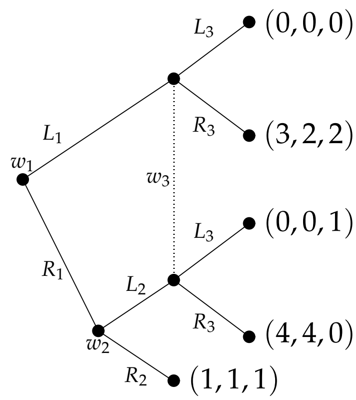


Figure 3.7.2. The game of Exercise 3.6.

3.7. Consider the game in Figure 3.6.4 (a). Since $A\Gamma$ and Γ coincide, there is only one possible definition for both perfect and proper equilibrium. Show the following:
(a) If $\alpha < 2$, then (R_1, R_2) is a perfect equilibrium.
(b) If $\alpha \leq 1$, then (R_1, R_2) is not a proper equilibrium.
(c) If $\alpha \in (1, 2)$, then (R_1, R_2) is a proper equilibrium.

3.8. Show that the unique limit behavior strategy induced by some proper equilibrium of the extensive game of Example 3.6.5 is the one in which player 1 chooses L_1 and r_1 at information sets w_1^1 and w_1^2, respectively, and player 2 plays L_2.

3.9. Show that we cannot replace sequential equilibrium with perfect equilibrium in Proposition 3.7.1.

3.10. Consider the game battle of the sexes with payoffs given in Figure 2.5.1 as a stage game that is repeated twice and payoffs are undiscounted.
(a) Represent the situation as an extensive game.
(b) Describe the set of Nash equilibria and the set of subgame perfect equilibria.

3.11. Consider the infinitely repeated game with the discounting criterion of the stage game depicted in Figure 2.11.1.
(a) Obtain the set of individually rational payoff vectors.
(b) Is $(6, 6)$ a Nash equilibrium payoff of the repeated game? If so, describe one such equilibrium.

3.12. Consider the infinitely repeated game with the discounting criterion whose stage game is the game depicted in Figure 3.7.3.

(a) Find the set of discount factors for which $(3,2)$ is a Nash equilibrium payoff of the repeated game.
(b) Find the set of discount factors for which $(3,2)$ is a subgame perfect equilibrium payoff of the repeated game.

	L	R
U	0,1	0,1
D	2,4	3,2

Figure 3.7.3. The game of Exercise 3.12.

3.13. Consider three companies that choose the number of units of a certain good they bring to the market. For each $i \in \{1,2,3\}$, firm i chooses q_i and faces production cost cq_i, for some $c > 0$; $q = q_1 + q_2 + q_3$ is the total production. There is $d > 0$ such that, if $q < d$, the price of a unit of the good in the market is $p = d - q$ and, in any other case, $p = 0$. The firms choose the quantity they produce as follows: firm 1 chooses its production first and then firms 2 and 3 observe the choice and, simultaneously, choose their level of production.
(a) Represent this situation as an extensive game.
(b) Find the set of subgame perfect equilibria. Compare the equilibrium payoff vectors and equilibrium prices with those obtained when all the firms simultaneously choose their production levels.

Chapter 4

Games with Incomplete Information

4.1. Incomplete Information: Introduction and Modeling

A game with incomplete information is one in which some important feature of the game is unknown to some of the players: a player might lack some information about the set of players, the sets of actions, or the payoff functions.[1] Note the difference between incomplete and imperfect information: in the latter, all the elements of the (extensive) game are commonly known, but a player may imperfectly observe the choices of the other players along the game tree. To study games with incomplete information we first need to find the appropriate mathematical model to describe them. We follow the approach initiated by Harsanyi (1967-68), where he showed that both strategic and extensive games with incomplete information can be represented as extensive games with imperfect information. The importance of Harsanyi's approach was acknowledged with the award of the Nobel Prize in Economics in 1994 (shared with John Nash and Reinhard Selten).

The richness and generality of games with incomplete information is such that virtually any strategic interaction can be accommodated within this approach. However, often the challenge when dealing with incomplete

[1]The reader should be aware that the exposition in this chapter slightly departs from the style in the rest of the book. Typically, the analysis of games with incomplete information is substantially involved and, therefore, for the sake of exposition, concise arguments and concise proofs are sometimes replaced by more discursive reasonings.

information is to find games that are faithful representations of the strategic interactions being modeled and, at the same time, are tractable in practice.

This chapter is structured in the following way. We start by defining the notions of Bayesian game and Bayesian Nash equilibrium, two fundamental concepts for the analysis of games with incomplete information. Then, we revisit the chain store paradox (Example 3.7.2) and show how the paradox can be better understood by introducing a bit of incomplete information into the underlying game. We also present two very important classes of games with incomplete information: auctions and mechanism design problems. Although we present them as applications of Bayesian games, it is worth noting that these classes of games are very relevant in economics and that entire monographs and books have been devoted to these topics. Near the end of the chapter we present a class of extensive games with incomplete information and compare two equilibrium concepts: sequential equilibrium and perfect Bayesian equilibrium.[2]

As usual, we denote the set of players of a game by $N := \{1, \dots, n\}$. Let the (finite) set Ω represent all the possible states of the world that are relevant for the game under study.

First of all, Harsanyi noted that, when studying games with incomplete information, it can be assumed that the only source of uncertainty comes from the payoffs of the game. Here, we do not prove the previous claim. Instead, we informally argue what the idea is. Suppose that there is incomplete information about the actions of some player. In this case, we can take an alternative representation of the game in which there is, for each player, a "universal set of actions" that includes all his potential actions; these sets are commonly known by all players. Suppose that, in the original game, there is a state of the world $\omega \in \Omega$ in which action a of player j is not present. Since player j knows what his set of available strategies at each state is, he can then distinguish those states in which a is present from those in which a is not present. Hence, it suffices to define the payoffs in the new game so that action a is strictly dominated in those states in which it was not present in the original game. On the other hand, suppose that there is uncertainty concerning the set of players. The procedure now is similar to the one above. Take a game with a "universal set of players" and add a strategy to each player saying "I am playing" or "I am not playing". Then, it suffices to define the payoffs so that they provide the appropriate incentives to the players at the different states of the world (so that "I am

[2] We do not cover in this book the epistemic foundations of the different solution concepts for incomplete information games. The interested reader may refer to Dekel et al. (2005) and Ely and Peski (2006)

not playing" is a strictly dominant strategy in those states in which they were not present in the original game).

Thus, as far as this book is concerned, a game with incomplete information is one in which some players have some uncertainty about the payoff functions; all the remaining elements of the game are common knowledge. Then, the set Ω contains the payoff-relevant states of the world. Hereafter, we assume that there is a common prior whose existence is common knowledge. As we previously argued in this book, the common prior assumption is quite natural, but its implications are not innocuous.[3]

Note that, because of its complexity, a full development of Harsanyi's approach is beyond the scope of this book. Yet, we refer the reader to Section 4.7 for an outline in which we describe the main challenges of games with incomplete information and the main ideas underlying Harsanyi's contribution.

4.2. Bayesian Games and Bayesian Nash Equilibrium

For the sake of exposition, in this section we restrict our attention to strategic games with incomplete information. All the analysis is very similar for extensive games. The concepts of Bayesian game and Bayesian equilibrium, which we introduce in this section, are general and can be defined for both strategic and extensive games with incomplete information.

The most important elements of Harsanyi's approach are the so called *type-spaces*, from which the types of the players are drawn before the game starts. The *type* of each player contains all the information related to what this player knows and, moreover, his payoff function may be contingent on his type. According to Harsanyi, a strategic game with incomplete information can be seen as a (two-stage) extensive game with imperfect information:

Nature moves: A state of the world $\omega \in \Omega$ and a type-profile $\theta \in \Theta$ are realized (with probabilities given by the common prior).

Players receive information: Each player is informed about his type and receives no further information (the type already contains the information about what he knows). In many applications of games with incomplete information and, in particular, in those we discuss in Section 4.5, it is assumed that, by knowing his type, each player knows his payoff function and nothing else, *i.e.*, he is given no further information about the payoff functions of the others. The model we present here is far more general and can also account for situations in which players have different information

[3] Refer to the discussion after Theorem 2.12.1 in Section 2.12.

about the types of the others; it can even be the case that a player does not know his own payoff function while the others do.

Players move: Each player chooses an action conditional on his own type.

Payoffs are realized: The players receive the payoffs corresponding with their types and the chosen actions at the realized state of the world.

We have not formally defined infinite extensive games and, therefore, it is convenient to assume in this section that the sets of actions, the sets of states of the world, and the sets of types are finite. However, it is important to note that the definitions of Bayesian game and Bayesian equilibrium below can be easily adapted to the infinite case; actually, in Sections 4.4 and 4.5 we discuss applications of Bayesian games with infinite sets of actions and types.

Remark 4.2.1. It is worth noting that, under Harsanyi's approach, by appropriately enriching the type-spaces of the players, the set Ω may be dispensed with. However, by preserving Ω to account for the payoff-relevant states of the world and using the types of the players to capture the information and beliefs they have, we may sometimes get a more transparent representation of the underlying incomplete information game. Nonetheless, as the reader will see in this chapter, most common applications of Bayesian games can be cleanly studied without the set Ω.

The extensive games obtained with the above construction are very special. They are simple two-stage games where the choices available to the players are the same regardless of their types, and because of this, they are often modeled through the so-called Bayesian games. The name Bayesian comes from the fact that players are assumed to use Bayesian updating to update their beliefs from the prior when they learn their types. When dealing with Bayesian games we also assume that the players have von Neumann and Morgenstern utility functions.

Definition 4.2.1. An n-player *Bayesian game* with set of players N is a 5-tuple $BG := (\Omega, \Theta, \rho, A, u)$ whose elements are the following:

The (payoff-relevant) states of the world: The elements of the finite set Ω.

The types of the players: The elements of the finite set defined by $\Theta := \prod_{i \in N} \Theta_i$.

The common prior: $\rho \in \Delta(\Theta \times \Omega)$.

The basic action profiles: The elements of the finite set defined by $A := \prod_{i=1}^{n} A_i$. A **pure strategy** of player i is a map $\hat{a}_i \colon \Theta_i \to A_i$.

Let $\hat{A}_i$ be the set of pure strategies of i and $\hat{A} := \prod_{i=1}^n \hat{A}_i$,[4] that is, each player has to choose, for each possible type he may have, what action to take.

The basic payoff functions: $u := \prod_{i=1}^n u_i$, where, for each $i \in N$, $u_i \colon \Omega \times \Theta_i \times A \to \mathbb{R}$. Now, for each $i \in N$, player i's **(Bayesian) payoff function** $\hat{u}_i \colon \hat{A} \to \mathbb{R}$ is defined, for each $\hat{a} \in \hat{A}$ as $\hat{u}_i(\hat{a}) := \sum_{(\theta,\omega)\in\Theta\times\Omega} \rho_i(\omega \,|_{\theta_i}) u_i(\omega, \theta_i, \hat{a}(\theta))$.[5]

All the elements of a Bayesian game are common knowledge. There is no incomplete information. We just have imperfect information (the different types of the players playing the role of the different information sets).

Example 4.2.1. Consider an economy with two firms. Firm 1 has the monopoly of a certain commodity in one particular country. Firm 2 is considering the possibility of entering the market. Firm 2 knows that, at the same time, firm 1 might perhaps prepare itself for an eventual duopolistic competition by making an investment to improve its distribution channels. However, firm 2 does not know whether the investment needed for this improvement is large or small. Firm 2 believes that the probability that firm 1 has to make a large investment is $p \in (0,1)$. This belief is based on objective facts that are publicly known (so p is also known by firm 1). The (publicly known) benefits for both firms in all possible scenarios are depicted in Figure 4.2.1: I and NI stand for "invest" and "not invest", respectively; E and NE stand for "enter" and "not enter", respectively; and L and S stand for "large" and "small", respectively.

L	E	NE
I	$0,-1$	$2,\ 0$
NI	$2,\ 1$	$3,\ 0$

S	E	NE
I	$3,-1$	$5,\ 0$
NI	$2,\ 1$	$3,\ 0$

Figure 4.2.1. The payoffs of the game described in Example 4.2.1.

This is a game with incomplete information, since player 2 does not know player 1's payoff function precisely. Suppose that the benefits for the players are given by von Neumann and Morgenstern utility functions. Then, this situation is a very simple one, and it can be analyzed by just making use of common sense (we do not need to use the machinery that has taken us so long to introduce). Firm 1 will choose NI if the needed

[4] Given an action profile $\hat{a} \in \hat{A}$ and a type-profile $\theta \in \Theta$, we slightly abuse notation and use $\hat{a}(\theta)$ to denote $(\hat{a}_1(\theta_1), \ldots, \hat{a}_n(\theta_n))$.

[5] In the case of infinite states of the world or infinite sets of types, the sums would have to be replaced with integrals.

investment is large, and will choose I if the needed investment is small, irrespective of whether firm 2 enters into the market or not. Hence, firm 2 will obtain $p - (1 - p) = 2p - 1$ when choosing E, and will get 0 when choosing NE. Hence, firm 2 will choose E if $p > 1/2$, NE when $p < 1/2$, and is indifferent if $p = 1/2$.

Below, we describe a Bayesian game $BG = (\Omega, \Theta, \rho, A, u)$ that corresponds with the game with incomplete information defined above.

The states of the world: $\Omega = \{L, S\}$.

The types of the players: The types of the players are $\Theta_1 = \{\theta_1\}$ and $\Theta_2 = \{\theta_2\}$. It is common knowledge that player 1 knows the state of the world that has been realized and player 2 ignores it. $\Theta = \Theta_1 \times \Theta_2$. Since the type-spaces are singletons, we can write the beliefs just in terms of Ω.

The common prior: $\rho(L) = p$ and $\rho(S) = 1 - p$. The updated beliefs are: $\rho_1(L\,|_L) = \rho_1(S\,|_S) = 1$, $\rho_1(S\,|_L) = \rho_1(L\,|_S) = 0$, $\rho_2(L\,|_L) = \rho_2(L\,|_S) = p$, $\rho_2(S\,|_L) = \rho_2(S\,|_S) = 1 - p$.

The basic action profiles: $A_1 = \{I, NI\}$ and $A_2 = \{E, NE\}$.

The basic payoff functions: They are described in Figure 4.2.1.

In this example, the set Ω could be easily dispensed by introducing a second type for player 1. The first of the two types would then be the one with payoffs associated to a large investment cost and the second type would have the payoffs corresponding with the small investment. Figure 4.2.2 contains the extensive game representation of BG, Γ^{BG} (we have chosen the representation of BG in which player 1 moves first, but we could also have chosen the equivalent one in which player 2 moved first). $\diamond$

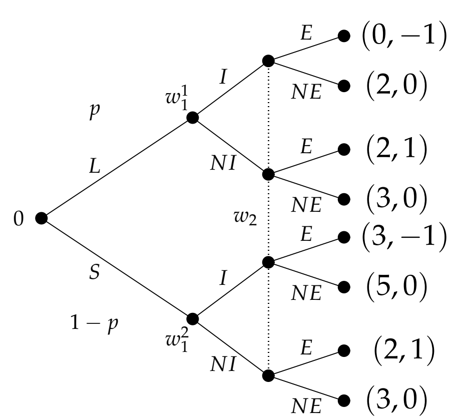


Figure 4.2.2. The extensive game with imperfect information of Example 4.2.1.

Example 4.2.2. Consider the following modification of the game in Example 4.2.1. Again, firm 1 is a monopolist, firm 2 has to decide whether to enter or not into the market, and firm 1 can make an investment to improve his distribution channels. The difference now is that no firm is necessarily informed about the cost of the investment. Yet, both firms have carried out independent studies to know the real cost of the investment. With probability p_1 (p_2) firm 1 (firm 2) knows if the cost is "large" or "small", that is, the game might be described as follows. There is a first stage where nature moves and selects L or S with probabilities p and $1-p$, respectively. Then, nature moves again and, with probability p_1 (p_2), firm 1 (firm 2) is informed about the cost of the investment (these probabilities being independent). Neither firm knows the information received by the other. Finally, the firms choose their actions.

In the corresponding Bayesian game, the set of states of the world is again $\Omega = \{L, S\}$. Now the type-spaces are different: $\Theta_1 = \{I, U\}$ and $\Theta_2 = \{I, U\}$, where I and U keep track of whether the players are informed or not about the cost of the investment. Also, note that a representation of this game without the set $\Omega = \{L, S\}$ would require us to enrich the type-space of player 1 to $\{(I, L), (I, S), (U, L), (U, S)\}$. ◇

Since a Bayesian game is simply a convenient representation of a special extensive game, we can also define behavior and mixed strategies.[6] Let $BG = (\Omega, \Theta, \rho, A, u)$ be a Bayesian game and let $i \in N$. A behavior strategy of player i is a function $b_i \colon \Theta_i \to \Delta A_i$ and a mixed strategy s_i of player i is an element of $\Delta \hat{A}_i$. The sets B_i, B, S_i, and S are defined as usual. Finally, we slightly abuse notation and use the $\hat{u}_i$ functions to denote the (expected) utilities of the players when using behavior and mixed strategies.

Given a Bayesian game BG, let Γ^{BG} be an extensive game with imperfect information defined under the approach described on page 165.[7] Then, each pure/mixed/behavior strategy in game BG induces a unique pure/mixed/behavior strategy in game Γ^{BG} in the obvious way (and *vice versa*).

Definition 4.2.2. Let $BG = (\Omega, \Theta, \rho, A, u)$ be a Bayesian game. A *Bayesian Nash equilibrium* in pure strategies is a strategy profile $\hat{a}^* \in \hat{A}$ such that, for each $i \in N$ and each $\hat{a}_i \in \hat{A}_i$,

$$\hat{u}_i(\hat{a}^*) \geq \hat{u}_i(\hat{a}^*_{-i}, \hat{a}_i).$$

[6] Bayesian games as we have just described are always extensive games with perfect recall and, hence, Kuhn theorem applies. A player learns his type and then he plays. There is nothing he can forget.

[7] Note that the game Γ^{BG} is not unique. After nature's move, in the second stage the players move simultaneously and this admits various (equivalent) representations by means of an extensive game (see Example 4.2.1).

The definitions of Bayesian Nash equilibrium in behavior strategies and in mixed strategies are analogous. By now, the next result should be quite apparent and, therefore, we leave its proof as an exercise.

Proposition 4.2.1. *Let BG be a Bayesian game.*

i) *If a pure/mixed/behavior strategy profile in BG is a Bayesian Nash equilibrium, then the induced strategy profile in* Γ^{BG} *is a Nash equilibrium.*
ii) *Conversely, if a pure/mixed/behavior strategy profile in* Γ^{BG} *is a Nash equilibrium, then the induced strategy profile in BG is a Bayesian Nash equilibrium.*

Moreover, if the common prior ρ *has full support,*[8] *then, when referring to* Γ^{BG}*, we can replace Nash equilibrium with sequential equilibrium in both i) and ii).*

Proof. Exercise 4.1. □

If the common prior does not have full support, then a Bayesian Nash equilibrium of *BG* needs not be a weak perfect Bayesian equilibrium of Γ^{BG}. To see this, consider the following situation. Suppose that there is a state $\omega \in \Omega$ that has probability 0 of being realized according to the common prior. Then, a strategy in which a player i chooses a strictly dominated action after observing state ω and a realization θ_i of his type can be part of a Nash equilibrium of Γ^{BG}. However, this equilibrium would never be weak perfect Bayesian since, no matter the beliefs player i might form after observing ω and θ_i, his action there would never be a best reply at the corresponding information set. On the other hand, provided that the common prior has full support, then every Nash equilibrium of Γ^{BG} is not only weak perfect Bayesian, but also sequential. The reason for this is that, under full support, every information set of Γ^{BG} is reached with positive probability and, hence, every Nash equilibrium is also sequential.

Corollary 4.2.2. *Every Bayesian game has at least one Bayesian Nash equilibrium in behavior strategies.*

Proof. It follows from the combination of Theorem 3.4.2, which states that every extensive game with perfect recall has, at least, a subgame perfect equilibrium, and Proposition 4.2.1 above. □

We now study the equilibria of Example 4.2.1. Exercise 4.2 asks the reader to complete the description and study the equilibria of the Bayesian game in Example 4.2.2.

[8] Since we are assuming that both Ω and Θ are finite, full support can be written as follows: for each $\omega \in \Omega$ and each $\theta \in \Theta$, $\rho(\theta, \omega) > 0$.

Example 4.2.3. Player 1 has two information sets: w_1^1 and w_1^2. Since both of them are reached with positive probability, for a strategy profile to be a Nash equilibrium of Γ^{BG}, player 1 should be best replying at both w_1^1 and w_1^2. At w_1^1, NI is a strictly dominant strategy and at w_1^2 the strictly dominant strategy is I. Player 2 has only one information set: w_2. Thus, the set of Nash equilibria of Γ^{BG} consists of all $b \in B$ such that $b_1(w_1^1) = NI$, $b_1(w_1^2) = I$, and:

$$b_2(w_2) = \begin{cases} NE & p < 1/2 \\ s_2 \in \Delta A_2 & p = 1/2 \\ E & p > 1/2. \end{cases} \qquad \diamond$$

4.3. The Chain Store Paradox in Perspective

This section has two main goals: first, we illustrate how a typical analysis of an extensive game with incomplete information might be carried out and, second, we show how the chain store paradox can be better understood when framing it in an incomplete information model. Also, the reader may be able to appreciate how a rigorous game theoretical analysis of a relatively simple strategic interaction is often far from trivial.

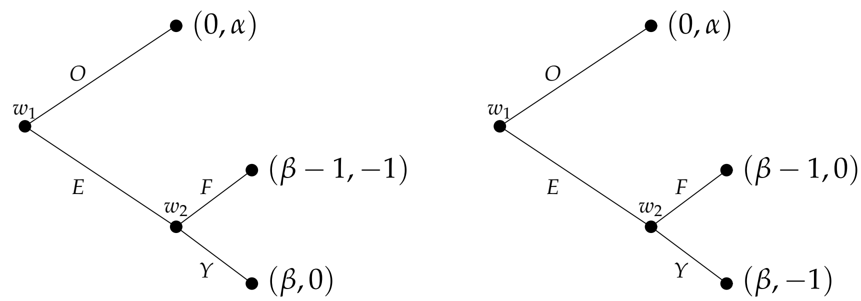
When studying repeated games in Section 3.7, we showed in Examples 3.7.1 and 3.7.2 that equilibrium strategies in the repeated prisoner's dilemma and in the repeated chain store game may be different from actual behavior observed in real-life situations or laboratory experiments. Though this may seem to undermine the potential of game theory to understand strategic interactions, in this section we show how games with incomplete information can be used to explain the behavior observed in those two settings. Since the analysis is similar for both examples, we just focus on one of them: the chain store paradox. The analysis for the repeated prisoner's dilemma can be found in Kreps et al. (1982).

In Figure 4.3.1 (a) below we again represent the chain store game as defined in Example 3.3.5. In Example 3.7.2 we considered a repeated version of this game in which the monopolist faced a (finite) succession of entrants. We argued why, in this repeated scenario, the unique subgame perfect equilibrium consists of all the entrants entering into the market and the monopolist yielding in every period. Thus, the game theoretical analysis of the perfect information model of Example 3.7.2 is unable to capture any reputation effect. Yet, we now show that this effect may come to life if the entrants perceive there is a chance that the monopolist may enjoy predatory behavior. The exposition below is based on Kreps and Wilson (1982a).[9]

[9] A highly related analysis is carried out in Milgrom and Roberts (1982).

The contention in Kreps and Wilson (1982a) is that the paradox arises because "the model does not capture a salient feature of realistic situations", which comes from the complete and perfect information formulation used in Selten (1978) when introducing the paradox.[10] In practical situations, the entrants will have some uncertainty about the monopolist's payoff function. They may be unsure about the costs of the monopolist or contemplate the possibility that the monopolist is not willing to share the market at any cost.

Based on the above considerations, we consider the following incomplete information game. There are two states of the world: S and W. The corresponding probabilities are q and $1-q$, with $q > 0$. In state W we have a *weak* monopolist, whose payoffs are the normal ones: 0 if he yields and -1 if he fights. In state S the monopolist is *strong* or tough, and his payoffs are 0 if he fights and -1 if he yields (see Figure 4.3.1 below); that is, a strong monopolist is better off by fighting. Only the monopolist knows the true state of the world. Since there is no other source of incomplete



(a) **State *W*.** Chain store game with a weak monopolist (Example 3.3.5).

(b) **State *S*.** Chain store game with a strong monopolist.

Figure 4.3.1. The payoff functions of the monopolist in the two possible states of the world. Recall that $\alpha > 1$ and $0 < \beta < 1$.

information, we do not need to specify types for the players.[11] We assume that the chain store game will be repeated T times, with a different entrant in each period. The action sets in the stage game are $A_1 := \{E, O\}$ for an entrant and $A_2 := \{Y, F\}$ for the monopolist; as usual, let $A := A_1 \times A_2$. We number the entrants according to the period in which they play, *i.e.*, entrant t is the entrant playing in period t. Let $A^0 := \{*\}$ denote the empty history and $h_t \in A^t$ denote the history of action profiles up to period t. Let

[10] Actually, the first one who made this contention was Rosenthal (1981), but he took a completely different approach to formalize this idea.

[11] Instead of having two states of nature we could have equivalently captured the incomplete information via two types for the monopolist: weak and strong.

$H := \bigcup_{t=1}^{T} A^{t-1}$ denote the set of all histories. We assume complete observability of past actions; in particular, each entrant is fully informed about the past behavior of the monopolist. Hence, when entrant t is given the chance to play, he conditions his action on the past history ($h_{t-1} \in A^{t-1}$). Since, in period t, the monopolist also knows what entrant t has played, he conditions his action on (h_{t-1}, e_t), where $h_{t-1} \in A^{t-1}$ and $e_t \in A_1$. Then, following Harsanyi's approach, the Bayesian game that corresponds with this situation can be defined as follows:

The (payoff-relevant) states of the world: $\Omega := \{S, W\}$. Only the monopolist is informed about the realized state of the world, *i.e.*, the entrants do not know the monopolist's payoff function.

The common prior: $\rho(S) := q$ and $\rho(W) := 1 - q$.

The basic action profiles: $A := A_1 \times A_2$. A **pure strategy** of entrant t is a map $\hat{e}_t \colon A^{t-1} \to A_1$. A **pure strategy** of the monopolist is a map $\hat{m} \colon \Omega \times H \times A_1 \to A_2$.

The payoff functions: The payoff of each entrant is his stage game payoff in the chain store game he plays. The payoff of the monopolist is the (undiscounted) sum of his stage game payoffs.

In Figure 4.3.2 below we represent the beginning of the corresponding extensive game with imperfect information, Γ^{CS}. Sequential equilibrium

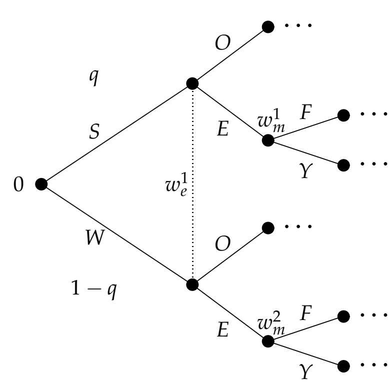


Figure 4.3.2. The beginning of the game Γ^{CS}.

seems to be an appropriate equilibrium concept for the analysis. Following the approach in the previous section, we could have chosen Nash equilibrium but, since we now have an extensive game, we would then allow for equilibria based on incredible threats.[12] To see this, just note that the strategy profile in which all the entrants stay out and the monopolist always fights (regardless of whether he is weak or strong) is a Nash equilibrium of Γ^{CS}. However, it is not even a subgame perfect equilibrium, since a weak monopolist, if given the chance to play in the last period, would yield instead of fight. Below we describe a sequential equilibrium of this game in which a strong monopolist will always fight entrants and a weak monopolist, depending on the parameters of the game, may decide to fight early entrants in order to build a reputation of being tough. Why is this possible now? *Ex ante*, all entrants think the monopolist is strong with probability $\rho(S) := q$. Suppose that entrant 2 has observed (E, F) in period 1. Then, he has to update his beliefs about the likelihood of state S using Bayes rule. Roughly speaking, since a strong monopolist always fights, having observed F in period 1 increases the likelihood of the monopolist being strong. If a weak monopolist builds enough reputation of being tough in the early periods, he may discourage future entrants from entering the market. Therefore, if the number of periods is large enough, the monopolist's losses in the early periods will be compensated by the lack of competitors in the long run. We devote the rest of this section to formalize this intuition.

The strategy profile we define has the property that the player's actions only depend on the beliefs of the players about the probability of states W and S given past history. Below we describe these beliefs and later on we show that given the strategies of the players, these beliefs can be derived using Bayes rule. Note that the monopolist has perfect information, so the beliefs only have to be defined for the entrants. Let $q_1 := q$ and, for each $t \in \{2, \dots, T\}$, let $q_t(h_{t-1})$ denote the probability entrant t attaches to the monopolist being strong after having observed history h_{t-1}. When no confusion arises, we just use q_t to denote $q_t(h_{t-1})$. In our construction q_t can be computed directly from q_{t-1}, irrespective of the specifics of the past history of play. More precisely, these probabilities can be recursively computed as follows:

i) If there is no entry at period $t-1$, then $q_t = q_{t-1}$.

ii) If there is entry at period $t-1$, the monopolist fights, and $q_{t-1} > 0$, then $q_t = \max\{\beta^{T-t+1}, q_{t-1}\}$.

iii) If there is entry at period $t-1$ and either the monopolist yields or $q_{t-1} = 0$, then $q_t = 0$.

To save notation, we write q_t instead of $q_t(h_{t-1})$, *i.e.*, q_t represents the beliefs of entrant t, computed from the past history of play as explained

[12]Put differently, the part of the statement of Proposition 4.2.1 involving sequential equilibrium does not hold anymore.

above. Now we can define the strategies of the players as functions of the q_t probabilities. We start by defining the (behavior) **strategy of the monopolist**:

- A strong monopolist always fights entry.
- Suppose that the monopolist is weak and that entrant t enters the market. Then, the strategy of the monopolist depends on t and q_t. If $t = T$, then the weak monopolist yields (no incentive to build reputation). If $t < T$ and $q_t \geq \beta^{T-t}$, the weak monopolist fights. If $t < T$ and $q_t < \beta^{T-t}$, the weak monopolist fights with probability
$$\frac{(1-\beta^{T-t})q_t}{(1-q_t)\beta^{T-t}},$$
and yields with the complementary probability.

The **strategies of the entrants**:

- If $q_t > \beta^{T-t+1}$, entrant t stays out.
- If $q_t < \beta^{T-t+1}$, entrant t enters the market.
- If $q_t = \beta^{T-t+1}$, entrant t stays out with probability $\frac{1}{\alpha}$.

Before proving that the above strategies and beliefs form a sequential equilibrium we need some preparation.

Lemma 4.3.1. *Let $t \in \{1, \ldots, T\}$.*

i) *If $q_t < \beta^{T-t+1}$, then, if the players follow the above strategies, the total expected payoff of a weak monopolist from period t onwards is 0.*

ii) *If $q_t = \beta^{T-t+1}$, then, if the players follow the above strategies, the total expected payoff of a weak monopolist from period t onwards is 1.*

Proof. The proof is made by (backward) induction on t. Suppose $t = T$ and $q_t < \beta^{T-t+1}$. Then, entrant T enters the market and the weak monopolist yields and gets payoff 0. This establishes claim i) for $t = T$. Suppose $t = T$ and $q_t = \beta^{T-t+1}$. Then, entrant T stays out with probability $1/\alpha$. If entrant T enters, then, as above, the weak monopolist yields and gets payoff 0. In this case, his overall expected payoff is $(1/\alpha)\alpha + (1-1/\alpha)0 = 1$. This establishes claim ii) for $t = T$.

Suppose that both claims are true for all $\bar{t} > t$, with $1 \leq t < T$. We show that the claims are also true for t. Suppose that $q_t < \beta^{T-t+1}$. Then, entrant t enters the market. If the weak monopolist yields, he gets 0 in the present period; also, $q_{t+1} = 0 < \beta^{T-t}$ so, by the induction hypothesis, he gets a total expected payoff of 0 from period $t+1$ onwards. If the weak monopolist fights, he gets -1 in the present period; also, $q_{t+1} = \beta^{T-t}$ and, by the induction hypothesis, he gets a total expected payoff of 1 from period $t+1$ onwards. Hence, his overall expected payoff by fighting is also 0. This establishes claim i) for t. Finally, the proof of claim ii) for t follows similar arguments and we leave it to the reader. $\square$

It is worth noting what the role was in the above proof of the randomization of the entrants when $q_t = \beta^{T-t+1}$. It ensures that, whenever $t < T$ and $q_t < \beta^{T-t}$, a weak monopolist is indifferent between Y and F. Similarly, in the proof of Proposition 4.3.2 below, we also show that the probabilities chosen for the mixing of the monopolist ensure that entrant t is indifferent between entering or staying out when $q_t = \beta^{T-t+1}$.

For each $t \in \{1, \dots, T\}$, let $u_t(q_t)$ denote the total expected payoff of the weak monopolist from periods t to T when the belief of entrant t is given by q_t and both players follow the above strategies. Let $k(q) := \min\{n \in \mathbb{N} : \beta^n < q\}$ if $q > 0$ and $k(0) = \infty$. The value $k(q)$ expresses that, given q, the monopolist will face entry for the first time when there are $k(q) - 1$ periods left, *i.e.*, the first entry will take place at period $T - k(q) + 2$.[13] Actually, if $\beta^{T-k(q)} = q$, this first entry occurs with probability $1 - 1/\alpha$. Based on the previous observations and on Lemma 4.3.1, it is not hard to see that the function u_t can be computed as follows:

$$(4.3.1) \quad u_t(q_t) = \begin{cases} \alpha(T - t - k(q_t) + 2) + 1 & \beta^{T-t+1} < q_t = \beta^{T-k(q_t)} \\ \alpha(T - t - k(q_t) + 2) & \beta^{T-t+1} < q_t < \beta^{T-k(q_t)} \\ 1 & q_t = \beta^{T-t+1} \\ 0 & q_t < \beta^{T-t+1}. \end{cases}$$

Proposition 4.3.2. *The above strategies and beliefs form a sequential equilibrium of the chain store game with incomplete information* Γ^{CS}.

Proof. We have to prove that these strategies and beliefs are consistent and sequentially rational.

Consistency of beliefs: To be completely formal and show that these beliefs are consistent in the sense of Definition 3.5.6, we should define a sequence of completely mixed strategy profiles converging to our strategy profile and such that the corresponding beliefs converge to our beliefs. Instead of doing so, we separate the analysis in two parts. We start by showing that, along the path of the strategy, the beliefs are computed using Bayes rule.[14] Then we discuss off-path beliefs. If there is no entry at period t, then nothing new is learned about the monopolist, so $q_{t+1} = q_t$. Suppose there is

[13] In particular, if $k(q) = 1$, no entry will ever occur and, if $k(q) - 1 > T$, the first entry will occur at period 1.

[14] Along the path of the strategy, any beliefs computed as limits of the beliefs associated to completely mixed strategy profiles converging to our strategy profile have the property that they can be directly computed using Bayes rule from the limit strategy.

entry at period t. If $q_t \geq \beta^{T-t}$, then the monopolist is supposed to fight, no matter whether he is weak or strong; again, nothing is learned in this case, so $q_{t+1} = q_t$. If $q_t = 0$, then the weak monopolist is supposed to yield and, once yield is played, Bayes rule again implies that $q_{t+1} = q_t$. All these cases agree with our definitions of the q_t probabilities. Now, let $q_t \in (0, \beta^{T-t})$. In this case, a weak monopolist mixes between F and Y. Therefore, if the monopolist plays Y, since a strong monopolist never yields, Bayes rule implies that $q_{t+1} = 0$. Below we denote by S and W the events in which the monopolist is strong and weak respectively. Similarly, F^t denotes the event in which the monopolist fights at period t. Then, if the monopolist fights then Bayes rule implies that

$$q_{t+1} = p(S\,|_{F^t}) = \frac{p(S \cap F^t)}{p(F^t)} = \frac{p(F^t\,|_S)p(S)}{p(F^t\,|_S)p(S) + p(F^t\,|_W)p(W)},$$

which reduces to

$$q_{t+1} = \frac{q_t}{q_t + \frac{(1-\beta^{T-t})q_t}{(1-q_t)\beta^{T-t}}(1-q_t)} = \beta^{T-t},$$

which also corresponds with our definition.

We now move to off-path beliefs, *i.e.*, beliefs at information sets where Bayes rule does not apply. There are two situations in which this can happen: i) $q_t \geq \beta^{T-t}$ and the monopolist yields after entry and ii) $q_t = 0$ and the monopolist fights. In both cases our beliefs assume that $q_{t+1} = 0$. One way to interpret these off-path beliefs is to think of the strong monopolist as being very determined in his (fighting) behavior and, therefore, all deviations are attributed to the "hesitation" of a weak monopolist. Besides, this belief is irreversible in the sense that, no matter what entrants observe later on in the game, they will always think that the monopolist is weak with probability one. In order to define the sequence of completely mixed strategy profiles whose limit beliefs would deliver our beliefs, we can just specify, for each information set in which the player is not mixing between his two choices there, the probability of the player not making the choice specified by our strategy. We refer to any such choice as a "mistake". Hence, it suffices to specify the probabilities of the mistakes in such a way that they converge to 0 and lead to the desired beliefs. The way in which we define the probabilities of entrants' mistakes is irrelevant; they never have an impact in the way future entrants update their beliefs about the monopolist. For each $n \in \mathbb{N}$, each mistake of a weak monopolist is made with probability $\frac{1}{n}$, whereas the mistakes of the strong monopolist are highly more unlikely, having probability $\frac{1}{n^2}$. We leave it up to the reader to verify that these probabilities deliver the desired beliefs.

Sequential Rationality: We start with the **incentives of the entrants**.

- If $q_t >= \beta^{T-t}$, entrant t expects entry to be fought and, therefore, stays out.
- If $\beta^{T-t+1} < q_t < \beta^{T-t}$, entrant t expects entry to be fought with probability $p(F\,|_S)p(S) + p(F\,|_W)p(W)$, which is given by

$$q_t + \frac{(1-\beta^{T-t})q_t}{(1-q_t)\beta^{T-t}}(1-q_t) > \beta^{T-t+1} + \frac{(1-\beta^{T-t})\beta^{T-t+1}}{(1-\beta^{T-t+1})\beta^{T-t}}(1-\beta^{T-t+1}) = \beta.$$

  Hence, the expected payoff after entry is less than $\beta(\beta-1)+(1-\beta)\beta = 0$. Therefore, entrant t stays out as well.
- If $q_t = \beta^{T-t+1}$, now entrant t expects entry to be fought with probability β and his expected payoff is exactly 0, so he is indifferent.
- If $q_t < \beta^{T-t+1}$, the expected payoff of entrant t by entering the market is now positive and so he enters.

The **incentives of a strong monopolist** are straightforward. Given the strategies of the entrants, yielding at any point will, if anything, increase the number of future entries. Hence, yielding at any period produces a short run loss and, possibly, a long run loss for the monopolist. Therefore, a strong monopolist is always better off by fighting. The **incentives of a weak monopolist** follow easily from the payoffs in Eq. (4.3.1). Suppose that there is entry in period t. If the monopolist yields, he gets 0 in this period and also in all future periods (because q_{t+1} becomes 0). If he fights, he gets -1 in the present period and his expected payoff in the future is: i) 0 if $q_t = 0$, ii) 1 if $0 < q_t < \beta^{T-t}$, and iii) more than 1 if $q_t > \beta^{T-t}$. Therefore, following the given strategy is always a best reply. □

Remarkably, given our equilibrium strategies, even a tiny amount of uncertainty may be enough for the reputation effect to predominate. Suppose that $\beta = 1/2$ and $q = 1/1000$. If we use the $k(\cdot)$ function defined above, we get that $k(q) = 10$ ($\beta^{10} = 1/1024 < 1/1000$). Hence, as long as there are more than 10 periods left in the game, all entrants stay out.

In Kreps and Wilson (1982a), the authors show that there are also sequential equilibria in which a strong monopolist yields. However, they also argue that these equilibria are based on implausible beliefs and show that any sequential equilibrium with plausible beliefs has to be of the same form as the one described above.

4.4. A First Application of Bayesian Games: Auctions

In this section we present a brief introduction to auction theory, a remarkable class of Bayesian games. Indeed, auction theory can be seen as a bridge between mathematical game theory and some fields in economics such as mechanism design, contract theory, and economics of information. In spite of being a relatively new field, the relevance of its applications has attracted many researchers and the literature on auctions has grown rapidly. For the reader willing to learn more about auction theory, two monographs on the topic are Krishna (2002) and Menezes and Monteiro (2005).[15] Since we only intend to give a few glimpses of the field, we restrict ourselves to very simple auctions, which can be tackled with the Bayesian games introduced in the previous section.

Informally, an auction consists of a set N of players bidding to get an object. There is $\bar{v} > 0$ such that each player has a valuation of the object that belongs to the interval $[0, \bar{v}]$; this valuation determines his payoff function. Moreover, we assume that the valuations are independently and identically distributed on $[0, \bar{v}]$ according to some distribution of probability ρ whose density is differentiable and has full support (in particular, ρ has no mass points). When submitting the bids, each player knows his valuation (type) $\theta_i \in [0, \bar{v}]$ and that the valuations of the other bidders are independently distributed according to ρ; that is, valuations are private. All the elements of the auction are common knowledge.[16] Following Harsanyi's approach, an auction can be modeled as follows:

Nature moves: There is a first stage in which nature makes its move and types are realized.

Players receive information: Each player is informed about his type and receives no further information.

Players move: Each player places his bid.

Payoffs are realized: The player with the highest bid gets the object. In case of a tie, each highest bidder receives the object with equal probability. For the payoffs, we distinguish here between two standard auctions:

- **First-price auction:** The bidder who gets the object pays his bid.
- **Second-price auction:** The bidder who gets the object pays the second highest bid.

Since each player does not know the valuations of the others, an auction is a game with incomplete information. One special feature of auctions is that, by knowing the valuation of a player, we also know his information (and *vice versa*). Therefore, when dealing with auctions, the type-spaces and the states of the world become the same thing (indeed, this also happens in

[15]The proofs of the results in this section follow the arguments of the corresponding results in Krishna's book.

[16]Recall that we already discussed auctions as strategic games with complete information in Chapter 2. The valuations were implicitly assumed to be publicly known. Yet, as it is natural, most applications of auction theory are to models where there is incomplete information about the valuations of the players. Thus, modeling auctions as Bayesian games seems more appropriate.

most applications of Bayesian games). Below we formally represent first-price and second-price auctions as Bayesian games.

Definition 4.4.1. An n-player *first-price auction* with set of players N is a 4-tuple (Θ, ρ, A, u^I) whose elements are the following:

The types of the players: $\Theta := \prod_{i\in N}\Theta_i$, where, for each $i \in N$, $\Theta_i = [0, \bar{v}]$.

The common prior: The distribution of probability over $[0, \bar{v}]$, denoted by ρ, from which the types are independently drawn.

The basic action profiles: $A := \prod_{i=1}^n A_i$, where $A_i := [0, \infty)$. A **pure strategy** of player i is a function $\hat{a}_i \colon \Theta_i \to A_i$. Let $\hat{A}_i$ be the set of pure strategies of i and $\hat{A} := \prod_{i=1}^n \hat{A}_i$.

The basic payoff functions: $u^I := \prod_{i=1}^n u_i^I$, where, for each $i \in N$, $u_i^I \colon [0, \bar{v}] \times A \to \mathbb{R}$ is defined as follows

$$u_i^I(\theta_i, a) = \begin{cases} \theta_i - a_i & a_i > \max_{j\neq i} a_j \\ \frac{\theta_i - a_i}{|\{j \,:\, a_j = \max_{k\in N} a_k\}|} & a_i = \max_{j\neq i} a_j \\ 0 & a_i < \max_{j\neq i} a_j. \end{cases}$$

For each $i \in N$, i's **(Bayesian) payoff function** $\hat{u}_i^I \colon \hat{A} \to \mathbb{R}$ is defined by $\hat{u}_i^I(\hat{a}) := \int_{\theta\in[0,\bar{v}]^n} u_i^I(\theta_i, \hat{a}(\theta))\, d\rho(\theta_1)\ldots d\rho(\theta_n)$.

Definition 4.4.2. An n-player *second-price auction* with set of players N is a 4-tuple $(\Theta, \rho, A, u^{II})$ whose elements are the same as in the first-price auction except for the following difference:

The basic payoff functions: $u^{II} := \prod_{i=1}^n u_i^{II}$, where, for each $i \in N$, $u_i^{II} \colon [0, \bar{v}] \times A \to \mathbb{R}$ is defined as follows

$$u_i^{II}(\theta_i, a) = \begin{cases} \theta_i - \max_{j\neq i} a_j & a_i > \max_{j\neq i} a_j \\ \frac{\theta_i - \max_{j\neq i} a_j}{|\{j \,:\, a_j = \max_{k\in N} a_k\}|} & a_i = \max_{j\neq i} a_j \\ 0 & a_i < \max_{j\neq i} a_j. \end{cases}$$

For each $i \in N$, i's **(Bayesian) payoff function** is defined analogously to first-price auction.

Remark 4.4.1. First-price and second-price auctions are commonly referred to as sealed-bid auctions, that is, the strategy of a player consists of a bid in a sealed envelope. There are also two *open* formats that are worth mentioning. The first, which is very common in real life, is the (English) ascending auction, where the auction begins with a low price, say 0, and then it rises gradually until there is only one interested bidder remaining. The second one is the (Dutch) descending auction, where the auction begins with a high price, say $\bar{v}$, and the price is lowered gradually until there is one bidder interested in getting the object at that price. From the strategic point of view, it can be shown that the first-price auction is equivalent to the descending auction and, as far as valuations are private information, the second-price auction is equivalent to the ascending auction.

The equilibrium definitions for first-price and second-price auctions are the usual ones (see Definition 4.2.2); moreover, behavior and mixed strategies are not discussed in this section. We begin the strategic analysis with the second-price auction, which is the less involved one.

Proposition 4.4.1. *Let* $(\Theta, \rho, A, u^{II})$ *be a second-price auction. Then, for each* $i \in N$*, the strategy defined by* $\hat{a}_i^{II}(\theta_i) := \theta_i$ *is a dominant strategy,* i.e., *for each* $\hat{a}_{-i} \in \hat{A}_{-i}$ *and each* $\hat{a}_i \in \hat{A}_i$*,* $\hat{u}_i^{II}(\hat{a}_{-i}, \hat{a}^{II}) \geq \hat{u}_i^{II}(\hat{a}_{-i}, \hat{a}_i)$*.*

Proof. Let $\hat{a}_{-i} \in A_{-i}$ and $\hat{a}_i \in A_i$. Let $\theta \in [0, \bar{v}]^n$. We now claim that θ_i is a weakly dominant strategy for i at θ. Let $\hat{b} := \max_{j \neq i} \hat{a}_j(\theta_j)$. We now show that it is not profitable for player i to bid $b < \theta_i$. We distinguish four cases:

- $\boldsymbol{\theta_i > b > \hat{b}}$**:** By bidding b, i gets the object with profit $\theta_i - \hat{b}$; the same profit he would get by bidding θ_i.
- $\boldsymbol{\theta_i > b = \hat{b}}$**:** By bidding b, i gets the object with profit $\frac{\theta_i - \hat{b}}{k}$, where $k \geq 2$ is the number of players submitting bid $\hat{b}$. Hence, he would be better off by bidding θ_i.
- $\boldsymbol{\hat{b} \geq \theta_i > b}$**:** Neither θ_i nor b suffice to get the object, so player i gets payoff 0 with both bids.
- $\boldsymbol{\theta_i > \hat{b} > b}$**:** Now, bid b is worse than θ_i.

Hence, bids below θ_i are never better than θ_i and sometimes they lead to a worse payoff. Similar arguments show that bids above θ_i are not profitable. □

Recall that, from the proof above, Proposition 4.4.1 holds in a more general setting than the one we have restricted attention to (we have not used any of the assumptions on ρ).

Below, we compare the two types of auctions defined above by studying their Bayesian equilibria. Often, the set of equilibria of a Bayesian game is very large. Thus, characterizing it completely is not an easy task, and because of this, it is quite standard to focus only on some specific Nash equilibria. In particular, we restrict to Bayesian equilibria that are i) symmetric, *i.e.*, given that all players are *ex ante* identical, it is natural to assume that they play the same strategy; and ii) differentiable and (weakly) increasing as functions from $[0, \bar{v}]$ to $[0, \infty)$.[17] When dealing with symmetric strategy

[17] We refer the reader to the specialized literature to see to what extent these assumptions are needed. Actually, the assumptions in ii) are only needed for the analysis of the first-price auction.

profiles as $(\hat{a}^{II}, \dots, \hat{a}^{II})$ we slightly abuse notation and use $\hat{a}^{II}$ to denote both the individual strategies and the strategy profile. Note that the strategy profile $\hat{a}^{II}$ is also differentiable and increasing.

Proposition 4.4.2. *The strategy profile $\hat{a}^{II}$ is the unique symmetric, differentiable, and increasing Bayesian Nash equilibrium of the second-price auction.*

Proof. Let $\hat{y}\colon [0,\bar{v}] \to [0,\infty)$ be a symmetric, differentiable, and increasing Bayesian Nash equilibrium of the second-price auction $(\Theta, \rho, A, u^{II})$. We show in two steps that, for each $i \in N$ and each $\theta \in [0,\bar{v}]$, submitting bid θ is a strictly dominant strategy for player i.

Step 1: We show that, for each $\theta \in (0,\bar{v})$, $\hat{y}(\theta) \geq \theta$. Suppose, on the contrary, that there is $\theta \in (0,\bar{v})$ such that $\hat{y}(\theta) > \theta$. By the continuity of $\hat{y}$, there is $\underline{b} \in (0,\theta)$, such that $\hat{y}(\underline{b}) > \theta$. Fix $i \in N$. Since ρ has full support, then, with positive probability, for each $j \neq i$, $\theta_j \in (\underline{b}, \theta)$. Hence, if i has type $\theta_i = \theta$, there is a positive probability that i wins the auction and pays something above θ when submitting bid $\hat{y}(\theta)$, incurring a loss with respect to what he would get by submitting bid θ (which always ensures a non-negative payoff). The latter observation, combined with Proposition 4.4.1, concludes this step.

Step 2: We show that, for each $\theta \in (0,\bar{v})$, $\hat{y}(\theta) \leq \theta$. Suppose, on the contrary, that there is $\theta \in (0,\bar{v})$ such that $\hat{y}(\theta) < \theta$. By the continuity of $\hat{y}$, there is $\bar{b} \in (\theta,\bar{v})$, such that $\hat{y}(\bar{b}) < \theta$. Fix $i \in N$. Since ρ has full support, then, with positive probability, for each $j \neq i$, $\theta_j \in (\theta, \bar{b})$. Hence, if player i has type $\theta_i = \theta$, with positive probability he will not win the auction when submitting bid $\hat{y}(\theta)$, whereas he would make a strictly positive profit submitting bid θ. The latter observation, combined with Proposition 4.4.1, concludes this step.

The result for types 0 and $\hat{v}$ follows from the continuity of $\hat{y}$. □

Equilibrium behavior is not so easy to characterize in the first-price auction. Let F denote the distribution function associated with ρ and let Z denote the random variable that gives the highest value of $n-1$ realizations of ρ. The distribution function associated with Z is $\hat{F}$, where, for each $v \in [0,\bar{v}]$, $\hat{F}(v) = F(v)^{n-1}$.

Proposition 4.4.3. *The strategy profile $\hat{a}^{I}$ given by $\hat{a}^{I}(\theta) := \mathbb{E}(Z\,|_{Z<\theta})$ is the unique symmetric, differentiable, and increasing Bayesian Nash equilibrium of the first-price auction.*

Proof. Let $\hat{y}\colon [0,\bar{v}] \to [0,\infty)$ be a symmetric, differentiable, and increasing Bayesian Nash equilibrium of the first-price auction (Θ, ρ, A, u^{I}). First, we claim that the distribution of probability induced by $\hat{y}$ on $[0,\infty)$ has no mass points, *i.e.*, ties have probability 0 according to $\hat{y}$. Suppose that the latter is not true and let $b \in [0,\infty)$ be a bid at which players can tie with positive probability when $\hat{y}$ is played. Thus, since ρ has no mass points, the set $D = \{\theta \in [0,\bar{v}]\backslash\{b\} : \hat{y}(\theta) = b\}$ has positive probability. Since ties happen with positive probability at bid b and $\hat{y}$ is an equilibrium, if $v \in D$, then $v > b$. Since D has positive probability, there is $v \in D$ such that $v > b$. Now, there is $\varepsilon > 0$ small enough so that every player can profitably deviate at type v to $\hat{y}(v) = b + \varepsilon$, and be the only winner of the auction whenever the types of all the other players fall inside D; in this case, he would get $v - b - \varepsilon$ without having to share this payoff with any other player. Hence, in the rest of the proof we do not care about ties since they will have probability 0. Note that the fact that $\hat{y}$ has no mass points implies that $\hat{y}$ is strictly increasing and, hence, invertible.

Now, we calculate the optimal bid $b \in [0,\infty)$ of an arbitrary player $i \in N$ of type θ_i against the profile $\hat{y}$. If i has type θ_i and submits a bid strictly greater than θ_i, then his expected payoff is negative. Hence, $\hat{y}(0) = 0$ and $\hat{y}(\bar{v}) \leq \bar{v}$. For each $j \neq i$, let $\theta_j \in [0,\bar{v}]$. Then, i wins the bid if $\max_{j\neq i} \hat{y}(\theta_j) < b$. Since $\hat{y}$ is increasing, $\max_{j\neq i} \hat{y}(\theta_j) = \hat{y}(\max_{j\neq i} \theta_j) = \hat{y}(Z)$. Hence, i wins if $\hat{y}(Z) < b$, *i.e.*, if $Z < \hat{y}^{-1}(b)$. Hence, i's expected payoff at type θ_i is $\hat{F}(\hat{y}^{-1}(b))(\theta_i - b)$. Let $\hat{f} = \hat{F}'$ be the density of Z. Maximizing with respect to b we get the first order condition

$$\frac{\hat{f}(\hat{y}^{-1}(b))}{\hat{y}'(\hat{y}^{-1}(b))}(\theta_i - b) - \hat{F}(\hat{y}^{-1}(b)) = 0.$$

Since we are restricting to symmetric equilibria, for $\hat{y}$ to be an equilibrium it has to be the case that $\hat{y}(\theta_i) = b$, *i.e.*, all the players submit bid b when they are type θ_i and, hence, $\hat{y}^{-1}(b) = \theta_i$. Then, a necessary condition for $\hat{y}$ to be an equilibrium is that it satisfies the differential equation $\hat{F}(\theta_i)\hat{y}'(\theta_i) + \hat{f}(\theta_i)\hat{y}(\theta_i) = \theta_i \hat{f}(\theta_i)$, which can be rewritten as

$$(\hat{F}(\theta_i)\hat{y}(\theta_i))' = \theta_i \hat{f}(\theta_i).$$

Since $\hat{y}(0) = 0$, we can integrate the above differential equation and get

$$\hat{y}(\theta_i) = \frac{1}{\hat{F}(\theta_i)} \int_0^{\theta_i} z\hat{f}(z)dz = \mathbb{E}[Z\,|_{Z<\theta_i}].$$

Hence, we have already shown that $\hat{a}^I$ is the unique candidate to be a symmetric, differentiable, and increasing Bayesian Nash equilibrium of the first-price auction. Below, we show that $\hat{a}^I$ is indeed an equilibrium. Let $i \in N$ and let θ_i be his type. We now show that $\hat{a}^I(\theta_i)$ is a best reply for i given that all other players play $\hat{a}^I$. Let $b \in [0,\bar{v}]$. If $b > \hat{a}^I(\bar{v})$, b is strictly dominated by $\frac{b+\hat{a}^I(\bar{v})}{2}$. Hence, assume that $b \in [0,\hat{a}^I(\bar{v})]$. Let $v = (\hat{a}^I)^{-1}(b)$, *i.e.*, $\hat{a}^I(v) = b$. Let $u_i^I(\theta_i,(\hat{a}_{-i},a_i))$ denote the expected payoff of player i

when he has already been informed that his type is θ_i, he is playing the basic action profile a_i, and the other players are following the action profile $\hat{a}_i$. Then,

$$\begin{aligned} u_i^I(\theta_i, (\hat{a}^I_{-i}, b)) &= \hat{F}(v)(\theta_i - \hat{a}^I(v)) = \hat{F}(v)\theta_i - \hat{F}(v)\mathbb{E}(Z\,|_{Z<v}) \\ &= \hat{F}(v)\theta_i - \int_0^v y\hat{f}(y)dy \overset{\substack{\text{int. by}\\ \text{parts}}}{=} \hat{F}(v)\theta_i - \hat{F}(v)v + \int_0^v \hat{F}(y)dy \\ &= \hat{F}(v)(\theta_i - v) + \int_0^v \hat{F}(y)dy. \end{aligned}$$

Below, we compare the payoff obtained with b $(= \hat{a}^I(v))$ and the one obtained with $\hat{a}^I(\theta_i)$. $u_i^I(\theta_i, (\hat{a}^I_{-i}, \hat{a}^I(\theta_i))) - u_i^I(\theta_i, (\hat{a}^I_{-i}, b)) = \int_0^{\theta_i} \hat{F}(y)dy + \hat{F}(v)(v - \theta_i) - \int_0^v \hat{F}(y)dy$. We distinguish two cases:

$v \geq \theta_i$: In this case we have $u_i^I(\theta_i, (\hat{a}^I_{-i}, \hat{a}^I(\theta_i))) - u_i^I(\theta_i, (\hat{a}^I_{-i}, b)) = \hat{F}(v)(v - \theta_i) - \int_{\theta_i}^v \hat{F}(y)dy$. Therefore, since $\hat{F}$ is increasing, we have $\int_{\theta_i}^v \hat{F}(y)dy \leq \hat{F}(v)(v - \theta_i)$.

$v \leq \theta_i$: In this case we have $u_i^I(\theta_i, (\hat{a}^I_{-i}, \hat{a}^I(\theta_i))) - u_i^I(\theta_i, (\hat{a}^I_{-i}, b)) = \int_v^{\theta_i} \hat{F}(y)dy - \hat{F}(v)(\theta_i - v)$. Therefore, since $\hat{F}$ is increasing, we have $\int_{\theta_i}^v \hat{F}(y)dy \geq \hat{F}(v)(\theta_i - v)$.

In both cases we get $u_i^I(\theta_i, (\hat{a}^I_{-i}, \hat{a}^I(\theta_i))) \geq u_i^I(\theta_i, (\hat{a}^I_{-i}, b))$. Hence, deviations from $\hat{a}^I(\theta_i)$ are not profitable. Therefore, $\hat{a}^I$ is a Bayesian Nash equilibrium of the first-price auction, which, moreover, is unique. □

Example 4.4.1. We now solve for the simplest case of all, where ρ corresponds with the uniform distribution over $[0,1]$. In this case, for each $v \in [0,1]$, we have $F(v) = v$ and $\hat{F}(v) = v^{n-1}$. Bidding behavior in the unique symmetric, differentiable, and increasing Bayesian Nash equilibrium of the second-price auction is straightforward: for each $\theta \in [0,1]$, $\hat{a}^{II}(\theta) = \theta$. On the other hand, for the first-price auction we get, for each $\theta \in [0,1]$,

$$\hat{a}^I(\theta) = \int_0^\theta \frac{z\hat{f}(z)}{\hat{F}(\theta)}dz = \frac{1}{\theta^{n-1}}\int_0^\theta z(n-1)z^{n-2}dz = \frac{n-1}{n}\theta;$$

that is, whereas in the second-price auction a player always bids his valuation, in the first-price auction each player bids a fix proportion of his valuation. ◇

Once we have characterized the equilibrium strategies of the two formats of auctions at hand, a natural question appears: Which of the two auctions yields a higher revenue to the seller? The next result shows that, from the point of view of the seller, both auction formats are equivalent.

Proposition 4.4.4 (Revenue equivalence principle)**.** *If the valuations of the players are private and independently and identically distributed, then the expected revenues for the seller in first-price and second-price auctions coincide.*

Proof. Note that the expected revenue of the seller coincides with the sum of the expected payments of the bidders. We derive these expected payments below. Let $v \in [0, \bar{v}]$. Let $\pi(v)$ be the expected payment of a bidder of type v.

Second-price auction: In equilibrium, a player of type v submits bid v and his probability of winning is $\hat{F}(v)$. The winner pays the second highest bid. Hence, $\pi(v) = \hat{F}(v)\mathbb{E}(Z\,|_{Z<v})$.

First-price auction: In equilibrium, a player of type v submits bid $\mathbb{E}(Z\,|_{Z<v})$ and, since all the other players follow the same strategy, his probability of winning is $\hat{F}(v)$. The winner pays his bid. Hence, $\pi(v) = \hat{F}(v)\mathbb{E}(Z\,|_{Z<v})$.

Now we are done. The expected payment of a player, before knowing his type, is $\int_0^{\bar{v}} \pi(v)d\rho(v)$, which is the same for first-price and second-price auctions. Therefore, in both cases, the expected revenue of the seller is $n\int_0^{\bar{v}} \pi(v)d\rho(v)$. □

It is worth mentioning that, as long as the types of the players are drawn independently from the same distribution of probability, the revenue equivalence principle holds for a wide variety of auction formats and not only for first-price and second-price auctions. For deeper discussions on the results we have presented in this section within similar settings, the reader may check the seminal papers by Vickrey (1961, 1962) and also the contributions by Riley and Samuelson (1981) and Myerson (1981) (these last two papers contain general versions of the revenue equivalence principle).

We have briefly analyzed the simplest models one can face when doing auction theory. There are many directions in which the framework can be enriched. Below we briefly describe the idea of some standard extensions:

Interdependent values: It is natural to think that the valuations of the object are not independent from each other; for instance, there might be some common sources of information about the quality of the object. Thus, because of this, there are many papers in which independence has been relaxed in the direction of positive correlation. Mathematically, this can be done by defining the common prior ρ over $[0, \bar{v}]^n$ so that the marginal distributions are correlated in the desired way. A classic paper in this setting is Milgrom and Weber (1982).

Asymmetries: It is also natural to consider scenarios in which there are asymmetries across players (Krishna (2002) devotes a whole chapter to this issue).

Multiple object auctions: There are also situations in which several objects are to be sold. They might be identical to each other or, at least, substitutes to some extent. The study of multiple object auctions goes back to Vickrey (1961) and they are thoroughly discussed in Krishna (2002).

All-pay auctions: Other less conventional allocation schemes have also been discussed in literature. In an all-pay auction, for instance, every player pays his bid irrespective of whether he gets the object or not. These apparently unnatural auctions can be applied to a wide variety of economic problems (Baye et al. 1993, 1996).

4.5. A Second Application of Bayesian Games: Mechanism Design and the Revelation Principle

Problems of mechanism design constitute one of the major fields of application of Bayesian games. Indeed, auctions themselves can be regarded as a special class of mechanisms. We now present an example to illustrate what a mechanism design problem is. The manager of a firm wants to hire a new employee and there are two candidates. Each of them can be either a high productivity type or a low productivity type and this information is unknown to the manager of the firm. The manager would like to hire a high productivity type. However, it is unclear how to achieve this goal. If he asks the candidates about the type they belong to, it is quite likely that low type candidates would pretend that they are high type, *i.e.*, they would misreport their types. Thus, the manager's problem is to design a mechanism (a contract in this example) that provides the candidates with appropriate incentives to tell the truth. Since, in general, this can only be done in a costly way, the problem is to design a mechanism that is as efficient as possible from the manager's point of view.

More generally, a problem of mechanism design can be described as follows. There is a distinguished player, the *principal*, and some other players, the *agents*. There is some information that is privately known by the agents. Both the principal and the agents face a strategic situation in which the principal would like to take an action conditional on the types of the agents. Hence, the problem of the principal is to design a mechanism that allows him to extract information from the agents in order to take the most profitable action for him. In most cases, it is costly to ensure that the received information is truthful. Thus, there is a trade off between the cost of the extraction of the information and the extra profit the principal can make by knowing such information. Mechanism design is concerned with the problem faced by the principal and has become extremely popular because it has many applications in different economic settings: price discrimination, optimal taxation, auction design, provision of public goods, among others. Refer to Green and Laffont (1979) for a complete monograph on mechanism design. Also, Chapter 23 in Mas-Colell et al. (1995) and Chapter 7 in Fudenberg and Tirole (1991a) deeply discuss mechanism design and some of its applications.[18]

We now describe the primitives of the model. There are $n+1$ players: the *n agents* ($i \in N = \{1, \ldots, n\}$) and the *principal* (player $n+1$). Each agent $i \in N$ has a type $\theta_i \in \Theta_i$ and the set of type profiles is $\Theta = \prod_{i \in N} \Theta_i$. Types are drawn from a distribution of probability ρ defined over Θ and each player is privately informed about his own type.[19] The principal has to make a choice in some compact, convex, and nonempty set $C \subset \mathbb{R}^n$ and select a vector $p \in \mathbb{R}^n$ of payments for the agents (monetary transfers that can be either negative or positive); that is, he has to propose an allocation $z = (c, p) \in C \times \mathbb{R}^n$. The agents can either accept or reject the allocation proposed by the principal, *i.e.*, each agent has an outside option that gives him his *reservation payoff*. Without loss of generality we assume that the reservation payoff of each agent is 0. All the players have (von Neumann and Morgernstern) payoff functions: for each $i \in \{1, \ldots, n+1\}$, $u_i \colon \Theta \times C \times \mathbb{R}^n \to \mathbb{R}$. All the above primitives are common knowledge.

Ideally, the principal would like to condition the allocation he chooses on the types of the agents so that he maximizes his profit. Since the information of the agents is private, the principal devises a mechanism according to which the agents send him messages that, eventually, might reveal some information. In view of the received messages, the principal chooses an allocation. A mechanism is a strategic game for the agents, where their sets of strategies are the possible messages they can send to the principal. Let $M = \prod_{i \in N} M_i$ be the set of profiles of messages. For each message-profile, the principal selects an allocation according to the mapping $\mathcal{S} \colon M \to C \times \mathbb{R}^n$, which we refer to as an *allocation scheme*. Importantly, the principal is not a player in the mechanism, *i.e.*, he commits to an allocation scheme $\mathcal{S}$ that is publicly announced to the agents before they play. The payoff functions are defined combining $\mathcal{S}$ and the u_i functions. The timing of the associated Bayesian game is as follows: i) the types

[18] The importance of mechanism design has been acknowledged with the award of the Nobel Prize in Economics in 2007 to L. Hurwicz, E. Maskin, and R. B. Myerson for their contributions to this field.

[19] The situation is very similar to the one we presented in Section 4.4. By knowing the type of a player we already know his information and *vice versa*. Hence we do not need the full generality of our definition of Bayesian games (type-spaces and the states of the world become the same entity).

are realized, each player is only informed about his type, and the principal receives no information, ii) the principal proposes a mechanism (commits to an allocation scheme), iii) the agents send their messages, and iv) the principal selects an allocation and the payoffs are realized. The decisions of the agents at step iii) are conditional on their types. Thus, at step iv), the allocation selected by the principal depends on the types only through the agents' messages. There are problems of mechanism design in which the agents can decide whether to participate in the game proposed by the principal or not (and get their reservation payoff). In the latter case, we assume that, for each $i \in N$, $r \in M_i$; the message r is the refusal to participate. Note that the players decide whether to participate or not at step iii). Once an agent decides to participate, he cannot reject the allocation proposed by the mechanism at step iv), even if the corresponding payoff is below his reservation payoff.

Definition 4.5.1. An n-player *mechanism* with set of players N is a 5-tuple $(\Theta, \rho, M, \mathcal{S}, u)$ whose elements are the following:

The types of the players: $\Theta := \prod_{i\in N} \Theta_i$.

The common prior: The distribution of probability ρ from which the types are drawn (not necessarily independently).

The basic action profiles: $M := \prod_{i=1}^{n} M_i$. A **pure strategy** of player i is a map $\hat{m}_i \colon \Theta_i \to M_i$. Let $\hat{M}_i$ be the set of pure strategies of i and $\hat{M} := \prod_{i=1}^{n} \hat{M}_i$.

The allocation scheme: $\mathcal{S} \colon M \to C \times \mathbb{R}^n$.

The basic payoff functions: $u := \prod_{i=1}^{n} u_i$, where, for each $i \in N$, $u_i \colon \Theta \times C \times \mathbb{R}^n \to \mathbb{R}$. For each $i \in N$, his **(Bayesian) payoff function** $\hat{u}_i \colon \hat{M} \to \mathbb{R}$ is defined by $\hat{u}_i(\hat{m}) := \int_{\theta\in\Theta} u_i(\theta, \mathcal{S}(\hat{m}(\theta)))d\rho(\theta)$.

A mechanism is a particular case of a Bayesian game and, hence, one would like to study its Bayesian Nash equilibria. Behavior and mixed strategies are not discussed in this section. A mechanism is said to be *direct* if $M = \prod_{i\in N} \Theta_i \cup \{r\}$, *i.e.*, the possible messages of the players coincide with their types plus the nonparticipation decision. Given a direct mechanism, if it is an equilibrium for the agents to reveal their true types, then the equilibrium is said to be *truthful* (in particular, in a truthful equilibrium all the agents agree to participate in the game).

The objective of the principal is to design a mechanism whose equilibrium allocations are profitable for him, that is, a mechanism should be seen as the second stage of an extensive game with incomplete information in which the principal is himself a player with some utility function defined over $\Theta \times C \times \mathbb{R}^n$. First the nature draws the types of the agents, then the principal, with no information about the realizations of the types, commits to a mechanism that is publicly observed and then the agents choose their actions. In this case, one should look for the *perfect Bayesian equilibria* of this *multistage game* with incomplete information. We develop these two concepts in the following section.

Yet, since the class of all mechanisms is very general, the difficulty of finding optimal mechanisms is a tough problem for the principal. Fortunately, the major result in mechanism design, the revelation principle, states that the principal can restrict attention to direct mechanisms.

Theorem 4.5.1 (The revelation principle)**.** *Let* $(\Theta, \rho, M, \mathcal{S}, u)$ *be a mechanism and* $\hat{m}$ *a Bayesian equilibrium. Then, there is a direct mechanism in which there is a truthful equilibrium whose outcome coincides with the outcome of the original equilibrium,* i.e., *for each* $\theta \in \Theta$*, it delivers* $\mathcal{S}(\hat{m}(\theta))$*.*

Proof. Consider the direct mechanism $(\Theta, \rho, \bar{M}, \bar{\mathcal{S}}, u)$ defined as follows: $\bar{M} := \prod_{i \in N} \Theta_i \cup \{r\}$ and $\bar{\mathcal{S}}$ is defined, for each $\theta \in \Theta$, as $\bar{\mathcal{S}}(\theta) := \mathcal{S}(\hat{m}(\theta))$. Now, it is easy to see that telling the truth is a Bayesian equilibrium of the new mechanism and it is also easy to see that it leads to the same outcomes of the original equilibrium. $\square$

The intuition behind the revelation principle is quite simple. Given a mechanism and an equilibrium $\hat{m}$, we can ask the players to report their types and, for each reported profile $\theta \in \Theta$, we select the allocation we would have selected in the original mechanism after observing the messages $\hat{m}(\theta)$. If $\hat{m}(\theta)$ was an equilibrium of the original game, then telling the truth must be an equilibrium of the direct mechanism. Clearly, the outcomes in both cases coincide.

Remark 4.5.1. Exercise 4.8 asks the reader to write first-price and second-price auctions as mechanisms (Definition 4.5.1). In this setting, the principal is the seller and the revenue equivalence principle says that, quite generally, he is indifferent between the two mechanisms, since his expected revenues in equilibrium coincide.

Remark 4.5.2. In spite of having a clean statement, there are some questions that can be raised on the relevance of the revelation principle. One major concern is the fact that the truthful equilibrium of the direct mechanism constructed in the proof of Theorem 4.5.1 may not be unique. We refer the reader to the specialized literature for a more careful analysis on the implications and limitations of the revelation principle.

4.6. Extensive Games with Incomplete Information: Multistage Games and Perfect Bayesian Equilibrium

We already know that, following the approach in Harsanyi (1967-68), an extensive game with incomplete information can be transformed in an extensive game with imperfect information. Recall that, in the representation of a strategic game with incomplete information as an extensive game with imperfect information, once the nature moves and the players are informed about their types, all the players simultaneously choose an action and the game finishes; that is, any such game can be seen as an extensive game with two stages: in the first one the nature moves and in the second one all the players move simultaneously. On the other hand, when the underlying game is an extensive game with incomplete information, there is much more freedom for the corresponding extensive game with imperfect information: the different players may have to play several times and they may get information about the actions taken by other players as the play unfolds. Indeed, the players may even get further information about the move made by nature in the initial period of the game.

The fact that the players can learn about the move made by nature in the initial period not only through the information they get when they learn their types, but also by processing the information they get as the game is played, is an issue of fundamental importance in extensive games with incomplete information. There are two different sources for the information the players can get about the move made by nature in the initial period as the game unfolds: firstly, they can get direct information through new signals that are (possibly privately) revealed to the players and, secondly, the players get indirect information about the state of nature and the types of the other players through the (Bayesian) inferences they can make based on their own (possibly imperfect) observation of the actions that have been played.

Sequential equilibrium is one of the main equilibrium concepts that has been used to study extensive games with incomplete information. However, many of the applications of these games deal with situations in which the players might have a continuum of strategies, and sequential equilibrium is not defined for these classes of games. Thus, because of this, *perfect Bayesian equilibrium* is widely used in this literature, since its extension to games of infinite width is not so difficult. The underlying idea of perfect Bayesian equilibrium is very similar to that of sequential equilibrium: an assessment that is sequentially rational and whose beliefs are derived by Bayes rule whenever possible. Yet, what does it mean "whenever possible"? For those games in which sequential equilibrium is well defined, consistency of beliefs stands out as a very compelling answer (recall that consistent beliefs are the limit of beliefs derived by using Bayes rule everywhere). Yet, when sequential equilibrium cannot be defined, the meaning of "whenever possible" has to be clarified. Thus, different versions of perfect Bayesian equilibrium have been defined depending on the structure of the incomplete information games to analyze.

The rest of this section is structured as follows. First, we present an important class of extensive games with incomplete information: multistage games with incomplete information. Then we formally define perfect Bayesian equilibrium within this class of games. Finally, we compare perfect Bayesian equilibrium and sequential equilibrium in this setting. In order to understand the similarities and the differences of the perfect Bayesian approach and that of sequential equilibrium, we restrict our attention to finite games, where sequential equilibrium is well defined. Most of the exposition below is based on Fudenberg and Tirole (1991b).

4.6.1. Multistage games with incomplete information. The set of players is $N := \{1, \ldots, n\}$. Each player $i \in N$ has a type coming from the finite set Θ_i. We assume that the types of the players are independent. More precisely, there is a common prior ρ over $\Theta := \prod_{i \in N} \Theta_i$ such that $\rho = \prod_{i \in N} \nu_i$, where ν_i is the marginal distribution over player i's type. At the beginning of the game, each player is told his own type, but is not given any information about the types of his opponents. Since the types are independent, for each $i \in N$ and each $\theta_i \in \Theta_i$, player i's initial beliefs about the types of his opponents are given, for each $\theta_{-i} \in \Theta_{-i}$, by $\mu_i(\theta_{-i} \,|_{\theta_i}) = \prod_{j \neq i} \nu_j(\theta_j)$, *i.e.*, the beliefs of player i about his opponents are independent of his own type. In addition, we assume that players do not receive any additional signal about the realization of the types of his opponents as the game unfolds. As in a repeated game (Section 3.7), the game is played in periods and at each stage t all the players simultaneously choose an action. We assume that actions are observable, *i.e.*, at the end of each stage, each player perfectly observes the actions played by all the other players. We also assume that the possible actions of a player are independent of his type.[20] This specification is more general than it may appear because the set of feasible actions can be time and history dependent, so that games with alternating moves are included. We assume that the length of the game is history independent and given by $T \in \mathbb{N} \cup \{\infty\}$; this assumption is without loss of generality as we can always artificially lengthen the game after a certain history by adding payoff irrelevant periods in which each player has only one action. To give a hint about the generality and applications of multistage games, note that they extend repeated games and also another widely applied class of games:

[20] Recall that there is no loss of generality in this assumption (see the discussion in Section 4.1).

signaling games (refer to Fudenberg and Tirole (1991a, Section 8.2) for an introduction to this class of games). Moreover, many problems of mechanism design can also be accommodated within this framework.

The formal definition of the sets of strategies in a multistage game resembles the one for a repeated game but is more involved. Let $h^0 = \{*\}$ and let $\mathscr{A}_i(h^0)$ denote the finite set of actions available for player i at stage 1. If the history of moves (other than natures choice of types) before stage t is h^{t-1}, then $\mathscr{A}_i(h^{t-1})$ denotes the actions available for player i at stage t. Each $\mathscr{A}_i$ is referred to as the *availability correspondence* of player i. Let $\mathscr{A} := \prod_{i\in N} \mathscr{A}_i$. For notational convenience, we assume that, at each stage at which the game is not over, each player always has at least one available action. Then, if $a \in \prod_{i\in N} \mathscr{A}_i(h^{t-1})$ is played at time t, the new history becomes $h^t := (h^{t-1}, a)$. Let H^t denote the set of all possible histories at stage t. Then, the information that player i has at stage t is an element of the set $H^{t-1} \times \Theta_i$. The set of all histories can be partitioned in *terminal histories*, *i.e.*, those in H^T, and *nonterminal histories*, *i.e.*, those in $H := \bigcup_{t=1}^{T} H^{t-1}$. For each $i \in \mathbb{N}$, let $A_i := \bigcup_{h\in H} \mathscr{A}_i(h)$ be the set of potential actions of player i. The set of pure strategy profiles is $\tilde{A} := \prod_{i\in N} \tilde{A}_i$, where $\tilde{A}_i := \{\tilde{a}_i \in A_i^{H\times\Theta_i} :$ for each $\theta_i \in \Theta_i$ and each $h \in H$, $\tilde{a}_i(h,\theta_i) \in \mathscr{A}_i(h)\}$, *i.e.*, the set of mappings from $H \times \Theta_i$ to A_i, but with the restriction that the players can only take actions that are available after the given history. Similarly, the set of behavior strategy profiles is defined by $B := \prod_{i\in N} B_i$, where $B_i := \{b_i \in (\Delta A_i)^{H\times\Theta_i} :$ for each $\theta_i \in \Theta_i$, each $h \in H$, and each $a_i \in A_i$, if $b_i(h,\theta_i)(a_i) > 0$, then $a_i \in \mathscr{A}_i(h)\}$, *i.e.*, the players can only play with positive probability those actions that are available after the given history.[21] For each history $h^T \in H^T$ and each distribution of types $\theta \in \Theta$, the final payoff of a player i is given by $u_i(\theta, h^T)$. In particular the payoffs need not be separable over periods as in repeated games.

Therefore, an n-player *multistage game with incomplete information (and observed actions)* with player set $N = \{1, \ldots, n\}$ can be characterized by a 4-tuple $(\Theta, \rho, \mathscr{A}, u)$. The length of the game, the set of histories, and the sets of strategies are uniquely determined via availability correspondences.

4.6.2. Perfect Bayesian equilibrium. Recall that an assessment is a par (b, μ) where $b \in B$ and μ is a system of beliefs. In multistage games a system of beliefs can be characterized by giving, for each player $i \in N$, each $\theta_i \in \Theta_i$, and each history $h \in H$, a distribution of probability over Θ_{-i}, the types of i's opponents; given $\theta_{-i} \in \Theta_{-i}$, we denote the latter probability by

[21] Recall that in Section 3.7, when defining the sets of strategies for repeated games, we did not make any distinction between pure and behavior strategies. The reason is that, in our analysis of repeated games, we assumed that mixed actions were observable (see Subsection 3.7.1).

$\mu_i(\theta_{-i}\,|_{\theta_i,h})$. To define perfect Bayesian equilibrium, we first introduce the notion of *reasonable* assessment (b,μ), meaning:

B(i) Posterior beliefs are independent: i) the beliefs of player i are independent of his own type and ii) the beliefs of player i about the types of the others are independent. Formally, for each $i \in N$, each $\theta \in \Theta$ and each $h \in H$, $\mu_i(\theta_{-i}\,|_{\theta_i,h}) = \prod_{j\neq i} \mu_i(\theta_j\,|_h)$. In particular, even if a player faces an expected event (a history with prior probability 0), he cannot interpret it as evidence that the types are correlated (because he knows they are not).

B(ii) Use Bayes rule whenever possible: Let $i,j \in N$ and $h,\bar{h} \in H$ be such that $\bar{h} = (h,a)$ with $a \in \mathscr{A}(h)$. If there is $\bar{\theta}_j \in \Theta_j$ such that $\mu_i(\theta_j\,|_h) > 0$ and $b_j(h,\theta_j)(a_j) > 0$, then, for each $\theta_j \in \Theta_j$,

$$\mu_i(\theta_j\,|_{\bar{h}}) = \frac{\mu_i(\theta_j\,|_h)\, b_j(h,\theta_j)(a_j)}{\sum_{\hat{\theta}_j\in\Theta_j} \mu_i(\hat{\theta}_j\,|_h)\, b_j(h,\hat{\theta}_j)(a_j)}.$$

This property implies that reasonable beliefs are weakly consistent with Bayes rule and, hence, perfect Bayesian equilibrium is a refinement of weak perfect Bayesian equilibrium (see Definition 3.5.5). Nonetheless, this property goes further than that, since it also implies that Bayes rule is also used off the equilibrium path, *i.e.*, the history h above might have probability 0 of being reached when playing according to b and yet, once a player forms some beliefs at the corresponding information set, he has to apply Bayes rule to those beliefs to compute his beliefs in the subsequent information sets (if possible).

B(iii) No signaling what you do not know: Let $i,j \in N$, $\theta_j \in \Theta_j$, $h \in H$ and $a,\bar{a} \in \mathscr{A}(h)$. If $a_j = \bar{a}_j$, $\mu_i(\theta_j\,|_{(h,a)}) = \mu_i(\theta_j\,|_{(h,\bar{a})})$. All the players $k \neq j$ have the same information about the type of player j and, hence, player i should not interpret actions of a player $k \neq j$ as indications of the type of player j.

B(iv) Common beliefs: Let $i,j \in N$ and let $h \in H$. Then, for each $k \in N\backslash\{i,j\}$ and each $\theta_k \in \Theta_k$, $\mu_i(\theta_k\,|_h) = \mu_j(\theta_k\,|_h) = \mu(\theta_k\,|_h)$, that is, after having observed the same history, each pair of players agree on the beliefs about any third player. This implies that the beliefs of the different players come from a common distribution. In Fudenberg and Tirole (1991a) the authors motivate this property "as being in the spirit of equilibrium analysis, since it supposes that players have the same beliefs about each other's strategies".

Definition 4.6.1. Let $(\Theta,\rho,\mathscr{A},u)$ be a multistage game with incomplete information. An assessment (b,μ) is *reasonable* if it satisfies B(i)-B(iv).

Definition 4.6.2. Let $(\Theta, \rho, \mathscr{A}, u)$ be a multistage game with incomplete information. A *perfect Bayesian equilibrium* is an assessment that is sequentially rational and reasonable.

Proposition 4.6.1. *Let $(\Theta, \rho, \mathscr{A}, u)$ be a multistage game with incomplete information. If an assessment (b, μ) is consistent, then it is reasonable.*

Proof. Exercise 4.9. □

Corollary 4.6.2. *Let $(\Theta, \rho, \mathscr{A}, u)$ be a multistage game with incomplete information. If an assessment (b, μ) is a sequential equilibrium then it is a perfect Bayesian equilibrium.*

Proof. Follows from Proposition 4.6.1. □

Corollary 4.6.3. *Let $(\Theta, \rho, \mathscr{A}, u)$ be a multistage game with incomplete information. Then, $(\Theta, \rho, \mathscr{A}, u)$ has at least one perfect Bayesian equilibrium in behavior strategies.*

Proof. Follows from the combination of Theorem 3.5.2 and Corollary 4.6.2. □

Moreover, in Fudenberg and Tirole (1991b), the following converses of Corollary 4.6.2 are proved (we refer the reader to the original publication for the proofs).

Proposition 4.6.4. *Let $(\Theta, \rho, \mathscr{A}, u)$ be a multistage game with incomplete information. Suppose that each player has only two possible types, that both types have nonzero prior probability, and that types are independent. Then, an assessment is consistent if and only if it is reasonable and therefore, the sets of sequential equilibria and perfect Bayesian equilibria coincide.*

Proposition 4.6.5. *Let $(\Theta, \rho, \mathscr{A}, u)$ be a multistage game with incomplete information. Suppose that $(\Theta, \rho, \mathscr{A}, u)$ has only two periods and that types are independent. Then, an assessment is consistent if and only if it is reasonable and therefore, the sets of sequential equilibria and perfect Bayesian equilibria coincide.*

4.6.3. Perfect Bayesian *vs.* sequential. Despite the two last results, the sets of perfect Bayesian equilibria and sequential equilibria do not coincide for arbitrary multistage games with incomplete information. By definition, these two equilibrium concepts are different from each other to the same extent to which reasonable beliefs and consistent beliefs differ. Let (b, μ) be an assessment. If μ is reasonable or if it is consistent, a player uses Bayes rule to form his beliefs after histories that, given the strategy b, have positive probability of being realized. Hence, the difference between both concepts stems from the restrictions that they impose on the beliefs to be formed after probability 0 histories. The idea of both reasonable and consistent assessments is that, once a player i finds himself at an information set w that had probability 0 of being reached according to b, he has to make a "story" and then derive his beliefs based on this "story"; for instance, with sequential equilibrium the "story" is given by the sequence of trembles and the beliefs are derived as the limit case of this "story". Thereafter, player i uses Bayes rule again, assuming that all the players will stick to b in the future. If later in the game player i finds himself again at an information set $\bar{w}$ that had probability 0 of being reached according to all his previously formed beliefs, then he has to come up with a new "story" to form his beliefs at $\bar{w}$.[22] The main difference between reasonable and consistent assessments lies in the kind of "stories" that players can use at $\bar{w}$. As far as reasonable beliefs are concerned, the "story" used by player i to compute his beliefs at $\bar{w}$ can be completely unrelated to the one used for information set w. Yet, in order to have a consistent assessment, player i has to come up with a "story" at information set $\bar{w}$ that is consistent with the one he had for information set w (*i.e.*, use the same sequence of trembles). Though related to the latter, there is another difference between reasonable and consistent assessments. Namely, if given a history there are two types that have probability 0, a reasonable assessment does not specify relative probabilities between them, whereas consistency (implicitly) does. We illustrate this fact in Example 4.6.1 below.

Example 4.6.1. Consider the fragment of a multistage game in Figure 4.6.1. It depicts the situation of a two player multistage game at stage t after a certain history h, where player 1 has three possible types and player 2 has a unique possible type θ_2.

Suppose that, using Bayes rule, player 2 infers that $\mu(\hat{\theta}_1 |_{\theta_2,h}) = 1$ and $\mu(\bar{\theta}_1 |_{\theta_2,h}) = \mu(\tilde{\theta}_1 |_{\theta_2,h}) = 0$. In addition, let h_l be the history in which, after observing h, player 2 knows that player 1 chose l at stage t; similarly, h_r corresponds with the choice r at stage t. Suppose that the beliefs of player 2 at stage $(t+1)$ are such that $\mu(\bar{\theta}_1 |_{\theta_2,h_l}) = \mu(\tilde{\theta}_1 |_{\theta_2,h_r}) = 1$, that is, after observing l, player 2 thinks that player 1 has type $\bar{\theta}_1$ and after observing r he thinks that player 1 has type $\tilde{\theta}_1$. Since there are only two players and, given the beliefs of player 2 at stage t, both h_l and h_r had probability 0 of being realized, it is clear that reasonable beliefs impose no restriction in the beliefs to be formed at stage $(t+1)$. Therefore, the above beliefs might be part of a reasonable assessment. On the other hand, it is not difficult to

[22]One important restriction of both reasonable and consistent assessments is that the "stories" made at the different information sets are the same for all players, *i.e.*, there is an agreement on the "stories" used to form the beliefs. As already argued when introducing property B(iv), although this might seem too demanding, it follows the spirit of Nash equilibrium, which assumes that all the players agree on the strategy profile that is to be played.

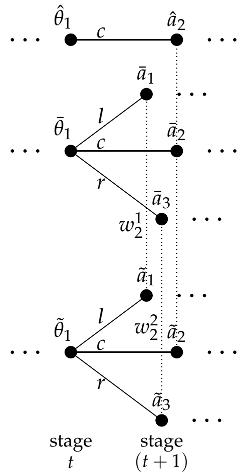


Figure 4.6.1. A difference between reasonable and consistent assessments.

check that they can never be part of a consistent assessment. Informally, just note that, when justifying his beliefs after history h_l, he has to come up with a "story" (sequence of trembles) for which the relative probabilities of $\bar{\theta}_1$ and $\tilde{\theta}_1$ at stage t are such that the former is infinitely more likely than the latter. Conversely, to justify his beliefs after history h_r, the "story" has to make $\bar{\theta}_1$ infinitely less likely than $\tilde{\theta}_1$ at stage t. However, these two "stories" are not consistent with each other. The key is that with a consistent assessment, even if two nodes in the same information set are given probability 0, any sequence of trembles implicitly pins down the relative probabilities between them. Exercise 4.10 asks the reader to formalize the latter argument. $\diamond$

Remark 4.6.1. It is worth mentioning that in Fudenberg and Tirole (1991b) the authors define *extended assessments* that can capture relative probabilities and extend conditions B(i)-B(iv) to get *reasonable extended assessments*, where Bayes rule is also used for relative probabilities. Then, they define *perfect extended Bayesian equilibrium* as a sequentially rational and reasonable extended assessment. This extended Bayesian approach also accounts for the issue of having inconsistent "stories" at different information sets in the sense discussed in the above example. However, even reasonable extended assessments are weaker than consistent assessments; this last result is shown in Battigalli (1996), where the author also studies more refinements of subgame perfection, based on different restrictions on players' assessments.

4.7. An Outline of Harsanyi's Approach

In this section we present a concise description of the main difficulties one has to face when working with games with incomplete information and the way Harsanyi managed to circumvent them.[23]

4.7.1. A first approach to incomplete information: Hierarchies of beliefs. Since the payoffs of a game with incomplete information might depend on the state of the world, in order to formally analyze a game with incomplete information it is important to know the beliefs of the players about the underlying uncertainty, *i.e.*, about Ω. In addition, i's beliefs about the other players' beliefs might also be relevant for him to make his choices, and also i's beliefs on the beliefs the others have about, for instance, i's own beliefs and so on. This infinite regress leads to the concept of *(infinite) hierarchies of beliefs* of the players. The beliefs of a player about the state of the world can be represented by an element of $\Delta\Omega$. Such beliefs are referred to as *first order beliefs*, *i.e.*, the direct beliefs of a player about what the state of the world is. As noted above, for the strategic analysis it is also important to know the beliefs of each player about the first order beliefs of the others at each state of the world. These new beliefs are called *second order beliefs*. The second order beliefs of a player are defined through an element of $\Delta\big((\Delta\Omega)^{n-1} \times \Omega\big)$. In order to completely specify the beliefs of the players about the uncertain elements of the game, we should define the infinite hierarchies of beliefs:

- **First order beliefs:** $\beta^1 \in \Delta\Omega = \mathscr{H}^1$.
- **Second order beliefs:** $\beta^2 \in \Delta\big((\Delta\Omega)^{n-1} \times \Omega\big) = \mathscr{H}^2$. Equivalently, $\mathscr{H}^2 = \Delta\big((\mathscr{H}^1)^{n-1} \times \Omega\big)$.
- **Third order beliefs:** $\beta^3 \in \Delta\Big(\big(\Delta((\Delta\Omega)^{n-1} \times \Omega)\big)^{n-1} \times \Omega\Big) = \mathscr{H}^3$. Equivalently, $\mathscr{H}^3 = \Delta\big((\mathscr{H}^2)^{n-1} \times \Omega\big)$.
- $\dots$
- **k-th order beliefs:** $\beta^k \in \Delta\big((\mathscr{H}^{k-1})^{n-1} \times \Omega\big) = \mathscr{H}^k$.
- $\dots$

In addition, the above hierarchies of beliefs must be consistent with one another. More precisely, for each k, the k order beliefs can be used to recover the $k-1$ order beliefs. For instance, if for a given player we take the marginal distribution over Ω of his second order beliefs, we should get his first order beliefs. Hence, inductively, from the k order beliefs we should be able to recover all lower order beliefs.

[23] The reader should be aware that, despite the absence of formal results and proofs, the exposition in this section is quite involved, with many notions condensed in just a couple of pages.

So defined, the hierarchies of beliefs provide an exhaustive description of all the uncertainty underlying a game with incomplete information. Yet, they are a mathematical object very difficult to deal with. Thus, because of this, Harsanyi (1967-68) devised an alternative approach that has been in the basis of the literature on games with incomplete information thereafter.

Yet, before trying to make this approach more tractable, it would be interesting to know to what extent we need to endow the players with such elaborate cognitive structures, *i.e.*, to know whether this unbounded sophistication of the players has relevant strategic implications.[24] Rubinstein (1989) introduced the so-called "electronic mail game" to show that there is a gap between the strategic behavior of (rational) players that can form infinite order beliefs, and the behavior when they can only form finite order beliefs (no matter how high). In this game there is a strategy profile that is a Nash equilibrium if the payoffs of the game are common knowledge and which, under "almost common knowledge", is never an equilibrium. The relevance of this "discontinuity" of strategic behavior with respect to knowledge and beliefs is formally discussed in Monderer and Samet (1989) and, more recently, in Dekel et al. (2006), Ely and Peski (2007), and Chen et al. (2009). Also, a related branch of the literature is that of bounded rationality, in which different bounds are imposed in the sophistication of the players (see Rubinstein (1998) for a complete reference on this topic).

4.7.2. Harsanyi's approach (1967-68). According to Harsanyi's interpretation, a game with incomplete information can be seen as an extensive game with imperfect information in the following way.

Nature moves: There is a first stage in which nature makes its move and a state of the world $\omega \in \Omega$ is realized.

Players receive information: Each player receives some information about ω. This information might vary from player to player. To model this differential information we complement Ω with an information model with common prior, namely $I = (\Omega, \rho, \{\mathcal{P}_i\}_{i \in N})$ (see Section 2.10), that is, for each $\omega \in \Omega$, each player i knows that he is at some state in $P_i(\omega) \in \mathcal{P}_i$. From the point of view of the underlying extensive game, each set in $\mathcal{P}_i$ corresponds with an information set of player i.

Players move: Players can choose different actions at the different states of the world as long as they are *consistent* with the information they have,[25] *i.e.*, each player has to play the same action at all the nodes within the same information set.

[24] This relates to the topics discussed in Section 2.10 (Example 2.10.2) and Section 2.12 (Remark 2.12.6).

[25] Consistent in the sense of Definition 2.11.1.

Payoffs are realized: The players receive the payoffs corresponding to the chosen actions at the state of the world realized (chosen by nature) in the first stage.

Remark 4.7.1. Consider a game with incomplete information in which, for each player $i \in N$, his set of strategies is A_i and his payoff function is $u_i \colon \Omega \times A \to \mathbb{R}$. Then, we can already define a *Bayesian game* as a 4-tuple (Ω, I, A, u). A *pure strategy* of player i would be a map $\hat{a}_i \colon \Omega \to A_i$ consistent with I (*i.e.*, if $\hat{\omega} \in P_i(\omega)$, then $\hat{a}_i(\omega) = \hat{a}_i(\hat{\omega})$). $\hat{A}_i$ being the set of pure strategies of i and $\hat{A} := \prod_{i=1}^{n} \hat{A}_i$. For each $i \in N$, his *(Bayesian) payoff function* $\hat{u}_i \colon \hat{A} \to \mathbb{R}$ being, for each $\hat{a} \in \hat{A}$, $\hat{u}_i(\hat{a}) := \sum_{\omega \in \Omega} u_i(\omega, \hat{a}(\omega))\rho_i(\omega)$. The definitions of behavior strategies, mixed strategies, and Nash equilibrium would be straightforward (recall the similarities between this model and the one we introduced when studying the correlated equilibrium in Section 2.11). However, under this approach, the information model has to be rich enough to capture all the information of the infinite hierarchies of beliefs described above. Harsanyi's idea was to define Bayesian games in a way in which dealing with hierarchies of beliefs or information models was not necessary, and this is what we present below.[26]

Harsanyi's approach consists of summarizing the whole system of beliefs each player might have at a given state of the world into a single entity: the *type* of the player. Formally, for each player $i \in N$, his type is an element of $\prod_{k=1}^{\infty} \mathscr{H}^k$. Let Θ_i denote the set of possible types for i in the given game and $\Theta := \prod_{i \in N} \Theta_i$. Now, each type of player i has beliefs about the types of the others and about the states of the world. These beliefs are modeled through the *belief mappings*:

$$\rho_i \colon \Theta_i \longrightarrow \Delta(\Theta_{-i} \times \Omega).$$

Although the beliefs of the players might be "incompatible" with each other,[27] if they share a common prior $\rho \in \Delta(\Theta \times \Omega)$, then $\rho_i(\theta_i) = \rho(\cdot\,|_{\theta_i})$.[28] Hence, under the type-spaces representation, the set Ω contains the payoff-relevant states of the world and the type-spaces contain all the information about the different (hierarchies of) beliefs the players might have. Further, Harsanyi showed that, by appropriately enriching the type-spaces of the players, the set Ω may be dispensed with in the description of a Bayesian game.

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[26] So far, we are using the expression *Bayesian game* to refer to general games in which, as the game unfolds, the players update their beliefs using Bayes rule.

[27] Refer to Example 2.10.1 for an explanation of what we mean by incompatible.

[28] In the previous equality we have slightly abused notation, since $\rho_i(\theta_i) \in \Delta(\Theta_{-i} \times \Omega)$ and $\rho(\cdot\,|_{\theta_i}) \in \Delta(\Theta \times \Omega)$. Nonetheless, there is no confusion about what the intended identification is.

A natural question now is whether, for each game with incomplete information in which players have infinite hierarchies of beliefs, there are Harsanyi type-spaces representing those beliefs, *i.e.*, whether the Harsanyi type-spaces suffice to capture all the uncertainty of any game with incomplete information. This is not a trivial problem and we do not tackle it in this book; for a formal treatment of hierarchies of beliefs and Harsanyi type-spaces we refer the reader to Harsanyi (1967-68), Mertens and Zamir (1985), and Brandenburger and Dekel (1993). Besides, the latter paper also relates the type-spaces framework with the standard models of knowledge we used in Sections 2.10-2.12.

Remark 4.7.2. It is worth noting that, despite the complexity of the above definitions, the players themselves need not have any idea about what a Harsanyi type-space is. We just think of the players as people with beliefs about the underlying game and we use Harsanyi type-spaces to model such a situation.

Exercises of Chapter 4

4.1. Prove Proposition 4.2.1.

4.2. Write formally the Bayesian game described in Example 4.2.2 and study its Bayesian equilibria.

4.3. Show that the expected revenue of the seller in first-price and second-price auctions is the expectation of the second highest valuation (type).

4.4. One seller and one buyer are about to trade one unit of a certain good. The seller (player 1) has an initial valuation of θ_s and the buyer (player 2) has a initial valuation of θ_b, where θ_s and θ_b are independently drawn from the uniform distribution in $[0,1]$. Valuations are private. Both players simultaneously choose bids $b_1, b_2 \in [0,1]$. If $b_1 < b_2$, then transaction does not take place and each one gets zero. If $b_1 \geq b_2$, then transaction occurs at price $(b_1+b_2)/2$.

- Formally describe this situation as a Bayesian game.
- Show that given $v \in (0,1)$, the strategy profile $(\tilde{a}_1, \tilde{a}_2)$ defined below is a Bayesian equilibrium:

$$\tilde{a}_1(\theta_s) = \begin{cases} v & \theta_s \leq v \\ 1 & \theta_s > v \end{cases} \quad \text{and} \quad \tilde{a}_2(\theta_b) = \begin{cases} v & \theta_b \geq v \\ 0 & \theta_b < v. \end{cases}$$

- Now, consider the strategies that, for each $i \in \{1,2\}$, are of the form $\tilde{a}_i(\theta) = \alpha_i + \beta_i\theta$, with $\alpha_i, \beta_i \geq 0$. Determine the values of the α_i and β_i parameters, such that $(\tilde{a}_1, \tilde{a}_2)$ is a Bayesian equilibrium.

4.5. Consider a first-price auction where there are only two possible valuations, $\underline{v}$ and $\overline{v}$, with $\underline{v} < \overline{v}$. Assume the valuations are private and independently drawn according to the probabilities $\rho(\overline{v}) = \rho(\underline{v}) = 1/2$. Assume that the admissible bids belong to the set $\{\overline{v}, \underline{v}, (\overline{v} + \underline{v})/2\}$.

- Describe the situation as a Bayesian game.
- Obtain a Bayesian equilibrium.

4.6. *War of attrition* (Maynard Smith 1974). There are two players. Each player chooses a nonnegative number; this number represents the moment in time he quits the game. Both players choose simultaneously. Each player $i \in \{1, 2\}$ has a type $\theta_i \geq 0$. Types are private information and independently drawn from an exponential distribution with mean 1, *i.e.*, for each $\theta_i \geq 0$, $f(\theta_i) = e^{-\theta_i}$. The player that quits last, wins. For each $i \in \{1, 2\}$, each type profile $\theta = (\theta_1, \theta_2)$, and each strategy profile $\tilde{a}$, i's payoffs are given by

$$u_i(\theta, \tilde{a}) = \begin{cases} -\tilde{a}_i(\theta_i) & \tilde{a}_i(\theta_i) \leq \tilde{a}_j(\theta_j) \\ \theta_i - \tilde{a}_j(\theta_j) & \tilde{a}_i(\theta_i) > \tilde{a}_j(\theta_j). \end{cases}$$

This specific war of attrition game follows the spirit of the *chicken game* (Example 2.11.4): Staying in the game is costly, but there is a price for staying longer than the opponent. Obtain a Bayesian equilibrium of the game.

4.7. Formally write the definitions of behavior and mixed strategies in a mechanism and check that Theorem 4.5.1 also holds for mixed and behavior Bayesian equilibria.

4.8. The problem of auction design is a particular case of mechanism design. Rewrite the definitions of first-price and second-price auctions in Section 4.4 as mechanisms (according to Definition 4.5.1). Discuss the implications of the revelation principle within this setting.

4.9. Prove Proposition 4.6.1.

4.10. Prove that the beliefs described in Example 4.6.1 are not consistent.

Chapter 5

Cooperative Games

5.1. Introduction to Cooperative Games

So far we have been concerned with noncooperative models, where the main focus is on the strategic aspects of the interaction among the players. The approach in cooperative game theory is different. Now, it is assumed that players can commit to behave in a way that is socially optimal. The main issue is how to share the benefits arising from cooperation. Important elements in this approach are the different subgroups of players, referred to as *coalitions*, and the set of outcomes that each coalition can get regardless of what the players outside the coalition do.[1] When discussing the different equilibrium concepts for noncooperative games, we were concerned about whether a given strategy profile was self-enforcing or not, in the sense that no player had incentives to deviate. We now assume that players can make binding agreements and, hence, instead of being worried about issues like self-enforceability, we care about notions like fairness and equity.

In this chapter, as customary, the set of players is denoted by $N := \{1, \ldots, n\}$. As opposed to noncooperative games, where most of the analysis was done at the individual level, coalitions are very important in cooperative models. For each $S \subset N$, we refer to S as a *coalition*, with $|S|$ denoting the number of players in S. Coalition N is often referred to as the *grand coalition*.

[1] In Peleg and Sudhölter (2003, Chapter 11), the authors discuss in detail some relations between the two approaches and, in particular, they derive the definition of *cooperative game without transferable utility* (Definition 5.2.1 below) from a strategic game in which the players are allowed to form coalitions and use them to coordinate their strategies through binding agreements.

We start this chapter by briefly describing the most general class of cooperative games, the so called *nontransferable utility games*. Then, we discuss two important subclasses: *bargaining problems* and *transferable utility games*. For each of these two classes we present the most important solution concepts and some axiomatic characterizations. Moreover, Section 5.9 is devoted to *implementation theory*, where we discuss how some of these solution concepts can be obtained as the equilibrium outcomes of different noncooperative games. Then, we discuss three relevant problems that can be studied with the tools of cooperative game theory: *airport problems*, *bankruptcy problems*, and *voting problems*. Finally, we conclude this chapter by presenting some applications of cooperative game theory to the study of operations research problems.

5.2. Nontransferable Utility Games

In this section we present a brief introduction to the most general class of cooperative games: *nontransferable utility games* or *NTU-game*. The main source of generality comes from the fact that, although binding agreements between the players are implicitly assumed to be possible, utility is not transferable across players. Below, we present the formal definition and then we illustrate it with an example.

Given $S \subset N$ and a set $A \subset \mathbb{R}^S$, we say that A is *comprehensive* if, for each pair $x, y \in \mathbb{R}^S$ such that $x \in A$ and $y \leq x$, we have that $y \in A$. Moreover, the *comprehensive hull* of a set A is the smallest comprehensive set containing A.

Definition 5.2.1. An n-player *nontransferable utility game* (*NTU-game*) is a pair (N, V), where N is the set of players and V is a function that assigns, to each coalition $S \subset N$, a set $V(S) \subset \mathbb{R}^S$. By convention, $V(\emptyset) := \{0\}$. Moreover, for each $S \subset N$, $S \neq \emptyset$:

i) $V(S)$ is a nonempty and closed subset of $\mathbb{R}^S$.
ii) $V(S)$ is *comprehensive*. Moreover, for each $i \in N$, $V(\{i\}) \neq \mathbb{R}$, *i.e.*, there is $v_i \in \mathbb{R}$ such that $V(\{i\}) = (-\infty, v_i]$.
iii) The set $V(S) \cap \{y \in \mathbb{R}^S : \text{ for each } i \in S, y_i \geq v_i\}$ is bounded.

Remark 5.2.1. In an NTU-game, the following elements are implicitly involved:

- For each $S \subset N$, R^S is the set of outcomes that the players in coalition S can obtain by themselves.

- For each $S \subset N$, $\{\succeq_i^S\}_{i\in S}$ are the preferences of the players in S over the outcomes in R^S. They are assumed to be complete, transitive, and representable through a utility function.[2]
- For each $S \subset N$, $\{U_i^S\}_{i\in S}$ are the utility functions of the players, which represent their preferences on R^S.

Hence, an NTU-game is a "simplification" in which, for each $S \subset N$ and each $x \in V(S)$, there is an outcome $r \in R^S$ such that, for each $i \in S$, $x_i = U_i^S(r)$.

Remark 5.2.2. It is common to find slightly different (and often equivalent) conditions for the $V(S)$ sets in the definition of an NTU-game. In each case, the authors choose the definition that is more suitable for their specific objectives. Nonemptiness and closedness are two technical requirements, which are also fairly natural. Requiring the $V(S)$ sets to be comprehensive is a convenient assumption, whose basic idea is that the players in coalition S can throw away utility if they want to. Moreover, it is worth mentioning that it is also often assumed that the $V(S)$ sets are convex, which would imply, in particular, that the players inside each coalition S can choose lotteries over the elements of R^S and that their utility functions are of the von Neumann and Morgenstern type.

Definition 5.2.2. Let (N,V) be an NTU-game. Then, the vectors in $\mathbb{R}^N$ are called *allocations*. An allocation $x \in \mathbb{R}^N$ is *feasible* if there is a partition $\{S_1,\dots,S_k\}$ of N satisfying that, for each $l \in \{1,\dots k\}$, there is $y \in V(S_l)$ such that, for each $i \in S_l$, $y_i = x_i$.

Example 5.2.1. (The banker game (Owen 1972)). Consider the NTU-game given by:

$$\begin{aligned} v(\{i\}) &= \{x_i : x_i \leq 0\}, \quad i \in \{1,2,3\}, \\ v(\{1,2\}) &= \{(x_1,x_2) : x_1 + 4x_2 \leq 1000 \text{ and } x_1 \leq 1000\}, \\ v(\{1,3\}) &= \{(x_1,x_3) : x_1 \leq 0 \text{ and } x_3 \leq 0\}, \\ v(\{2,3\}) &= \{(x_2,x_3) : x_2 \leq 0 \text{ and } x_3 \leq 0\}, \\ v(N) &= \{(x_1,x_2,x_3) : x_1 + x_2 + x_3 \leq 1000\}. \end{aligned}$$

One can think of this game in the following way. On its own, no player can get anything. Player 1, with the help of player 2, can get 1000 dollars. Player 1 can reward player 2 by sending him money, but the money sent is lost or stolen with probability 0.75. Player 3 is a banker, so player 1 can ensure his transactions are safely delivered to player 2 by using player 3 as an intermediary. Hence, the question is how much should player 1 pay to player 2 for his help to get the 1000 dollars and how much to player 3 for

[2] There is a countable subset of R^S that is order dense in R^S (see Theorem 1.2.3).

helping him to make costless transactions to player 2. The reason for referring to these games as nontransferable utility games is that some transfers among the players may not be allowed. In this example, for instance, $(1000, 0)$ belongs to $v(\{1,2\})$, but players 1 and 2 cannot agree to the share $(500, 500)$ without the help of player 3. In Section 5.4 we define games with transferable utility, in which all transfers are assumed to be possible. $\diamond$

The main objective of the theoretical analysis in this field is to find appropriate rules for choosing feasible allocations for the general class of NTU-games. These rules are referred to as *solutions* and aim to select allocations that have desirable properties according to different criteria such as equity, fairness, and stability. If a solution selects a single allocation for each game, then it is commonly referred to as an *allocation rule*. The definition of NTU-game allows us to model a wide variety of situations and yet, at the same time, because of its generality, the study of NTU-games quickly becomes mathematically involved. Thus, because of this, the literature has focused more on studying some special cases than on studying the general framework. In this book we follow the same approach and do not cover NTU-games in full generality. For discussions and characterizations of different solution concepts in the general setting, the reader may refer, for instance, to Aumann (1961, 1985), Kalai and Samet (1985), Hart (1985), Borm et al. (1992), and to the book by Peleg and Sudhölter (2003). We discuss the two most relevant subclasses of NTU-games: *bargaining problems* and *games with transferable utility* (TU-games).

5.3. Bargaining

In this section we study a special class of NTU-games, referred to as *bargaining problems*, originally studied in Nash (1950a). In a bargaining problem, there is a set of possible allocations, the *feasible set* F, and one of them has to be chosen by the players. Importantly, all the players have to agree on the chosen allocation; otherwise, the realized allocation is d, the *disagreement point*.

Definition 5.3.1. An n-player *bargaining problem* with set of players N is a pair (F, d) whose elements are the following:

- **Feasible set:** F is the comprehensive hull of a compact and convex subset of $\mathbb{R}^N$.
- **Disagreement point:** d is an allocation in F. It is assumed that there is $x \in F$ such that $x > d$.

Remark 5.3.1. An n-player bargaining problem (F, d) can be seen as an NTU-game (N, V), where $V(N) := F$ and, for each nonempty coalition $S \neq N$, $V(S) := \{y \in \mathbb{R}^S : \text{ for each } i \in S, y_i \leq d_i\}$.

Hence, given a bargaining problem (F,d), the feasible set represents the utilities the players get from the outcomes associated with the available agreements. The disagreement point delivers the utilities in the case in which no agreement is reached. The assumptions on the feasible set are mathematically convenient and, at the same time, natural and not too restrictive. As we already pointed out when introducing NTU-games, the convexity assumption can be interpreted as the ability of the players to choose lotteries over the possible agreements, with the utilities over lotteries being derived by means of von Neumann and Morgenstern utility functions.

We denote the set of n-player bargaining problems by B^N. Moreover, given a bargaining problem $(F,d) \in B^N$, we define the compact set $F_d := \{x \in F : x \geq d\}$. Given two allocations $x, y \in F$, we say that x is *Pareto dominated by y* or that *y Pareto dominates x* if $x \leq y$ and $x \neq y$, *i.e.*, for each $i \in N$, $x_i \leq y_i$, with strict inequality for at least one player. An allocation $x \in F$ is *Pareto efficient in F*, or just *efficient*, if no allocation in F Pareto dominates x.[3]

We now discuss several solution concepts for n-player bargaining problems. More precisely, we study *allocation rules*.

Definition 5.3.2. An *allocation rule* for n-player bargaining problems is a map $\varphi\colon B^N \to \mathbb{R}^N$ such that, for each $(F,d) \in B^N$, $\varphi(F,d) \in F_d$.

In the seminal paper by Nash (1950a), he gave some appealing properties that an allocation rule should satisfy and then proved that they characterize a unique allocation rule, which is known as the *Nash solution*.[4]

Let φ be an allocation rule and consider the following properties we may impose on it.

Pareto Efficiency (EFF): The allocation rule φ satisfies EFF if, for each $(F,d) \in B^N$, $\varphi(F,d)$ is a Pareto efficient allocation.

Symmetry (SYM): Let π denote a permutation of the elements of N and, given $x \in \mathbb{R}^N$, let x^π be defined, for each $i \in N$, by $x_i^\pi := x_{\pi(i)}$. We say that a bargaining problem $(F,d) \in B^N$ is *symmetric* if, for each permutation π of the elements of N, we have that i) $d^\pi = d$ and ii) for each $x \in F$, $x^\pi \in F$.

Now, φ satisfies SYM if, for each symmetric bargaining problem $(F,d) \in B^N$, we have that, for each pair $i,j \in N$, $\varphi_i(F,d) = \varphi_j(F,d)$.

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[3] Pareto efficiency is also known as Pareto optimality, the term being named after the studies of the economist V. Pareto in the early twentieth century.

[4] Nash introduced his allocation rule for the two-player case; here, we present a straightforward generalization to the n-player case.

Covariance with positive affine transformations (CAT): $f^A\colon \mathbb{R}^N \to \mathbb{R}^N$ is a *positive affine transformation* if, for each $i \in N$, there are $a_i, b_i \in \mathbb{R}$, with $a_i > 0$, such that, for each $x \in \mathbb{R}^N$, $f_i^A(x) = a_i x_i + b_i$. Now, φ satisfies CAT if, for each $(F,d) \in B^N$ and each positive affine transformation f^A,

$$\varphi(f^A(F), f^A(d)) = f^A(\varphi(F,d)).$$

Independence of irrelevant alternatives (IIA): φ satisfies IIA if, for each pair of problems $(F,d), (\hat{F},d) \in B^N$, with $\hat{F} \subset F$, $\varphi(F,d) \in \hat{F}$ implies that $\varphi(\hat{F},d) = \varphi(F,d)$.

The four properties above are certainly appealing. EFF and SYM are very natural and CAT states that the choice of the utility representations should not affect the allocation rule (see Theorem 1.3.3). The most controversial one, as we will see later on, is IIA. However, the interpretation of this property is clear and reasonable: If the feasible set is reduced and the proposal of the allocation rule for the original problem is still feasible in the new one, then the allocation rule has to make the same proposal in the new problem. Before giving the definition of the Nash solution we need an auxiliary result.

Throughout this chapter, given $(F,d) \in B^N$, let $g^d\colon \mathbb{R}^N \to \mathbb{R}$ be defined, for each $x \in \mathbb{R}^N$, by $g^d(x) := \prod_{i\in N}(x_i - d_i)$, *i.e.*, if $x > d$, $g^d(x)$ represents the product of the gains of the players at x with respect to their utilities at d.

Proposition 5.3.1. *Let $(F,d) \in B^N$. Then, there is a unique $z \in F_d$ that maximizes the function g^d over the set F_d.*

Proof. Since g^d is continuous and F_d is compact, g^d has a maximum in F_d. Suppose that there are $z, \hat{z} \in F_d$, with $z \neq \hat{z}$, such that

$$\max_{x\in F_d} g^d(x) = g^d(z) = g^d(\hat{z}).$$

For each $i \in N$, $z_i > d_i$ and $\hat{z}_i > d_i$. By the convexity of F_d, $\bar{z} := \frac{z}{2} + \frac{\hat{z}}{2} \in F_d$. We now show that $g^d(\bar{z}) > g^d(z)$, which is a contradiction with the fact that z is a maximum. We have that $\ln(g^d(\bar{z})) = \sum_{i\in N}\ln(\bar{z}_i - d_i) = \sum_{i\in N}\ln(\frac{z_i-d_i}{2} + \frac{\hat{z}_i-d_i}{2})$, which, by the strict concavity of the logarithmic functions, is strictly larger than $\sum_{i\in N}(\frac{1}{2}\ln(z_i - d_i) + \frac{1}{2}\ln(\hat{z}_i - d_i)) = \frac{1}{2}\ln(g^d(z)) + \frac{1}{2}\ln(g^d(\hat{z})) = \ln(g^d(z))$ and, hence, $g^d(\bar{z}) > g^d(z)$. □

Definition 5.3.3. The Nash solution, NA, is defined, for each bargaining problem $(F,d) \in B^N$, by $\mathrm{NA}(F,d) := z$, where $g^d(z) = \max_{x\in F_d} g^d(x) = \max_{x\in F_d}\prod_{i\in N}(x_i - d_i)$.

Proposition 5.3.1 ensures that the Nash solution is a well defined allocation rule. Given a bargaining problem $(F,d) \in B^N$, the Nash solution selects the unique allocation in F_d that maximizes the product of the gains of the players with respect to the disagreement point. This allows for a nice geometric interpretation, which, for $n = 2$, says that the Nash solution is the point z in F_d that maximizes the area of the rectangle with vertices z, d, (z_1, d_2) and (d_1, z_2). Below, we present the axiomatic characterization that Nash (1950a) provided for this allocation rule. First, we need an auxiliary lemma.

Lemma 5.3.2. *Let $(F,d) \in B^N$ and let $z := \mathrm{NA}(F,d)$. For each $x \in \mathbb{R}^N$, let $h(x) := \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) x_i$. Then, for each $x \in F$, $h(x) \leq h(z)$.*

Proof. Suppose that there is $x \in F$ with $h(x) > h(z)$. For each $\varepsilon \in (0,1)$, let $x^\varepsilon := \varepsilon x + (1-\varepsilon) z$. By the convexity of F, $x^\varepsilon \in F$. Since $z \in F_d$ and $z > d$ then, for sufficiently small ε, $x^\varepsilon \in F_d$. Moreover,

$$\begin{aligned} g^d(x^\varepsilon) &= \prod_{i \in N} (z_i - d_i + \varepsilon(x_i - z_i)) \\ &= \prod_{i \in N} (z_i - d_i) + \varepsilon \sum_{i \in N} \prod_{j \neq i} (z_j - d_j)(x_i - z_i) + \sum_{i=2}^{n} \varepsilon^i f_i(x,z,d) \\ &= g^d(z) + \varepsilon(h(x) - h(z)) + \sum_{i=2}^{n} \varepsilon^i f_i(x,z,d), \end{aligned}$$

where every $f_i(x,z,d)$ is a function that only depends on x, z, and d. Then, since $h(x) > h(z)$, $g^d(x^\varepsilon)$ is greater than $g^d(z)$ for a sufficiently small ε, which contradicts that $z = \max_{x \in F_d} g^d(x)$. $\square$

Theorem 5.3.3. *The Nash solution is the unique allocation rule for n-player bargaining problems that satisfies* EFF, SYM, CAT, *and* IIA.

Proof. It can be easily checked that NA satisfies EFF, SYM, CAT, and IIA. Let φ be an allocation rule for n-player bargaining problems that satisfies the four properties and let $(F,d) \in B^N$. Let $z := \mathrm{NA}(F,d)$. We now show that $\varphi(F,d) = z$. Let $U := \{x \in \mathbb{R}^N : h(x) \leq h(z)\}$, where h is defined as in Lemma 5.3.2, which, in turn, ensures that $F \subset U$. Let f^A be the positive affine transformation that associates, to each $x \in \mathbb{R}^N$, the vector $(f_1^A(x), \ldots, f_n^A(x))$ where, for each $i \in N$,

$$f_i^A(x) := \frac{1}{z_i - d_i} x_i - \frac{d_i}{z_i - d_i}.$$

Now, we compute $f^A(U)$.

$$\begin{aligned} f^A(U) &= \{y \in \mathbb{R}^N : (f^A)^{-1}(y) \in U\} \\ &= \{y \in \mathbb{R}^N : h((f^A)^{-1}(y)) \leq h(z)\} \\ &= \{y \in \mathbb{R}^N : h((z_1 - d_1)y_1 + d_1, \dots, (z_n - d_n)y_n + d_n) \leq h(z)\} \\ &= \{y \in \mathbb{R}^N : \sum_{i \in N} \prod_{j \neq i} (z_j - d_j)((z_i - d_i)y_i + d_i) \leq \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) z_i\}, \end{aligned}$$

which, after straightforward algebra, leads to

$$\begin{aligned} f^A(U) &= \{y \in \mathbb{R}^N : \sum_{i \in N} \prod_{j \in N} (z_j - d_j) y_i \leq \sum_{i \in N} \prod_{j \in N} (z_j - d_j)\} \\ &= \{y \in \mathbb{R}^N : \prod_{j \in N} (z_j - d_j) \sum_{i \in N} y_i \leq \prod_{j \in N} (z_j - d_j) \sum_{i \in N} 1\} \\ &= \{y \in \mathbb{R}^N : \sum_{i \in N} y_i \leq n\}. \end{aligned}$$

Note that $f^A(d) = (0, \dots, 0)$ and, hence, $(f^A(U), f^A(d))$ is a symmetric bargaining problem. Since φ satisfies EFF and SYM, $\varphi(f^A(U), f^A(d)) = (1, \dots, 1)$. Since φ also satisfies CAT, $\varphi(U, d) = (f^A)^{-1}((1, \dots, 1)) = z$. Finally, since $z \in F$, $F \subset U$, and φ satisfies IIA, $\varphi(F, d) = z$. □

After the characterization result above, it is natural to wonder if the result is tight or, in other words, if any of the axioms is superfluous.

Proposition 5.3.4. *None of the axioms used in the characterization of the Nash solution given by Theorem 5.3.3 is superfluous.*

Proof. We show that, for each of the axioms in the characterization, there is an allocation rule different from the Nash solution that satisfies the remaining three.

Remove EFF: The allocation rule φ defined, for each bargaining problem (F, d), by $\varphi(F, d) := d$ satisfies SYM, CAT, and IIA.

Remove SYM: Let φ be the allocation rule defined as follows. Let (F, d) be a bargaining problem. Then, $\varphi_1(F, d) := \max_{x \in F_d} x_1$. For each $i > 1$, $\varphi_i(F, d) := \max_{x \in F_d^i} x_i$, where $F_d^i := \{x \in F_d : \text{ for each } j < i, x_j = \varphi_j(F, d)\}$. These kind of solutions are known as *serial dictatorships*, since there is an ordering of the players and each player chooses the allocation he prefers among those that are left when he is given the turn to choose. This allocation rule satisfies EFF, CAT, and IIA.

Remove CAT: Let φ be the allocation rule that, for each bargaining problem (F, d), selects the allocation $\varphi(F, d) := d + \bar{t}(1, \dots, 1)$, where $\bar{t} := \max\{t \in \mathbb{R} : d + t(1, \dots, 1) \in F_d\}$ (the compactness of F_d ensures that $\bar{t}$ is well defined). This allocation rule, known as the *egalitarian solution* (Kalai 1977), satisfies EFF, SYM, and IIA. Moreover, since it is generally different from the Nash solution, by Theorem 5.3.3, it cannot satisfy CAT. Exercise 5.1 asks the reader to provide a direct proof of the above fact.

Remove IIA: The *Kalai-Smorodinsky solution*, which we present below, satisfies EFF, SYM, and CAT. Since this solution is different from the Nash solution, by Theorem 5.3.3, it cannot satisfy IIA. The latter is also illustrated in Example 5.3.2. □

As we have already mentioned, the only property in the characterization of the Nash solution that might be controversial is IIA. Now, we show an example that illustrates the main criticism against this property.

Example 5.3.1. Consider the two-player bargaining problem (F, d), where $d = (0, 0)$ and F is the comprehensive hull of the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 2\}$. Since NA satisfies SYM and EFF, then $\mathrm{NA}(F, d) = (1, 1)$. Consider now the problem $(\tilde{F}, d)$, where $\tilde{F}$ is the intersection of F with the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 1\}$. Since NA satisfies IIA, $\tilde{F} \subset F$, and $\mathrm{NA}(F, d) \in \tilde{F}$, then $\mathrm{NA}(\tilde{F}, d) = (1, 1)$. These two bargaining problems and the corresponding proposals made by the Nash solution are depicted in Figure 5.3.1.

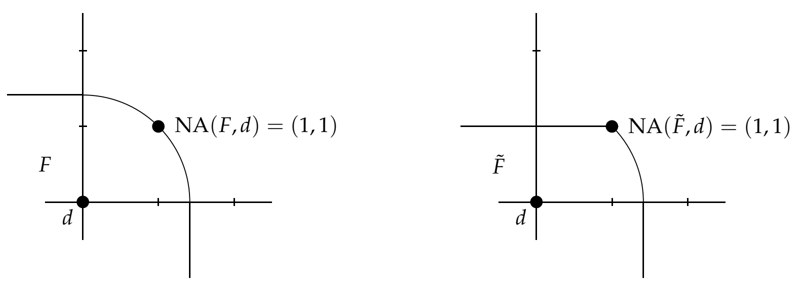


Figure 5.3.1. The Nash solution in Example 5.3.1.

This example suggests that the Nash solution is quite insensitive to the *aspirations* of the players. The maximal aspiration of player 2 in F is larger than in $\tilde{F}$ (meanwhile, the maximal aspiration of player 1 does not change), but the Nash solution proposes the same allocation for player 2 in both problems; indeed, in $(\tilde{F}, d)$, it proposes his maximal aspiration. ◇

In Kalai and Smorodinsky (1975), an alternative allocation rule was introduced for two-player bargaining problems. This new solution strongly depends on the aspiration levels of the players[5]. These levels give rise to the so-called *utopia point*, which we define below. Then, we introduce the Kalai-Smorodinsky solution for n-player bargaining problems.

[5] This solution had also been discussed in Raiffa (1953). For this reason, some texts refer to it as the Raiffa-Kalai-Smorodinsky solution.

Definition 5.3.4. The *utopia point* of a bargaining problem $(F, d) \in B^N$ is given by the vector $b(F, d) \in \mathbb{R}^N$ where, for each $i \in N$, $b_i(F, d) = \max_{x \in F_d} x_i$.

Therefore, for each $i \in N$, $b_i(F, d)$ denotes the largest utility that player i can get in F_d.

Definition 5.3.5. The Kalai-Smorodinsky solution, KS, is defined, for each $(F, d) \in B^N$, by

$$\mathrm{KS}(F, d) := d + \bar{t}(b(F, d) - d),$$

where $\bar{t} := \max\{t \in \mathbb{R} : d + t(b(F, d) - d) \in F_d\}$.

Note that, in the above definition, the compactness of F_d ensures that $\bar{t}$ is well defined.

Example 5.3.2. Consider again the bargaining problems described in Example 5.3.1. Now, $b(F, d) = (\sqrt{2}, \sqrt{2})$ and $b(\tilde{F}, d) = (\sqrt{2}, 1)$. Then, it is easy to check that $\mathrm{KS}(F, d) = (1, 1)$ and $\mathrm{KS}(\tilde{F}, d) = (\sqrt{2/3}, \sqrt{1/3})$. Thus, the Kalai-Smorodinsky solution is sensitive to the less favorable situation of player 2 in $(\tilde{F}, d)$. The computation of the Kalai-Smorodinsky solution for these two bargaining problems is illustrated in Figure 5.3.2. Note that this example also illustrates that the Kalai-Smorodinsky solution does not satisfy IIA. $\diamond$

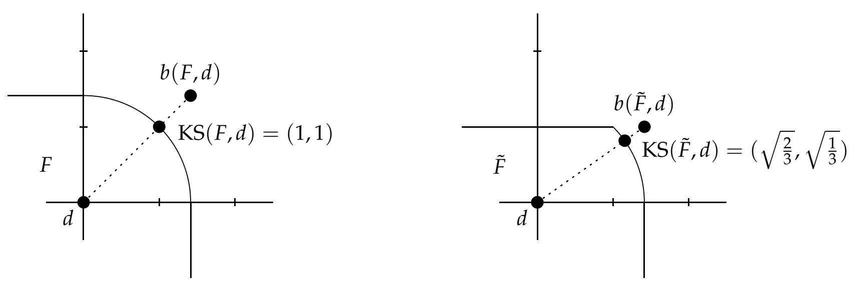


Figure 5.3.2. The Kalai-Smorodinsky solution in Example 5.3.2.

Kalai and Smorodinsky (1975) provided a characterization of their allocation rule for two-player bargaining problems that is based on the following monotonicity property.

Individual Monotonicity (IM): Let $(F, d), (\hat{F}, d) \in B^N$ be a pair of bargaining problems such that $\hat{F}_d \subset F_d$. Let $i \in N$ be such that, for each $j \neq i$, $b_j(\hat{F}, d) = b_j(F, d)$. If φ is an allocation rule for n-player bargaining problems that satisfies IM, then $\varphi_i(\hat{F}, d) \leq \varphi_i(F, d)$.

Theorem 5.3.5. *The Kalai-Smorodinsky solution is the unique allocation rule for two-player bargaining problems that satisfies* EFF, SYM, CAT, *and* IM.

Proof. It is easy to check that KS satisfies EFF, CAT, SYM, and IM. Let φ be an allocation rule for two-player bargaining problems that satisfies the four properties and let $(F,d) \in B^2$. We now show that $\varphi(F,d) = \mathrm{KS}(F,d)$. Since φ and KS satisfy CAT, we can assume, without loss of generality, that $d = (0,0)$ and that $b(F,d) = (1,1)$. Then, $\mathrm{KS}(F,d)$ lies in the segment joining $(0,0)$ and $(1,1)$ and, hence, $\mathrm{KS}_1(F,d) = \mathrm{KS}_2(F,d)$. Define $\hat{F}$ by:

$$\hat{F} := \{x \in \mathbb{R}^2 : \text{there is } y \in \operatorname{conv}\{(0,0),(0,1),(1,0),\mathrm{KS}(F,d)\} \text{ with } x \leq y\}.$$

Since φ satisfies IM, $\hat{F} \subset F$, and $b(\hat{F},d) = b(F,d)$, we have that $\varphi(\hat{F},d) \leq \varphi(F,d)$. Since $\hat{F}$ is symmetric, by SYM and EFF, $\varphi(\hat{F},d) = \mathrm{KS}(F,d)$. Then, $\varphi(\hat{F},d) = \mathrm{KS}(F,d) \leq \varphi(F,d)$ but, since $\mathrm{KS}(F,d)$ is Pareto efficient, we have $\mathrm{KS}(F,d) = \varphi(F,d)$. □

A minor modification of the proof of Proposition 5.3.4 can be used to show that none of the axioms in Theorem 5.3.5 is superfluous; Exercise 5.2 asks the reader to formally show this.

Theorem 5.3.5 provides a characterization of the Kalai-Smorodinsky solution for two-player bargaining problems. Unfortunately, this characterization cannot be generalized to $n \geq 3$, as the following proposition, taken from Roth (1979), shows.

Proposition 5.3.6. *Let $n \geq 3$. Then, there is no solution for n-player bargaining problems satisfying* EFF, SYM, *and* IM.

Proof. Let $n \geq 3$ and suppose that φ is a solution for n-player bargaining problems satisfying EFF, SYM, and IM. Let $d = (0,\ldots,0)$ and

$$\hat{F} := \{x \in \mathbb{R}^N : \text{there is } y \in \operatorname{conv}\{(0,1,\ldots,1),(1,0,1,\ldots,1)\} \text{ with } x \leq y\}.$$

By EFF, $\varphi(\hat{F},d)$ belongs to the segment joining $(0,1,\ldots,1)$ and $(1,0,1,\ldots,1)$ and, hence, $\varphi_3(\hat{F},d) = 1$. Let

$$F = \{x \in \mathbb{R}^N : \sum_{i=1}^{N} x_i \leq n-1 \text{ and, for each } i \in N, x_i \leq 1\}.$$

Since φ satisfies EFF and SYM then, for each $i \in N$, $\varphi_i(F,d) = \frac{n-1}{n}$. However, $\hat{F} \subset F$ and $b(\hat{F},d) = b(F,d) = (1,\ldots,1)$. Then, by IM, $\varphi(\hat{F},d) \leq \varphi(F,d)$, which is a contradiction with $\varphi_3(\hat{F},d) = 1 > \frac{n-1}{n} = \varphi_3(F,d)$. □

In spite of this negative result, Thomson (1980) showed that EFF, SYM, and IM characterize the Kalai-Smorodinsky solution if we restrict attention to a certain (large) domain of n-player bargaining problems.

The two allocation rules we have presented here for bargaining problems are the ones that have been most widely studied. There are, however, other proposals in the game theoretical literature. Those willing to study this topic in depth may refer to Peters (1992).

5.4. Transferable Utility Games

We now move to the most widely studied class of cooperative games: those with *transferable utility*, in short, *TU-games*. The situation is very similar to the one described in Section 5.2. The different coalitions that can be formed among the players in N can enforce certain allocations (possibly through binding agreements); the problem is to decide how the benefits generated by the cooperation of the players (formation of coalitions) have to be shared among them. However, there is one important departure from the general NTU-games framework. In a TU-game, given a coalition S and an allocation $x \in V(S) \subset \mathbb{R}^S$ that the players in S can enforce, all the allocations that can be obtained from x by transfers of utility among the players in S also belong to $V(S)$. Hence, $V(S)$ can be characterized by a single number, given by $\max_{x \in V(S)} \sum_{i \in S} x_i$. We denote the last number by $v(S)$, the *worth* of coalition S. The transferable utility assumption has important implications, both conceptually and mathematically. From the conceptual point of view, it implicitly assumes that there is a *numéraire* good (for instance, money) such that the utilities of all the players are linear with respect to it and that this good can be freely transferred among players. From the mathematical point of view, since the description of a game consists of a number for each coalition of players, TU-games are much more tractable than general NTU-games.

Definition 5.4.1. A TU-game is a pair (N, v), where N is the set of players and $v\colon 2^N \to \mathbb{R}$ is the *characteristic function* of the game. By convention, $v(\emptyset) := 0$.

In general, we interpret $v(S)$, the worth of coalition S, as the benefit that S can generate. When no confusion arises, we denote the game (N, v) by v. Also, we denote $v(\{i\})$ and $v(\{i, j\})$ by $v(i)$ and $v(ij)$, respectively. Let G^N be the class of TU-games with n players.

Remark 5.4.1. A TU-game (N, v) can be seen as an NTU-game (N, V) by defining, for each nonempty coalition $S \subset N$, $V(S) := \{y \in \mathbb{R}^S : \sum_{i \in S} y_i \leq v(S)\}$.

Definition 5.4.2. Let $(N, v) \in G^N$ and let $S \subset N$. The restriction of (N, v) to the coalition S is the TU-game (S, v_S), where, for each $T \subset S$, $v_S(T) := v(T)$.

Now, we show some examples.

Example 5.4.1. (Divide a million). A wealthy man dies and leaves one million euros to his three nephews, with the condition that at least two of them must agree on how to divide this amount among them; otherwise, the million euros will be burned. This situation can be modeled as the TU-game (N, v), where $N = \{1,2,3\}$, $v(1) = v(2) = v(3) = 0$, and $v(12) = v(13) = v(23) = v(N) = 1$. ◇

Example 5.4.2. (The glove game). Three players are willing to divide the benefits of selling a pair of gloves. Player 1 has a left glove and players 2 and 3 have one right glove each. A left-right pair of gloves can be sold for one euro. This situation can be modeled as the TU-game (N, v), where $N = \{1,2,3\}$, $v(1) = v(2) = v(3) = v(23) = 0$, and $v(12) = v(13) = v(N) = 1$. ◇

Example 5.4.3. (The Parliament of Aragón). This example illustrates that TU-games can also be used to model coalitional bargaining situations in which players negotiate with something more abstract than money. In this case we consider the Parliament of Aragón, one of the regions in which Spain is divided. After the elections which took place in May 1991, its composition was: PSOE (Socialist Party) had 30 seats, PP (Conservative Party) had 17 seats, PAR (Regionalist Party of Aragón) had 17 seats, and IU (a coalition mainly formed by communists) had 3 seats. In a Parliament, the most relevant decisions are made using the simple majority rule. We can use TU-games to measure the power of the different parties in a Parliament. This can be seen as "dividing" the power among them. A coalition is said to have the power if it collects more than half of the seats of the Parliament, 34 seats in this example. Then, this situation can be modeled as the TU-game (N, v), where $N = \{1,2,3,4\}$ (1=PSOE, 2=PP, 3=PAR, 4=IU), $v(S) = 1$ if there is $T \in \{\{1,2\},\{1,3\},\{2,3\}\}$ with $T \subset S$ and $v(S) = 0$ otherwise. In Section 5.12, we discuss a special class of TU-games, known as weighted majority games, that generalize the situation we have just described. The objective when dealing with these kind of games is to define *power indices* that measure how the total power is divided among the players. ◇

Example 5.4.4. (The visiting professor). We now illustrate how TU-games can also model situations that involve costs instead of benefits. Three research groups, from the universities of Milano (group 1), Genova (group 2), and Santiago de Compostela (group 3), plan to invite a Japanese professor to give a course on game theory. To minimize the costs, they coordinate the courses, so that the professor makes a tour visiting Milano, Genova, and Santiago de Compostela. Then, the groups want to allocate the cost of the tour among themselves. For this purpose they have estimated the travel cost (in euros) of the visit for all the possible coalitions of groups: $c(1) = 1500$, $c(2) = 1600$, $c(3) = 1900$, $c(12) = 1600$, $c(13) = 2900$, $c(23) = 3000$, $c(N) = 3000$ (for each S, $c(S)$ indicates the minimum cost of the tour that the professor should make to visit all the groups in S). Let $N = \{1,2,3\}$. Note that (N,c) is a TU-game. However, it is what we usually call a *cost game* since, for each coalition S, $c(S)$ does not represent the benefits that S can generate, but the cost that must be covered. The *saving game* associated to this situation (displaying the benefits generated by each coalition) is (N,v) where, for each $S \subset N$,

$$v(S) = \sum_{i \in S} c(i) - c(S).$$

Thus, $v(1) = v(2) = v(3) = 0$, $v(12) = 1500$, $v(13) = 500$, $v(23) = 500$, and $v(N) = 2000$. $\diamond$

These examples show that a wide range of situations can be modeled as TU-games. Now, we define a class of TU-games which is especially important.

Definition 5.4.3. A TU-game $v \in G^N$ is *superadditive* if, for each pair $S, T \subset N$, with $S \cap T = \emptyset$,

$$v(S \cup T) \geq v(S) + v(T).$$

We denote by SG^N the set of n-player superadditive TU-games. Note that a TU-game is superadditive when the players have real incentives for cooperation, that is, the union of any pair of disjoint coalitions of players never diminishes the total benefits. Hence, when dealing with superadditive games, it is natural to assume that the grand coalition will form and so, the question is how to allocate $v(N)$ among the players. In fact, a relevant part of the theory of TU-games is actually developed having superadditive games as the benchmark situation although, for simplicity, we always deal with the whole class G^N. Note that all the games in the examples above, with the only exception of (N,c) in Example 5.4.4, are superadditive.

We now present some other interesting classes of TU-games.

Definition 5.4.4. A TU-game $v \in G^N$ is *weakly superadditive* if, for each player $i \in N$ and each coalition $S \subset N \backslash \{i\}$, $v(S) + v(i) \leq v(S \cup \{i\})$.

Definition 5.4.5. A TU-game $v \in G^N$ is *additive* if, for each player $i \in N$ and each coalition $S \subset N \backslash \{i\}$, $v(S) + v(i) = v(S \cup \{i\})$. In particular, for each $S \subset N$, $v(S) = \sum_{i \in S} v(\{i\})$.

Definition 5.4.6. A TU-game is *monotonic* if, for each pair $S, T \subset N$ with $S \subset T$, we have $v(S) \leq v(T)$.

Definition 5.4.7. A TU-game $v \in G^N$ is *zero-normalized* if, for each player $i \in N$, $v(i) = 0$.

Given a game $v \in G^N$, the *zero-normalization* of v is the zero-normalized game w defined, for each $S \subset N$, by $w(S) := v(S) - \sum_{i \in S} v(i)$.

Definition 5.4.8. A TU-game $v \in G^N$ is *zero-monotonic* if its zero-normalization is a monotonic game.

The following result is straightforward.

Lemma 5.4.1. *Let $v \in G^N$. Then, v is weakly superadditive if and only if it is zero-monotonic.*

Proof. Exercise 5.5. □

The main goal of the theory of TU-games is to define solutions (and, in particular, allocation rules) that select, for each TU-game, sets of allocations (singletons in the case of allocation rules) that are admissible for the players. There are two important approaches in developing the previous task. One approach is based on *stability*, where the objective is to find solutions that pick sets of allocations that are stable according to different criteria. This is the approach underlying, for instance, the *core* (Gillies 1953), the *stable sets* (von Neumann and Morgenstern 1944), and the *bargaining set* (Aumann and Maschler 1964). The second approach is based on *fairness*: it aims to find allocation rules that propose, for each TU-game, an allocation that represents a fair compromise for the players. This is the approach underlying, for instance, the *Shapley value* (Shapley 1953), the *nucleolus* (Schmeidler 1969), and the *τ-value*, also known as *compromise value* and *Tijs value* (Tijs 1981). In this book we only cover the most important of the above solution concepts.

5.5. The Core and Related Concepts

In this section we study the most important concept dealing with stability: the *core*. First, we introduce some properties of the allocations associated with a TU-game. Let $v \in G^N$. Let $x \in \mathbb{R}^N$ be an allocation. Then, x is *efficient* if $\sum_{i \in N} x_i = v(N)$. Hence, provided that v is a superadditive game, efficiency just requires that the total benefit from cooperation is actually shared among the players. The allocation x is *individually rational* if, for each $i \in N$, $x_i \geq v(i)$, that is, no player gets less than what he can get by himself. The *set of imputations* of a TU-game, $I(v)$, consists of all the efficient and individually rational allocations.

Definition 5.5.1. Let $v \in G^N$. The *set of imputations* of v, $I(v)$, is defined by

$$I(v) := \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{and, for each } i \in N,\ x_i \geq v(i)\}.$$

The set of imputations of a superadditive game is always nonempty. The individual rationality of the allocations in $I(v)$ ensures that no player will individually block an allocation in $I(v)$, since he cannot get anything better by himself. Yet, it can be the case that a coalition of players has incentive to block some of the allocations in $I(v)$. To account for this, the core is defined by imposing a coalitional rationality condition on the allocations of $I(v)$. Given $v \in G^N$, an allocation $x \in \mathbb{R}^N$ is *coalitionally rational* if, for each $S \subset N$, $\sum_{i \in S} x_i \geq v(S)$. The core (Gillies 1953) is the set of efficient and coalitionally rational allocations.

Definition 5.5.2. Let $v \in G^N$. The *core* of v, $C(v)$, is defined by

$$C(v) := \{x \in I(v) : \text{ for each } S \subset N, \sum_{i \in S} x_i \geq v(S)\}.$$

The elements of $C(v)$ are usually called *core allocations*. The core is always a subset of the set of imputations. By definition, in a core allocation no coalition receives less than what it can get on its own (coalitional rationality). Hence, core allocations are stable in the sense that no coalition has incentives to block any of them. We now show that core allocations are also stable in a slightly different sense.

Definition 5.5.3. Let $v \in G^N$. Let $S \subset N$, $S \neq \emptyset$, and let $x, y \in I(v)$. We say that *y dominates x through S* if

i) for each $i \in S$, $y_i > x_i$, and

ii) $\sum_{i \in S} y_i \leq v(S)$.

We say that *y dominates x* if there is a nonempty coalition $S \subset N$ such that y dominates x through S. Finally, x is an *undominated* imputation of v if there is no $y \in I(v)$ such that y dominates x.

Observe that y dominates x through S if players in S prefer y to x and, moreover, y is a reasonable claim for S. Besides, if y dominates x, then there are coalitions willing and able to block x. Thus, a stable allocation should be undominated. This is in fact the case for core allocations.[6]

Proposition 5.5.1. *Let $v \in G^N$. Then,*

i) *If $x \in C(v)$, x is undominated.*

ii) *If $v \in SG^N$, $C(v) = \{x \in I(v) : x \text{ is undominated }\}$.*

[6]The set of undominated imputations of a TU-game v is often called the D-core of v.

Proof. i) Let $x \in C(v)$ and suppose there is $y \in I(v)$ and $S \subset N$, $S \neq \emptyset$, such that y dominates x through S. Then, $v(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i \geq v(S)$, which is a contradiction.

ii) Let $x \in I(v) \backslash C(v)$. Then, there is $S \subset N$ such that $\sum_{i \in S} x_i < v(S)$. Let $y \in \mathbb{R}^N$ be defined, for each $i \in N$, by

$$y_i := \begin{cases} x_i + \frac{v(S) - \sum_{j \in S} x_j}{|S|} & i \in S \\ v(i) + \frac{v(N) - v(S) - \sum_{j \in N \backslash S} v(j)}{|N \backslash S|} & i \notin S. \end{cases}$$

Since v is superadditive, $v(N) - v(S) - \sum_{j \in N \backslash S} v(j) \geq 0$ and, hence, $y \in I(v)$. Therefore, y dominates x through S. $\square$

Both the set of imputations and the core of a TU-game are bounded sets. Moreover, since they are defined as the intersection of a series of half-spaces, they are convex polytopes. Indeed, the set of imputations of a superadditive game is either a point (if the TU-game is additive) or a simplex with n extreme points.

Next, we provide several examples. The first one illustrates that non-superadditive games may have undominated imputations outside the core. The other examples study the cores of the games introduced in Examples 5.4.1, 5.4.2, 5.4.3, and 5.4.4.

Example 5.5.1. Consider the TU-game (N, v), where $N = \{1, 2, 3\}$, $v(1) = v(2) = 0$, $v(3) = 1$, $v(12) = 2$, $v(13) = v(23) = 1$, and $v(N) = 2$. Since $v(N) < v(12) + v(3)$, this is not a superadditive game and, moreover, it has an empty core. However, among many others, $(1, 0, 1)$ is an undominated imputation. $\diamond$

Example 5.5.2. It is easy to check that the core of the divide a million game is empty. This shows that the bargaining situation modeled by this game is strongly unstable. $\diamond$

Example 5.5.3. The core of the glove game is $\{(1, 0, 0)\}$. It may seem strange that the unique core allocation of this game consists of giving all the benefits to the player who has the unique left glove. However, this is the unique undominated imputation of this game. An interpretation of this is that the price of the right gloves becomes zero because there are too many right gloves available in the market. $\diamond$

Before moving to the game in Example 5.4.3, we introduce a new class of games.

Definition 5.5.4. A TU-game $v \in G^N$ is a *simple game* if i) it is monotonic, ii) for each $S \subset N$, $v(S) \in \{0, 1\}$, and iii) $v(N) = 1$.

We denote by S^N the class of simple games with n players. Notice that, to characterize a simple game v, it is enough to specify the collection W of its *winning coalitions* $W := \{S \subset N : v(S) = 1\}$ or, equivalently, the collection W^m of its *minimal winning coalitions* $W^m := \{S \in W : \text{ for each } T \in W, \text{ if } T \subset S, \text{ then } T = S\}$.

Definition 5.5.5. Let $v \in S^N$. Then, a player $i \in N$ is a *veto player* in v if $v(N \backslash \{i\}) = 0$.

Proposition 5.5.2. *Let $v \in S^N$ be a simple game. Then, $C(v) \neq \emptyset$ if and only if there is at least one veto player in v. Moreover, if $C(v) \neq \emptyset$, then*

$$C(v) = \{x \in I(v) : \text{ for each nonveto player } i \in N,\ x_i = 0\}.$$

Proof. Let $v \in S^N$. Let $x \in C(v)$ and let A be the set of veto players. Suppose that $A = \emptyset$. Then, for each $i \in N$, $v(N \backslash \{i\}) = 1$ and, hence,

$$0 = v(N) - v(N \backslash \{i\}) \geq \sum_{j \in N} x_j - \sum_{j \in N \backslash \{i\}} x_j = x_i \geq 0,$$

which is incompatible with the efficiency of x. The second part of the statement is straightforward. □

Example 5.5.4. In view of the results above, since the game of Example 5.4.3 is a simple game with an empty set of veto players, it has an empty core. Note that the games in Examples 5.4.1 and 5.4.2 are also simple games. ◇

Example 5.5.5. The core of the saving game v associated with the visiting professor allocation problem is given by the nonempty set $\{x \in I(v) : x_1 + x_2 \geq 1500, x_1 + x_3 \geq 500, x_2 + x_3 \geq 500\}$. Figure 5.5.1 depicts the core of this game inside the set of imputations.

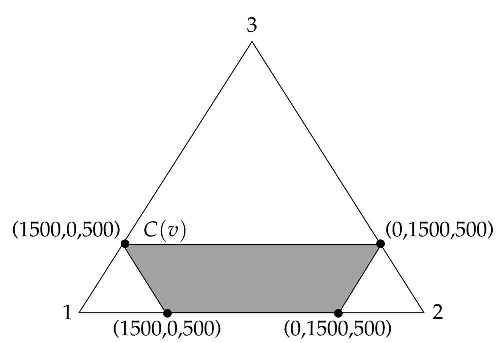


Figure 5.5.1. The core of the visiting professor game.

Actually, what is represented in Figure 5.5.1 are $I(v)$ and $C(v)$ as subsets of the efficiency hyperplane, which is given by $\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)\}$. The vertex of $I(v)$ with label i represents the allocation x such that, for each $j \neq i$, $x_j = v(j) = 0$ and $x_i = v(N) - \sum_{j \neq i} v(j) = 2000$. ◇

As we have seen, there are coalitional bargaining situations that are highly unstable and, hence, the corresponding games have an empty core. Next, we provide a necessary and sufficient condition for a game to have a nonempty core. The corresponding theorem was independently (although not simultaneously) proved in Bondareva (1963) and Shapley (1967), and is known as the Bondareva-Shapley theorem.

Definition 5.5.6. A family of coalitions $\mathcal{F} \subset 2^N \backslash \{\emptyset\}$ is *balanced* if there are positive real numbers $\{\alpha_S : S \in \mathcal{F}\}$ such that, for each $i \in N$,

$$\sum_{\substack{S \in \mathcal{F} \\ i \in S}} \alpha_S = 1.$$

The numbers $\{\alpha_S : S \in \mathcal{F}\}$ are called *balancing coefficients*.

Definition 5.5.7. A TU-game $v \in G^N$ is *balanced* if, for each balanced family $\mathcal{F}$, with balancing coefficients $\{\alpha_S : S \in \mathcal{F}\}$,

$$\sum_{S \in \mathcal{F}} \alpha_S v(S) \leq v(N).$$

A TU-game $v \in G^N$ is *totally balanced* if, for each $S \subset N$, the TU-game (S, v_S) is balanced.

Remark 5.5.1. Each partition of N is a balanced family of coalitions, with all the balancing coefficients being equal to 1. Then, given a TU-game $v \in G^N$, a necessary condition for v to be a balanced game is that $\sum_{i \in N} v(i) \leq v(N)$.

Given a balanced family of coalitions $\mathcal{F}$, the balancing coefficients can be interpreted in the following way. For each $S \in \mathcal{F}$, the coefficient α_S represents the time the players in S are allocating to S; the balancing condition then requires that each player has a unit of time to allocate among the different coalitions in $\mathcal{F}$. Now, the balancing condition of a game can be interpreted as the impossibility of the players to allocate their time among the different coalitions in a way that yields an aggregate payoff higher than $v(N)$. This suggests that, for a TU-game to be balanced, the worth of the coalitions different from N has to be relatively small when compared to the worth of the grand coalition. The last observation is, in some sense, related to the stability of the coalitional bargaining situation modeled by the TU-game. This connection is precisely established by the Bondareva-Shapley theorem. In practice, this result is especially useful as a tool to prove that a given game or class of games has an nonempty core.

Theorem 5.5.3 (Bondareva-Shapley theorem)**.** *Let $v \in G^N$. Then, $C(v) \neq \emptyset$ if and only if v is balanced.*

Proof.[7] Let $v \in G^N$ be such that $C(v) \neq \emptyset$. Let $x \in C(v)$ and let $\mathcal{F}$ be a balanced family with balancing coefficients $\{\alpha_S : S \in \mathcal{F}\}$. Then,

$$\sum_{S\in\mathcal{F}} \alpha_S v(S) \leq \sum_{S\in\mathcal{F}} \sum_{i\in S} \alpha_S x_i = \sum_{i\in N} \Big(x_i \sum_{\substack{S\in\mathcal{F}\\ i\in S}} \alpha_S \Big) = \sum_{i\in N} x_i = v(N).$$

Conversely, suppose that v is balanced and consider the following linear programming problem (P):

$$\begin{array}{ll} \text{Minimize} & \sum_{i\in N} x_i \\ \text{subject to} & \sum_{i\in S} x_i \geq v(S), \quad \forall S \in 2^N\backslash\{\emptyset\}. \end{array}$$

Clearly, $C(v) \neq \emptyset$ if and only if there is $\bar{x}$ an optimal solution of (P) with $\sum_{i\in N} \bar{x}_i = v(N)$. The dual of (P) is the following linear programming problem (D):[8]

$$\begin{array}{ll} \text{Maximize} & \sum_{S\in 2^N\backslash\{\emptyset\}} \alpha_S v(S) \\ \text{subject to} & \sum_{\substack{S\in 2^N\backslash\{\emptyset\}\\ i\in S}} \alpha_S = 1, \quad \forall i \in N, \\ & \alpha_S \geq 0, \quad \forall S \in 2^N\backslash\{\emptyset\}. \end{array}$$

Since the set of feasible solutions of (D) is nonempty and compact and the objective function is continuous, (D) has, at least, one optimal solution $\bar{\alpha}$. Let $\bar{\mathcal{F}} := \{S \subset N : \bar{\alpha}_S > 0\}$. Then, $\bar{\mathcal{F}}$ is a balanced family with balancing coefficients $\{\bar{\alpha}_S : S \in \bar{\mathcal{F}}\}$. Since (D) has an optimal solution, then, by the duality theorem, (P) also has an optimal solution $\bar{x}$ and, moreover,

$$\sum_{i\in N} \bar{x}_i = \sum_{S\in 2^N\backslash\{\emptyset\}} \bar{\alpha}_S v(S).$$

Hence, since v is balanced and $\bar{x}$ is an optimal solution of (P), $v(N) = \sum_{i\in N} \bar{x}_i$. Therefore, $C(v) \neq \emptyset$. $\square$

Different from the proof of Bondareva-Shapley theorem above, which relies on the duality theorem, there is an alternative proof for the "if" part that uses the minimax theorem, *i.e.*, that uses that every matrix game is strictly determined (von Neumann (1928)). This argument was developed by Aumann and nicely connects a classic result in noncooperative game

[7] The "if" part of this proof uses some linear programming results. We refer the reader to Section 2.8 for the basics of linear programming.

[8] See footnote 20 in Section 2.8 for a derivation of the dual problem when the primal has no nonnegativity constraints.

theory and a classic result in cooperative game theory. Before formally presenting it, we need some definitions and auxiliary lemmas.

Definition 5.5.8. A TU-game $v \in G^N$ is a 0-1-*normalized game* if and only if $v(N) = 1$ and, for each $i \in N$, $v(i) = 0$.

Definition 5.5.9. Let $v, \hat{v} \in G^N$. The games v and $\hat{v}$ are *S-equivalent* if there are $k > 0$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that, for each $S \subset N$,

$$\hat{v}(S) = kv(S) + \sum_{i \in S} a_i.$$

The next result is straightforward and the proof is left to the reader.

Lemma 5.5.4. *Let $v \in G^N$ be such that $v(N) > \sum_{i \in N} v(i)$. Then, there is a unique* 0-1-*normalized game $\hat{v}$ such that v and $\hat{v}$ are S-equivalent.*

Proof. Exercise 5.6. □

Lemma 5.5.5. *Let $v, \hat{v} \in G^N$ be S-equivalent and such that $v(N) > \sum_{i \in N} v(i)$. Then, v is balanced if and only if $\hat{v}$ is balanced.*

Proof. Let $k > 0$ and $a_1, \ldots, a_n \in \mathbb{R}$ be such that, for each $S \subset N$, $\hat{v}(S) = kv(S) + \sum_{i \in S} a_i$. Suppose that v is balanced. Let $\mathcal{F}$ be a balanced family of coalitions with balancing coefficients $\{\alpha_S : S \in \mathcal{F}\}$. Then,

$$\begin{aligned} \sum_{S \in \mathcal{F}} \alpha_S \hat{v}(S) &= \sum_{S \in \mathcal{F}} \alpha_S (kv(S) + \sum_{i \in S} a_i) \;=\; k \sum_{S \in \mathcal{F}} \alpha_S v(S) + \sum_{S \in \mathcal{F}} \Big(\alpha_S \sum_{i \in S} a_i \Big) \\ &= k \sum_{S \in \mathcal{F}} \alpha_S v(S) + \sum_{i \in S} \Big(a_i \sum_{\substack{S \in \mathcal{F} \\ i \in S}} \alpha_S \Big) \;=\; k \sum_{S \in \mathcal{F}} \alpha_S v(S) + \sum_{i \in S} a_i. \end{aligned}$$

Since $k > 0$ and v is balanced,

$$\sum_{S \in \mathcal{F}} \alpha_S \hat{v}(S) \leq kv(N) + \sum_{i \in N} a_i = \hat{v}(N).$$

Then, $\hat{v}$ is balanced. The converse is analogous. □

Lemma 5.5.6. *Let $v, \hat{v} \in G^N$ be S-equivalent and such that $v(N) > \sum_{i \in N} v(i)$. Then, $C(v) \neq \emptyset$ if and only if $C(\hat{v}) \neq \emptyset$.*

Proof. It immediately follows from definitions of core and S-equivalence (definitions 5.5.2 and 5.5.9). □

Alternative proof of the "if" part in Bondareva-Shapley theorem: *i.e.*, we again prove that, for each balanced game $v \in G^N$, $C(v) \neq \emptyset$. Let $v \in G^N$ be balanced. Then, $\sum_{i \in N} v(i) \leq v(N)$. We distinguish two cases.

Case 1: $v(N) = \sum_{i \in N} v(i)$. For each $S \subsetneq N$, the partition of N given by S and the singletons of the players outside S is a balanced family of coalitions (all the balancing coefficients being 1); then, since v is balanced, $v(S)+\sum_{i\notin S} v(i) \leq v(N)=\sum_{i\in N} v(i)$ and, hence, $v(S) \leq \sum_{i\in S} v(i)$. Therefore, $(v(1),\ldots,v(n)) \in C(v)$.

Case 2: $v(N) > \sum_{i\in N} v(i)$. By Lemmas 5.5.4, 5.5.5, and 5.5.6, we can restrict attention to the unique 0-1-normalized game that is S-equivalent to v. For the sake of notation, suppose that v itself is a 0-1-normalized game. Suppose that $C(v)=\emptyset$. Let $\mathcal{F}_1 := \{S \subset N : v(S) > 0\}$. Let $\mathcal{A}$ be the matrix game having entries a_{iS}, with $i \in N$ and $S \in \mathcal{F}_1$, where

$$a_{iS} := \begin{cases} 1/v(S) & i \in S \\ 0 & i \notin S. \end{cases}$$

By the minimax theorem, the game $\mathcal{A}$ is strictly determined. Let V be its value. For the sake of notation, let X and Y denote the sets of mixed strategies of the row player and the column player, respectively, in $\mathcal{A}$. Recall that, for each $x \in X$ and each $y \in Y$, the payoff of player 1 in $\mathcal{A}$ is given by $u_1(x,y) = x\mathcal{A}y^t$. Let $x_0 := (\frac{1}{n},\ldots,\frac{1}{n}) \in X$. Now, $\min_{y\in Y} u_1(x_0,y) > 0$ and, hence, $V > 0$. Next, we show that $V < 1$. Let $x \in X$. Then, $\sum_{i\in N} x_i = 1 = v(N)$ and, for each $i \in N$, $x_i \geq 0 = v(i)$. Since $C(v)=\emptyset$, there is $S_x \subset N$ such that $0 \leq \sum_{i\in S_x} x_i < v(S_x)$. Let $y^x \in Y$ be defined by $y^x_S = 0$ if $S \neq S_x$ and $y^x_S = 1$ if $S = S_x$. Then, for each $x \in X$, $\min_{y\in Y} u_1(x,y) = \min_{y\in Y} x\mathcal{A}y^t \leq x\mathcal{A}(y^x)^t = \sum_{i\in S_x} x_i/v(S_x) < 1$. Hence, $V < 1$.

Let $\bar{y} \in Y$ be an optimal strategy of player 2 and let $\mathcal{F}_2$ be the family of coalitions given by $\mathcal{F}_2 := \mathcal{F}_1 \cup \{\{1\},\ldots,\{n\}\}$. For each $S \in \mathcal{F}_2$, we define $\alpha_S \in \mathbb{R}$ as follows

$$\alpha_S := \begin{cases} \dfrac{\bar{y}_S}{Vv(S)} & S \in \mathcal{F}_1 \\ 1 - \displaystyle\sum_{\substack{S\in\mathcal{F}_1 \\ i\in S}} \alpha_S & S = \{i\}. \end{cases}$$

Since $V > 0$, v is a 0-1-normalized game, and $\bar{y}_S \in Y$, we have that, for each $S \in \mathcal{F}_1$, $\alpha_S \geq 0$. For each $i \in N$, let $x^i \in X$ be defined by $x^i_i := 1$ and, for each $j \neq i$, $x^i_j := 0$. Note that

$$\frac{1}{V} u_1(x^i,\bar{y}) = \frac{1}{V} \sum_{\substack{S\in\mathcal{F}_1 \\ i\in S}} \frac{\bar{y}_S}{v(S)} = \sum_{\substack{S\in\mathcal{F}_1 \\ i\in S}} \alpha_S.$$

Since $\bar{y}$ is an optimal strategy for player 2, we have that, for each $i \in N$, $V = \sup_{x\in X} u_1(x,\bar{y}) \geq u_1(x^i,\bar{y})$. Hence, for each $i \in N$, $\alpha_i \geq 0$.

Let $\mathcal{F} := \{S \in \mathcal{F}_2 : \alpha_S > 0\}$. By definition, the $\{\alpha_S\}_{S\in\mathcal{F}}$ are balanced coefficients for $\mathcal{F}$. Since $V < 1$ and $\bar{y} \in Y$,

$$\sum_{S\in\mathcal{F}} \alpha_S v(S) = \sum_{S\in\mathcal{F}_1} \alpha_S v(S) = \sum_{S\in\mathcal{F}_1} \frac{\bar{y}_S}{Vv(S)} v(S) = \frac{1}{V} > 1 = v(N).$$

Therefore, we have that v is not a balanced game and we reach a contradiction. $\square$

To conclude this section, we present a result, taken from Kalai and Zemel (1982b), that relates the class of additive games and the class of totally balanced games. First, we need to introduce two more concepts.

Definition 5.5.10. Let $v, \hat{v} \in G^N$. The *maximum game of v and $\hat{v}$, $v \vee \hat{v}$,* is defined, for each $S \subset N$, by $v \vee \hat{v}(S) := \max\{v(S), \hat{v}(S)\}$. Analogously, the *minimum game of v and $\hat{v}$, $v \wedge \hat{v}$,* is defined, for each $S \subset N$, by $v \wedge \hat{v}(S) := \min\{v(S), \hat{v}(S)\}$.

Lemma 5.5.7. *Let $v, \hat{v} \in G^N$. If v and $\hat{v}$ are totally balanced, then $v \wedge \hat{v}$ is totally balanced.*

Proof. Let $S \subset N$. Assume, without loss of generality, that $v(S) \leq \hat{v}(S)$. Since v is totally balanced, $C(v_S) \neq \emptyset$. Let $x \in C(v_S)$. Then, it is straightforward to see that $x \in C((v \wedge \hat{v})_S)$. $\square$

The result below shows that the class of nonnegative additive games, together with the minimum operator, spans the class of nonnegative totally balanced games.

Theorem 5.5.8. *A nonnegative TU-game $v \in G^N$ is totally balanced if and only if it is the minimum game of a finite collection of nonnegative additive games.*

Proof. A nonnegative additive game is totally balanced and, hence, the "if" part follows from Lemma 5.5.7. Conversely, let $v \in G^N$ be totally balanced. For each nonempty coalition $S \subset N$, we define $v^S \in G^N$ as follows. Let $x^S \in C(v_S)$. For each $i \in N \backslash S$, let $x_i^S \in \mathbb{R}$ be such that $x_i^S > v(N)$. Let v^S be the additive game such that, for each $i \in N$, $v^S(i) = x_i^S$. For each $S \subset N$ and each $T \subset N$, $v^S(T) \geq v(T)$ and $v^S(S) = v(S)$. Hence, for each $T \subset N$,

$$\min_{S\in 2^N\backslash\{\emptyset\}} v^S(T) = v(T).$$

Therefore, v is the minimum game of the finite collection $\{v^S\}_{S\in 2^N\backslash\{\emptyset\}}$. $\square$

5.6. The Shapley Value

In the previous section we studied the core of a TU-game, which is the most important set-valued solution concept for TU-games. Now, we present the most important allocation rule: the *Shapley value* (Shapley 1953). Formally, an allocation rule is defined as follows.

Definition 5.6.1. An *allocation rule* for n-player TU-games is just a map $\varphi\colon G^N \to \mathbb{R}^N$.

Shapley (1953), following an approach similar to the one taken by Nash when studying the Nash solution for bargaining problems, gave some appealing properties that an allocation rule should satisfy and proved that they characterize a unique allocation rule. First, we need to introduce two other concepts.

Definition 5.6.2. Let $v \in G^N$.

i) A player $i \in N$ is a *null player* if, for each $S \subset N$, $v(S \cup \{i\}) - v(S) = 0$.
ii) Two players i and j are *symmetric* if, for each coalition $S \subset N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

Let φ be an allocation rule and consider the following properties we might impose on it.

Efficiency (EFF): The allocation rule φ satisfies EFF if, for each $v \in G^N$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Null Player (NPP): The allocation rule φ satisfies NPP if, for each $v \in G^N$ and each null player $i \in N$, $\varphi_i(v) = 0$.

Symmetry (SYM): The allocation rule φ satisfies SYM if, for each $v \in G^N$ and each pair $i, j \in N$ of symmetric players, $\varphi_i(v) = \varphi_j(v)$.[9]

Additivity (ADD): The allocation rule φ satisfies ADD if, for each pair $v, w \in G^N$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Property EFF requires that φ allocates the total worth of the grand coalition, $v(N)$, among the players. Property NPP says that players that contribute zero to every coalition, *i.e.*, that do not generate any benefit, should

[9] Note that this symmetry property is slightly different from the one we introduced for bargaining problems. Roughly speaking, the property for bargaining problems said that, if all the players are symmetric, then they get the same utility. The symmetry property says that, for each pair of symmetric players, both of them get the same utility.

receive nothing. Property SYM asks φ to treat equal players equally.[10] Finally, ADD is the unique controversial of these axioms. Despite being a natural requirement, ADD is not motivated by any fairness notion.

We now present the definition of the Shapley value as originally introduced in Shapley (1953).

Definition 5.6.3. The *Shapley value*, Φ, is defined, for each $v \in G^N$ and each $i \in N$, by

$$\Phi_i(v) := \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!}(v(S \cup \{i\}) - v(S)). \tag{5.6.1}$$

Therefore, in the Shapley value, each player gets a weighted average of the contributions he makes to the different coalitions. Actually, the formula in Eq. (5.6.1) can be interpreted as follows. The grand coalition is to be formed inside a room, but the players have to enter the room sequentially, one at a time. When a player i enters, he gets his contribution to the coalition of players that are already inside (*i.e.*, if this coalition is S, he gets $v(S \cup \{i\}) - v(S)$). The order of the players is decided randomly, with all the $n!$ possible orderings being equally likely. It is easy to check that the Shapley value assigns to each player his expected value under this random ordered entry process.

The above discussion suggests an alternative definition of the Shapley value, based on the so-called *vectors of marginal contributions*. Let $\Pi(N)$ denote the set of all permutations of the elements in N and, for each $\pi \in \Pi(N)$, let $P^\pi(i)$ denote the set of predecessors of i under the ordering given by π, *i.e.*, $j \in P^\pi(i)$ if and only if $\pi(j) < \pi(i)$.

Definition 5.6.4. Let $v \in G^N$ be a TU-game. Let $\pi \in \Pi(N)$. The *vector of marginal contributions* associated with π, $m^\pi(v) \in \mathbb{R}^N$, is defined, for each $i \in N$, by $m_i^\pi(v) := v(P^\pi(i) \cup \{i\}) - v(P^\pi(i))$.

The convex hull of the set of vectors of marginal contributions is commonly known as the *Weber set*, formally introduced as a solution concept by Weber (1988). It is clear from the random order story that the formula of the Shapley value is equivalent to

$$\Phi_i(v) := \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v). \tag{5.6.2}$$

[10] Actually, this property is often referred to as *equal treatment of equals*. Sometimes the symmetry property is replaced by a similar property called anonymity, which requires that a relabeling of the players has to induce the same relabeling in the proposed allocation.

Exercise 5.7 asks the reader to formally show the equivalence between the two formulas. Before presenting the classic characterization of the Shapley value we introduce a class of TU-games that plays an important role in its proof.

Definition 5.6.5. Inside the class G^N, given $S \subset N$, the *unanimity game of coalition* S, w^S, is defined as follows. For each $T \subset N$, $v(T) := 1$ if $S \subset T$ and $v(T) := 0$ otherwise.

Theorem 5.6.1. *The Shapley value is the unique allocation rule in* G^N *that satisfies* EFF, NPP, SYM, *and* ADD.[11]

Proof. First, Φ satisfies both NPP and ADD. Moreover, it should be clear at this point that it also satifies EFF and SYM. Each vector of marginal contributions is an efficient allocation and, hence, EFF follows from Eq. (5.6.2). Also, SYM can be easily derived from Eq. (5.6.2).

Now, let φ be an allocation rule satisfying EFF, NPP, SYM, and ADD. Recall that each $v \in G^N$ can be seen as the vector $\{v(S)\}_{S \in 2^N \setminus \{\emptyset\}} \in \mathbb{R}^{2^n-1}$. Then, G^N can be identified with a $2^n - 1$ dimensional vector space. Now, we show that the unanimity games $U(N) := \{w^S : S \in 2^N \backslash \{\emptyset\}\}$ are a basis of such a vector space, *i.e.*, we show that $U(N)$ is a set of linearly independent vectors. Let $\{\alpha_S\}_{S \in 2^N \setminus \{\emptyset\}} \subset \mathbb{R}$ be such that $\sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S w^S = 0$ and suppose that there is $T \in 2^N \backslash \{\emptyset\}$ with $\alpha_T \neq 0$. We can assume, without loss of generality, that there is no $\hat{T} \subsetneq T$, such that $\alpha_{\hat{T}} \neq 0$. Then, $0 = \sum_{S \in 2^N \setminus \{\emptyset\}} \alpha_S w^S(T) = \alpha_T \neq 0$ and we have a contradiction. Since φ satisfies EFF, NPP, and SYM, we have that, for each $i \in N$, each $\emptyset \neq S \subset N$, and each $\alpha_S \in \mathbb{R}$,

$$\varphi_i(\alpha_S w^S) = \begin{cases} \frac{\alpha_S}{|S|} & i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then, if φ also satisfies ADD, φ is uniquely determined, because $U(N)$ is a basis of G^N. □

The proof above relies on the fact that a TU-game can be expressed as a linear combination of unanimity games. In Exercise 5.8 the reader is asked to obtain an explicit formula for the corresponding coefficients. As we did when characterizing the Nash solution for bargaining problems, we show that there is no superfluous axiom in the above characterization.

Proposition 5.6.2. *None of the axioms used in the characterization of the Shapley value given by Theorem 5.6.1 is superfluous.*

[11] These are not exactly the properties used by Shapley to characterize his value. He considered a property called *support property* instead of EFF and NPP. The support property is equivalent to EFF plus NPP.

Proof. We show that, for each of the axioms in the characterization, there is an allocation rule different from the Shapley value that satisfies the remaining three.

Remove EFF: The allocation rule φ defined, for each $v \in G^N$, by $\varphi(v) := 2\Phi(v)$ satisfies NPP, SYM, and ADD.

Remove NPP: Let φ be the allocation rule defined as follows. For each $v \in G^N$ and each $i \in N$, $\varphi_i(v) = v(N)/n$. This allocation rule is known as the *equal division rule* and it satisfies EFF, SYM, and ADD.[12]

Remove SYM: Let φ be the allocation rule defined as follows. Let $v \in G^N$ and let $\Pi^1(N)$ be the set of orderings of the players in N in which player 1 is in the first position, *i.e.*, $\pi \in \Pi^1(N)$ if and only if $\pi(1) = 1$. Then, for each $i \in N$, $\varphi_i(v) := \frac{1}{(n-1)!} \sum_{\pi \in \Pi^1(N)} m_i^\pi(v)$. This allocation rule satisfies EFF, NPP, and ADD.

Remove ADD: Let φ be the allocation rule defined as follows. Let $v \in G^N$ and let d be the number of null players in game v. Then, for each $i \in N$, $\varphi_i(v) = 0$ if i is a null player and $\varphi_i(v) = \frac{v(N)}{n-d}$ otherwise. This allocation rule satisfies EFF, NPP, and SYM. □

As we have already pointed out, the additivity axiom does not come from any fairness consideration and, because of this, it has received some criticisms in the literature. These criticisms motivated the appearance of alternative characterizations of the Shapley value. The most important ones are due to Young (1985), Hart and Mas-Colell (1989), and Chun (1989).

Remark 5.6.1. Let $v \in SG^N$. Then, for each $i \in N$ and each $\pi \in \Pi(N)$, we have $m_i^\pi(v) \geq v(i)$. Then, if $v \in SG^N$, for each $i \in N$,

$$\Phi_i(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v) \geq \frac{1}{n!} \sum_{\pi \in \Pi(N)} v(i) = v(i).$$

So, for the class of superadditive games, the Shapley value belongs to the set of imputations.

Below we compute the Shapley value for the games discussed in Examples 5.4.1, 5.4.2, 5.4.3, and 5.4.4. We omit the detailed computations, which can be easily done using the vectors of marginal contributions and Eq. (5.6.2).

Example 5.6.1. The Shapley value for the divide a million game is the allocation $(1/3, 1/3, 1/3)$. Note that, although the core of this game is empty,

[12] We refer the reader to van den Brink (2007) for a characterization of the equal division rule through the same axioms of the above characterization of the Shapley value with the exception that the null player property is replaced by a property called *nullifying player property*.

the Shapley value proposes an allocation for it. The core looks for allocations satisfying some stability requirements and, hence, it can be empty. $\diamond$

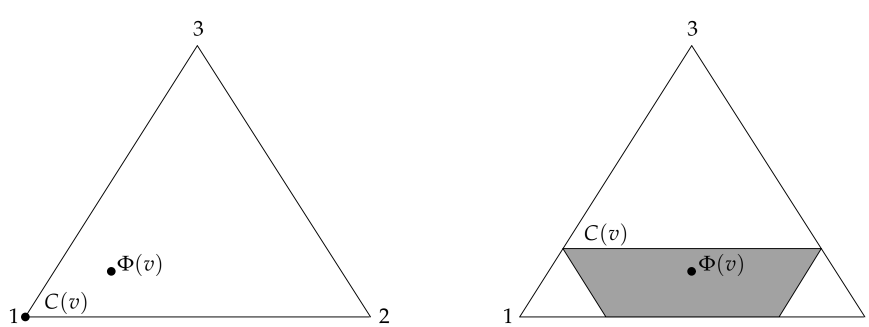
Example 5.6.2. The Shapley value for the glove game is $(2/3, 1/6, 1/6)$. Remember that the core of this game is $\{(1,0,0)\}$. The core and the Shapley value of this game are represented in Figure 5.6.1(a). Hence, even when the core is nonempty, the Shapley value may not be a core allocation. $\diamond$

Example 5.6.3. The Shapley value for the Parliament of Aragón is the allocation $(1/3, 1/3, 1/3, 0)$. This is a measure of the power of the four political parties in this Parliament. Note that IU is a null player and that the other three parties are symmetric in the simple game, which only takes into account their voting power. $\diamond$

Example 5.6.4. The Shapley value in the visiting professor game discussed in Example 5.4.4 is $\Phi(v) = (5000/6, 5000/6, 2000/6)$. These are the savings for the players. According to this allocation of the savings, the players have to pay $(4000/6, 4600/6, 9400/6)$. Note that the last vector is precisely $\Phi(c)$. This relationship holds for any pair of cost/benefit games:

$$\begin{aligned}
\Phi_i(v) &= \sum_{S \subset N \backslash \{i\}} \frac{s!(n-s-1)!}{n!}(v(S \cup \{i\}) - v(S)) \\
&= \sum_{S \subset N \backslash \{i\}} \frac{s!(n-s-1)!}{n!}(c(S) - c(S \cup \{i\}) + c(i)) \\
&= c(i) - \Phi_i(c).
\end{aligned}$$

Finally, observe that, for this game, $\Phi(v) \in C(v)$. In Figure 5.6.1(b) we represent the core and the Shapley value of this game. $\diamond$



(a) Core and Shapley value for the glove game.

(b) Core and Shapley value for the visiting professor game.

Figure 5.6.1. Two Examples of the Shapley value.

Example 5.6.2 shows that the Shapley value for a game can lie outside its core, even when the latter is nonempty. In Shapley (1971), a class of games that satisfy that the Shapley value is always a core allocation is discussed. This is the class of *convex games*, which has received a lot of attention because of its important properties. We study this class in Section 5.8.

The Shapley value is arguably the most widely studied concept in cooperative game theory. Winter (2002) and Moretti and Patrone (2008) are two surveys on the Shapley value, with the latter one being especially devoted to its applications.

5.7. The Nucleolus

In this section we present the nucleolus (Schmeidler 1969), perhaps the second most important allocation rule for TU-games, just behind the Shapley value. We present its definition, an existence result, and a procedure to compute it. Despite following a natural fairness principle, the definition of the nucleolus is already somewhat involved. Hence, the analysis of its properties is harder than the analysis we developed for the Shapley value and we do not cover it in detail here. For an axiomatic characterization of the nucleolus we refer the reader to Snijders (1995).

Let $v \in G^N$ and let $x \in \mathbb{R}^N$ be an allocation. Given a coalition $S \subset N$, the *excess of coalition S with respect to x* is defined by

$$e(S,x) := v(S) - \sum_{i \in S} x_i.$$

This is a measure of the degree of dissatisfaction of coalition S when the allocation x is realized. Note that, for each $x \in I(v)$, $e(N,x) = 0$. Moreover, if $x \in C(v)$, then, for each $S \subset N$, $e(S,x) \leq 0$. Now, we define the *vector of ordered excesses* $\theta(x) \in \mathbb{R}^{2^N}$ as the vector whose components are the excesses of the coalitions in 2^N arranged in nonincreasing order.

Given $x, y \in \mathbb{R}^N$, y is larger than x according to the lexicographic order, denoted $\theta(y) \succ_L \theta(x)$, if there is $l \in \mathbb{N}$, $1 \leq l \leq 2^n$, such that, for each $k \in \mathbb{N}$ with $k < l$, $\theta_k(y) = \theta_k(x)$ and $\theta_l(y) > \theta_l(x)$. We write $\theta(y) \succeq_L \theta(x)$ if either $\theta(y) \succ_L \theta(x)$ or $\theta(y) = \theta(x)$. We use the game in Example 5.5.1 to illustrate the above concepts.

Example 5.7.1. Let $v \in G^N$, where $N = \{1,2,3\}$ and $v(1) = v(2) = 0$, $v(3) = 1$, $v(12) = 2$, $v(13) = v(23) = 1$, and $v(N) = 2$. Let $x = (1,0,1)$ and $y = (3/2, 1/2, 0)$. The excess of each coalition with respect to the allocations x and y is depicted in Figure 5.7.1. Then, $\theta(x) = (1,0,0,0,0,0,-1,-1)$ and $\theta(y) = (1, 1/2, 0, 0, 0, -1/2, -1/2, -3/2)$ and, hence, $\theta(y) \succ_L \theta(x)$; that is, the most dissatisfied coalitions with respect to x and y are equally dissatisfied, but the second most dissatisfied coalition is more dissatisfied in y than it is in x. The idea of the nucleolus is to lexicographically minimize the degree of dissatisfaction of the different coalitions. With this fairness idea in mind, x would be more desirable than y. $\diamond$

Coalitions	$\emptyset$	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v(S)$	0	0	0	1	2	1	1	2
$e(S,x)$	0	-1	0	0	1	-1	0	0
$e(S,y)$	0	-3/2	-1/2	1	0	-1/2	1/2	0

Figure 5.7.1. Excesses of the coalitions with respect to x and y.

The next result concerning the function θ will prove to be very useful to analyze the nucleolus.

Lemma 5.7.1. *Let $v \in G^N$. Let $x, y \in \mathbb{R}^N$ be such that $x \neq y$ and $\theta(x) = \theta(y)$. Let $\alpha \in (0,1)$. Then, $\theta(x) \succ_L \theta(\alpha x + (1-\alpha)y)$.*

Proof. Note that, for each $S \subset N$, $e(S, \alpha x + (1-\alpha)y) = \alpha(v(S) - \sum_{i\in S} x_i) + (1-\alpha)(v(S) - \sum_{i \in S} y_i) = \alpha e(S,x) + (1-\alpha)e(S,y)$. Assume that

$$\theta(x) = (e(S_1,x), e(S_2,x), \ldots, e(S_{2^n},x)),$$

where $S_1, \ldots, S_{2^n}$ is a permutation of all the coalitions defined as follows. For each $k \in \{1, \ldots, 2^n - 1\}$,

- $e(S_k,x) \geq e(S_{k+1},x)$, and
- if $e(S_k,x) = e(S_{k+1},x)$, then $e(S_k,y) \geq e(S_{k+1},y)$.

Since $x \neq y$, there is $S \subset N$ such that $e(S,x) \neq e(S,y)$. Let k be the smallest index such that $e(S_k,x) \neq e(S_k,y)$. Then, for each $l \in \{1,\ldots,k-1\}$, $e(S_l, \alpha x + (1-\alpha)y) = e(S_l,x)$. Since $\theta(x) = \theta(y)$, it has to be the case that $e(S_k,y) < e(S_k,x)$. Moreover, the choice of the ordering $S_1,\ldots,S_{2^n}$ ensures that, for each $l > k$ such that $e(S_l,x) = e(S_k,x)$, we have $e(S_l,y) \leq e(S_k,y) < e(S_k,x)$. Hence, for each $l > k$, either i) $e(S_l,x) = e(S_k,x)$ and $e(S_l,y) < e(S_k,x)$ or ii) $e(S_l,x) < e(S_k,x)$ and $e(S_l,y) \leq e(S_k,x)$. Then, for each $l \geq k$,

$$e(S_l, \alpha x + (1-\alpha)y) = \alpha e(S_l,x) + (1-\alpha)e(S_l,y) < e(S_k,x).$$

Hence, for each $l \in \{1,\ldots,k-1\}$, $\theta_l(x) = \theta_l(\alpha x + (1-\alpha)y)$ and, for each $l \geq k$, $\theta_l(x) > \theta_k(\alpha x + (1-\alpha)y)$. Therefore, $\theta(x) \succ_L \theta(\alpha x + (1-\alpha)y)$. $\square$

Definition 5.7.1. Let $v \in G^N$ be such that $I(v) \neq \emptyset$. The *nucleolus* of v, that we denote by $\eta(v)$, is the set

$$\eta(v) := \{x \in I(v) : \text{ for each } y \in I(v),\ \theta(y) \succeq_L \theta(x)\}.$$

The nucleolus consists of those imputations that minimize the vector of nonincreasingly ordered excesses according to the lexicographic order within the set of imputations, that is, the nucleolus is not defined for games with an empty set of imputations. At the beginning of this section we said that the nucleolus is an allocation rule and yet, we have defined it as a set. Next, we show that the nucleolus is actually a singleton, *i.e.*, it is never empty and it contains a unique allocation. Therefore, with a slight abuse of language, we identify the unique element of the nucleolus with the nucleolus itself and refer to $\eta(v)$ as the nucleolus of v.

Theorem 5.7.2. *Let $v \in G^N$ be such that $I(v) \neq \emptyset$. Then, the set $\eta(v)$ contains a unique allocation.*

Proof. Let $I^0 := I(v)$. For each $k \in \{1, \ldots, 2^n\}$, let I^k be the set

$$I^k := \{x \in I^{k-1} : \text{ for each } y \in I^{k-1}, \theta_k(x) \leq \theta_k(y)\}.$$

For each $k \in \{1, \ldots, 2^n\}$, the function that assigns, to each $x \in I(v)$, the k-th coordinate of the vector $\theta(x)$ is continuous. Hence, since $I(v)$ is a nonempty and compact set, I^1 is also nonempty and compact and so are all the I^k sets. We claim that $\eta(v) = I^{2^n}$. Let $x \in I^{2^n}$ and let $y \in I(v)$. If $y \in I^{2^n}$, then $\theta(x) = \theta(y)$ and, hence, $\theta(y) \succeq_L \theta(x)$. Hence, suppose that $y \in I(v) \backslash I^{2^n}$. Let k be the smallest index such that $y \notin I^k$. Then, $\theta_k(y) > \theta_k(x)$. Since $x, y \in I^{k-1}$, for each $l \in \{1, \ldots, k-1\}$, $\theta_l(x) = \theta_l(y)$. Hence, $\theta(y) \succ_L \theta(x)$. Thus, $\eta(v)$ is a nonempty set.

We now show that $\eta(v)$ is a singleton. Suppose it is not. Let $x, y \in \eta(v)$, with $x \neq y$. Then, $\theta(x) = \theta(y)$. Since $I(v)$ is a convex set, for each $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y \in I(v)$. By Lemma 5.7.1, $\theta(x) \succ_L \theta(\alpha x + (1-\alpha)y)$, which contradicts that $x \in \eta(v)$. $\square$

Remark 5.7.1. It is easy to see that if a TU-game has a nonempty core, then the nucleolus is a core element. Just note that the maximum excess at a core allocation can never be positive and that, in any allocation outside the core, there is a positive excess for at least one coalition. Hence, the vector of ordered excesses cannot be lexicographically minimized outside the core.

There are several procedures to compute the nucleolus. A review can be found in Maschler (1992). Under certain conditions of the characteristic function v, some more specific methods have been given, for instance, in Kuipers et al. (2000), Faigle et al. (2001), and Quant et al. (2005). Below we describe a procedure provided by Maschler et al. (1979) as an application of the Kopelowitz algorithm (Kopelowitz 1967). Let $v \in G^N$ be such that

$I_0 := I(v) \neq \emptyset$. Consider the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_1 \\ \text{subject to} & \sum_{i\in S} x_i + \alpha_1 \geq v(S), \quad \emptyset \neq S \subsetneq N \\ & x \in I_0. \end{array}$$

This problem has at least one optimal solution. Let $\bar{\alpha}_1$ be the minimum of this linear programming problem. The set of optimal solutions is of the form $\{\bar{\alpha}_1\} \times I_1$. If this set is a singleton, then I_1 coincides with the nucleolus. Otherwise, let $\mathcal{F}_1$ be the collection of coalitions given by

$$\mathcal{F}_1 := \{S \subset N : \text{ for each } x \in I_1, \sum_{i\in S} x_i + \bar{\alpha}_1 = v(S)\}.$$

For $k > 1$, solve the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_k \\ \text{subject to} & \sum_{i\in S} x_i + \alpha_k \geq v(S), \quad \emptyset \neq S \subsetneq N, S \notin \cup_{l<k}\mathcal{F}_l \\ & x \in I_{k-1}, \end{array}$$

where, for each $l < k$, $\mathcal{F}_l$ is the collection of coalitions given by

$$\mathcal{F}_l := \{S \subset N : \text{ for each } x \in I_l, \sum_{i\in S} x_i + \bar{\alpha}_l = v(S)\}.$$

This algorithm finishes once the optimal solution set is a singleton. At the optimal solution at least one of the inequality restrictions has to be binding, *i.e.*, for each $k \geq 1$ and each $x \in I_k$, there is $S \notin \cup_{l<k}\mathcal{F}_l$ such that $\sum_{i\in S} x_i + \bar{\alpha}_k = v(S)$. Hence, at each step, the set $\mathcal{F}_k$ contains, at least, one new coalition and so the algorithm finishes in at most $2^n - 1$ steps. Note that $\bar{\alpha}_1$ is the largest excess and $\mathcal{F}_1$ is the collection of coalitions with the largest excess, $\bar{\alpha}_2$ is the second largest excess and $\mathcal{F}_2$ is the collection of coalitions with the second largest excess, and so on.

Example 5.7.2. Consider again the TU-game $v \in G^N$ given by $N = \{1, 2, 3\}$ and $v(1) = v(2) = 0, v(3) = 1, v(12) = 2, v(13) = v(23) = 1$, and $v(N) = 2$ (introduced in Example 5.5.1 and already discussed in this section in Example 5.7.1). First, we solve the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_1 \\ \text{subject to} & x_i + \alpha_1 \geq 0, \quad i \in \{1,2\} \\ & x_3 + \alpha_1 \geq 1 \\ & x_1 + x_2 + \alpha_1 \geq 2 \\ & x_1 + x_3 + \alpha_1 \geq 1 \\ & x_2 + x_3 + \alpha_1 \geq 1 \\ & x_i \geq 0, \quad i \in \{1,2\} \\ & x_3 \geq 1 \\ & x_1 + x_2 + x_3 = 2. \end{array}$$

The optimal solution set is given by $\bar{\alpha}_1 = 1$, $x_1 + x_2 = 1$, and $x_3 = 1$. The collection of coalitions $\mathcal{F}_1$ is given by $\mathcal{F}_1 = \{\{1,2\}\}$. Next, we solve the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_2 \\ \text{subject to} & x_i + \alpha_2 \geq 0, \quad i \in \{1,2\} \\ & x_3 + \alpha_2 \geq 1 \\ & x_1 + x_3 + \alpha_2 \geq 1 \\ & x_2 + x_3 + \alpha_2 \geq 1 \\ & x_i \geq 0, \quad i \in \{1,2\} \\ & x_1 + x_2 = 1 \\ & x_3 = 1. \end{array}$$

The optimal solution set is given by $\bar{\alpha}_2 = 0$, $x_1 + x_2 = 1$, and $x_3 = 1$. The collection of coalitions $\mathcal{F}_2$ is given by $\mathcal{F}_2 = \{\{3\}\}$. Finally, after some simplifications, we get to the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_3 \\ \text{subject to} & x_1 + \alpha_3 \geq 0 \\ & x_2 + \alpha_3 \geq 0 \\ & x_i \geq 0, \quad i \in \{1,2\} \\ & x_1 + x_2 = 1 \\ & x_3 = 1. \end{array}$$

The optimal solution set is given by $\bar{\alpha}_3 = -1/2$, $x_1 = x_2 = 1/2$, and $x_3 = 1$. The collection of coalitions $\mathcal{F}_3$ is given by $\mathcal{F}_3 = \{\{1\},\{2\},\{1,3\},\{2,3\}\}$. Then, the nucleolus of this game is $\eta(v) = (1/2, 1/2, 1)$. $\diamond$

5.8. Convex Games

This section is devoted to the study of the class of *convex games*, introduced in Shapley (1971). These games have several interesting properties; probably the most important one is that the Shapley value is always an element of the core in this class of games.

Definition 5.8.1. A TU-game $v \in G^N$ is convex if, for each $i \in N$ and each pair $S, T \subset N\backslash\{i\}$ with $S \subset T$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T). \tag{5.8.1}$$

It can be easily checked that every convex game is superadditive. We devote the rest of this section to establish two important properties of convex games. First, the core of a convex game is always nonempty. Second, the Shapley value of a convex game is always an element of its core. The vectors of marginal contributions introduced in Definition 5.6.4 are a very useful tool in this section.

Example 5.8.1. Consider the visiting professor game introduced in Example 5.4.4, where $N = \{1,2,3\}$, $v(1) = v(2) = v(3) = 0$, $v(12) = 1500$, $v(13) = 500$, $v(23) = 500$, and $v(N) = 2000$. This game is convex. Moreover, as we have already seen, its core is nonempty and the Shapley value belongs to the core (see Figure 5.6.1(b)). We represent the vectors of marginal contributions associated with the different orderings of the players in the table below. As we already know from Eq. (5.6.2), the Shapley value can be computed as the average of these vectors. $\diamond$

Permutation π	$m_1^\pi(v)$	$m_2^\pi(v)$	$m_3^\pi(v)$
123	0	1500	500
132	0	1500	500
213	1500	0	500
231	1500	0	500
312	500	1500	0
321	1500	500	0

Figure 5.8.1. Vectors of marginal contributions in the visiting professor game.

We now present the most important result for convex games. It relates convex games, vectors of marginal contributions, and the core.

Theorem 5.8.1. *Let $v \in G^N$. The following statements are equivalent:*

i) *The game v is convex.*

ii) *For each $\pi \in \Pi(N)$, $m^\pi(v) \in C(v)$.*

iii) $C(v) = \operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$, i.e., *the core and the Weber set coincide.*

Proof.[13] i) $\Rightarrow$ ii). Let $\pi \in \Pi(N)$. We have that $\sum_{i\in N} m_i^\pi(v) = v(N)$. Let $S \subsetneq N$. Let $i \in N \backslash S$ be such that, for each $j \in N\backslash(S \cup \{i\})$, $\pi(i) < \pi(j)$. Then, $P^\pi(i)$ is a subset of S. Since v is convex, $v(S \cup \{i\}) - v(S) \geq v(P^\pi(i) \cup \{i\}) - v(P^\pi(i)) = m_i^\pi(v) = \sum_{j\in S\cup\{i\}} m_j^\pi(v) - \sum_{j\in S} m_j^\pi(v)$. Hence,

$$\sum_{j\in S\cup\{i\}} m_j^\pi(v) - v(S \cup \{i\}) \leq \sum_{j\in S} m_j^\pi(v) - v(S).$$

Now, we can apply to $S \cup \{i\}$ the same argument we used above for S. By doing this repeatedly until we get coalition N, we get

$$0 = \sum_{j\in N} m_j^\pi(v) - v(N) \leq \sum_{j\in S} m_j^\pi(v) - v(S).$$

[13]The statement i) $\Rightarrow$ ii) was proved by Shapley (1971), ii) $\Rightarrow$ iii) by Weber (1988), and iii) $\Rightarrow$ i) by Ichiishi (1981).

Hence, for each $S \subset N$, $\sum_{j\in S} m_j^\pi(v) \geq v(S)$. Therefore, $m^\pi(v) \in C(v)$.

ii) $\Rightarrow$ iii). By ii) we already have $\operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\} \subset C(v)$. We now show that $C(v) \subset \operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$. We proceed by induction on the number of players. If $n = 1$, the result is straigthforward. Assume that the result is true for all games with less than n players and let $v \in G^N$. Since $C(v)$ is a convex set, it suffices to prove that all the points in the boundary of $C(v)$ belong to the set $\operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$. Let x be a point in the boundary of $C(v)$. Then, there is $S \subsetneq N$, $S \neq \emptyset$, such that $\sum_{i\in S} x_i = v(S)$. Recall that $v_S \in G^S$ is defined, for each $T \subset S$, by $v_S(T) := v(T)$. Let $v_1 \in G^{N\backslash S}$ be defined, for each $T \subset N\backslash S$, by $v_1(T) := v(T \cup S) - v(S)$. Let x_S be the restriction of x to the players in S. Clearly, $x_S \in C(v_S)$. Since $\sum_{i\in S} x_i = v(S)$ and $x \in C(v)$, we have that, for each $T \subset N\backslash S$,

$$\sum_{i\in T} x_i = \sum_{i\in T\cup S} x_i - \sum_{i\in S} x_i \geq v(T \cup S) - v(S).$$

Hence, $x_{N\backslash S} \in C(v_1)$. By the induction hypothesis,

$$x_S = \sum_{\pi\in\Pi(S)} \lambda_\pi m^\pi(v_S) \text{ and } x_{N\backslash S} = \sum_{\pi\in\Pi(N\backslash S)} \gamma_{N\backslash S} m^\pi(v_1),$$

where, for each $\pi \in \Pi(S)$, $\lambda_\pi \geq 0$ and $\sum_{\pi\in\Pi(S)} \lambda_\pi = 1$ and, for each $\pi \in \Pi(N\backslash S)$, $\gamma_\pi \geq 0$ and $\sum_{\pi\in\Pi(N\backslash S)} \gamma_\pi = 1$. Now, for each $\pi_S \in \Pi(S)$ and each $\pi_{N\backslash S} \in \Pi(N\backslash S)$, let $\pi^* \in \Pi(N)$ be defined by $\pi^* := (\pi_S, \pi_{N\backslash S})$. So defined, we have that, for each $i \in S$, $m_i^{\pi^*}(v) = m_i^{\pi_S}(v_S)$ and, for each $i \in N\backslash S$, $m_i^{\pi^*}(v) = m_i^{\pi_{N\backslash S}}(v_1)$. Hence,

$$x = \sum_{\pi_S\in\Pi(S)} \sum_{\pi_{N\backslash S}\in\Pi(N\backslash S)} \lambda_{\pi_S} \gamma_{\pi_{N\backslash S}} m^{\pi^*}(v),$$

which implies that $x \in \operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$.

iii) $\Rightarrow$ i). Let $i \in N$ and let $S, T \subset N\backslash\{i\}$ be such that $S \subset T$. Let $\pi \in \Pi(N)$ be such that $\pi = (\pi_S, \pi_{T\backslash S}, \pi_{N\backslash T})$ and, for each $j \in N\backslash T$, $j \neq i$, $\pi(i) < \pi(j)$. Then, $P^\pi(i) = T$, $m_i^\pi(v) = v(P^\pi(i) \cup \{i\}) - v(P^\pi(i)) = v(T \cup \{i\}) - v(T)$. Also, $\sum_{j\in S} m_j^\pi = v(S)$. Since $m^\pi \in C(v)$, then

$$v(S \cup \{i\}) \leq \sum_{j\in S\cup\{i\}} m_j^\pi(v) = v(S) + m_i^\pi(v).$$

Hence, $v(S \cup \{i\}) - v(S) \leq m_i^\pi(v) = v(T \cup \{i\}) - v(T)$. Therefore, v is convex. □

Corollary 5.8.2. *Let $v \in G^N$ be a convex game. Then, $\Phi(v) \in C(v)$.*

Proof. It immediately follows from the formula of the Shapley value (given in Eq. (5.6.2)) and the statement iii) in Theorem 5.8.1 above. □

We now show that the condition above is sufficient but it is not necessary.

Example 5.8.2. Consider the TU-game (N, v), where $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 2$, $v(23) = 6$ and $v(N) = 7$. This game is not convex because $v(23) - v(2) = 6 > 5 = v(N) - v(12)$. However, the reader can easily verify that $\Phi(v) = (1, 3, 3) \in C(v)$. $\diamond$

As we have seen above, when restricting attention to convex games one can get results that do not hold in general. The main advantage of dealing with convex games is that, mainly because of Theorem 5.8.1, their cores are easier to study. We refer the reader to Shapley (1971) and González-Díaz and Sánchez-Rodríguez (2008) for two papers in which the special structure of convex games is used to derive a series of geometric and game theoretical properties of their cores.

5.9. Noncooperative Models in Cooperative Game Theory: Implementation Theory

As we already discussed in the previous sections, a standard way to evaluate different solution concepts for cooperative games is through axiomatic characterizations. *Noncooperative implementation* provides an alternative approach in which the details of some negotiation process among the players participating in the cooperative game are made explicit and then, the tools of noncooperative game theory are used to analyze the new situation. Noncooperative implementation is often referred to as the *Nash program*. In the words of Serrano (2008), "in game theory, 'Nash program' is the name given to a research agenda, initiated in Nash (1953), intended to bridge the gap between the cooperative and noncooperative approaches to the discipline". In this section we provide some examples of key developments in the above agenda. We refer the reader to Serrano (2005) for a comprehensive survey. In all the games we describe in this section we restrict attention to pure strategies.

Remark 5.9.1. Suppose that we want to implement the Shapley value of a given TU-game and consider the following noncooperative game. All the players simultaneously make a proposal. If all the proposals coincide with the Shapley value, then this is what they get; otherwise, they get some bad outcome. The payoff in the unique Nash equilibrium of this game coincides with the Shapley value. Of course, this mechanism would work for any solution and it does not help to get any new insight or motivation. The problem in this mechanism is that the designer needs to know the characteristic function (or at least the corresponding Shapley value) to verify that the players are actually following the rules. In general, one should look at mechanisms that can be monitored even by someone who does not know the particular parameters of the underlying cooperative game.[14] Moreover, one should try to find noncooperative games that are easy to describe and intuitive given the solution concept we are trying to implement.

5.9.1. A first implementation of the Nash bargaining solution.[15] In the seminal paper on the Nash program, Nash (1953) proposed a noncooperative negotiation process to support the Nash bargaining solution, which he had introduced in Nash (1950a). This negotiation process runs as follows. Let $(F, d) \in B^N$.

- Simultaneously and independently, each player $i \in N$ chooses a demand, $a_i \geq d_i$.
- If the allocation $a = (a_1, \ldots, a_n)$ is feasible, *i.e.*, $a \in F$, then each player receives his demand. Otherwise there is no agreement and each player $i \in N$ gets d_i.

This corresponds with the strategic game $G^{\mathrm{NA}} = (N, A, u)$, with $A_i := \{a_i : a_i \geq d_i\}$ and, for each $i \in N$,

$$u_i(a) := \begin{cases} a_i & a \in F \\ d_i & \text{otherwise.} \end{cases}$$

Proposition 5.9.1. *The set of pure Nash equilibrium payoff vectors of the negotiation game G^{NA} consists of all Pareto efficient allocations of F along with the disagreement point.*

Proof. Let $a \in A$. If a belongs to F but it is not Pareto efficient, then there are $i \in N$ and $\hat{a}_i \in A_i$ such that $(a_{-i}, \hat{a}_i) \in F$ and $\hat{a}_i = u_i(a_{-i}, \hat{a}_i) > u_i(a) = a_i$. Hence, no such strategy profile is a Nash equilibrium of G^{NA}. Suppose that a is such that $|\{i \in N : a_i > \max_{\hat{a} \in F} u_i(\hat{a})\}| \geq 2$. Then, it is straightforward to check that a is a pure Nash equilibrium of G^{NA} with payoff d. Suppose that a is a Pareto efficient allocation. For each, $i \in N$ and each $\hat{a} \in A_i$, $a_i = u_i(a) \geq u_i(a_{-i}, \hat{a}_i)$; indeed, if $\hat{a}_i \neq a_i$, $u_i(a) > u_i(a_{-i}, \hat{a}_i)$. Hence, a is a (strict) pure Nash equilibrium whose payoff vector is a. □

Hence, although the Nash solution is a Nash equilibrium of game G^{NA}, the multiplicity of equilibria poses a serious problem. Actually, all the efficient Nash equilibria are strict equilibria, which is the more demanding

[14] A related issue is that of the representation of the preferences of the players. The predictions of the proposed mechanism should be essentially unaffected by changes in the representation of the preferences of the players. For instance, a change in the units in which we measure the utility of a player (*i.e.*, a rescaling of his utility function), should induce an equivalent change in the predicted payoffs. A simple way of doing this is defining the mechanism directly over the space of outcomes, instead of doing it over the set of payoffs induced by some utility representation.

[15] In this subsection we partially follow the exposition in van Damme (1991, Section 7.5).

refinement of Nash equilibrium that we have discussed in this book. Yet, Nash proposed an elegant way to refine the set of equilibria of the above game. He proposed, in his own words, "to smooth the game G^{NA} to obtain a continuous payoff function and then to study the equilibrium outcomes of the smoothed game as the amount of smoothing approaches 0". For each $\varepsilon > 0$, let B^ε be the set of functions $f : A \to (0, 1]$ such that

i) the function f is continuous,
ii) for each $a \in F$, $f(a) = 1$, and
iii) for each $a \notin F$, if $\min_{\hat{a}\in F} \sum_{i\in N}(a_i - \hat{a}_i)^2 > \varepsilon$, *i.e.*, if the distance between a and F is greater than ε, then

$$\max\{f(a), \prod_{i\in N}(a_i - d_i)f(a)\} < \varepsilon.$$

Let $B := \bigcup_{\varepsilon>0} B^\varepsilon$. We denote by $G^{\text{NA}}(f)$ the strategic game (N, A, u^f), where, for each $i \in N$,

$$u_i^f(a) := d_i + (a_i - d_i)f(a),$$

Condition iii) above implies that $f(a)$ decreases as a moves away from F. Moreover, as ε goes to 0, the functions in B^ε become steeper and, hence, the game $G^{\text{NA}}(f)$ gets closer and closer to the original game G^{NA}. One interpretation of the game $G^{\text{NA}}(f)$ might be that the players think that the feasible set may actually be larger than F; this uncertainty being captured by the function f, which would then represent the probability that a given pair of demands belongs to the real feasible set; just note that $d_i + (a_i - d_i)f(a) = a_i f(a) + d_i(1 - f(a))$.

Definition 5.9.1. Let a^* be a Nash equilibrium of the Nash negotiation game G^{NA}. Consider a pair of sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \to 0$ and $\{f^{\varepsilon_k}\}_{k\in\mathbb{N}}$, where, for each $k \in \mathbb{N}$, $\varepsilon_k > 0$ and $f^{\varepsilon_k} \in B^{\varepsilon_k}$. Then, a^* is a *B-essential* equilibrium if there is a third sequence $\{a^{\varepsilon_k}\} \subset A$ such that $\{a^{\varepsilon_k}\} \to a^*$ and, for each $k \in \mathbb{N}$, a^{ε_k} is a Nash equilibrium of $G^{\text{NA}}(f^{\varepsilon_k})$.[16]

Theorem 5.9.2. *The Nash solution is a B-essential equilibrium of the negotiation game G^{NA}.*

Proof. Let $z := \text{NA}(F, d)$ and recall that $\prod_{i\in N}(z_i - d_i) > 0$. Suppose that $0 < \varepsilon < \prod_{i\in N}(z_i - d_i)$. Then, for each $f^\varepsilon \in B^\varepsilon$ there is $\bar{a}^\varepsilon \in A$ that maximizes the function $\prod_{i\in N}(a_i - d_i)f^\varepsilon(a)$. For each $a \in A$ such that $\min\{\sum_{i\in N}(a_i - \hat{a}_i)^2 : \hat{a} \in F\} > \varepsilon$, we have $\prod_{i\in N}(a_i - d_i)f^\varepsilon(a) < \varepsilon <$

[16] The name B-essential equilibrium comes from the similarity with the notion of *essential equilibrium* (Wen-Tsun and Jia-He 1962). The difference is that the latter allows for any kind of trembles in the utility function of the game, whereas the former only considers trembles defined via the functions in B.

$\prod_{i\in N}(z_i - d_i)f^{\varepsilon}(z) = \prod_{i\in N}(z_i - d_i)$. Hence, $\min\{\sum_{i\in N}(\bar{a}_i^{\varepsilon} - \hat{a}_i)^2 : \hat{a} \in F\} \leq \varepsilon$, *i.e.*, $\bar{a}^{\varepsilon}$ gets arbitrarily close to F as ε goes to 0. For each $a_i \in A_i$, $\big(\prod_{j\neq i}(\bar{a}_j^{\varepsilon} - d_j)\big)(a_i - d_i)f^{\varepsilon}(\bar{a}_{-i}^{\varepsilon}, a_i) \leq \prod_{j\in N}(\bar{a}_j^{\varepsilon} - d_j)f^{\varepsilon}(\bar{a}^{\varepsilon})$ and, since $\bar{a}^{\varepsilon} > d$, we have

$$(a_i - d_i)f^{\varepsilon}(\bar{a}_{-i}^{\varepsilon}, a_i) \leq (\bar{a}_i^{\varepsilon} - d_i)f^{\varepsilon}(\bar{a}^{\varepsilon}).$$

Therefore, $\bar{a}^{\varepsilon}$ is a Nash equilibrium of $G^{\mathrm{NA}}(f^{\varepsilon})$. Since, for each $a \notin F$, $0 < f^{\varepsilon}(a) \leq 1$ and, for each $a \in F$, $f^{\varepsilon}(a) = 1$, then

$$\prod_{i\in N}(z_i - d_i) = \prod_{i\in N}(z_i - d_i)f^{\varepsilon}(z) \leq \prod_{i\in N}(\bar{a}_i^{\varepsilon} - d_i)f^{\varepsilon}(\bar{a}^{\varepsilon}) \leq \prod_{i\in N}(\bar{a}_i^{\varepsilon} - d_i).$$

Recall that, within F, the product $\prod_{i\in N}(a_i - d_i)$ is maximized at $a = z$. Now, as ε goes to 0 the distance between $\bar{a}^{\varepsilon}$ and F goes to 0 and, hence, $\lim_{\varepsilon\to 0}\bar{a}^{\varepsilon} = z$. □

Remark 5.9.2. The above proof shows that all the points at which the function $\prod_{i\in N}(a_i - d_i)f(a)$ gets maximized are equilibrium points of the game $G^{\mathrm{NA}}(f)$. Moreover, as ε goes to zero, all these equilibrium points converge to the Nash solution. However, there might be other equilibrium points of the $G^{\mathrm{NA}}(f)$ games, which might converge to a point in F other than the Nash solution. One may wonder if the Nash solution is actually the unique B-essential equilibrium of G^{NA} or not. Nash informally argued that if the smoothing function f goes to zero with some regularity, then the corresponding smoothed games would have a unique equilibrium, which would correspond with the unique maximum of $\prod_{i\in N}(a_i - d_i)f(a)$. Hence, in such a case, the Nash bargaining solution would be the unique B-essential equilibrium of Nash negotiation game. A pair of formalizations of this statement and the corresponding proofs can be found in (Binmore 1987a,b) and (van Damme 1991).

5.9.2. A second implementation of the Nash bargaining solution. Now, we present the implementation based on the *alternating offers game* developed by Rubinstein (1982).[17] A deep discussion on the connections between the alternating offers game and the Nash bargaining solution can be found in Binmore et al. (1986). This implementation has become much more popular than Nash's implementation, essentially because it is done through an extensive game and, therefore, it can capture some dynamic aspects of negotiation that Nash's static approach cannot. For the sake of exposition, we present a simplified version of the two-player model in Rubinstein (1982). Moreover, extensions of Rubinstein's results to n-players can be found in Chae and Yang (1994) and Krishna and Serrano (1996).

[17] A very similar game had already been introduced years earlier in Ståhl (1972, 1977). The main difference is that Ståhl's model restricts attention to finite horizon games, whereas Rubinstein's approach does not.

The alternating offers game. There are two players bargaining to share a pie. Let $R := \{(r_1, r_2) : r_1 \geq 0, r_2 \geq 0, \text{ and } r_1 + r_2 \leq 1\}$ denote the set of possible shares of the pie. The game starts in period 0, in which player 1 proposes a share of the pie $(r_1, r_2) \in R$. This proposal can be either accepted or rejected by player 2. Upon acceptance, the proposed share is enforced and the game is over. If the proposal is rejected, period 2 starts, with player 2 making a proposal and player 1 deciding whether to accept or not. As long as no offer has been accepted, player 1 proposes in the even periods and player 2 in the odd ones. Moreover, player i's preferences over the different shares are represented by the utility function $u_i \colon [0,1] \to \mathbb{R}$, which is such that $u_i(0) = 0$. The players are not perfectly patient, in the sense that they discount future utilities according to a discount factor $\delta \in (0,1)$, that is, if share (r_1, r_2) is accepted in period t, the payoffs to the players are given by $(u_1(r_1)\delta^t, u_2(r_2)\delta^t)$. The discount factor δ may not necessarily be attributed to patience. It can also be interpreted as the probability that negotiations continue after a rejection. The u_i functions are assumed to be strictly increasing, concave, and continuous. The first assumption just reflects the fact that the pie is a desirable good. The other two assumptions ensure that the set $F := \{(u_1(r_1), u_2(r_2)) : (r_1, r_2) \in R)\}$ is convex and compact. Therefore, if we define $\bar{F}$ as the comprehensive hull of F, we get that $(\bar{F}, (0,0))$ is a well defined bargaining problem. Recall that $\bar{F}_{(0,0)} := \{x \in \bar{F} : x \geq (0,0)\}$. In addition, there is no loss of generality in assuming the normalizations $u_1(1) = u_2(1) = 1$. For the sake of exposition, we do not explicitly describe all the elements of the alternating offers game such as strategy profiles and histories. In Figure 5.9.1 we represent an example of a bargaining set $\bar{F}_{(0,0)}$ with its Nash solution.

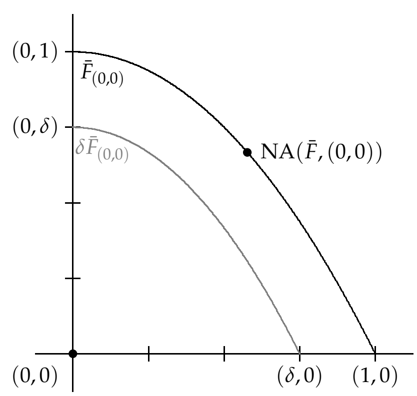


Figure 5.9.1. The bargaining set associated with an alternating offers game.

Remark 5.9.3. Although it may seem that the class of bargaining sets that can be defined through alternating offers games is quite narrow, this is not the case. The normalizations on the utility functions and the choice of the disagreement point are virtually without loss of generality. On the other hand, despite the symmetry of the strategy sets, the shape of the bargaining set still depends on the utility functions of the players, which are allowed to be very general.[18]

To familiarize the reader with the model and some of the intuitions behind the ensuing implementation, we start by solving a couple of simplifications of the alternating offers game. In both examples we consider that, for each $i \in \{1,2\}$ and each $r_i \in [0,1]$, $u_i(r_i) = r_i$. Moreover, there is a finite-horizon. If no offer is accepted in the first T periods, then the game is over and both players get 0.

Example 5.9.1. (The ultimatum game).[19] Suppose that $T = 0$, that is, if the proposal of player 1 in period 0 is not accepted, then both players get 0. It is easy to see that this game has an infinite number of Nash equilibria. We now show that all but one are based on incredible threats. More precisely, we show that there is a unique subgame perfect equilibrium. We proceed backwards. In any subgame perfect equilibrium, when asking whether to accept or not, player 2 will accept any share (r_1, r_2) with $r_2 > 0$. However, no such share can be part of a subgame perfect equilibrium since, given proposal (r_1, r_2) with $r_2 > 0$, player 1 can profitably deviate by offering $(r_1 + \frac{r_2}{2}, \frac{r_2}{2})$. Then, in any subgame perfect equilibrium, player 1 proposes the share $(1,0)$. Player 2 is then indifferent between accepting and rejecting. If he accepts, we have a subgame perfect equilibrium. If he rejects, we would not have an equilibrium, since player 1 would deviate to some proposal (r_1, r_2) with $r_2 > 0$, which, as we have already seen, would be accepted. Therefore, in the unique subgame perfect equilibrium of the ultimatum game, player 1 proposes the share $(1,0)$ and player 2 accepts. $\diamond$

Example 5.9.2. (A simple bargaining game). Suppose that $T = 1$, *i.e.*, the game has two periods, labeled 0 and 1. Again, we proceed backwards to show that this game also has a unique subgame perfect equilibrium. If period 1 is reached, the corresponding subgame is an ultimatum game. Thus, in equilibrium, if period 1 is reached, player 2 offers the share $(0,1)$ and player 1 accepts, which leads to payoff 0 for player 1 and δ for player 2. Hence, in period 0, player 1 knows that player 2 will accept any offer (r_1, r_2) with $r_2 > \delta$. However, repeating the arguments above, no such share can

[18] Note that instead of having the players offering shares of the pie, we could have equivalently defined the alternating offers game with the proposals belonging to a given bargaining set. Actually, this is the approach taken in Myerson (1991). He shows that, for Rubinstein's result to go through, he just needs to impose a mild regularity condition on the bargaining set; this regularity condition plays the role of our assumptions on the utility functions.

[19] This game is often called divide-the-dollar game.

be part of a subgame perfect equilibrium since, given proposal (r_1, r_2) with $r_2 > \delta$, player 1 can profitably deviate by offering $(r_1 + \frac{r_2-\delta}{2}, r_2 - \frac{r_2-\delta}{2})$. Thus, it is easy to see that the unique subgame perfect equilibrium of this game has player 1 offering the share $(1-\delta, \delta)$ and player 2 accepting the offer. Again, in order to have a subgame perfect equilibrium, the player who accepts the offer has to be indifferent between accepting and rejecting. Note the role of patience in this example. If the players are very patient, then player 2 gets almost 1, whereas if the players are very impatient, then player 1 gets almost all the pie (actually, in this case the argument only depends on the patience of player 2). $\diamond$

Note that an allocation $x \in \bar{F}$ is Pareto efficient in $\bar{F}$ if and only if $x = (u_1(r_1), u_2(r_2))$, with $(r_1, r_2) \in R$ and $r_1 + r_2 = 1$. The next result highlights the importance of having the parameter δ introducing some "friction" in the bargaining process, in the sense that delay is costly.

Proposition 5.9.3. *Consider the alternating offers game with perfectly patient players* (i.e., $\delta = 1$)*. Then, an allocation $x \in \bar{F}_{(0,0)}$ is a subgame perfect equilibrium payoff if and only if x is Pareto efficient.*

Proof. Let $(r_1, r_2) \in R$ be such that $x = (u_1(r_1), u_2(r_2))$ is Pareto efficient in $\bar{F}$, *i.e.*, $r_1 + r_2 = 1$. Consider the following strategy profile. Both players, when playing in the role of the proposer, propose the share (r_1, r_2). When asked to accept or reject, player i accepts any offer that gives him at least r_i and rejects any other proposal. It is easy to verify that this strategy profile is a subgame perfect equilibrium of the game when the players are perfectly patient. Conversely, suppose that we have a subgame perfect equilibrium delivering an inefficient allocation x. Let $(r_1, r_2) \in R$ be such that $x = (u_1(r_1), u_2(r_2))$ and $r_1 + r_2 < 1$. Consider the proposal $(\bar{r}_1, \bar{r}_2)$ defined, for each $i \in \{1,2\}$, by $\bar{r}_i := r_i + \frac{1-r_1-r_2}{2}$. Then, there has to be a subgame where the proposer at the start of the subgame can profitably deviate by proposing $(\bar{r}_1, \bar{r}_2)$ instead of (r_1, r_2). $\square$

One of the main virtues of Rubinstein's game is that any degree of impatience reduces the set of subgame perfect equilibrium payoff vectors to a single allocation. Moreover, it can be easily checked that even for impatient players, the alternating offers game has a continuum of Nash equilibria. Hence, it is worth noting the strength of subgame perfect equilibrium as a refinement of Nash equilibrium in this particular game. Consider the strategy profile in which player 1 always proposes a share $r^1 \in R$ and player 2 always proposes a share $r^2 \in R$. Player 1 accepts any offer that gives him at least $u_1(r^2_1)$ and player 2 accepts any offer that gives him at least $u_2(r^1_2)$. Note, in particular, that the action of a player at a given period depends neither on the past history of play nor on the number of periods that have elapsed. Moreover, according to this strategy, the first proposal is accepted, so the impatience of the players does not lead to inefficient allocations. Can there be a subgame perfect equilibrium characterized by this simple behavior? The answer is positive and, in addition, the unique subgame perfect equilibrium of the alternating offers game is of this form. Let (s_{r^1}, s_{r^2}) denote the above strategy profile.

Theorem 5.9.4. *The alternating offers game with discount $\delta \in (0,1)$ has a unique subgame perfect equilibrium (s_{r^1}, s_{r^2}), where the shares r^1 and r^2 lead to Pareto efficient allocations satisfying that*

$$u_1(r_1^2) = \delta u_1(r_1^1) \quad \text{and} \quad u_2(r_2^1) = \delta u_2(r_2^2).$$

In particular, the first proposal is accepted and the equilibrium payoff vector is given by $(u_1(r_1^1), u_2(r_2^1))$.

Proof.[20] Note that all the subgames that begin at an even period (with player 1 offering first) have the same set of subgame perfect equilibria as the whole game. Similarly, all the subgames that begin at an odd period (with player 2 offering first) have the same set of subgame perfect equilibria. Let M_1 and m_1 be the supremum and the infimum of the set of subgame perfect equilibrium payoffs for player 1 in subgames where he proposes first. Similarly, let L_2 and l_2 be the supremum and the infimum of the set of subgame perfect equilibrium payoffs for player 2 in subgames where he proposes first. We now show that $M_1 = m_1$ and $L_2 = l_2$. We also need to define two auxiliary functions h_1 and h_2. Given $z \in [0,1]$, let $h_1(z) := \max\{x_1 : (x_1, z) \in \bar{F}_{(0,0)}\}$ and $h_2(z) := \max\{x_2 : (z, x_2) \in \bar{F}_{(0,0)}\}$, *i.e.*, $h_i(z)$ denotes the maximum utility player i can get conditional on the other player getting z.

In any subgame perfect equilibrium, player 2 always accepts any offer giving him more than δL_2, which is the best payoff he can get (in equilibrium) by rejecting the present proposal. Hence, in any equilibrium where player 1 is the first proposer he gets, at least, $h_1(\delta L_2)$. Moreover, by the definition of L_2, for each $\varepsilon > 0$, there is a subgame perfect equilibrium of the alternating offers game in which player 2 gets at least $L_2 - \varepsilon$ in all the subgames starting at period 1. In such an equilibrium, player 2 can ensure for himself at least payoff $\delta(L_2 - \varepsilon)$ by rejecting the proposal in period 0. Therefore, in this equilibrium player 1 gets, at most, $h_1(\delta(L_2 - \varepsilon))$. Since we have already seen that $h_1(\delta L_2)$ is a lower bound for the payoff of player 1 in any equilibrium when he is the first proposer, by letting ε go to 0, we get that $m_1 = h_1(\delta L_2)$. By a similar argument, we get $l_2 = h_2(\delta M_1)$.

[20] This proof is an adaptation of the arguments in the proof of Theorem 8.3 in Myerson (1991).

In any subgame perfect equilibrium, player 2 never accepts any offer giving him less than δl_2, since (in equilibrium) he can get δl_2 by rejecting the present proposal. Hence, in any equilibrium where player 1 is the first proposer he gets, at most, $h_1(\delta l_2)$. Moreover, by the definition of l_2, for each $\varepsilon > 0$, there is a subgame perfect equilibrium of the alternating offers game in which player 2 gets at most $l_2 + \varepsilon$ in all the subgames starting at period 1. Therefore, in this equilibrium player 1 gets, at least, $h_1(\delta(l_2 + \varepsilon))$. Again, by letting ε go to 0, we get that $M_1 = h_1(\delta l_2)$. By a similar argument, we get $L_2 = h_2(\delta m_1)$.

We complete the definitions of the allocations M, m, L, and l. Let $M_2 := h_2(M_1)$, $m_2 := h_2(m_1)$, $L_1 := h_1(L_2)$, and $l_1 := h_1(l_2)$. So defined, these four vectors are Pareto efficient. We claim that, by construction, $M_2 = \delta l_2$, $l_1 = \delta M_1$, $m_2 = \delta L_2$, and $L_1 = \delta m_1$. We do the proof for M_2, with the other cases being analogous. Note that $M_2 = h_2(h_1(\delta l_2))$. By definition, $(h_1(\delta l_2), \delta l_2)$ and $(h_1(\delta l_2), h_2(h_1(\delta l_2)))$ are Pareto efficient allocations in $\bar{F}_{(0,0)}$, but this can only happen if $h_2(h_1(\delta l_2)) = \delta l_2$. Hence, we have

$$M_2 = \delta l_2 \quad \text{and} \quad l_1 = \delta M_1, \tag{5.9.1}$$

$$m_2 = \delta L_2 \quad \text{and} \quad L_1 = \delta m_1. \tag{5.9.2}$$

We now show that there is a unique pair of Pareto efficient allocations x and y in $\bar{F}_{(0,0)}$ such that (see Figure 5.9.2)

$$y_1 = \delta x_1 \quad \text{and} \quad x_2 = \delta y_2.$$

First, note that the above equations can be equivalently rewritten as

$$x_1 - y_1 = (1-\delta)x_1 \quad \text{and} \quad y_2 - x_2 = \frac{(1-\delta)x_2}{\delta}.$$

Note that if $x_1 = 0$, then $x_1 - y_1 = 0$. Since both x and y are Pareto efficient, $x_1 - y_1 = 0$ implies that $y_2 - x_2 = 0$. Hence, based on the above equations, monotonically increasing x_1 from 0 to 1 has the following effects. First, $x_1 - y_1$ also increases monotonically from 0. Second, since $x_1 - y_1$ increases, using Pareto efficiency again, $y_2 - x_2$ also increases monotonically from 0.[21] Third, x_2 decreases monotonically from 1 to 0. Therefore, since everything varies continuously, by monotonically increasing x_1 from 0 to 1, there is exactly one value at which the equation $y_2 - x_2 = \frac{(1-\delta)x_2}{\delta}$ is satisfied. Hence, Eqs. (5.9.1) and (5.9.2) immediately imply that $M = m = x$ and $L = l = y$. Therefore, x is the unique equilibrium payoff in each subgame

[21] In words, efficiency implies that, the better player 1 is in x relative to y, the better player 2 is in y relative to x.

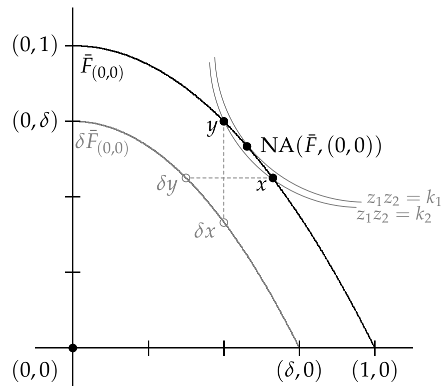


Figure 5.9.2. Subgame perfect equilibrium payoff vectors for a given δ; x is the unique equilibrium payoff vector in the subgames where player 1 is the first proposer and, similarly, y is the unique equilibrium payoff vector in the subgames where player 2 is the first proposer.

in which player 1 is the first proposer. Similarly, y is the unique equilibrium payoff in each subgame in which player 2 is the first proposer.[22]

Let r^1 and r^2 be the unique shares in R such that $(u_1(r^1_1), u_2(r^1_2)) = x$ and $(u_1(r^2_1), u_2(r^2_2)) = y$, respectively. It is now straightforward to check that (s_{r^1}, s_{r^2}) is a subgame perfect equilibrium of the alternating offers game.

We have already proved the uniqueness of the equilibrium payoff vectors. What is left to show is the uniqueness of the equilibrium strategies. Suppose there is another subgame perfect equilibrium of the game with the property that x is the payoff vector at all subgames where player 1 is the first proposer, and y is the payoff vector at all subgames where player 2 is the first proposer. By the above property, in each subgame perfect equilibrium, in each subgame, player 2 accepts any proposal giving him more than r^1_2 and rejects any proposal giving him less. Moreover, player 2 also accepts when the proposal gives him r^1_2 since, if he were to reject such an offer at a given subgame, then player 1 would profitably deviate by proposing an efficient allocation giving player 2 something slightly above r^1_2.[23] Similarly, in each subgame perfect equilibrium, player 1 accepts a proposal if and only if it gives him at least r^2_1. Suppose now that there is a subgame in which player 1 is the first proposer and offers less than r^2_1 to player 2; player 2 rejects the proposal and, hence, player 1 gets, at most, δr^2_1. Thus, player 1 can profitably deviate in that subgame by making the proposal r^1,

[22]Note that we may not get $y_1 = x_2$ and $x_2 = y_1$. Whether these extra equalities are satisfied depends on the shape of the Pareto efficient boundary of $\bar{F}$. For instance, for the bargaining set depicted in Figure 5.9.2, we have that $y_2 > x_1$ and $x_2 > y_1$.

[23]This is exactly the same argument we made for the indifferences in Examples 5.9.1 and 5.9.2.

which would be accepted by player 2, and get share r_1^1. Therefore, in equilibrium, player 1 never offers less than r_2^1 to player 2. Also, player 1 never offers more than r_1^2 to player 2, since such an offer would be accepted and player 1 would be better off by proposing r^1 instead. Hence, the only option is that player 1 makes proposal r^1 whenever he is the proposer. Similarly, player 2 makes proposal r^2 whenever he is the proposer. □

The above result highlights a natural property of the equilibrium shares r^1 and r^2. Since $u_1(r_1^2) = \delta u_1(r_1^1)$ and $u_2(r_2^1) = \delta u_2(r_2^2)$, the first mover has an advantage in the alternating offers game. Moreover, this advantage is larger the more patient the players are. Actually, in the limit, as both players become perfectly patient, the first mover advantage disappears.

In spite of the strength of Theorem 5.9.4, we have not yet found any link between the equilibrium payoff vectors of the alternating offers game and the Nash solution. Actually, Figure 5.9.2 represents a situation in which the equilibrium payoff vectors do not coincide with the Nash solution. Nonetheless, the product of the gains of the players with respect to the disagreement point, the one that the Nash solution maximizes, is implicit in the conditions

$$u_1(r_1^2) = \delta u_1(r_1^1) \quad \text{and} \quad u_2(r_2^1) = \delta u_2(r_2^2).$$

To see why, just note that these conditions imply that

$$\frac{u_1(r_1^2)}{u_1(r_1^1)} = \frac{u_2(r_2^1)}{u_2(r_2^2)} = \delta$$

and, therefore, $u_1(r_1^1)u_2(r_2^1) = u_1(r_1^2)u_2(r_2^2)$ (this feature is illustrated in Figure 5.9.2). Since the disagreement point in the alternating offers game is $(0,0)$, the Nash solution is the share $r^* \in R$ that maximizes the above product of utilities. The result below shows that, as the players become more and more patient, the proposals r^1 and r^2 collapse into r^*. Therefore, the more patient the players are, the closer the unique equilibrium payoff vector is to the Nash solution.

Corollary 5.9.5. *The equilibrium allocation of the alternating offers game converges to the Nash solution of the bargaining problem $(\bar{F}, (0,0))$ as δ converges to* 1.

Proof. Given an alternating offers game with discount factor δ, let x^δ be the equilibrium payoff vector when player 1 proposes first, and let y^δ be the equilibrium payoff vector when player 2 proposes first. These payoff vectors satisfy that $x_1^\delta x_2^\delta = y_1^\delta y_2^\delta$. Recall that, by definition, the Nash solution is defined as the unique maximizer of the previous product inside the set $\bar{F}$.

Hence, for each discount factor δ, the Nash solution lies in between the corresponding equilibrium allocations x^δ and y^δ (see Figure 5.9.2). Moreover, since $y_1^\delta = \delta x_1^\delta$ and $x_2^\delta = \delta y_2^\delta$, as δ goes to 1, $x_1^\delta - y_1^\delta$ and $x_2^\delta - y_2^\delta$ converge to 0. Therefore, as δ goes to 1, both x^δ and y^δ converge to the Nash solution. $\square$

The relevance of this last result is nicely summarized in the following passage taken from Serrano (2005): "This remarkable result provides a sharp prediction to the bilateral bargaining problem. But unlike Nash's simultaneous-move game, Rubinstein's uniqueness is a result of the use of credible threats in negotiations, together with the assumption of impatience (or breakdown probabilities in the negotiations)".

5.9.3. Implementing the Kalai-Smorodinsky solution. We now present an extensive game introduced by Moulin (1984) to implement the Kalai-Smorodinsky solution. Although the original paper considered general n-player bargaining problems, we restrict attention to the two-player case. Moulin (1984) considered the following noncooperative negotiation process, given by a three-stage extensive game that we call Γ^{KS}.

Stage 1: Simultaneously and independently, each player $i \in \{1,2\}$ submits a bid a_i, with $0 \leq a_i \leq 1$.

Stage 2: The player with the highest bid becomes the proposer. If $a_1 = a_2$, the proposer is randomly chosen. Say i is the proposer and j is not. Player i proposes an allocation $x \in F$. If player j accepts, then the outcome x is realized. If player j rejects x, then with probability $1 - a_i$ the final outcome is the disagreement point and with probability a_i we go to Stage 3.

Stage 3: Player j proposes an allocation. If it is accepted by player i, the proposal is realized, otherwise, the realized outcome is the disagreement point.

In this game both players bid to have the advantage of being the first proposer. Note that, whenever a player is given the chance to make an offer at Stage 3, he is actually a dictator, since the opponent will have to choose between the offer and the disagreement point. Hence, the higher the winning bid is, the less advantageous it is to win the auction, since the probability that the other player becomes a dictator in case of rejection in Stage 2 becomes larger.[24]

Proposition 5.9.6. *Let $(F,d) \in B^2$ and let $x \in F_d$. Then, there is a Nash equilibrium of Γ^{KS} whose payoffs equal x.*

[24] The alternative game in which the probability of player j of becoming a proposer at Stage 3 is given by a_j instead of a_i would lead to the same results (see Exercise 5.12).

Proof. Let $x \in F$. Consider the following strategy profile:

Stage 1: Both players bid 0.

Stage 2: The proposer chooses x. The player who is not the proposer accepts x and rejects any other proposal.

Stage 3: The proposer chooses the disagreement point.

This is a Nash equilibrium of Γ^{KS} and the payoffs are given by x. □

The above result implies that Nash equilibrium does not discriminate among the outcomes in F_d. In addition, in the subgames where player i is a dictator, *i.e.*, when he is given the chance to propose at Stage 3, he can enforce any feasible outcome. Hence, the Nash equilibrium described in Proposition 5.9.6 is not subgame perfect. The next result shows that under a reasonably mild condition, the use of subgame perfect equilibrium is enough to pin down a unique outcome, which, moreover, coincides with the Kalai-Smorodinsky solution.

Recall that, given a bargaining problem $(F,d) \in B^2$, $b(F,d)$ is the vector of utopia payoffs of the different players. Consider the following assumption, which we call A1. Let i and j be the two players in (F,d). For each $x \in F_d$, if $x_i = b_i(F,d)$, then $x_j = d_j$. This assumption is met, for instance, when the players are bargaining on how to share some desirable good. If one player gets everything, then the other player gets nothing, which would give him the same payoff as if no agreement were reached. In the two bargaining problems in Figure 5.3.2 (Section 5.3), the left one satisfies A1 and the right one does not.

Theorem 5.9.7. *Let $(F,d) \in B^2$ be a bargaining problem satisfying* A1. Then, the Kalai-Smorodinsky solution of (F,d) is the unique subgame perfect equilibrium payoff of game Γ^{KS}.

Proof. Since the Kalai-Smorodinsky solution satisfies CAT, we can restrict attention, without loss of generality, to bargaining problems where, for each $i \in N$, $d_i = 0$ and $b_i(F,d) = 1$. In a game where the utilities have been normalized in this way, the Kalai-Smorodinsky solution is the unique Pareto efficient allocation that gives both players the same payoff.

For each payoff x_2 that player 2 can get in F_d, let $g_1(x_2)$ be the maximum payoff that player 1 can get in F_d conditional on player 2 getting at least x_2. Define $g_2(x_1)$ in an analogous manner. Suppose that the bids at Stage 0 are such that $a_1 > a_2$. Then, player 2 can ensure himself payoff a_1 by rejecting the offer of player 1 and making offer $(0,1)$ if he is given the chance to (which happens with probability a_1). Moreover, by A1, the only way in which player 2 can offer 1 to himself is by offering 0 to player 1. Hence, the optimal offer of player 1 at Stage 2 is $(g_1(a_1), a_1)$. Therefore, by bidding a_1

at Stage 1, player 1 gets a_2 if $a_1 < a_2$, $g_1(a_1)$ if $a_1 > a_2$, and something in between a_1 and $g_1(a_1)$ if $a_2 = a_1$. Now, let a^* be the unique number such that (a^*, a^*) is efficient, *i.e.*, (a^*, a^*) is the Kalai-Smorodinsky solution. We now claim that, $g_1(a^*) = a^*$. Suppose, on the contrary that there is $\alpha > 0$ such that $g_1(a^*) = a^* + \alpha$. Then, $(a^* + \alpha, a^*) \in F$. Also, $(0,1) \in F$ and, since $a^* + \alpha > 0$, A1 ensures that $a^* < 1$. Let $t := \frac{\alpha}{1+\alpha}$ and consider the allocation $t(0,1) + (1-t)((a^* + \alpha, a^*) = (\frac{a^*+\alpha}{1+\alpha}, \frac{a^*+\alpha}{1+\alpha})$. Since F is convex and $t \in (0,1)$, this allocation belongs to F and, moreover, since $\frac{a^*+\alpha}{1+\alpha} > a^*$, we get a contradiction with the fact that (a^*, a^*) is efficient. Then, since $g_1(a^*) = a^*$, player 1 is guaranteed payoff a^* by bidding a^*. A symmetric argument shows that player 2 is also guaranteed payoff a^* by bidding a^*. Hence, the unique candidate to be a subgame perfect equilibrium payoff of game Γ^{KS} is (a^*, a^*), *i.e.*, the Kalai-Smorodinsky solution. One such subgame perfect equilibrium is the one in which both players submit bid a^* at Stage 0 and the proposal a^* is made and accepted at Stage 2. □

5.9.4. Implementing the Shapley value. Because of the popularity of the Shapley value, a large number of mechanisms implementing it have been discussed in the literature. Here, we restrict attention to the one proposed by Pérez-Castrillo and Wettstein (2001). Other relevant noncooperative approaches to the Shapley value can be found, for instance, in Gul (1989), Evans (1990), Hart and Moore (1990), Winter (1994), Hart and Mas-Colell (1996), Dasgupta and Chiu (1996) and, for more recent contributions, Mutuswami et al. (2004) and Vidal-Puga (2004). It is worth emphasizing the generality of the approach in Hart and Mas-Colell (1996), since they developed their analysis for NTU-games, where their noncooperative game implements a well known extension of the Shapley value. Thus, when restricting attention to TU-games, their approach implements the Shapley value and, moreover, when restricting attention to bargaining problems, it implements the Nash solution. Again, we refer the reader to Serrano (2005) for a detailed survey. Pérez-Castrillo and Wettstein (2001) proposed a bidding mechanism that implements the Shapley value in the class of zero-monotonic games (see Definition 5.4.8). Next, we describe the mechanism in a recursive way as the number of players increases. First, if there is a unique player, the player obtains his value. Suppose that the mechanism has been described when the number of players is at most $n-1$. Now, take a zero-monotonic game $(N, v) \in G^N$ with $N = \{1, \ldots, n\}$. Consider the extensive game Γ^{Φ} defined as follows.

Stage 1: Simultaneously and independently, each player $i \in N$ submits offers to the players in $N \backslash \{i\}$. These offers are bids to become the proposer at Stage 2. The set of actions of player i at Stage 1 is

$\mathbb{R}^{n-1}$. Let b^i_j denote the amount that player i offers to player $j \neq i$. For each $i \in N$, let

$$\alpha^i := \sum_{j \neq i} b^i_j - \sum_{j \neq i} b^j_i.$$

Let p be such that $\alpha^p = \max_{i \in N} \alpha^i$. If there are several maximizers, p is chosen randomly among them. Once p has been chosen, he pays b^p_i to each player $i \neq p$.

Stage 2: Player p makes an offer to any player other than p, that is, his strategy set is again $\mathbb{R}^{n-1}$. Suppose that the offer is a^p.

Stage 3: Players other than p sequentially accept or reject the offer made in Stage 2. The offer is accepted if and only if all the other players accept it. If the offer is rejected, player p leaves the game and players in $N \backslash \{p\}$ play the mechanism again. In this case, player p obtains $v(p) - \sum_{i \neq p} b^p_i$ and each player $i \neq p$ obtains b^p_i plus what he obtains in the ensuing game. If the offer is accepted, the game finishes. Each player $i \neq p$ obtains $b^p_i + a^p_i$ and player p obtains $v(N) - \sum_{i \neq p} b^p_i - \sum_{i \neq p} a^p_i$.

Theorem 5.9.8. *Let (N, v) be a zero-monotonic TU-game. Then, the Shapley value of (N, v) is the unique subgame perfect equilibrium payoff of game Γ^Φ.*

Proof. First, note that the formula for the Shapley value can be equivalently rewritten, for each $v \in G^N$ and each $i \in N$, as

$$\Phi_i(v) = \frac{1}{n}(v(N) - v(N \backslash \{i\})) + \frac{1}{n} \sum_{j \neq i} \Phi_i(v_{N \backslash \{j\}}). \tag{5.9.3}$$

We proceed by induction on the number of players. The case $n = 1$ is straightforward. Assume that the result holds for zero-monotonic games with at most $n - 1$ players. Let (N, v) be a zero-monotonic game. For the sake of notation, for each $S \subset N$, we denote $\Phi(v_S)$ by $\Phi(S)$. Since (N, v) is zero-monotonic, all the v_S games are zero-monotonic as well.

Claim 1. The Shapley value is the payoff vector of some subgame perfect equilibrium of Γ^Φ.

Consider the following strategies.

- At Stage 1, each player $i \in N$ bids b^i, where, for each $j \neq i$ $b^i_j := \Phi_j(v) - \Phi_j(N \backslash \{i\})$.
- At Stage 2, if the proposer is p, he offers, to each $i \neq p$, $a^p_i := \Phi_i(N \backslash \{p\})$.

- At Stage 3, if the proposer is p, every player $i \neq p$ accepts any proposal greater or equal than $\Phi_i(N\backslash\{p\})$ and rejects any proposal smaller than $\Phi_i(N\backslash\{p\})$.

Under this strategy profile, if p is the proposer, then each player $i \neq p$ gets $\Phi_i(v) - \Phi_i(N\backslash\{p\}) + \Phi_i(N\backslash\{p\}) = \Phi_i(v)$. Then, since Φ satisfies EFF, the proposer p also gets $\Phi_p(v)$. For each $i \in N$,

$$\alpha^i = \sum_{j\neq i}(\Phi_j(v) - \Phi_j(N\backslash\{i\})) - \sum_{j\neq i}(\Phi_i(v) - \Phi_i(N\backslash\{j\})).$$

Now, the first sum equals $\sum_{j\neq i}\Phi_j(v) - \sum_{j\neq i}\Phi_j(N\backslash\{i\}) = v(N) - \Phi_i(v) - v(N\backslash\{i\})$. By Eq. (5.9.3), the second equals $\sum_{j\neq i}\Phi_i(v) - \sum_{j\neq i}\Phi_i(N\backslash\{j\}) = (n-1)\Phi_i(v) - n\Phi_i(v) + v(N) - v(N\backslash\{i\})$. Hence, $\alpha^i = 0$.

We now show that the above strategy profile is a subgame perfect equilibrium. The induction hypothesis ensures that no player $i \neq p$ can gain by deviating at Stage 3, whatever the offer made by p is. At Stage 2, if p offers $\hat{a}^p \in \mathbb{R}^{n-1}$, with $\hat{a}^p_i < \Phi_i(N\backslash\{p\})$ for some $i \neq p$, then i rejects the offer. In this case, p gets $v(p) - \sum_{i\neq p}(\Phi_i(v) - \Phi_i(N\backslash\{p\}))$, which, by Eq. (5.9.3), equals $v(p) - v(N) + \Phi_p(v) + v(N\backslash\{p\})$ and, by zero-monotonicity, this is at most $\Phi_p(v)$. Hence, this deviation is not profitable for p. If p's deviation is such that there is $i \neq p$ with $\hat{a}^p_i > \Phi_i(N\backslash\{p\})$, and there is no $i \neq p$ who gets offered less than $\Phi_i(N\backslash\{p\})$, then p strictly decreases his payoff. Finally, it is straightforward to check that deviations at Stage 1 are not profitable.

Claim 2. Any subgame perfect equilibrium of Γ^Φ yields the Shapley value.

In the rest of the proof, whenever an arbitrary subgame perfect equilibrium of Γ^Φ is considered, p denotes the proposer at Stage 2, b^p denotes his bid at Stage 1, and a^p denotes his offer at Stage 2. We proceed by proving several new claims.

Claim 2.1. In any subgame perfect equilibrium, all the players in $N\backslash\{p\}$ accept the offer if, for each $i \neq p$, $a^p_i > \Phi_i(N\backslash\{p\})$. Moreover, if there is $i \neq p$ with $a^p_i < \Phi_i(N\backslash\{p\})$, then the proposal is rejected.

Let $i \neq p$. By the induction hypothesis, if there is a rejection at Stage 3, then player i gets $b^p_i + \Phi_i(N\backslash\{p\})$. Let l be the last player who decides to accept or reject the offer. If l is given the chance to accept or reject, *i.e.*, if there has been no prior rejection, at a best reply l accepts any offer above $\Phi_l(N\backslash\{p\})$ and rejects any offer below $\Phi_l(N\backslash\{p\})$. Let k be the second to last player. Since player k is also best replying, if $a^p_k > \Phi_k(N\backslash\{p\})$, $a^p_l > \Phi_l(N\backslash\{p\})$, and the game reaches k, then player k accepts the offer. If $a^p_k < \Phi_k(N\backslash\{p\})$ and $a^p_l > \Phi_l(N\backslash\{p\})$, then player k rejects the offer.

If $a_l^p < \Phi_l(N\backslash\{p\})$, then player k is indifferent to rejecting or accepting since he knows that player l will reject anyway. Hence, we can just proceed backwards and repeat the same argument to get the proof of Claim 2.1.

Claim 2.2. If $v(N) > v(N\backslash\{p\}) + v(p)$, then, starting at Stage 2, in any subgame perfect equilibrium we have:

- At Stage 2, player p offers, to each $i \neq p$, $a_i^p = \Phi_i(N\backslash\{p\})$.
- At Stage 3, each $i \neq p$ accepts any offer $\hat{a}^p \in \mathbb{R}^{n-1}$ with $\hat{a}_i^p \geq \Phi_i(N\backslash\{p\})$. Any offer that gives less than $\Phi_i(N\backslash\{p\})$ to some $i \neq p$ is rejected.

If $v(N) = v(N\backslash\{p\}) + v(p)$, there are other types of subgame perfect equilibria starting at Stage 2 in which:

- At Stage 2, player p offers $a^p \in \mathbb{R}^{n-1}$, with $a_i^p \leq \Phi_i(N\backslash\{p\})$ for some $i \neq p$.
- At Stage 3, each $i \neq p$ rejects any offer $\hat{a}^p \in \mathbb{R}^{n-1}$ with $\hat{a}_i^p \leq \Phi_i(N\backslash\{p\})$.

All the above equilibria lead to the same payoff vector. The proposer p gets $v(N) - v(N\backslash\{p\}) - \sum_{i\neq p} b_i^p$ and each $i \neq p$ gets $\Phi_i(N\backslash\{p\}) + b_i^p$.

It is now easy to see that all the strategy profiles we have described above are subgame perfect equilibria of the subgame starting at Stage 2. Suppose that we are at a subgame perfect equilibrium. First, suppose that $v(N) > v(N\backslash\{p\}) + v(p)$. If a player $i \neq p$ rejects the offer, then p gets $v(p) - \sum_{i\neq p} b_i^p$. In such a case, if p deviates and proposes, to each $i \neq p$, $\hat{a}_i^p = \Phi_i(N\backslash\{p\}) + \varepsilon/(n-1)$, with $0 < \varepsilon < v(N) - v(N\backslash\{p\}) - v(p)$, he increases his payoff because, by Claim 2.1, $\hat{a}^p$ is accepted at Stage 3. Therefore, at Stage 3, the proposal gets accepted in any subgame perfect equilibrium. This implies that player p offers, to each $i \neq p$, $a_i^p \geq \Phi_i(N\backslash\{p\})$. However, if there is $i \neq p$ such that $a_i^p > \Phi_i(N\backslash\{p\})$, then p increases his payoff by slightly decreasing the offer to i, but still offering him something above $\Phi_i(N\backslash\{p\})$. Hence, in any subgame perfect equilibrium, for each $i \neq p$ $a_i^p = \Phi_i(N\backslash\{p\})$. Finally, we have already shown that, at Stage 3, the offer gets accepted and, hence, every offer $a^p \in \mathbb{R}^{n-1}$ such that, for each $i \neq p$, $a_i^p \geq \Phi_i(N\backslash\{p\})$ gets accepted.

If $v(N) = v(N\backslash\{p\}) + v(p)$, then any proposal $a^p \in \mathbb{R}^{n-1}$ such that $\sum_{i\neq p} a_i^p < v(N\backslash\{p\})$ is rejected in any subgame perfect equilibrium (by Claim 2.1). Otherwise, by the induction hypothesis, each player i with $a_i^p < \Phi_i(N\backslash\{p\})$ increases his payoff by rejecting the offer. Thus, at Stage 3 an offer would be accepted if $\sum_{i\neq p} a_i^p \geq v(N\backslash\{p\})$. Paralleling the argument above, acceptance at Stage 3 implies that, for each $i \neq p$, $a_i^p = \Phi_i(N\backslash\{p\})$.

Also, since p gets $v(p)$ in case of a rejection, any offer that leads to a rejection is also part of a subgame perfect equilibrium.

Claim 2.3. In any subgame perfect equilibrium, for each $i, j \in N$, $\alpha^i = \alpha^j$. Moreover, since $\sum_{i\in N} \alpha^i = 0$, we have that, for each $i \in N$, $\alpha_i = 0$.

For each $i \in N$, let $b^i \in \mathbb{R}^{n-1}$ be the bid of player i at Stage 1 in a subgame perfect equilibrium. Let $S := \{i \in N : \alpha^i = \max_{j\in N} \alpha^j\}$. If $S = N$, since $\sum_{j\in N} \alpha^j = 0$, we have that, for each $i \in N$, $\alpha^i = 0$. Suppose that $S \subsetneq N$. Let $k \notin S$ and let $i \in S$. Let $\varepsilon > 0$ and consider the bid of player i given by $\hat{b}^i \in \mathbb{R}^{n-1}$, where, for each $l \in S\backslash\{i\}$, $\hat{b}^i_l = b^i_l + \varepsilon$, $\hat{b}^i_k = b^i_k - |S|\varepsilon$, and, otherwise, $\hat{b}^i_l = b^i_l$. Hence, for each $l \in S$, $\hat{\alpha}^l = \alpha^l - \varepsilon$, $\hat{\alpha}^k = \alpha^k + |S|\varepsilon$, and, otherwise, $\hat{\alpha}^l = \alpha^l$. Therefore, if ε is small enough, the set S does not change. However, player i increases his payoff, since

$$\sum_{l\neq i} \hat{b}^i_l = \sum_{l\neq i} b^i_l - \varepsilon.$$

Claim 2.4. In any subgame perfect equilibrium, independently of who is chosen as the proposer, each player gets the same payoff.

By Claim 2.3, we know that, for each $i, j \in N$, $\alpha^i = \alpha^j$. Note that Claim 2.2 implies that if the proposer pays b^p at Stage 1, then the rest of the payoffs only depend on who the proposer is, *i.e.*, they are independent of the specific bids made at Stage 1. Hence, if a player i had strict incentives to be the proposer, he could increase his payoff by slightly increasing his bid. Similarly, if i strictly preferred player $j \neq i$ to be the proposer, he could increase his payoff by slightly decreasing b^i_j. Therefore, in any equilibrium all the players have to be indifferent to the identity of the proposer.

Claim 2.5. In any subgame perfect equilibrium, the realized payoff coincides with the Shapley value.

Consider a subgame perfect equilibrium and let $i \in N$. By Claim 2.1, if i is the proposer then he gets $x^i_i = v(N) - v(N\backslash\{i\}) - \sum_{k\neq i} b^i_k$; if the proposer is $j \in N\backslash\{i\}$, then player i gets $x^j_i = b^j_i + \Phi_i(N\backslash\{j\})$. Then, using that $\alpha_i = 0$ first and then Eq. (5.9.3), we have

$$\begin{aligned}
\textstyle\sum_{j\in N} x^j_i &= v(N) - v(N\backslash\{i\}) - \sum_{j\neq i} b^i_j + \sum_{j\neq i} b^j_i + \sum_{j\neq i} \Phi_i(N\backslash\{j\}) \\
&= v(N) - v(N\backslash\{i\} + \sum_{j\neq i} \Phi_i(N\backslash\{j\}) \\
&= n\Phi_i(v).
\end{aligned}$$

Moreover, by Claim 2.4, for each $j \in N$, $x^i_i = x^j_i$. Hence, for each $j \in N$, $x^j_i = \Phi_i(v)$. □

Remark 5.9.4. The last step of the proof above implies that the game Γ^Φ is not just an implementation of the Shapley value as the expected payoff in any subgame perfect equilibrium. In any subgame perfect equilibrium, the realized payoffs always coincide with the Shapley value.

5.10. Airport Problems and Airport Games

In this section we present an application of TU-games. This model has been proposed by Littlechild and Owen (1973) as an example of a situation where the Shapley value is especially easy to compute and, moreover, allows for a very natural interpretation. The idea was suggested by the work of Baker (1965) and Thompson (1971) on charging airport movements for different types of aircrafts, where a movement means a take-off or a landing. Now, we present and study the game theoretic setting.

Assume that there are m different types of aircrafts. For each type l, we know the number of movements, n_l, and the overall runway costs, c_l, associated with type l aircrafts. Since we want to find out how much each individual movement should be charged, we interpret each of them as a single player. Hence, let N_l be the set of "players" associated with movements of type l aircrafts.

We assume that types are arranged in an increasing order with respect to the costs, *i.e.*, $0 < c_1 < \ldots < c_m$. Let N be the (finite) set of all movements, namely $N := \cup_{l=1}^{m} N_l$. The elements we have described fully characterize the so-called *airport problems*. The *airport game* associated with this cost allocation problem is the TU-game with player set N and characteristic function c defined, for each $S \subset N$, $S \neq \emptyset$, by

$$v(S) := -\max\{c_l : S \cap N_l \neq \emptyset\}. \tag{5.10.1}$$

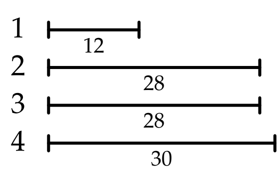
The interpretation here is that larger aircrafts need longer runways. Thus, a coalition S should not be responsible for costs associated with parts of the runway that are never used by the movements in S. In particular, if a runway is suitable for type l, then it is also suitable for smaller types. Hence, the cost of building a suitable runway for a set of movements S coincides with the cost of building the suitable runway for the largest type with some movement in S. We change the sign in order to keep the interpretation of the characteristic function as the benefit of a coalition S.

As defined above, any airport game is superadditive and convex (Exercise 5.13 asks the reader to prove this assertion). Then, by Corollary 5.8.2, the core is nonempty, with the Shapley value being one of its elements. For this problem, the proposal of an allocation rule corresponds with the losses of the players, *i.e.*, the opposite of the Shapley value can be seen as the fee to be paid for each movement.

In the early works by Baker (1965) and Thompson (1971), they proposed the following sequential allocation scheme for the fees in an airport problem. First, the runway costs induced by the smallest type of aircraft are equally divided among the number of movements of all aircrafts. This first part of the allocation scheme seems fair, since all the movements use that part of the runway. Next, divide the incremental cost for the runway for the second smallest type of aircraft equally among the number of movements of all but the smallest type of aircraft. Continue until the incremental cost of the largest type of aircraft is divided equally among the number of movements made by the largest aircraft type. Example 5.10.1 below illustrates this allocation scheme. Note that this allocation scheme is defined over the original airport problem and, hence, at this point, this idea is independent of the airport game associated with the airport problem.

Example 5.10.1. Consider $m = 3$, $N_1 = \{1\}$, $N_2 = \{2,3\}$, $N_3 = \{4\}$ and the costs are given by 12 units for each movement of type 1, 28 units for each movement of type 2, and 30 units for each movement of type 3. Then, the characteristic function of the airport game is given, for each $S \subset N = \{1,2,3,4\}$, by

$$v(S) := \begin{cases} -12 & S = \{1\} \\ -28 & S \cap N_2 \neq \emptyset \text{ and } 4 \notin S \\ -30 & \text{otherwise.} \end{cases}$$



$$x_1 = -\tfrac{12}{4} = -3$$
$$x_2 = -\tfrac{12}{4} - \tfrac{16}{3} = -\tfrac{25}{3}$$
$$x_3 = -\tfrac{12}{4} - \tfrac{16}{3} = -\tfrac{25}{3}$$
$$x_4 = -\tfrac{12}{4} - \tfrac{16}{3} - 2 == -\tfrac{31}{3}$$

Figure 5.10.1. Illustration of the sequential allocation scheme.

In Figure 5.10.1, we illustrate the allocation scheme described above and present the allocation it would propose. ◇

In Littlechild and Owen (1973), the authors prove that the above sequential allocation scheme coincides with the Shapley value of the associated airport game. Given an airport problem, for each type of aircraft $l \in \{1,\ldots,m\}$, let $r_l := \sum_{j=l}^{m} |N_j|$ denote the total number of movements of aircrafts larger or equal than those of type l.

Theorem 5.10.1. *Let (N,v) be an airport game. Let $c_0 := 0$ and, for each $i \in N$, let l_i be the type of aircraft of player i. Then, the Shapley value of (N,v) can be* *computed, for each $i \in N$, by*

$$\Phi_i(v) = \sum_{l=1}^{l_i} \frac{-(c_l - c_{l-1})}{r_l}.$$

Proof. For each $l \in \{1, \ldots, m\}$, we define the set $R_l := \cup_{j=l}^{m} N_j$ and the TU-game v_l given, for each $S \subset N$, by

$$v_l(S) := \begin{cases} 0 & S \cap R_l = \emptyset \\ -(c_l - c_{l-1}) & S \cap R_l \neq \emptyset. \end{cases}$$

Let $k \in \{1, \ldots, m\}$ and let $S \subset N$ be such that $\max\{c_l : S \cap N_l \neq \emptyset\} = c_k$. Then, $S \cap R_l = \emptyset$ if $l > k$ and $S \cap R_l \neq \emptyset$ if $l \leq k$. Hence, $\sum_{l=1}^{m} v_l(S) = \sum_{l=1}^{k} -(c_l - c_{l-1}) = -(c_k - c_0) = v(S)$. Therefore, for each $S \subset N$, $v(S) = \sum_{l=1}^{m} v_l(S)$. Since the Shapley value satisfies ADD, we have that, for each $i \in N$,

$$\Phi_i(v) = \sum_{l=1}^{m} \Phi_i(v_l).$$

Let $l \in \{1, \ldots, m\}$ and consider the game v_l. Each $i \notin R_l$ is a null player and any two players $i, j \in R_l$ are symmetric. Then, since the Shapley value satisfies NPP, SYM, and EFF, we have that, for each $i \in N$,

$$\Phi_i(v_l) = \begin{cases} 0 & i \notin R_l \\ \frac{-(c_l - c_{l-1})}{r_l} & i \in R_l. \end{cases}$$

For each $i \in N$, if l_i is the type of player i, then $i \in R_l$ if and only if $l \leq l_i$. Therefore, $\Phi_i(v) = \sum_{l=1}^{l_i} \frac{-(c_l - c_{l-1})}{r_l}$. □

Another reasonable option to choose the fees is through the nucleolus of the airport game. Littlechild (1974) used the special structure of airport games to simplify the procedure described in Section 5.7 to compute the nucleolus. Recall that, within this approach, several linear programming problems have to be solved to compute the nucleolus. For each $S \subset N$, let $l(S) := \max\{l : S \cap N_l \neq \emptyset\}$. We start by solving the linear programming problem P_1 given by

$$\begin{array}{ll} \text{Minimize} & \alpha_1 \\ \text{subject to} & \sum_{i \in S} x_i + \alpha_1 \geq -c_{l(S)}, \quad \emptyset \neq S \subsetneq N \\ & \sum_{i \in N} x_i = -c_m \\ & x_i \geq -c_{l(i)}. \end{array}$$

First, we claim that any optimal solution (α_1, x) satisfies that $\alpha_1 \leq 0$ and, for each $i \in N$, $x_i \leq 0$. A feasible point is given by $\alpha_1 = 0$, for each

$i \notin N_m$, $x_i = 0$, and, for each $i \in N_m$, $x_i = -c_m/|N_m|$. Thus, $\alpha_1 \leq 0$ in any optimal solution. Now, if there is $i \in N$ such that $x_i > 0$, then

$$-c_{l(N\backslash\{i\})} \leq -c_{l(N\backslash\{i\})} - \alpha_1 \leq \sum_{j \in N\backslash\{i\}} x_j < -c_m \leq -c_{l(N\backslash\{i\})},$$

which is impossible and, hence, the claim is proved.

Next, we show that some constraints in problem P_1 can be removed. To begin with, for each $i \in N$, since $\alpha_1 \leq 0$, the constraint $x_i + \alpha_1 \geq -c_{l(i)}$ is more restrictive than the restriction $x_i \geq -c_{l(i)}$, which can therefore be removed. Let $S \subsetneq N$ be such that $l(S) = m$. The constraint associated with S is given by

$$\sum_{j \in S} x_j + \alpha_1 = \sum_{j \in N} x_j - \sum_{j \notin S} x_j + \alpha_1 \geq -c_m.$$

Since we also have the constraint $\sum_{j \in N} x_j = -c_m$, the above constraint is equivalent to $\alpha_1 - \sum_{j \in S} x_j \geq 0$. Now, if we let $i \notin S$ we have that $l(N\backslash\{i\}) = l(S) = m$. Then, the constraint for coalition $N\backslash\{i\}$ reduces to $\alpha_1 - x_i \geq 0$, which is at least as restrictive as the previous constraint. Hence, all the constraints associated with coalitions such that $l(S) = m$ can be replaced by the constraints $\alpha_1 - x_i \geq 0$, with $i \in N$.

Let $S \subset N$ be such that $l(S) = k < m$. Consider as well the coalition $\cup_{l=1}^k N_l$. The constraints in P_1 associated with S and $\cup_{l=1}^k N_l$ are given by

$$\sum_{j \in S} x_j + \alpha_1 \geq -c_k, \quad \text{and} \quad \sum_{j \in \cup_{l=1}^k N_l} x_j + \alpha_1 \geq -c_k.$$

Since $S \subseteq \cup_{l=1}^k N_l$ and, for each $i \in N$, $x_i \leq 0$, the constraint for $\cup_{l=1}^k N_l$ is more restrictive than the one for S.

Hence, it suffices to solve the linear programming problem P_1^* given by

$$\begin{array}{ll} \text{Minimize} & \alpha_1 \\ \text{subject to} & \sum_{i \in \cup_{l=1}^k N_l} x_i + \alpha_1 \geq -c_k, \quad k \in \{1, \ldots, m-1\} \\ & \alpha_1 - x_i \geq 0, \quad i \in N \\ & \sum_{i \in N} x_i = -c_m. \end{array}$$

Since at an optimal solution, for each $i \in N$, $\alpha_1 \geq x_i$, we have that $\alpha_1 \geq \frac{-c_m}{n}$ and, for each $k \in \{1, \ldots, m-1\}$, $\alpha_1 \geq \frac{-c_k}{|\cup_{l=1}^k N_l|+1}$. Hence,

$$-\alpha_1 \leq \min\left\{\frac{c_m}{n}, \min_{k \in \{1,\ldots,m-1\}} \frac{c_k}{|\cup_{l=1}^k N_l| + 1}\right\}.$$

Let k_1 be the largest index at which this minimum is attained and define

$$\bar{\alpha}_1 := \begin{cases} \frac{-c_{k_1}}{|\cup_{l=1}^{k_1} N_l|+1} & k_1 < m \\ \frac{-c_m}{n} & k_1 = m. \end{cases}$$

Also, define $\bar{x} \in \mathbb{R}^N$ as follows. For each $i \notin N_m$, $\bar{x}_i := \bar{\alpha}_1$ and, for each $i \in N_m$, $\bar{x}_i := -\frac{c_m + \bar{\alpha}_1(n-|N_m|)}{|N_m|}$. So defined, $(\bar{\alpha}_1, \bar{x})$ is a feasible point and, since α_1 cannot be smaller than $\bar{\alpha}_1$, $(\bar{\alpha}_1, \bar{x})$ is an optimal solution of P_1^*. Moreover, any optimal solution $(\bar{\alpha}_1, x)$ has to satisfy that, for each $i \in \cup_{l=1}^{k_1} N_l$, $x_i = \bar{\alpha}_1$. Suppose, on the contrary, that there is $i \in \cup_{l=1}^{k_1} N_l$ such that $x_i < \bar{\alpha}_1$ (recall that $x_i \leq \bar{\alpha}_1$ always holds). Then,

- if $k_1 < m$, $\sum_{i \in \cup_{l=1}^{k_1} N_l} x_i + \bar{\alpha}_1 < -c_{k_1}$, and
- if $k_1 = m$, $\sum_{i \in \cup_{l=1}^{k_1} N_l} x_i < -c_m$,

and we get a contradiction in both cases. Therefore, in any optimal solution, the constraints associated with a coalition S such that $l(S) \leq k_1$ are binding. Therefore, if either $k_1 = m$ or $k_1 = m-1$, P_1^* has a unique optimal solution, the nucleolus. Otherwise, we solve the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_2 \\ \text{subject to} & \sum_{i \in \cup_{l=k_1+1}^{k} N_l} x_i + \alpha_2 \geq -c_k - \bar{\alpha}_1 |\cup_{l=1}^{k_1} N_l|, \\ & \qquad\qquad\qquad\qquad k \in \{k_1+1, \ldots, m-1\} \\ & \alpha_2 - x_i \geq 0, \quad i \in N \backslash (\cup_{l=1}^{k_1} N_l) \\ & \sum_{i=k_1+1}^{m} x_i = -c_m - \bar{\alpha}_1 |\cup_{l=1}^{k_1} N_l|. \end{array}$$

This linear programming problem has a similar structure to the one of P_1^*. Hence, the optimal value is

$$-\bar{\alpha}_2 = \min\left\{ \frac{c_m + \bar{\alpha}_1 |\cup_{l=1}^{k_1} N_l|}{n - |\cup_{l=1}^{k_1} N_l|}, \min_{k \in \{k_1+1,\ldots,m-1\}} \frac{c_k + \bar{\alpha}_1 |\cup_{l=1}^{k_1} N_l|}{|\cup_{l=k_1+1}^{k} N_l| + 1} \right\}.$$

Let k_2 be the largest index at which this minimum is attained. Then, any optimal solution of this problem satisfies that, for each $i \in \cup_{l=k_1}^{k_2} N_l$, $x_i = \bar{\alpha}_2$. Then, we can inductively obtain values $\bar{\alpha}_0 = 0 > \bar{\alpha}_1 > \bar{\alpha}_2 > \ldots > \bar{\alpha}_q$ and indices $k_0 = 0 < k_1 < k_2 < \ldots < k_q = m$ such that, for each $p = 0, \ldots, q-1$, the value $-\bar{\alpha}_{k_{p+1}}$ is given by

$$\min\left\{ \frac{c_m + \sum_{r=1}^{p} \bar{\alpha}_{k_r} t_{k_{r-1}+1}^{k_r}}{n - t_1^{k_p}}, \min_{k \in \{k_p+1,\ldots,m-1\}} \frac{c_k + \sum_{r=1}^{p} \bar{\alpha}_{k_r} t_{k_{r-1}+1}^{k_r}}{t_{k_p+1}^{k} + 1} \right\}, \tag{5.10.2}$$

where $t_1^{k_p} := \sum_{l=1}^{k_p} |N_l|$, $t_{k_{r-1}+1}^{k_r} := \sum_{l=k_{r-1}+1}^{k_r} |N_l|$, and $t_{k_p+1}^{k} := \sum_{l=k_p+1}^{k} |N_l|$. The index k_{p+1} is the largest index at which the minimum in Eq. (5.10.2) is attained. We illustrate the above procedure to compute the nucleolus in Example 5.10.2.

Example 5.10.2. Consider the airport problem of Example 5.10.1. In the first step we get

$$\min\left\{\frac{30}{4}, \min\left\{\frac{12}{2}, \frac{28}{3}\right\}\right\},$$

and, hence, $\bar{\alpha}_1 = -6$ and $k_1 = 1$. Then, we have

$$\min\left\{\frac{30-6}{4-1}, \min\left\{\frac{28-6}{3}\right\}\right\},$$

which leads to $\bar{\alpha}_2 = -\frac{28-6}{3} = -\frac{22}{3}$ and $k_2 = 2$. Finally, $\bar{\alpha}_3 = -\frac{28}{3}$. Then, the nucleolus of the airport game is the allocation $(-6, -\frac{22}{3}, -\frac{22}{3}, -\frac{28}{3})$. $\diamond$

Several extensions of the class of airport games have been studied in the literature. For instance, in Vázquez-Brage et al. (1997) the authors consider that aircrafts are organized in airlines and in Brânzei et al. (2006) the benefits generated by the aircraft movements are taken into account.

5.11. Bankruptcy Problems and Bankruptcy Games

The terms *bankruptcy problem* and *claim problem* are used to refer to a situation in which several agents claim a portion of a resource larger than the amount available. The standard example, from which these kind of situations borrowed the name, is the case of the creditors of a bankrupt company claiming their shares. The literature on bankruptcy problems has focused on defining *division rules* that divide the available amount, usually referred to as the *estate*, among the claimants. These division rules should satisfy some fairness principles. This literature started with the seminal paper by O'Neill (1982). For a complete survey, see Thomson (2003). In this section we present some of the division rules defined for bankruptcy problems and establish a connection between bankruptcy problems and a special class of TU-games, known as *bankruptcy games*.

The claim problems discussed in Example 5.11.1 below are taken from the Talmud.[25] They were discussed as bankruptcy problems in Aumann and Maschler (1985).

Example 5.11.1. After the death of a man his creditors demand 100, 200, and 300. The estate left is insufficient to meet all the claims. Three cases are considered: the estate is 100, 200, and 300. The Talmud stipulates, for each of the cases, the divisions depicted in Figure 5.11.1. Aumann and Maschler (1985) studied whether there was some common rationale for the divisions proposed in the Talmud. We present one of their main findings in Theorem 5.11.2, later in this section. $\diamond$

[25] The Talmud is a record of rabbinic discussions pertaining to Jewish law, ethics, customs, and history.

Demands / Estate	100	200	300
100	$33+\frac{1}{3}$	$33+\frac{1}{3}$	$33+\frac{1}{3}$
200	50	75	75
300	50	100	150

Figure 5.11.1. Allocations proposed in the Talmud.

We now move to the formal definition of a bankruptcy problem.

Definition 5.11.1. A *bankruptcy problem* with set of claimants N is a pair (E, d), where $E \in \mathbb{R}$ and $d \in \mathbb{R}^N$ are such that, for each $i \in N$, $d_i \geq 0$ and $0 \leq E \leq \sum_{i=1}^{n} d_i$.

Thus, in a bankruptcy problem (E, d), E denotes the estate and each d_i represents the demand of claimant i. We assume, without loss of generality, that the demands are ordered so that $0 \leq d_1 \leq d_2 \leq \ldots \leq d_n$.

Several cooperative games have been used to model bankruptcy problems. Here, we follow the most common of these approaches, initiated by O'Neill (1982). O'Neill associated a TU-game to each bankruptcy problem. Once this has been done, the division of the estate among the claimants can be done using some allocation rule for TU-games such as the Shapley value or the nucleolus. For a review of other game theoretic approaches to bankruptcy problems, refer to Thomson (2003).

Definition 5.11.2. Given a bankruptcy problem (E, d) with set of players N, the associated *bankruptcy game* is the TU-game (N, v) defined, for each $S \subset N$ by

$$v(S) := \max\{0, E - \sum_{i \notin S} d_i\}.$$

The interpretation here is that the claimants take a pessimistic point of view. The worth of a coalition is what is left, if anything, once the players outside the coalition have received their own claims. It is easy to see that bankruptcy games are convex (see Exercise 5.14) and, hence, they have a nonempty core, which contains both the Shapley value and the nucleolus.

Example 5.11.2. Consider the bankruptcy problem in Example 5.11.1 when the estate is $E = 200$. The players are $N = \{1, 2, 3\}$ and the demands are $d_1 = 100$, $d_2 = 200$, and $d_3 = 300$. The characteristic function assigns $v(23) = \max\{0, 200 - 100\} = 100$ to coalition $\{2, 3\}$. The whole characteristic function is given by $v(23) = 100$, $v(N) = 200$, and $v(S) = 0$ otherwise. The Shapley value proposes the division $\Phi(v) = (33 + \frac{1}{3}, 83 + \frac{1}{3}, 83 + \frac{1}{3})$

and the nucleolus $\eta(v) = (50, 75, 75)$. In particular, the proposal of the nucleolus coincides with the proposal made in the Talmud (see Figure 5.11.1). $\diamond$

We now present some division rules that can be directly defined over the bankruptcy problem itself and we then establish some connections between these rules and the ones we have already studied for general TU-games.

Definition 5.11.3. A *division rule* f is a function that assigns, to each bankruptcy problem (E, d) a division $f(E, d) \in \mathbb{R}^N$ such that $\sum_{i \in N} f_i(E, d) = E$ and, for each $i \in N$, $0 \leq f_i(E, d) \leq d_i$.

Many division rules have been proposed in the literature and here we only focus on three notable cases. We discuss the *random arrival rule* (O'Neill 1982), the *Talmud rule* (Aumann and Maschler 1985), and one rule often applied in practice: the *proportional rule*.

The random arrival rule is based on the following procedure. Assume that the agents arrive one at a time to make their claims. When an agent arrives, he receives the minimum between his claim and what is left. The resulting division depends on the order of arrival of the claimants. To introduce some fairness into the process, the random arrival rule assumes that all the orders of the players are equally likely and then takes the expected division.

Recall that $\Pi(N)$ denotes the set of all permutations of the elements in N and, for each $\pi \in \Pi(N)$, $P^{\pi}(i)$ denotes the set of predecessors of i under π.

Definition 5.11.4. Let (E, d) be a bankruptcy problem. The random arrival rule, f^{RA}, assigns, to each claimant $i \in N$,

$$f_i^{\mathrm{RA}}(E, d) := \frac{1}{n!} \sum_{\pi \in \Pi(N)} \min\{d_i, \max\{0, E - \sum_{j \in P^{\pi}(i)} d_j\}\}.$$

Before defining the Talmud rule, we define two other standard rules: the *constrained equal awards rule* and the *constrained equal losses rule*. The idea of the former is to give the same to each claimant, but conditional on no one getting more than his claim. The idea of the latter is, in a sense, the dual one: the difference between the claim of a player and what he gets should be equal across players, conditional on no player getting less than 0.

Definition 5.11.5. Let (E, d) be a bankruptcy problem. The *constrained equal awards rule*, f^{CEA}, assigns, to each claimant $i \in N$, $f_i^{\mathrm{CEA}}(E, d) := \min\{d_i, \lambda\}$, where λ is chosen so that $\sum_{i \in N} \min\{d_i, \lambda\} = E$.

The *constrained equal losses rule*, f^{CEL}, assigns, to each claimant $i \in N$, $f_i^{\text{CEL}}(E,d) := \max\{0, d_i - \lambda\}$, where λ is chosen so that $\sum_{i\in N} \max\{0, d_i - \lambda\} = E$.

The motivation for the Talmud rule was to find a natural division rule whose proposals matched the ones made by the Talmud in the bankruptcy problems of Example 5.11.1. This rule is defined as follows. If the estate equals the half-sum of the demands, then each claimant receives half of his demand. If the estate is smaller than the half-sum of the demands, the rule proceeds as the constrained equal awards rule, but with all the demands divided by two (*i.e.*, no claimant gets more that half of his demand). If the estate is greater than the half-sum of the demands, then the rule assigns to each claimant half of his claim and the rest is allocated using the constrained equal losses rule, but with all the demands divided by two (*i.e.*, no claimant gets less than half of his claim).

Definition 5.11.6. Let (E,d) be a bankruptcy problem. The *Talmud rule*, f^{TR}, assigns, to each claimant $i \in N$,

$$f_i^{\text{TR}}(E,d) := \begin{cases} \min\{\frac{d_i}{2}, \lambda\} & E \leq \sum_{i\in N} \frac{d_i}{2} \\ \frac{d_i}{2} + \max\{0, \frac{d_i}{2} - \lambda\} = d_i - \min\{\frac{d_i}{2}, \lambda\} & E \geq \sum_{i\in N} \frac{d_i}{2}, \end{cases}$$

where in both cases λ is chosen so that $\sum_{i\in N} f_i^{\text{TR}}(E,d) = E$.

One of the most natural rules is the proportional rule, in which each agent receives a portion of the estate given by the weight of his claim with respect to the total amount being claimed.

Definition 5.11.7. Let (E,d) be a bankruptcy problem. The *proportional rule*, f^{PR}, assigns, to each claimant $i \in N$,

$$f_i^{\text{PR}}(E,d) := \frac{d_i}{\sum_{j\in N} d_j} E,$$

unless all the demands are zero, in which case all players get zero.

Example 5.11.3. We now use the bankruptcy problems in Example 5.11.1 to illustrate the divisions proposed by the random arrival rule, the Talmud rule, and the proportional rule. Note that, as intended, the Talmud rule equals the divisions proposed in the Talmud. $\diamond$

In order to discriminate between the different division rules, one can take any of the approaches we have discussed in some of the previous sections. Both the axiomatic analysis and the theory of implementation have been widely used in this setting. At the end of this section, we present a couple of properties that a division rule may satisfy and whose motivations are different from the kind of properties we discussed for bargaining

Estate	100			200			300		
Rule	1	2	3	1	2	3	1	2	3
Random arrival	$33+\frac{1}{3}$	$33+\frac{1}{3}$	$33+\frac{1}{3}$	$33+\frac{1}{3}$	$83+\frac{1}{3}$	$83+\frac{1}{3}$	50	100	150
Talmud	$33+\frac{1}{3}$	$33+\frac{1}{3}$	$33+\frac{1}{3}$	50	75	75	50	100	150
Proportional	$16+\frac{2}{3}$	$33+\frac{1}{3}$	50	$33+\frac{1}{3}$	$66+\frac{2}{3}$	100	50	100	150

Figure 5.11.2. Illustration of the divisions proposed by different rules.

problems and TU-games. In the rest of this section we mainly concentrate on the connections between division rules for bankruptcy problems and allocation rules for bankruptcy games.

The motivation for the random arrival rule we presented above resembles the interpretation of the Shapley value we made after its definition in Section 5.6. Actually, the division proposed by the random arrival rule coincides with the Shapley value of the associated bankruptcy game. The proof of this result is quite straightforward and is left as an exercise.

Proposition 5.11.1. *Let (E,d) be a bankruptcy problem and let v be the associated bankruptcy game. Then, $f^{RA}(E,d) = \Phi(v)$.*

Proof. Exercise 5.15. □

The next result shows that the Talmud rule coincides with the nucleolus of the bankruptcy game. This result was originally proved by Aumann and Maschler (1985); here, we present an adaptation of the proof proposed by Benoît (1997).

Theorem 5.11.2. *Let (E,d) be a bankruptcy problem and let v be the associated bankruptcy game. Then, $f^{TR}(E,d) = \eta(v)$.*

Proof. The case $n = 1$ is immediate and hence, we assume that $n \geq 2$. We begin by computing the excess of a coalition at any imputation in the bankruptcy game. Let $S \subset N$ and let $x \in I(v)$. Then, $e(S,x) = v(S) - \sum_{i\in S} x_i$. If $E - \sum_{i\notin S} d_i \geq 0$, then $v(S) = E - \sum_{i\notin S} d_i$ and, hence, since $E = \sum_{i\in N} x_i$, $e(S,x) = (E - \sum_{i\notin S} d_i) - \sum_{i\in S} x_i = -\sum_{i\notin S}(d_i - x_i)$. On the other hand, if $E - \sum_{i\notin S} d_i < 0$, then $v(S) = 0$, and $e(S,x) = -\sum_{i\in S} x_i$. We distinguish two cases.

Case 1: $\sum_{i\in N} d_i/2 \geq E$. In this case, by the definition of f^{TR}, for each $i \in N$, $f^{\mathrm{TR}}(E,d) \leq d_i/2$. We now show that $f^{\mathrm{TR}}(E,d) \in I(v)$. For each $i \neq n$, $v(i) = \max\{0, E - \sum_{j\in N\setminus\{i\}} d_j\} \leq \max\{0, \sum_{j\in N} d_j/2 - \sum_{j\in N\setminus\{i\}} d_j\} = \max\{0, \sum_{j\in N\setminus\{i,n\}}(d_j/2 - d_j) + d_i/2 + d_n/2 - d_n\}$. Since $d_i \leq d_{n-1} \leq d_n$, the previous maximum equals 0. Then, for each $i \neq n$, $v(i) = 0$ and, hence,

$f_i^{\mathrm{TR}}(E,d) \geq v(i)$. Moreover, since, for each $i \neq n$, $f_i^{\mathrm{TR}}(E,d) \leq d_i/2$,

$$f_n^{\mathrm{TR}}(E,d) \geq \max\{0, E - \sum_{j=1}^{n-1} \frac{d_j}{2}\} \geq \max\{0, E - \sum_{j=1}^{n-1} d_j\} = v(n).$$

Then, $f^{\mathrm{TR}}(E,d) \in I(v)$. We distinguish two subcases.

Case 1.1: For each $i \in N$, $f_i^{\mathrm{TR}}(E,d) < d_i/2$. Then, for each $i \in N$, $f_i^{\mathrm{TR}}(E,d) = E/n$. In this case $v(n) = 0$. Then, for each $i \in N$, we have that $e(\{i\}, f^{\mathrm{TR}}(E,d)) = -E/n$. Let $S \subsetneq N$ be such that $|S| \geq 2$. If $v(S) = 0$, then $e(S, f^{\mathrm{TR}}(E,d)) = -\sum_{i \in S} f_i^{\mathrm{TR}}(E,d) = -|S|E/n \leq -E/n$. If $v(S) > 0$, then

$$e(S, f^{\mathrm{TR}}(E,d)) = -\sum_{i \notin S}(d_i - \frac{E}{n}) \leq -\sum_{i \notin S} \frac{d_i}{2} \leq -|N \backslash S| \frac{E}{n} \leq -\frac{E}{n}.$$

Let $x \in I(v)$ be such that $x \neq f^{\mathrm{TR}}(E,d)$. Then, there is $j \in N$ such that $x_j < f_j^{\mathrm{TR}}(E,d)$. Hence, $e(\{j\}, x) = -x_j > -E/n = e(\{j\}, f^{\mathrm{TR}}(E,d))$. Since, for each $S \subset N$, $-E/n \geq e(S, f^{\mathrm{TR}}(E,d))$, then $x \succ_L f^{\mathrm{TR}}(E,d)$ and, hence, $\eta(v) = f^{\mathrm{TR}}(E,d)$.

Case 1.2: There is $i \in N$, such that $f_i^{\mathrm{TR}}(E,d) = d_i/2$. Because of the ordering of the players, we have $f_1^{\mathrm{TR}}(E,d) = d_1/2$ and, for each $j \neq 1$, $f_j^{\mathrm{TR}}(E,d) \geq d_1/2$. Clearly, $v(N \backslash \{1\}) = E - d_1 > 0$. Hence,

$$e(\{1\}, f^{\mathrm{TR}}(E,d)) = -\frac{d_1}{2} = (E - d_1) - (E - \frac{d_1}{2}) = e(N \backslash \{1\}, f^{\mathrm{TR}}(E,d)).$$

Let $S \subset N$, $S \neq \{1\}$ and $S \neq N \backslash \{1\}$. If $v(S) = 0$, then $e(S, f^{\mathrm{TR}}(E,d)) = -\sum_{i \in S} f_i^{\mathrm{TR}}(E,d) \leq -|S|d_1/2 \leq -d_1/2$, because $f_i^{\mathrm{TR}}(E,d) \geq d_1/2$, for every $i \in N$. If $v(S) > 0$, then

$$e(S, f^{\mathrm{TR}}(E,d)) = -\sum_{i \notin S}(d_i - f_i^{\mathrm{TR}}(E,d)) \leq -|N \backslash S| \frac{d_1}{2} \leq -\frac{d_1}{2}.$$

Let $x \in I(v)$ be such that $x \neq f^{\mathrm{TR}}(E,d)$. If, for each $j \in N$, $x_j > f_1^{\mathrm{TR}}(E,d)$, then $e(N \backslash \{1\}, x) = -(d_1 - x_1) > -d_1 + d_1/2 = -d_1/2$. If there is $j \in N$ such that $x_j < f_1^{\mathrm{TR}}(E,d)$, then $e(\{j\}, x) = v(j) - x_j > v(j) - f_1^{\mathrm{TR}}(E,d) \geq -f_1^{\mathrm{TR}}(E,d) = -d_1/2$. In either case, $x \succ_L f^{\mathrm{TR}}(E,d)$ and, hence, $\eta(v) = f^{\mathrm{TR}}(E,d)$.

Case 2: $\sum_{i \in N} d_i/2 < E$. Let $\hat{E} := \sum_{i \in N} d_i - E$ and consider the bankruptcy problem $(\hat{E}, d)$. Now, $\sum_{i \in N} d_i/2 > \hat{E}$. We claim that the divisions proposed by the Talmud rule in this problem and in the original one are related by

$$f^{\mathrm{TR}}(E,d) = d - f^{\mathrm{TR}}(\hat{E}, c). \qquad (5.11.1)$$

Since, $\sum_{i\in N} d_i/2 > \hat{E}$, by the definition of f^{TR}, $f_i^{\mathrm{TR}}(\hat{E},d) = \min\{d_i/2,\lambda\}$, where λ is chosen so that $\sum_{i\in N}\min\{d_i/2,\lambda\} = \hat{E} = \sum_{i\in N} d_i - E$. On the other hand, since $\sum_{i\in N} d_i/2 < E$, we have $f_i^{\mathrm{TR}}(E,d) = d_i - \min\{d_i/2,\tilde{\lambda}\}$, where $\tilde{\lambda}$ is chosen so that $\sum_{i\in N} d_i - \sum_{i\in N}\min\{d_i/2,\tilde{\lambda}\} = E$ or, equivalently, $\sum_{i\in N}\min\{d_i/2,\tilde{\lambda}\} = \sum_{i\in N} d_i - E = \hat{E}$. Hence, the restrictions for λ and $\tilde{\lambda}$ are indeed the same one. Therefore, the desired relation between $f^{\mathrm{TR}}(E,d)$ and $f^{\mathrm{TR}}(\hat{E},c)$ follows.

Let v and $\hat{v}$ denote the bankruptcy games corresponding to the bankruptcy problems (E,d) and $(\hat{E},d)$ and let e and $\hat{e}$ denote the corresponding excess functions. We now show that an analogous relationship to the one in Eq. (5.11.1) holds for the nucleoli of the games v and $\hat{v}$. If $\hat{x} \in I(v)$, then $x := d - \hat{x} \in I(v)$, and *vice versa*. Let $S \subset N$ and $x \in I(v)$. Then, $\hat{x} = d - x \in I(\hat{v})$. If $v(S) = 0$, then $E - \sum_{j\notin S} d_j \leq 0$, $\hat{v}(N\backslash S) = \sum_{j\notin S} d_j - E$, and

$$v(S) - \sum_{j\in S} x_i = -\sum_{j\in S}(d_j - \hat{x}_j) = -\sum_{j\in S} d_j + \hat{E} - \sum_{j\notin S}\hat{x}_j = \hat{v}(N\backslash S) - \sum_{j\notin S}\hat{x}_j.$$

Hence, $e(S,x) = \hat{e}(N\backslash S,\hat{x})$. On the other hand, if $v(S) > 0$, then $E - \sum_{j\in N\backslash S} d_j > 0$, $\hat{v}(N\backslash S) = 0$, and

$$v(S) - \sum_{j\in S} x_j = E - \sum_{j\notin S} d_j - \sum_{j\in S}(d_j - \hat{x}_j) = \hat{v}(N\backslash S) - \sum_{j\notin S}\hat{x}_j.$$

Then, the vectors of ordered excesses coincide for x and $\hat{x}$, *i.e.*, $\theta(x) = \theta(\hat{x})$. Hence, $\eta(v) = d - \eta(\hat{v})$. Therefore, by applying the Case 1 to the problem $(\hat{E},d)$ and using Eq. (5.11.1), we get $f^{\mathrm{TR}}(E,d) = d - f^{\mathrm{TR}}(\hat{E},c) = d - \eta(\hat{v}) = \eta(v)$. □

A natural question is whether each division rule for bankruptcy problems corresponds to some allocation rule for TU-games. The answer is negative and the reason is the following. Given a bankruptcy problem in which a player demands more than the estate, if we further increase the demand of this player, we get a different bankruptcy problem, but both problems have associated the same bankruptcy game. Therefore, a division rule that assigns different divisions to these two problems can never be represented by an allocation rule. Further elaborating on the above discussion, Curiel et al. (1987) proposed a necessary and sufficient condition for such correspondence to exist. Given a bankruptcy problem (E,d), we define the problem with *truncated demands* associated with (E,d) as (E,d^T), where, for each $i \in N$, $d_i^T := \min\{d_i, E\}$.

Theorem 5.11.3. *A division rule f for banckruptcy problems can be represented through an allocation rule for bankruptcy games if and only if, for each bankruptcy problem (E,d), $f(E,d) = f(E,d^T)$.*

Proof. The "only if" part can be easily shown by example (refer, for instance, to Example 5.11.4 below). We prove the "if" part. Let φ be the allocation rule on SG^N defined as follows.[26] For each $v \in SG^N$, $\varphi(v) := f(v(N), d^v)$, where

$$d_i^v := v(N) - v(N\backslash\{i\}) + \frac{1}{n}\max\{0, v(N) - \sum_{j\in N}(v(N) - v(N\backslash\{j\})\}.$$

It is easy to see that, if $v \in SG^N$, then $(v(N), d^v)$ is a bankruptcy problem. Moreover, given a bankruptcy problem (E, d) with associated bankruptcy game v, we have $(v(N), d^v) = (E, d^T)$ and, hence, $\varphi(v) = f(E, d^T) = f(E, d)$. □

Example 5.11.4. The proportional rule, for instance, does not satisfy the necessary and sufficient condition in Theorem 5.11.3. Consider the bankruptcy problems (E, d) and $(E, \hat{d})$, with $E = 200$, $d_1 = 100$, $d_2 = 200$, $d_3 = 500$, $\hat{d}_1 = 100$, $\hat{d}_2 = 200$, and $\hat{d}_3 = 200$. The proportional rule gives the allocations $f^{\text{PR}}(E, d) = (25, 50, 125) \neq (40, 80, 80) = f^{\text{PR}}(E, \hat{d})$. ◇

Before concluding this section we want to present two more axioms that may be considered desirable for bankruptcy problems. These axioms belong to a class of axioms whose spirit is different from the spirit of the ones discussed in some of the previous sections. They focus on how a rule behaves with respect to some variations in the number of claimants.

Consistency (CONS): Let f be a division rule. Let (E, d) be a bankruptcy problem with set of claimants N. Let $S \subset N$ and let $(\hat{E}, \hat{d})$ be the bankruptcy problem with set of claimants S given by $\hat{E} := E - \sum_{j\notin S} f_j(E, d)$ and, for each $i \in S$, $\hat{d}_i := d_i$. Then, f satisfies CONS if, for each $i \in S$,

$$f_i(\hat{E}, \hat{d}) = f_i(E, d).$$

No advantageous merging or splitting (NAMS): Let f be a division rule. Let (E, d) be a bankruptcy problem with set of claimants N. Let $S \subset N$ and let $(\hat{E}, \hat{d})$ be the bankruptcy problem with set of claimants S given by: i) $\hat{E} := E$ and ii) there is $i \in S$ satisfying that $\hat{d}_i := d_i + \sum_{k\notin S} d_k$ and, for each $j \in S$, $j \neq i$, $\hat{d}_j := d_j$. Then, f satisfies NAMS if

$$f_i(\hat{E}, \hat{d}) = f_i(E, d) + \sum_{j\notin S} f_j(E, d).$$

[26] Recall that bankruptcy games are convex and, hence, superadditive.

Axiom CONS requires that, if a group of claimants leaves the problem with their part of the division as proposed by the rule, then the division proposed by the rule for the bankruptcy problem with the remaining estate and the remaining claimants should not change, that is, each remaining claimant receives the same amount as in the original problem. This axiom is very natural. Suppose that the estate has been deposited in a bank and that the claimants can withdraw what the division rule proposes for them whenever they want. Yet, whenever a claimant or group of claimants go to the bank, they get what the rule proposes for the bankruptcy problem given by the remaining estate and claimants. In this case, CONS implies that regardless of the timing chosen by the claimants to go to the bank, the final division of the initial estate will be the same. On the other hand, NAMS is one of the axioms that target the robustness of the division rules with respect to possible strategic manipulations on the side of the players. In particular, NAMS deals with situations where a group of claimants can join together and show up as a single claimant or, on the contrary, one claimant can split his claim among several new claimants. This property establishes that none of these two manipulations should ever be beneficial for the manipulating agents.

Table 5.11.3 below shows how the Talmud rule, the random arrival rule, and the proportional rule behave with respect to these properties. Example 5.11.5 below illustrates the negative results. The proof that the Talmud rule satisfies CONS can be found in Aumann and Maschler (1985). Concerning the results for the proportional rule, the reader may refer to Young (1987) for the proof that it satisfies CONS and to de Frutos (1999) for a proof that it satisfies NAMS.[27]

Properties / Rules	CONS	NAMS
Random arrival	No	No
Talmud	Yes	No
Proportional	Yes	Yes

Figure 5.11.3. Bankruptcy rules and properties.

Example 5.11.5. Let (E, d) be the bankruptcy problem given by $E = 200$, $d_1 = 100$, $d_2 = 200$, and $d_3 = 400$. The proposal of the random arrival rule is $(33 + 1/3, 83 + 1/3, 83 + 1/3)$ and the one of the Talmud rule is $(50, 75, 75)$. Suppose that claimant 2 takes his part from the proposal of the random arrival rule, $83 + 1/3$, and leaves. The new bankruptcy situation is

[27] We refer the reader to Thomson (2003) for some characterizations of allocation rules based on NAMS and on consistency-like properties.

$(\hat{E},\hat{d})$ with $\hat{E} = 116 + 2/3$, $\hat{d}_1 = 100$, and $\hat{d}_3 = 400$. Now, $f_1^{\mathrm{RA}}(\hat{E},\hat{d}) = 50$ and $f_3^{\mathrm{RA}}(\hat{E},\hat{d}) = 66 + \frac{2}{3}$. Hence, the random arrival rule does not satisfy CONS.

Now, let $(E,\hat{d})$ be the problem in which player 3 splits into two players with demands 200 and 200. Then, $\hat{d}_1 = 100$, $\hat{d}_2 = 200$, $\hat{d}_3 = 200$, and $\hat{d}_4 = 200$; in particular, $d_3 = \hat{d}_3 + \hat{d}_4$. For this new bankruptcy problem, the random arrival rule and the Talmud rule propose, $f^{\mathrm{RA}}(E,\hat{d}) = (25, 58 + \frac{1}{3}, 58 + \frac{1}{3}, 58 + \frac{1}{3})$ and $f^{\mathrm{TR}}(E,\hat{d}) = (50,50,50,50)$. In both cases player 3 ends up better off after the split. Hence, neither the random arrival rule nor the Talmud rule satisfy NAMS. $\diamond$

5.12. Voting Problems and Voting Games: Power Indices

In Section 5.5 we introduced the class of simple games. These games are often used to model voting situations or, more specifically, to measure the power of the different members of a committee. Thus, because of this, when restricting attention to simple games, the allocation rules are commonly referred to as *power indices*. Suppose that the set N represents the members of a committee and they are deciding whether to approve some action or not. Now, the fact that a coalition S is such that $v(S) = 1$ means that, if the players in S vote in favor of the action, then the action is undertaken. In this context, power indices can be interpreted as the *a priori* ability of the players to change the outcome. Here, we present a pair of axiomatic characterizations, one for each of the two most widely used power indices: the *Shapley-Shubik index* (Shapley and Shubik 1954) and the *Banzhaf index* (Banzhaf 1965). For an extensive review of definitions and characterizations of power indices the reader is referred to Laruelle (1999) and Laruelle and Valenciano (2001).

Sometimes in voting problems different agents have different weights. This situation can be represented by a special class of simple games: the *weighted majority games*. Recall that the class of simple games with n players is denoted by S^N and that W and W^m denote the sets of winning coalitions and minimal winning coalitions, respectively.

Definition 5.12.1. A simple game $v \in S^N$ is a *weighted majority game* if there are a *quota*, $q > 0$, and a system of nonnegative weights, $p_1, \ldots, p_n$, such that

$$v(S) = 1 \text{ if and only if } \sum_{i \in S} p_i \geq q.$$

Example 5.12.1. A classic example of a committee that can be modeled by a weighted majority game is the United Nations Security Council. This council has fifteen members. Five of them are permanent members (China, France, Russia, United Kingdom, and USA). The remaining ten are nonpermanent members for a two-year term. A proposal is accepted if at least nine of the fifteen members are in favor and, in addition, the five permanent members are in favor too. This voting rule can be modeled by a weighted majority rule where the quota is $q = 39$, the weight of each permanent member is 7, and the weight of each nonpermanent member is 1. Then, the set of minimal winning coalitions is formed by all coalitions with nine members where all five permanent members are present. ◇

Example 5.12.2. Consider the TU-game described in Example 5.4.3, which represents the composition of the Parliament of Aragón after the elections in May 1991. The quota is $q = 34$ and the weights are given by the number of seats of each party. Then, $p_1 = 30$, $p_2 = 17$, $p_3 = 17$, and $p_4 = 3$. The set of minimal winning coalitions is $W^m = \{\{1,2\},\{1,3\},\{2,3\}\}$. ◇

Example 5.12.3. Consider the glove game in Example 5.4.2. The set of players is $N = \{1,2,3\}$ and the characteristic function is given by $v(1) = v(2) = v(3) = v(23) = 0$ and $v(12) = v(13) = v(N) = 1$. The set of minimal winning coalitions is $W^m = \{\{1,2\},\{1,3\}\}$ and player 1 is a veto player. This game can be seen as a weighted majority game with $q = 3$, $p_1 = 2$, $p_2 = 1$, and $p_3 = 1$. ◇

Not all simple games are weighted majority games. We illustrate this in the following example.

Example 5.12.4. Consider the simple game given by $N = \{1,2,3,4,5\}$ and $W^m = \{\{1,2,3\},\{4,5\}\}$. This simple game does not arise from any quota q and system of weights $p_1,\ldots,p_5$. Suppose, on the contrary, that there is a quota $q > 0$ and a system of weights $p_1,\ldots,p_5$ for this simple game. Then, $p_1 + p_2 + p_3 \geq q$ and $p_4 + p_5 \geq q$. Moreover, $q > p_1 + p_2 + p_4 \geq q - p_3 + p_4$ and, hence, $p_3 > p_4$. Now, $p_3 + p_5 > p_4 + p_5 \geq q$ and $\{3,5\}$ would be a winning coalition, which is not the case. ◇

The restriction of the Shapley value to the class of simple games is known as the Shapley-Shubik index, which therefore is also denoted by Φ. Although we have already given an axiomatic characterization of the Shapley value for the class of games G^N, this characterization is not valid when restricting attention to S^N. The main reason is that the sum of simple games is never a simple game and, hence, the additivity axiom is senseless in S^N. Dubey (1975) proposed a characterization of the Shapley-Shubik index replacing ADD with what he called the *transfer property* (TF). This property relates the power indices assigned to the maximum and the minimum games (see Definition 5.5.10) of two simple games. Let $v, \hat{v} \in S^N$. The set of winning coalitions of the simple game $v \vee \hat{v}$ is the union of the sets of winning coalitions of v and $\hat{v}$. In the simple game $v \wedge \hat{v}$, the set of winning coalitions is the intersection of the sets of winning coalitions of v and $\hat{v}$. Recall that, given a coalition S, w^S denotes the unanimity game of coalition S. Besides, given a simple game $v \in S^N$ with minimal winning coalition set $W^m = \{S_1, \ldots, S_k\}$, then $v = w^{S_1} \vee w^{S_2} \vee \ldots \vee w^{S_k}$. We now present the transfer property.

Transfer (TF): A power index φ satisfies TF if, for each pair $v, \hat{v} \in S^N$, $\varphi(v \vee \hat{v}) + \varphi(v \wedge \hat{v}) = \varphi(v) + \varphi(\hat{v})$.

The role of this property in the characterization result is similar to that of ADD in the characterization of the Shapley value. However, axiom TF is more obscure and difficult to justify.[28]

Theorem 5.12.1. *The Shapley-Shubik index is the unique allocation rule in S^N that satisfies* EFF, NPP, SYM, *and* TF.

Proof. From Theorem 5.6.1, the Shapley-Shubik index satisfies EFF, NPP, and SYM. Moreover, given two simple games v and $\hat{v}$, $v + \hat{v} = v \vee \hat{v} + v \wedge \hat{v}$ and, hence, since the Shapley value satisfies ADD, the Shapley-Shubik index satisfies TF.

We prove the uniqueness by induction on the number of minimal winning coalitions, *i.e.*, on $|W^m|$. Let φ be a power index satisfying EFF, NPP, SYM, and TF. Let v be a simple game. If $|W^m| = 1$, then v is the unanimity game for some coalition S, *i.e.*, $v = w^S$. By EFF, NPP, and SYM $\varphi(v) = \Phi(v)$. Assume that, for each simple game v such that $|W^m| \leq k - 1$, $\varphi(v) = \Phi(v)$. Let v be a simple game $|W^m| = k$, *i.e.*, $W^m = \{S_1, \ldots, S_k\}$ and

$$v = w^{S_1} \vee w^{S_2} \vee \ldots \vee w^{S_k}.$$

Let $\hat{v} = w^{S_2} \vee \ldots \vee w^{S_k}$. Then, $w^{S_1} \vee \hat{v} = v$ and

$$w^{S_1} \wedge \hat{v} = \vee_{T \in W^m \setminus \{S_1\}} w^{T \cup S_1}.$$

The number of minimal winning coalitions of each of the simple games w^{S_1}, $\hat{v}$, and $w^{S_1} \wedge \hat{v}$ is smaller than k. Then, by the induction hypothesis, $\varphi(w^{S_1}) = \Phi(w^{S_1})$, $\varphi(\hat{v}) = \Phi(\hat{v})$, and $\varphi(w^{S_1} \wedge \hat{v}) = \Phi(w^{S_1} \wedge \hat{v})$. By TF, $\varphi(w^{S_1} \vee \hat{v}) + \varphi(w^{S_1} \wedge \hat{v}) = \varphi(w^{S_1}) + \varphi(\hat{v})$. Hence, since $w^{S_1} \vee \hat{v} = v$,

$$\begin{aligned} \varphi(v) &= \varphi(w^{S_1}) + \varphi(\hat{v}) - \varphi(w^{S_1} \wedge \hat{v}) \\ &= \Phi(w^{S_1}) + \Phi(\hat{v}) - \Phi(w^{S_1} \wedge \hat{v}) \\ &= \Phi(w^{S_1} + \hat{v} - w^{S_1} \wedge \hat{v}) = \Phi(v). \end{aligned}$$

□

[28] We have presented the transfer axiom as originally introduced in Dubey (1975). We refer the reader to Laruelle and Valenciano (2001) for an alternative (and equivalent) formulation that is more transparent. Besides, Dubey followed the steps of Shapley and also considered a property called *support property* instead of the properties of EFF and NPP, which have become more standard.

Recall that the Shapley value can be seen as a weighted average of the marginal contributions of the players to the different coalitions. Hence, when working with simple games, to compute the Shapley-Shubik index we can restrict attention to those pairs of coalitions $S, S \cup \{i\}$ such that S is not a winning coalition but $S \cup \{i\}$ is. A coalition S like this is called a *swing* for player i.

Definition 5.12.2. Let $v \in S^N$ and let $i \in N$. A *swing* for player i is a coalition $S \subset N \backslash \{i\}$ such that $S \notin W$ and $S \cup \{i\} \in W$.

We denote by $\mu_i(v)$ the number of swings of player i and by $\bar{\mu}(v)$ the total number of swings, *i.e.*,

$$\bar{\mu}(v) := \sum_{i \in N} \mu_i(v).$$

If we look at how the Shapley-Shubik index relates to the idea of a swing, we have that this index is a weighted average of the swings of each player, where the weight of each swing depends on its size. One can argue that all the swings should be given the same relevance and, indeed, this is the idea of the index introduced by Banzhaf (1965), who proposed the number of swings of each player as a new power index, $\mu(v)$. Following Dubey and Shapley (1979), we call this index the *"raw" Banzhaf index*. In many applications, the principal interest lies more in the ratios than in the actual magnitudes and, hence, it is common practice to normalize this index so that its components add up to 1. We denote this index by β, which is defined, for each $v \in S^N$ and each $i \in N$, by

$$\beta_i(v) := \frac{\mu_i(v)}{\bar{\mu}(v)}.$$

This index is sometimes referred to as the *normalized Banzhaf index*. However, one has to be careful with these kind of normalizations since, in general, they will not be innocuous, for instance, from the point of view of the axioms they satisfy. The first characterization of the "raw" Banzhaf index appeared in Dubey and Shapley (1979).

Theorem 5.12.2. *The "raw" Banzhaf index is the unique allocation rule φ in S^N that satisfies* NPP, SYM, TF, *and that, for each $v \in S^N$,*

$$\sum_{i \in N} \varphi_i(v) = \bar{\mu}(v). \tag{5.12.1}$$

Proof. It follows similar lines to the proof of Theorem 5.12.1 and it is left to the reader (see Exercise 5.16). □

Now, EFF and the property implied by Eq. (5.12.1) are very similar properties and they only differ in the total power to be shared among the agents in any given game. The index β satisfies NPP, SYM, and EFF but does not satisfy the property given by Eq. (5.12.1). Also, as we show in one of the examples below, in general $\Phi \neq \beta$ and, hence, by Theorem 5.12.1, we have that β does not satisfy TF. Therefore, the normalization of the "raw" Banzhaf index recovers EFF, but at the cost of TF.

There is another variation of the "raw" Banzhaf index, which consists of a different kind of normalization and that has become the most popular one. For each $v \in S^N$ and each $i \in N$,

$$\mathrm{Bz}_i(v) := \frac{\mu_i(v)}{2^{n-1}}.$$

This new index is what is commonly referred to as the *Banzhaf index*.[29] This normalization admits a natural interpretation. For each coalition $S \subset N\backslash\{i\}$ we say that i is a *pivot* for S if S is a swing for i. $\mathrm{Bz}_i(v)$ is the probability that player i is a pivot when a coalition $S \subset N\backslash\{i\}$ is chosen at random. Different from the Shapley-Shubik index, which always allocates the same power among the agents, the total power allocated by the Banzhaf index may be different for different games.

Theorem 5.12.3. *The Banzhaf index is the unique allocation rule φ in S^N that satisfies* NPP, SYM, TF, *and that, for each $v \in S^N$,*

$$\sum_{i\in N} \varphi_i(v) = \frac{\bar{\mu}(v)}{2^{n-1}}. \tag{5.12.2}$$

Proof. It follows similar lines to the proof of Theorem 5.12.1 and it is left to the reader (see Exercise 5.16). □

The property defined by Eq. (5.12.2) is usually referred to as the *Banzhaf total power property* and plays the role of EFF in the case of the Shapley-Shubik power index. We now present some examples of the indices discussed above.

Example 5.12.5. If we again consider the simple game in Example 5.4.3, we obtain $\beta(v) = (1/3, 1/3, 1/3, 0) = \Phi(v)$ and $\mathrm{Bz}(v) = (1/2, 1/2, 1/2, 0)$. This example illustrates that the total power allocated by the Banzhaf index, *i.e.*, $\sum_{i\in N} \mathrm{Bz}_i(v)$, can be greater than 1. ◇

Example 5.12.6. We now show that, in general, the power indices β and Φ do not coincide. Consider the simple game given by $N = \{1,2,3,4\}$ and $W^m = \{\{1,2,3\},\{1,2,4\}\}$. The swings of player 1 are $(\{1,2,3\},\{2,3\})$, $(\{1,2,4\},\{2,4\})$, and $(\{1,2,3,4\},\{2,3,4\})$ and, hence, $\mu_1(v) = 3$. Similarly, we obtain $\mu_2(v) = 3$, $\mu_3(v) = 1$, and $\mu_4(v) = 1$. Then, $\beta(v) =$

[29]Because of the similar approach taken by Coleman (1971), it is often called Banzhaf-Coleman index. However, as argued in Laruelle and Valenciano (2001), both approaches are not exactly equivalent.

$\text{Bz}(v) = (3/8, 3/8, 1/8, 1/8)$. The Shapley-Shubik index for this game is $\Phi(v) = (5/12, 5/12, 1/12, 1/12)$. $\diamond$

Example 5.12.7. If we go back to the United Nations Security Council (Example 5.12.1), we also get that the power indices β and Φ do not coincide. The set of players N is the set of the fifteen members. The swings of each nonpermanent member $i \in N$ are given by $(S \cup \{i\}, S)$ where $S \subset N \setminus \{i\}$ is a coalition containing all five permanent members and exactly three nonpermanent members. Given a permanent member $i \in N$, a swing is given by $(S \cup \{i\}, S)$, where $S \subset N \setminus \{i\}$ is a coalition formed by four permanent members and at least four nonpermanent members. Then, the Shapley-Shubik index assigns to each permanent member a value 0.196 and to each nonpermanent member the value 0.00187. The Banzhaf index assigns to each permament member the value 0.194 and to each nonpermanent member 0.00275.[30] $\diamond$

A good number of alternative characterizations of the Shapley-Shubik index and Banzhaf index have been discussed in the literature. Most of them focused on replacing the TF property in the case of the Shapley-Shubik power index (Khmelnitskaya 1999), and both TF and Banzhaf total power property in the case of the Banzhaf index (Nowak 1997). Also, in Laruelle and Valenciano (2001), the authors provide new characterizations for both indices in the class of superadditive simple games.

The Shapley-Shubik index and the Banzhaf index are by far the most important power indices, but there are other relevant indices that have also been widely studied. Just to cite a few, the Deegan-Pakel index (Deegan and Packel 1978), the Johnston index (Johnston 1978), and the public good index (Holler 1982) are well-known power indices. The reader interested in further readings on the measurement of voting power can refer to Felsenthal and Machover (1998) and Laruelle and Valenciano (2008).

5.13. Cooperation in Operations Research Models

In this section we present a series of applications of cooperative game theory to the analysis of operations research models in which there are several agents involved. In these situations, it is often the case that cooperation is profitable for the agents, and the idea of the so-called *cooperative operations research games* is to study how the benefits generated by cooperation should be allocated among the agents. Hence, the standard approach is to associate an appropriate class of cooperative games with each operations research model and then study the allocations proposed by the different solutions concepts. Curiel (1997) presents a detailed analysis of cooperative

[30]We have rounded both the Shapley-Shubik and the Banzhaf indices.

operations research games and, more recently, Borm et al. (2001) published a survey on the topic.

Here, we cover three classes of cooperative operations research games: *linear production games* (Owen 1975), *maximum flow games* (Kalai and Zemel 1982b), and *inventory games* (Meca et al. 2004).

5.13.1. Linear production games. These games were introduced by Owen (1975). Let $N := \{1, \ldots, n\}$ represent a set of producers. Each producer has a bundle of l commodities, represented by a nonnegative vector $c^i := (c^i_1, \ldots, c^i_l) \in \mathbb{R}^l$. From these commodities, m different goods can be obtained by a linear production process given by an $l \times m$ nonnegative matrix $\mathcal{A}$, where each row has at least a nonzero entry; the entry a_{jk} represents the number of units of commodity j that are needed in the production of one unit of a good k.[31] Each commodity is worthless in itself. The profit of each producer is obtained by selling the produced goods. The market price of each good is given by a nonnegative vector $b := (b_1, \ldots, b_m) \in \mathbb{R}^m$. The goal of each producer is to maximize his profit. Therefore, each producer $i \in N$ faces an optimization problem that can be described through the following linear programming problem P^i:[32]

$$\begin{array}{ll} \text{Maximize} & by^t \\ \text{subject to} & \mathcal{A}y^t \leq (c^i)^t, \\ & y \geq 0. \end{array}$$

The overall production process can then be characterized by the 4-tuple $(N, \mathcal{A}, b, c)$. Suppose that a group of producers decides to cooperate by joining their commodity bundles and then producing the goods in a unique production center. In such a case, the total amount of commodities available for the producers in S is given by $c^S \in \mathbb{R}^l$, where, for each commodity j, $c^S_j := \sum_{i \in S} c^i_j$. Then, the profit that coalition S can obtain is given by the optimal value of the linear programming problem P^S below:

$$\begin{array}{ll} \text{Maximize} & by^t \\ \text{subject to} & \mathcal{A}y^t \leq (c^S)^t, \\ & y \geq 0. \end{array}$$

This approach already suggests a natural way to associate a TU-game with each production process $(N, \mathcal{A}, b, c)$. This game has N as the set of players and the characteristic function assigns, to each coalition $S \subset N$, the worth $v(S)$ given by the optimal value of the problem P^S. Any such game

[31] We do not restrict attention to integer problems. One can use, for instance, 1.5 units of a commodity or produce 3.7 units of a good.

[32] Note that, different from Section 2.8, here we take the primal problem to be a maximization problem and the dual one to be a minimization problem.

is referred to as a *linear production game*. Clearly, linear production games are superadditive. Moreover, they have nonempty cores and, indeed, not only are they balanced, but also totally balanced.

Theorem 5.13.1. *For each linear production process $(N, \mathcal{A}, b, c)$, the associated linear production game is totally balanced.*

Proof. Let $(N, \mathcal{A}, b, c)$ be a linear production process and let v be the characteristic function of the associated linear production game. For each coalition S, the restriction of the characteristic function to S is also a linear production game. When combined with Bondareva-Shapley theorem, the last observation implies that, to show that a linear production game is totally balanced, it suffices to show that it has a nonempty core. Now, we find an allocation in the core of the linear production game. Below we present the dual of the linear programming problem P^N, which we denote by D^N,

$$\begin{array}{ll} \text{Minimize} & c^N z^t \\ \text{subject to} & z\mathcal{A} \geq b, \\ & z \geq 0. \end{array}$$

By the duality theorem, the optimal solution set is nonempty and the optimal value is $v(N)$. Let $\bar{z}$ be an optimal solution of D^N. For each $i \in N$, let $\bar{x}_i := c^i \bar{z}^t$. We now show that $\bar{x} \in C(v)$. Let $S \subset N$ and let D^S be the dual of P^S. Using again the duality theorem, we get that the optimal value of D^S is $v(S)$. Now, $\bar{z}$ is feasible for D^S, $\sum_{i \in S} \bar{x}_i = \sum_{i \in S} c^i \bar{z}^t = c^S \bar{z}^t \geq v(S)$, and $\sum_{i \in N} \bar{x}_i = \sum_{i \in N} c^i \bar{z}^t = c^N \bar{z}^t = v(N)$. Then, $\bar{x} \in C(v)$. $\square$

The set of core allocations defined through the optimal solutions of D^N as in the proof above is known as the *Owen set* of the linear production game. This set has been characterized by van Gellekom et al. (2000). In general, this set is a proper subset of the core as the following example illustrates. This example also illustrates that linear production games need not be convex.

Example 5.13.1. Consider the linear production process $(N, \mathcal{A}, b, c)$ with $N = \{1, 2, 3\}$,

$$\mathcal{A} = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \end{pmatrix},\ b = (3, 1, 8),\ c_1 = (1, 0),\ c_2 = (2, 2), \text{ and } c_3 = (0, 2).$$

The linear programming problem D^N is given by

$$\begin{array}{ll} \text{Minimize} & 3z_1 + 4z_2 \\ \text{subject to} & z_1 + z_2 \geq 3, \\ & z_2 \geq 1, \\ & 2z_1 + 3z_2 \geq 8, \\ & z_1, z_2 \geq 0. \end{array}$$

The optimal value is 11 and the optimal solution set is $\{(1,2)\}$. Thus, $v(N) = 11$. The complete characteristic function of the linear production problem is given by

$$\begin{array}{lll} v(1) = 0, & v(2) = 6, & v(3) = 2, \\ v(12) = 6, & v(13) = 9/2, & v(23) = 9, \text{ and} \\ v(N) = 11. & & \end{array}$$

Then, the Owen set has a unique element, the allocation $\bar{x} = (\bar{x}_1, \bar{x}_2)$ with $\bar{x}_1 = (1,0)(1,2)^t = 1$, $\bar{x}_2 = (2,2)(1,2)^t = 6$ and $\bar{x}_3 = (0,2)(1,2)^t = 4$. Nevertheless, the core of this linear production game is given by

$$C(v) = \operatorname{conv}\{(2,6,3), (0,6,5), (0,13/2,9/2), (2,13/2,5/2)\}.$$

Moreover, this game is not convex. For instance, taking $S = \{3\}$, $T = \{2,3\}$ and $i = 1$, we have $v(T \cup \{1\}) - v(T) = 2 < 5/2 = v(S \cup \{1\}) - v(S)$. The Shapley value is given by $\Phi(v) = (13/12, 19/3, 43/12)$ and it belongs to the core. The nucleolus is $\eta(v) = (1, 25/4, 15/4)$. $\diamond$

To conclude this section we prove that the converse of Theorem 5.13.1 is also true.

Theorem 5.13.2. *Let $v \in G^N$ be a nonnegative TU-game. If v is totally balanced, then v is a linear production game.*

Proof. Let $v \in G^N$ be a nonnegative totally balanced game. We now show that v can be seen as the game associated with a linear production process. The set of agents is N. There are n commodities and each agent $i \in N$ only has a unit of commodity i, that is, $c_i^i := 1$ and $c_j^i := 0$ if $j \neq i$. The set of goods is given by the set of nonempty coalitions. The linear production matrix $\mathcal{A}$ is defined, for each $i \in N$ and each $S \in 2^N \setminus \{\emptyset\}$, by

$$a_{iS} := \begin{cases} 1 & i \in S \\ 0 & \text{otherwise.} \end{cases}$$

The (nonnegative) price vector is given by b, where, for each coalition $S \in 2^N \setminus \{\emptyset\}$, $b_S := v(S)$. Then, the characteristic function of the linear production game associated with this linear production process is obtained by solving the linear programming problems

$$\begin{array}{ll} \text{Maximize} & by^t \\ \text{subject to} & \mathcal{A}y^t \leq (c^S)^t, \\ & y \geq 0, \end{array}$$

for the different coalitions $S \subset N$. Note that $c_i^S = 1$ if $i \in S$ and $c_i^S = 0$ if $i \notin S$. We now show that the optimal values of each of these linear programming problems coincides with the worth of the corresponding coalition in game v, *i.e.*, with $v(S)$. If we take the dual linear programming problems we get

$$\begin{array}{ll} \text{Minimize} & \sum_{i\in S} x_i \\ \text{subject to} & \sum_{i\in T} x_i \geq v(T), \quad \forall T \in 2^S\backslash\{\emptyset\} \\ & x \geq 0. \end{array}$$

Since v is a nonnegative game, the nonnegativity constraints on x are redundant. Then, since v is totally balanced, we can, for each coalition S, use the same arguments in the proof of the Bondareva-Shapley theorem and get that the optimal value of the corresponding minimization problem is $v(S)$. Moreover, by the duality theorem, each $v(S)$ is also the optimal value of the corresponding primal problem and, therefore, v is the TU-game associated with the linear production process we have defined. □

Corollary 5.13.3. *The class of nonnegative totally balanced TU-games coincides with the class of linear production games.*

Proof. Follows from the combination of Theorems 5.13.1 and 5.13.2. □

Remark 5.13.1. The conditions imposed on the matrix $\mathcal{A}$ and the vector $b \in \mathbb{R}^m$ are sufficient to guarantee that each linear programming problem P^S is feasible and bounded. However, these conditions are not necessary. There are several extensions of this model such as linear programming games (Kalai and Zemel 1982a).[33] The extra generality comes from the fact that the nonnegativity restrictions on the parameters are dropped. These games are totally balanced and, moreover, it can be shown that a game is totally balanced if and only if it is a linear programming game (Curiel 1997).[34]

5.13.2. Maximum flow games. These games were introduced by Kalai and Zemel (1982b) in the context of maximum flow problems.[35] First, we describe maximum flow problems and then we define maximum flow games.

Consider a network given by a set of nodes X and set of edges E. We consider *directed* networks, in the sense that the existence of an edge from node x to node $\hat{x}$ does not imply the existence of an edge from $\hat{x}$ to x. We distinguish two nodes: the origin node or *source*, $x_o \in X$, and the arrival node or *sink*, $x_a \in X$. For each $x \in X$, we denote by $F(x)$ the set of edges starting at x and by $B(x)$ the set of edges ending at x. Each edge $e \in E$ has associated a certain *capacity*, $\mathrm{cap}(e) > 0$. Hence, a maximum flow problem is given by a triple (X, E, cap). A *flow* from source to sink in this network is a function f from E to $\mathbb{R}$ such that

[33] Other extensions can be found in Dubey and Shapley (1984) and Samet and Zemel (1984).

[34] Actually, another widely studied family of games is the class of market games (Shapley and Shubik 1969), for which the same result also holds, *i.e.*, a TU-game is totally balanced if and only if it is a market game.

[35] We refer the reader to Kalai and Zemel (1982a) for generalizations of this model.

- for each $e \in E$, $0 \leq f(e) \leq \text{cap}(e)$,
- for each $x \in X \backslash \{x_o, x_a\}$, $\sum_{e \in B(x)} f(e) = \sum_{e \in F(x)} f(e)$, and
- $\sum_{e \in F(x_o)} f(e) = \sum_{e \in B(x_a)} f(e)$.

The first task in a maximum flow problem is to determine the maximum flow that can be transferred through the network. Let $\hat{X} \subset X$ be a subset of nodes such that $x_o \in \hat{X}$ and $x_a \notin \hat{X}$. A *cut* associated with $\hat{X}$ in the maximum flow problem (X, E, cap) is the subset of edges starting at a node in $\hat{X}$ and ending at a node in $X \backslash \hat{X}$, that is,

$$\{e \in E : e \in F(\hat{x}) \text{ for some } \hat{x} \in \hat{X}, \text{ and } e \in B(x) \text{ for some } x \in X \backslash \hat{X}\}.$$

We denote this cut by $(\hat{X}, X \backslash \hat{X})$. The capacity of the cut is the sum of the capacities of all the edges it contains. Ford and Fulkerson (1962) proposed an efficient algorithm to find the maximum flow that can be transferred through a given network. Moreover, they showed that the maximum flow in a network is the capacity of a cut with minimum capacity; any such cut is called *minimum cut*.[36]

We now present an example to illustrate the above concepts.

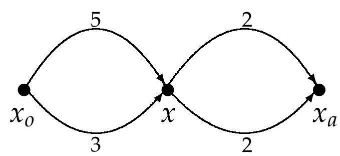
Example 5.13.2. Consider the network depicted in Figure 5.13.1 (a). The set of nodes is $X = \{x_o, x, x_a\}$ and there are 4 edges. The number at an edge indicates its capacity. The cut $(\{x_o\}, \{x, x_a\})$ has capacity 8. There is only a minimun cut, $(\{x_o, x\}, \{x_a\})$, and its capacity is 4. Therefore, it corresponds with the maximum flow for the given network. $\diamond$

Suppose that there is a set of agents $N = \{1, \ldots, n\}$ such that each edge belongs to one agent in N (each agent possibly owning more than one edge). Suppose as well that each unit of flow that is transferred from source to sink generates one unit of profit. Then, a natural question is how to allocate the total benefit generated in the network among the agents. We now show how TU-games can be useful to address this issue.

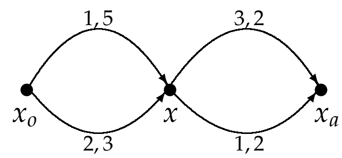
Let (X, E, cap) be a maximum flow problem and let $N = \{1, \ldots, n\}$ be a set of players. Assume that there is an *ownership function* o^E from E to N such that $o^E(e) = i$ indicates that the edge e is owned by player i. Given a coalition $S \subset N$, the restriction of the maximum flow problem to S is given by (X, E_S, cap_S), where E_S is the set of edges owned by some player in S and, for each $e \in E_S$, $\text{cap}_S(e) := \text{cap}(e)$.

Definition 5.13.1. Let (X, E, cap) be a maximum flow problem and let o^E be an ownership function for the edges in E. Then, the associated maximum flow game, $v \in G^N$, assigns, to each $S \subset N$, the maximum flow in the problem (X, E_S, cap_S).

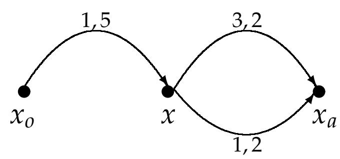
[36] We refer the reader to the original paper for a proof of this statement.



(a) Maximum flow problem.



(b) Maximum flow problem and ownership function.



(c) Network restricted to $\{1,3\}$.

Figure 5.13.1. Maximum flow problem and maximum flow game.

Example 5.13.3. Figure 5.13.1 (b) shows a function o^E associated to the network with $N = \{1,2,3\}$. The first number at an edge represents its owner and the second number represents the capacity. In Figure 5.13.1 (c), we depict the restriction of the network to coalition $\{1,3\}$. We now present the maximum flow game. The set of players is $N = \{1,2,3\}$. For coalition $S = \{1,3\}$, a minimum cut has capacity 4 and, hence, $v(13) = 4$. In a similar way we can get $v(N) = v(13) = 4$, $v(2) = v(3) = 0$, and $v(S) = 2$ otherwise. $\diamond$

Kalai and Zemel (1982b) showed that maximum flow games are closely related to totally balanced games. We present their main finding in Theorem 5.13.6.

Lemma 5.13.4. *Every nonnegative additive game is a maximum flow game.*

Proof. Let $v \in G^N$ be a nonnegative additive game given by the nonnegative real numbers $v(1), \ldots, v(n)$. Let $X := \{x_o, x_a\}$ and, for each $i \in N$, introduce an edge from x_o to x_a, owned by player i and with capacity $v(i)$. The maximum flow game associated with this maximum flow problem coincides with v. $\square$

Lemma 5.13.5. *Let v^1 and v^2 be two maximum flow games. Then, the minimum game $v^1 \wedge v^2$ is a maximum flow game.*

Proof. Let $(X^l, E^l, \mathrm{cap}^l)$ and o^{E^l} be the maximum flow problem and the ownership function, respectively, of game v^l, with $l \in \{1,2\}$. Now, let (X, E, cap) be the maximum flow problem whose network is obtained by concatenating the two previous networks, *i.e.*, identifying the sink in X_1 and the source in X_2. The ownership function o^E is defined in the obvious way, $o(e) = o^l(e)$ if $e \in E^l$, with $l \in \{1,2\}$. The flow game corresponding to this situation is the TU-game $v^1 \wedge v^2$. □

Theorem 5.13.6. *A nonnegative TU-game $v \in G^N$ is totally balanced if and only if it is a maximum flow game.*

Proof. From Theorem 5.5.8, Lemma 5.13.4, and Lemma 5.13.5, every nonnegative totally balanced game is a maximum flow game. It remains to prove the converse. Given a maximum flow game v its restriction to a coalition $S \subset N$, v_S, is also a maximum flow game. Hence, to get the result it suffices to show that each maximum flow game is balanced, which, by Bondareva-Shapley theorem is equivalent to show that its core is nonempty. Let v be the maximum flow game associated with a maximum flow problem (X, E, cap) and ownership function o^E. Let $(\hat{X}, X\backslash\hat{X})$ be a minimum cut. Consider now the allocation $x \in \mathbb{R}^N$ given, for each $i \in N$, by

$$x_i := \begin{cases} 0 & \text{if, for each } e \in (\hat{X}, X\backslash\hat{X}),\ o^E(e) \neq i \\ \displaystyle\sum_{\substack{e\in(\hat{X}, X\backslash\hat{X})\\ o^E(e)=i}} \text{cap}(e) & \text{otherwise.} \end{cases}$$

Since the maximum flow of (X, E, cap) equals the capacity of any minimum cut, $\sum_{i\in N} x_i = v(N)$. For each $S \subset N$, $(\hat{X}, X\backslash\hat{X})$ also defines a cut in (X, E_S, cap_S). The capacity of this cut in the restricted problem equals $\sum_{i\in S} x_i$. However, this capacity is greater than or equal to the capacity of the minimum cut of (X, E_S, cap_S), which, by definition, equals $v(S)$. Hence, $x \in C(v)$. □

Note that the above result also implies that the two classes of operations research games we have discussed so far coincide.

Corollary 5.13.7. *A nonnegative TU-game is a linear production game if and only if it is a maximum flow game.*

Proof. Follows immediately from the combination of Corollary 5.13.3 and Theorem 5.13.6. □

5.13.3. Inventory games. These games were first introduced by Meca et al. (2004). They are motivated by the problem of saving costs in a simple inventory management setting. Several firms order the same commodity to a unique supplier. The ordering cost is identical for all firms. Firms can cooperate by placing orders simultaneously, which would then lead to some savings. The issue here is how to share the total cost among them in a fair way.

Here, we present the *economic order quantity inventory model*; for a more detailed discussion of this and other inventory models the reader is referred to Tersine (1994). In this model each firm sells a unique commodity. The demand of the commodity that each firm faces is continuous over time and at a constant rate. The time between the placement of the order to the supplier and arrival of the commodity is deterministic and, without loss of generality, it can be assumed to be zero. Inventory costs are time-invariant and there are no constraints on the quantity ordered and stored. Each firm bears two different types of costs. For each placed order, a firm has to pay a fixed cost, a, independent of the ordered quantity. The holding costs for having one unit of the commodity in stock during one time unit is constant and is denoted by h. We assume that the firms face a demand of d units of the commodity per time unit. Each firm would like to optimize the timing and quantity of its orders.

Consider now the problem of a single firm trying to find the optimal quantity Q to order from a supplier. Let m denote the number of orders placed per time unit, *i.e.*, $m := d/Q$. A cycle is a time interval of length Q/d starting at a point in time when a firm places an order. Within a cycle, the inventory level decreases at a constant rate. Given Q, the average ordering cost per time unit is then ad/Q. Since we are assuming that the order is completed immediately, it is optimal to place an order whenever the firm runs out of stock. Then, it is easy to compute that the average stock per time unit is given by $Q/2$ and, therefore, the average holding cost per time unit is $hQ/2$. Then, the overall average cost of the firm per time unit, $\mathrm{AC}(Q)$, equals

$$\mathrm{AC}(Q) = a\frac{d}{Q} + h\frac{Q}{2}.$$

Figure 5.13.2 illustrates some of the points we have just discussed.

If we minimize the average cost over $Q > 0$, we get

- the optimal ordering cost is $\bar{Q} = \sqrt{2ad/h}$,
- the optimal number of orders per time unit is given by $\bar{m} = d/\bar{Q} = \sqrt{dh/(2a)}$, and
- the minimal average cost per time unit is $\mathrm{AC}(\bar{Q}) = 2a\bar{m}$.

Now, consider the n-firm *inventory problem*, (N, d, h, a), where $N = \{1, \ldots, n\}$ is the set of firms, $d \in \mathbb{R}^N$ is the vector of demand levels, $h \in \mathbb{R}^N$ is the vector of holding costs, and $a > 0$ is the common ordering cost. By Q_i we denote the size of an order of firm i. Now, if two firms decide to make an order at the same time, they can make a joint order instead of independent

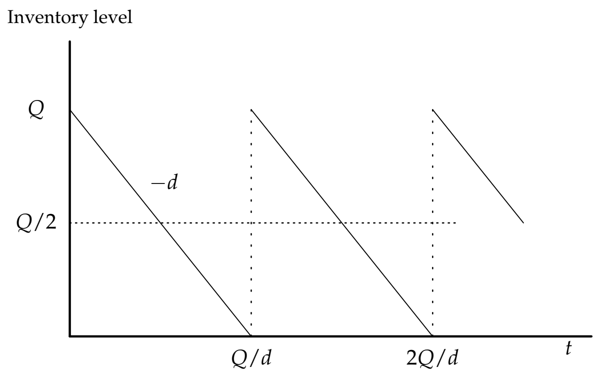
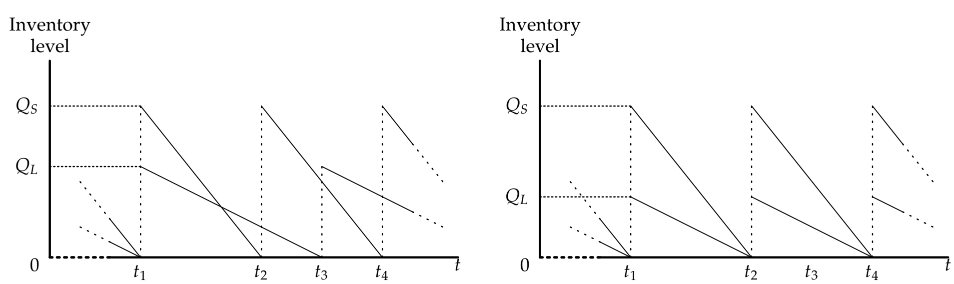


Figure 5.13.2. Change in the inventory level over time.

orders, which would lead to a saving of a. The objective is to optimize the timing and quantity of the orders of the different firms, taking into account that they can coordinate their orders. We claim that, at the optimum, the firms have cycles of the same length and make orders simultaneously. For simplicity, we only present the argument for the case $n = 2$. Suppose that firm 1 has a longer cycle than firm 2. Then, overall costs decrease when firm 1 shortens its cycle length to that of firm 2. Indeed, the overall ordering cost decreases because less orders are placed; holding costs remain unchanged for firm 2 but they decrease for firm 1, since the level of the inventory of firm 1, at any moment in time, falls. This reasoning is illustrated in Figure 5.13.3.



(a) Unequal cycles. (b) Equal cycles

Figure 5.13.3. Equal cycles save ordering costs.

Therefore, in the optimum we have that, for each firm $i \in N$, $\frac{Q_1}{d_1} = \frac{Q_i}{d_i}$. Hence, the (aggregate) average costs per time unit for the firms reduce to

$$\mathrm{AC}(Q_1, \ldots, Q_n) = a\frac{d_1}{Q_1} + \sum_{i \in N} h_i \frac{Q_i}{2} = a\frac{d_1}{Q_1} + \frac{Q_1}{2d_1} \sum_{i \in N} h_i d_i.$$

Thus, after minimizing over $Q_1 > 0$, we get that, for each $i \in N$, the optimal ordering level is

$$\bar{Q}_i = \sqrt{\frac{2ad_i^2}{\sum_{j\in N} d_j h_j}}.$$

The optimal number of orders per time unit is then given by

$$m_N = \frac{d_i}{\bar{Q}_i} = \sqrt{\frac{\sum_{j\in N} h_j d_j}{2a}} = \sqrt{\sum_{j\in N} m_j^2}$$

and the minimal average cost is $\mathrm{AC}(\bar{Q}_1) = 2am_N$. In the optimum, both ordering costs and holding costs equal am_N. Note that the minimal costs only depend on the parameter a and the optimal number of orders, m_i, of the different firms. Hence, in order to compute the minimal costs, it suffices that each firm $i \in N$ reveals its optimal value m_i, and keeps private the parameters d_i and h_i. When all firms place a unique order, the size of the order of firm $i \in N$ is d_i/m_N, which, since $m_N > \bar{m}_i$, is smaller than its individual optimal size $d_i/\bar{m}_i$; this implies a reduction on the average inventory level of firm i and, hence, a reduction on its own holding costs. We assume that each firm pays its own holding cost and then the question is how to divide the optimal ordering cost, the amount am_N.

Then, we have an *ordering cost problem*, which is characterized by a triple (N, a, m), where $N = \{1, \ldots, n\}$, $a > 0$, and $m \in \mathbb{R}^m$ is such that, for each $i \in N$, $m_i > 0$. To each ordering cost problem we associate a TU-game, which we call *inventory game*,[37] given by (N, c), where, for each $S \subset \mathbb{N}$, $c(S) := am_S$ with

$$m_S = \sqrt{\sum_{i\in S} m_i^2}.$$

Note that, for each $S \subset N$, $c^2(S) = \sum_{i\in S} c^2(i)$ and, hence, an inventory game is fully characterized by defining the individual costs. It is easy to see that inventory games have a nonempty core. Indeed, we present below an allocation rule for this class of games that always selects core allocations. Since we are working with a cost game, an allocation $x \in \mathbb{R}^N$ belongs to the core of an inventory game if $\sum_{i\in N} x_i = c(N)$ and, for each $S \subset N$, $\sum_{i\in S} x_i \leq c(S)$.

We now present the *share the ordering cost* rule, or the SOC rule for short. Given an inventory game (N, c), the SOC rule is defined, for each $i \in N$, by

$$\mathrm{SOC}_i(N, c) := \frac{c(i)^2}{\sum_{j\in N} c(j)^2} c(N) = \frac{c(i)^2}{c(N)},$$

[37] In this model it is more natural to deal with costs than with benefits.

i.e., this rule divides the total costs proportionally to the squared individual costs. Moreover, it can be rewritten as

$$\mathrm{SOC}_i(N,c) = \frac{m_i^2}{\sum_{j\in N} m_j^2} c(N) = \frac{a m_i^2}{\sqrt{\sum_{j\in N} m_j^2}}.$$

Proposition 5.13.8. *Given an inventory game (N,c), the SOC rule gives an allocation in the core of the game.*

Proof. Exercise 5.20. □

Several characterizations of this rule have appeared in the literature (Meca et al. 2003, Mosquera et al. 2008). We present the characterization of the SOC rule that appears in Mosquera et al. (2008). The main axiom of this characterization involves changes in the set of players of an inventory game. When this is the case, it is commonly understood that the set of *potential players* is given by $\mathbb{N}$ and, hence, to define an inventory game, one first needs to fix a finite set $N \subset \mathbb{N}$.

Definition 5.13.2. Let (N,c) and $(\hat{N},\hat{c})$ be two inventory games and let $S \subset N$. We say that $(\hat{N},\hat{c})$ is an S-manipulation of (N,c) if

i) there is $i \in S$ such that $\hat{N} = \{i\} \cup (N \backslash S)$; we denote player i by i_S,
ii) for each $T \subset \hat{N} \backslash \{i_S\}$, $\hat{c}(T) = c(T)$, and
iii) for each $T \subset \hat{N}$ with $i_S \in T$, $\hat{c}(T) = c(S \cup T)$.

The key property for the characterization is the following.

Immunity to coalitional manipulation (ICM): A cost allocation rule φ satisfies ICM if, for each inventory game (N,c), given $(\hat{N},\hat{c})$ an S-manipulation of (N,c), then

$$\varphi_{i_S}(\hat{N},\hat{c}) = \sum_{j\in S} \varphi_j(N,c).$$

This property describes how the rule behaves if a player splits in several players or several players merge into a unique player.[38] In both situations, the splitting or merging players cannot benefit with respect to the original situation. Hence, in situations where strategic merging and strategic splitting may be an issue, ICM seems to be a desirable property. However, ICM turns out to be a very demanding property and, indeed, only the SOC rule satisfies both ICM and EFF. Before presenting this characterization result, we show that EFF and ICM imply the following anonymity property.

[38] Actually, ICM is the natural translation to this setting of the NAMS property introduced in Section 5.11 for bankruptcy problems.

Anonymity (ANO): A cost allocation rule φ satisfies ANO if, for each inventory game (N,c), each permutation $\pi \in \Pi(N)$, and each $i \in N$,

$$\varphi_i(N,c) = \varphi_{\pi(i)}(N,c^{\pi}),$$

where, for each $S \subset N$, $c^{\pi}(S) := c(\{\pi^{-1}(j) : j \in S\})$.

This is a standard property that only requires that a relabeling of the players induces the same relabeling on the allocation proposed by the allocation rule.

Proposition 5.13.9. *Let φ be an allocation rule in the class of inventory games satisfying* EFF *and* ICM. *Then, φ satisfies* ANO.

Proof. Let φ satisfy EFF and ICM. Suppose that φ does not satisfy ANO. Then, there are $N \subset \mathbb{N}$, an inventory game (N,c), and a permutation $\pi \in \Pi(N)$ such that $\varphi_i(N,c) > \varphi_{\pi(i)}(N,c^{\pi})$. Let $j := \pi(i)$. We distinguish two cases:

Case 1: $|N| > 2$. Let $k \in N\backslash\{i,j\}$ and let $(\{i,k\},\hat{c})$ be an $N\backslash\{i\}$-manipulation of (N,c). Also, let $(\{j,k\},\hat{c}^{\pi})$ be an $N\backslash\{j\}$-manipulation of (N,c^{π}). Then, by EFF and ICM,

$$\varphi_i(\{i,k\},\hat{c}) = \hat{c}(N) - \varphi_k(\{i,k\},\hat{c}) = c(N) - \sum_{l \neq i} \varphi_l(N,c) = \varphi_i(N,c)$$

and

$$\varphi_j(\{j,k\},\hat{c}^{\pi}) = \hat{c}^{\pi}(N) - \varphi_k(\{j,k\},\hat{c}^{\pi}) = c^{\pi}(N) - \sum_{l \neq j} \varphi_l(N,\hat{c}^{\pi}) = \varphi_j(N,c^{\pi}).$$

Hence, $\varphi_i(\{i,k\},\hat{c}) > \varphi_j(\{j,k\},\hat{c}^{\pi})$. Let (M,d) be the inventory game such that $M := \{i,j,k\}$, $d^2(i) := d^2(j) := c^2(i)/2$, $d^2(k) := c^2(N) - c^2(i)$, and $d(M) = c(N)$. The $\{i,j\}$-manipulation of (M,d) given by $(\{i,k\},\hat{d})$ coincides with $(\{i,k\},\hat{c})$. By ICM, $\varphi_i(\{i,k\},\hat{d}) = \varphi_i(M,d) + \varphi_j(M,d)$. Also, the $\{i,j\}$-manipulation of (M,d) given by $(\{j,k\},\hat{d})$ coincides with $(\{j,k\},\hat{c}^{\pi})$. Then, we have $\varphi_j(\{j,k\},\hat{d}) = \varphi_j(\{j,k\},\hat{c}^{\pi}) < \varphi_i(\{i,k\},\hat{c}) = \varphi_i(\{i,k\},\hat{d}) = \varphi_i(M,d) + \varphi_j(M,d)$, which contradicts the fact that φ satisfies ICM.

Case 2: $|N| = 2$. Then, we split both players i and j in two players, i_1, i_2 and j_1, j_2, respectively. Then, let (M,d) be the inventory game such that $M := \{i_1,i_2,j_1,j_2\}$, $d^2(i_1) := d^2(i_2) := c^2(i)/2$, and $d(j_1)^2 := d^2(j_2) := c^2(j)/2$. Since φ satisfies ICM and ANO when there are three or more players, it can be easily checked that φ satisfies ANO for $|N| = 2$. □

Theorem 5.13.10. *The SOC rule is the unique allocation rule defined in the class of inventory games that satisfies* EFF *and* ICM.

Proof. The SOC rule satisfies EFF. We now prove that this rule satisfies ICM. Let (N,c) and $(\hat{N},\hat{c})$ be two inventory games such that $(\hat{N},\hat{c})$ is the S-manipulation of (N,c) for some $S \subset N$. Then,

$$\mathrm{SOC}_{i_S}(\hat{N},\hat{c}) = \frac{\hat{c}(i_S)^2}{\hat{c}(N)} = \frac{c(S)^2}{c(N)} = \sum_{i\in S}\frac{c(i)^2}{c(N)} = \sum_{i\in S}\mathrm{SOC}_i(N,c).$$

Now, we prove uniqueness. Let φ be an allocation rule in the class of inventory games satisfying EFF and ICM. We show that, for each inventory game (N,c) and each $i \in N$, $\varphi_i(N,c)$ only depends on $c(N)^2$ and $c(i)^2$. If $n=2$, $c(N)^2 = c(1)^2 + c(2)^2$ and the claim is trivially true. If $n > 2$, take the $N\backslash\{i\}$ manipulation of (N,c), $(\hat{N},\hat{c})$, where $\hat{N} = \{i, i_{N\backslash\{i\}}\}$. Since $(\hat{N},\hat{c})$ is a two-player game, $\varphi_i(\hat{N},\hat{c})$ only depends on $\hat{c}(\hat{N})^2$ and $\hat{c}(i)^2$. Now, $\hat{c}(i) = c(i)$ and $\hat{c}(\hat{N}) = c(N)$ and, since φ satisfies EFF and ICM,

$$\varphi_i(N,c) = c(N) - \sum_{j\in N\backslash\{i\}}\varphi_j(N,c) = \hat{c}(\hat{N}) - \varphi_{i_{N\backslash\{i\}}}(\hat{N},\hat{c}) = \varphi_i(\hat{N},\hat{c}).$$

Hence, $\varphi_i(N,c)$ only depends on $c(i)^2$ and $c(N)^2$. Therefore, for each $i \in N$, there is a function g_i such that, for each inventory game (N,c), $\varphi_i(N,c) = g_i(c(N)^2, c(i)^2)$. Moreover, by Proposition 5.13.9, we know that φ satisfies ANO and, hence, there is a function g such that, for each $i \in N$, $g_i = g$. Suppose that g is a linear function in the second coordinate. Then, there is another function $\tilde{g}$ such that, for each inventory game (N,c) and each $i \in N$, $\varphi_i(N,c) = \tilde{g}(c(N)^2)c(i)^2$. Since

$$c(N) = \sum_{i\in N}\varphi_i(N,c) = \tilde{g}(c(N)^2)\sum_{i\in N}c(i)^2 = \tilde{g}(c(N)^2)c(N)^2,$$

we have $\tilde{g}(c(N)^2) = \frac{1}{c(N)}$. Hence, $\varphi_i(N,c) = \frac{c(i)^2}{c(N)} = \mathrm{SOC}_i(N,c)$.

We now show that g is indeed linear in the second component. Note that we have a collection of functions

$$\{g(\alpha,\cdot) : \alpha \in (0,+\infty)\}$$

such that, for each $\alpha \in (0,+\infty)$, $g(\alpha,\cdot) : (0,\alpha] \to [0,\alpha]$. Let $\alpha, x, y \in (0,+\infty)$ be such that $x + y \leq \alpha$. Then, there is an inventory game $(N,\tilde{c})$ such that $\alpha = \tilde{c}(N)^2$, $x = \tilde{c}(1)^2$ and $y = \tilde{c}(2)^2$. Let $S = \{1,2\}$ and let $(\{i_S\}\cup(N\backslash S),\hat{c})$ be the S-manipulation of $(N,\tilde{c})$. Then,

$$\begin{aligned}
g(\alpha, x+y) &= g(\tilde{c}(N)^2, \textstyle\sum_{i\in S}\tilde{c}(i)^2) = g(\tilde{c}(N)^2, \tilde{c}(S)^2)\\
&= g(\hat{c}(\{i_S\}\cup(N\backslash S))^2, \tilde{c}(i_S)^2)\\
&= \varphi_{i_S}(\{i_S\}\cup(N\backslash S),\hat{c}) = \textstyle\sum_{i\in S}\varphi_i(N,\tilde{c})\\
&= \textstyle\sum_{i\in S} g(\tilde{c}(N)^2, \tilde{c}(i)^2) = g(\alpha,x) + g(\alpha,y).
\end{aligned}$$

Hence, for each $\alpha \in (0,+\infty)$, $g(\alpha,\cdot)$ is additive. Moreover, since $g(\alpha,\cdot)$ is nonnegative, it is clear that it is also nondecreasing. Finally, it is easy to prove that every nondecreasing additive function $h : (0, \alpha] \to [0, \alpha]$ is also linear. $\square$

Exercises of Chapter 5

5.1. Let φ be the allocation rule that, for each bargaining problem (F, d), selects the allocation $\varphi(F, d) := d + \bar{t}(1, \ldots, 1)$, where $\bar{t} := \max\{t \in \mathbb{R} : d + t(b(F, d) - d) \in F_d\}$. Show that this allocation rule does not satisfy CAT. In particular, show that a player can benefit by changing the scale in the utility function used to represent his preferences.

5.2. Show that none of the axioms used in Theorem 5.3.5 to characterize the Kalai-Somorodinsky solution is superfluous.

5.3. Consider the bargaining problem (F, d) with $d = (1, 0)$ and

$$F_d = \{(x, y) \in \mathbb{R}^2 : x^2 + y \leq 4, 0 \leq y \leq 3, 1 \leq x\}.$$

Obtain $\mathrm{NA}(F, d)$ and $\mathrm{KS}(F, d)$.

5.4. Let (F, d) be the bargaining problem with $d = (0, 0)$ and

$$F = \{(x, y) \in \mathbb{R}^2 : 4x + 2y \leq 2\}.$$

Derive $\mathrm{NA}(F, d)$ and $\mathrm{KS}(F, d)$ using the properties that characterize these solutions.

5.5. Prove Lemma 5.4.1.

5.6. Prove Lemma 5.5.4.

5.7. Show that the two formulas of the Shapley value given in Eqs. (5.6.1) and (5.6.2) are equivalent.

5.8. Suppose that $v \in G^N$ can be decomposed as $v = \sum_{S \in 2^N \backslash \{\emptyset\}} \alpha_S w^S$, where $\alpha_S \in \mathbb{R}$ and, for each $S \in 2^N \backslash \{\emptyset\}$, w^S is the unanimity game of coalition S. Now, for the coalitions in $2^N \backslash \{\emptyset\}$, we recursively define the following real numbers, called Harsanyi dividends (Harsanyi 1959),

$$d_S := \begin{cases} v(S) & |S| = 1 \\ \frac{v(S) - \sum_{T \subsetneq S} |T| d_T}{|S|} & |S| > 1. \end{cases}$$

Prove that, for each $S \in 2^N \backslash \{\emptyset\}$, $\alpha_S = |S| d_S$.

5.9. Let (N, v) be the TU-game where $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 4$, $v(23) = 2$, and $v(N) = 8$.

(a) Is (N, v) a convex game?

(b) Obtain the Shapley value.

(c) Compute the core and the nucleolus.

5.10. Describe the core of the weighted majority game introduced in Example 5.12.1.

5.11. Two house owners, players 1 and 2, are trying to sell their houses. There are three potential buyers, players 3, 4, and 5. Each buyer is only interested in buying one house. Players 1 and 2 value their houses in 40 and 60 units, respectively. The buyers valuations are given in the following table.

Buyer	House 1	House 2
3	45	70
4	50	50
5	50	65

(a) Determine the characteristic function of the corresponding game.
(b) Obtain the Shapley value.
(c) Obtain the nucleolus.

5.12. Consider the following modification of the mechanism proposed by Moulin to implement the Kalai-Smorodinsky solution (Section 5.9.3). If player j rejects the offer of player i at Stage 2, then the probability that he becomes a proposer is a_j instead of a_i; with the remaining probability, the disagreement point is the realized outcome. Show that Proposition 5.9.6 and Theorem 5.9.7 also hold for this game.

5.13. Show that airport games are superadditive and convex.

5.14. Show that the TU-game associated with a bankruptcy problem is convex.

5.15. Prove Proposition 5.11.1.

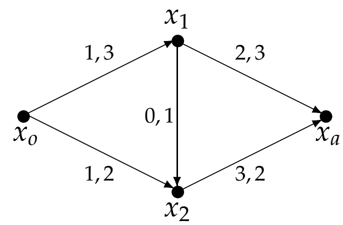
5.16. Prove Theorems 5.12.2 and 5.12.3.

5.17. Let $(N, \mathcal{A}, b, c)$ be a linear production problem, where $N = \{1,2,3\}$, $b_1 = (1,1,1)$, $b_2 = (2,1,1)$, $b_3 = (1,2,1)$, $c = (2,3,1)$, and

$$\mathcal{A} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & 3 \\ 0 & 2 & 2 \end{pmatrix}.$$

(a) Determine the characteristic function of the corresponding linear production game.
(b) Obtain the Owen set.
(c) Obtain the Shapley value. Does it belong to the core?

5.18. Consider the following network:



For each edge, the first number indicates the owner and the second number indicates the capacity. A public edge is any edge whose owner is 0.

(a) Determine the characteristic function of the maximum flow game.
(b) Is it a balanced game?
(c) Find an imputation that dominates $(1,1,1)$.

5.19. Show that any inventory game is zero-monotonic.

5.20. Prove Proposition 5.13.8.

5.21. Consider an inventory situation with 3 players, $N = \{1,2,3\}$, the fixed ordering cost is $a = 20$ units and the individual optimal number of orders are $m_1 = 4$, $m_2 = 3$, and $m_3 = 5$.

(a) Determine the characteristic function.
(b) Find the allocation given by the SOC rule.
(c) Does the Shapley value satisfy ICM for this particular inventory game?

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Notations

We do not pretend to provide a comprehensive list of all the notations used in this book. We just intend to provide a brief reference list containing those notations that might be unclear to some readers and also those conventions that are not completely standard.

$\mathbb{N}$	The set of natural numbers $\{1, 2, \ldots\}$
$B \subset A$	Set B is a subset of A (possibly equal)
$B \subsetneq A$	Set B is a subset of A and B is not equal to A
$\lvert A \rvert$	The number of elements of set A
$\operatorname{conv}(A)$	The convex hull of set A, *i.e.*, $\operatorname{conv}(A) := \{\sum_{i=1}^{k} \alpha_i x_i : k \in \mathbb{N},\ x_i \in A,\ \alpha_i \in \mathbb{R}, \alpha_i \geq 0,$ and $\sum_{i=1}^{k} \alpha_i = 1\}$
$\operatorname{ext}(A)$	The extreme points of set A, *i.e.*, $\operatorname{ext}(A) := \{x \in A :$ for each pair $y, z \in A$ and each $\alpha \in (0,1), x = \alpha y + (1-\alpha) z$ implies $x = y = z\}$
2^A	The set of all subsets of set A
B^A	Maps from the set A to the set B
ΔA	$\{x \in [0,1]^A : \lvert\{a \in A : x(a) > 0\}\rvert < \infty$ and $\sum_{a \in A} x(a) = 1\}$
$\lvert \mathcal{A} \rvert$	The determinant of matrix $\mathcal{A}$
$\mathcal{A}^t$	The transpose of matrix $\mathcal{A}$
I_m	The identity $m \times m$ matrix
$\mathbb{1}_m$	The vector $(1, \ldots, 1) \in \mathbb{R}^m$
$\mathbb{E}(X)$	The expectation of random variable X
$\mathbb{E}(X\,\vert_{Y})$	The expectation of variable X conditional on Y
$P(A)$	The probability of event A
$P(A\,\vert_{B})$	The probability of event A conditional on event B

$\{x^k\}$	Abbreviation for $\{x^k\}_{k\in\mathbb{N}}$
$\{x^k\} \to x$	Abbreviation for $\lim_{k\to\infty}\{x^k\} = x$
$\operatorname{argmin}_{x\in\Omega} f(x)$	The set $\{y \in \Omega : f(y) = \min_{x\in\Omega} f(x)\}$
Let $a, b \in \mathbb{R}^n$:	
$a \geq b$	For each $i \in \{1, \ldots, n\}$, $a_i \geq b_i$
$a > b$	For each $i \in \{1, \ldots, n\}$, $a_i > b_i$
(a_{-i}, b_i)	The vector $(a_1, \ldots, a_{i-1}, b_i, a_{i+1}, \ldots, a_n)$

Index of Authors

Index of Solution Concepts

Boldface page numbers are used to indicate the pages where the concepts are defined.

Subject Index

Boldface page numbers are used to indicate the pages where the concepts are defined.

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Game theory provides a mathematical setting for analyzing competition and cooperation in interactive situations. The theory has been famously applied in economics, but is relevant in many other sciences, such as political science, biology, and, more recently, computer science. This book presents an introductory and up-to-date course on game theory addressed to mathematicians and economists, and to other scientists having a basic mathematical background. The book is self-contained, providing a formal description of the classic game-theoretic concepts together with rigorous proofs of the main results in the field. The theory is illustrated through abundant examples, applications, and exercises.

The style is distinctively concise, while offering motivations and interpretations of the theory to make the book accessible to a wide readership. The basic concepts and results of game theory are given a formal treatment, and the mathematical tools necessary to develop them are carefully presented. Cooperative games are explained in detail, with bargaining and TU-games being treated as part of a general framework. The authors stress the relation between game theory and operations research.

The book is suitable for a graduate or an advanced undergraduate course on game theory.

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