Binary Relation

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What is Binary Relation ?

Given a set $\mathbb{S}$, $\boldsymbol{R} \subseteq \mathbb{S} \times \mathbb{S}$ is said to be a Binary Relation. In other words a binary relation should be a subset of cartesian product of the ground set.

Example: less than or equal to $\leq$, divisibility, etc.

$\mathbb{S} = \{1,2,3,4,5,6,7,8\}$

$\mathbb{S} \times \mathbb{S} =$
$\{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{1, 6\}, \{1, 7\}, \{1, 8\},$
$\{2, 1\}, \{2, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{2, 7\}, \{2, 8\},$
$\{3, 1\}, \{3, 2\}, \{3, 3\}, \{3, 4\}, \{3, 5\}, \{3, 6\}, \{3, 7\}, \{3, 8\},$
$\{4, 1\}, \{4, 2\}, \{4, 3\}, \{4, 4\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{4, 8\},$
$\{5, 1\}, \{5, 2\}, \{5, 3\}, \{5, 4\}, \{5, 5\}, \{5, 6\}, \{5, 7\}, \{5, 8\},$
$\{6, 1\}, \{6, 2\}, \{6, 3\}, \{6, 4\}, \{6, 5\}, \{6, 6\}, \{6, 7\}, \{6, 8\},$
$\{7, 1\}, \{7, 2\}, \{7, 3\}, \{7, 4\}, \{7, 5\}, \{7, 6\}, \{7, 7\}, \{7, 8\},$
$\{8, 1\}, \{8, 2\}, \{8, 3\}, \{8, 4\}, \{8, 5\}, \{8, 6\}, \{8, 7\}, \{8, 8\}\}$

If a $\leq$ b defines a relation on set $\mathbb{S} \times \mathbb{S}$, then,
(a,b) $\leftarrow R_{(\leq)} \iff$ a $\leq$ b, and the set containing R is
$R_{(s)} = \{\{1, 1\},\{1, 2\},\{1, 3\},...,\{2,3\},...,\{5,6\},...\}$.

Let R be the relation that a divides b, then, (a,b) $\leftarrow R_1 \iff$ a | b.
$R_1 = \{\{1, 1\}, \{1, 2\}, ...\ \{1, 8\},$
$\{2, 2\}, \{2, 4\}, ...$
$\{3, 3\}, \{3, 6\}, ...$
$\{4, 8\}, \{5, 5\}, \{6, 6\}, \{7, 7\}, \{8,8\}\}$

Types of Relation

Reflexive Relation

For all a, (a, a) $\in \boldsymbol{R}$.

Symmetric Relation

If (a, b) $\in \boldsymbol{R} \implies$ (b, a) $\in \boldsymbol{R}$.

AntiSymmetric Relation

If (a, b) $\in \boldsymbol{R} \wedge$ (b, a) $\in \boldsymbol{R} \implies$ a = b.

Contrapositive Relation

If p $\implies$ q ,Then, $\neg$ q = $\neg$ p.

Demorgan's law

$\neg(a \wedge b) = \bar{a} \vee \bar{b}$.
$\neg(a \vee b) = \bar{a} \wedge \bar{b}$.

Transitive Relation

If (a, b) $\in \boldsymbol{R} \wedge$ (b, c) $\in \boldsymbol{R} \longrightarrow$ (a, c) $\in \boldsymbol{R}$.

Partial Order Binary Relation

A partial order is a binary relation which is reflexive, antisymmetric and transitive. E.g. divisibility ($\leq$) subset containment.

Strict Partial Order Binary Relation

A strict partial order is a binary relation which is irreflexive, antisymmetric and transitive. E.g. less than ($<$).

We denote a partial order on a ground set X by (X, $<$) where
a $<$ b $\iff$ (a, b) $\in \boldsymbol{R}$.

Covering of elements

An element **a** is covered by **b** in a partial order (X, $<$)
if $\nexists$ y $\in$ X such that a $<$ y $<$ b.
E.g. 3 is covered by 6, but 2 is not covered by 8 under relation | for set S.

Hasse Diagrams

Conditions for drawing Hasse Diagrams

A Hasse diagram for a partial order (X,<) is drawn such that the following conditions are satisfied:

- Each element in X is represented by a point on the plane.
- If a < b, then point representing a is below the point representing b.
- We put an edge between two elements a, b iff a $<_c b$.

Drawing Hasse Diagrams

Draw the Hasse diagram for $R_|$, given A = {1,2,3,4,5,6,7,8}.

The hasse diagram is drawn as shown in Figure 1.

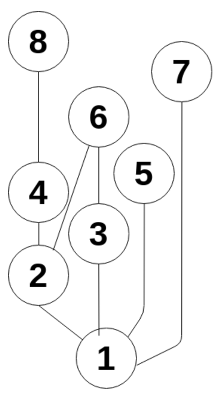

Figure 1: Hasse Diagram for $R_|$

Linear Order

A linear order is a partial order (X, <) such that
a < b or b < a $\forall$ (a, b) $\leftarrow$ X $\times$ X, a $\neq$ b.

Questions

1. X is a subset of a finite set. Prove that this induces a POSET (Partial order set) over a ground set
OR
Prove that (2^S,$\subseteq$) is a POSET.

For a set to be POSET it has to be Reflexive, Antisymmetric and Transitive.

S $\times$ S = { {1, 1}, {1, 2}, {1, 3}, ... {n, n}}.
2^S = { ϕ, {1}, {2}, ... {n}, {1, 2}, {1, 3}, ... , {1, n}, {2, 3}, ...}

From the power set we found out that:

- $2^S \nRightarrow$ Reflexive, so is Irreflexive.
- $2^S \nRightarrow$ Symmetric, so is Antisymmetric.
- $2^S \Rightarrow$ Transitive: If {a, b} $\subseteq 2^S$ and {b, c} $\subseteq 2^S$, then {a, c} $\subseteq 2^S$, $\forall$ a,b,c $\in$ {1, 2, 3, ... n}

Reflexive

$\forall$ a where a $\in 2^S$. (a, a) $\in$ R [As a $\subseteq$ a]

Antisymmetric

If a $\in 2^S$, b $\in 2^S$
if (a, b) $\in$ R $\wedge$ (b, a) $\in$ R.
$\Rightarrow$ a $\subseteq$ b [As a $\subseteq$ b and b $\subseteq$ a]
$\Rightarrow$ a = b

Transitive

If a,b $\in 2^S$
If (a, b) $\in$ R $\wedge$ (b, c) $\in$ R.
$\Rightarrow$ a $\subseteq$ c
$\Rightarrow$ (a, c) $\in$ R

Hence, (2^S,$\subseteq$) is a POSET

2. A note on $R_{\leqslant}$

Let S = {1, 2, 3, 4}.
2^S = {ϕ, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {2, 3, 4}, {1, 3, 4}, {1, 2, 3, 4}}.

A relation is defined on the ground set.
$\leqslant \subseteq$ (X $\times$ X), then this relation is a subset of (X $\times$ X).
$P_{\leqslant} \subseteq (2^S \times 2^S)$. (a, b) where a, b $\in 2^S$

- $2^S \nRightarrow$ Reflexive, so is Irreflexive.
- $2^S \nRightarrow$ Symmetric, so is Antisymmetric.
- $2^S \Rightarrow$ Transitive: If {a, b} $\subseteq 2^S$ and {b, c} $\subseteq 2^S$, then {a, c} $\subseteq 2^S$, $\forall$ a,b,c $\in$ {1, 2, 3, ... n}

3. R denotes any binary relation on 2^S. Suppose R denotes the disjointness relation. Which means if a $\cap$ b = ϕ for some (a, b) $\in {}_2S$ [we say a and b have disjointness relation]. List all the elements of R_d

R_d = { ({ ϕ}, { ϕ }), ({ ϕ}, { 1}), ({ ϕ}, { 2}), ({ ϕ}, { 3}), ({ ϕ}, { 4}), ... ({ ϕ}, { n}), ({ 1}, { 2}), ({ 1}, { 3}), ({ 1}, { 4}), ... ({ 1}, { n}), ... ({ 1, 2}, { 3, 4}), ({ 1, 2}, { 4, 5}), ... }

4. List all the elements of R_d. Consider R_c a binary relation on $2^S \times 2^S$ where, (a, b) $\in R_c \iff$ | a | $< \dfrac{|\,b\,|}{2}$, (a, b) $\in 2^S$.

Elements of 2^S = { { ϕ}, {1}, {2}, {3}, ... {n }, {1, 1}, {1, 2}, {1, 3}, .. {1, n}, {2, 1}, {2, 2}, ... {2, n}, ... }
For simplicity, Let us consider the set S = {1, 2, 3, 4}.

2^S = { {ϕ}, {1}, {2}, {3}, ... {1, 2}, {1, 3}, {1, 4}, {2, 1}, {2, 3} ..., {3, 1}, {3, 2}, ... {4, 1},... {1, 2, 3}, {2, 3, 4}... {1, 2, 3, 4}, }

(a, b) $\in R_c \iff$ | a | $< \dfrac{|\,b\,|}{2}$
Elements of the relation = { ({ϕ}, {1}), ({ϕ}, {2}), ({ϕ}, {3}), ... ({1}, {2, 3}), ({1}, {3, 4}), ({2}, {1, 3}), ({2}, {1, 4}), ..., ({1}, {2, 3, 4}), ({2}, {1, 3, 4}), ... }

A note on the above relation

- $2^S \nRightarrow$ Reflexive, so is Irreflexive.
- $2^S \nRightarrow$ Symmetric, so is Antisymmetric.
- $2^S \Rightarrow$ Transitive:
  (a, b) $\in R_c \iff$ | a | $< \dfrac{|\,b\,|}{2}$ ..... (i)
  (b, c) $\in R_c \iff$ | b | $< \dfrac{|\,c\,|}{2}$ ..... (ii)

(a, c) $\in R_c \iff$ | a | $< \frac{|c|}{4}$ ..... (iii)
From Eq. (iii) we get to know that $\frac{|b|}{2} < \frac{|c|}{4}$,
Hence | a | $< \frac{|c|}{4}$

Maximal and Minimal Subsets

Let S = {1, 2, 3, 4}

2^S = { ϕ, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4}, {2, 4}, {3, 4},{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}}

Maximal Family of Subset

Maximal family of the subset of 2^S, so that any two elements of the subset are not related. [No two elements of the subset are comparable.]
We define the binary relation of S by the subset relation.
(A, B) $\leftarrow$ R if A $\subseteq$ B.

Find the maximal subset of 2^S.

Maximal subset = {{1}, {2}, {3}, {4}}, which is an **anti-chain**.

Q. Find a maximal subset of 2^S, such that every two elements of the subset are comparable or related. (2 elements are comparable)

{ ϕ, {1}, {1, 2}, {1, 2, 3}, {1, 2, 3, 4}} which is a **chain**.

Chains and Anti-Chains

Consider a partial order set (X, $\leqslant$). Any subset of X is called a **chain** if any two elements of the subsets are **related**. Any subset of X is called an **anti-chain** if any two elements of the subsets are **not related**

Theorem 1: Let (X, $\leqslant$) be a finite partially ordered set, and let r be the size of a largest chain. Then r can be partitioned into at least r antichains.

Q. Partition 2^S into 5 antichains.

{ { ϕ}, {{1}, {2}, {3}, {4}}, {{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}}, {{1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}}, {{1, 2, 3, 4}} }

Q. Suppose A is an antichain on X, C is a chain on X. Then prove that $\mid$ A $\cap$ B $\mid$ $\leqslant$ 1

Suppose for contradiction $\mid$ A $\cap$ B $\mid$ $\geqslant$ 2 is true. Therefore, there exists 2 elements which belongs to both A and C. Since the pair belongs to A, the elements are not related but at the same time they belongs to C and they should be related, so this forms a contradiction. If it belongs to C then the elements are comparable but if it doesn't belongs to C then the elements are not comparable.
Hence $\mid$ A $\cap$ B $\mid$ $\leqslant$ 1 is true.

Partition the set X into maximal chains.

A minimal element α of a chain C $\subseteq$ X is that element which satisfies the following, $\alpha \leq x$, $\forall$ $x \in$ C.
$\alpha \mid x \iff \alpha \leq x$
$x \mid \alpha \iff x \leq \alpha$ ($\leq$ is a binary relation)
Minimum element of C is { ϕ} Consider the set with the divisibility relation. D = {3, 9, 18, 24, 21, 30}. $\alpha \subset$ X. A maximal element α of a chain $C \subseteq$ X is that element which satisfies the following:

- $x \leq \alpha$. $\forall$ $x \in C$.
- Maximal element of C is {1, 2, 3, 4}.
- Consider the set with divisibility relation. D = {3, -9, 27, -81}. Note that D is a chain under divisibility and -81 is the maximal element.

Can Hasse diagrams always be planar ?

No.
Consider the subset relation where elements are: $\{A\}$, $\{B\}$, $\{C\}$, $\{A, B, C, P\}$, $\{A, B, C, Q\}$ and $\{A, B, C, R\}$, The Hasse diagram for this type of relation is K-33 as shown in Figure 2 .
Consider the divisibility relation where elements are: 1, 2, 3, 12, 18 and 30, The Hasse diagram for this type of relation is K-33 as shown in Figure 3 .
Hence we can see that the Hasse diagram can not always be planar.

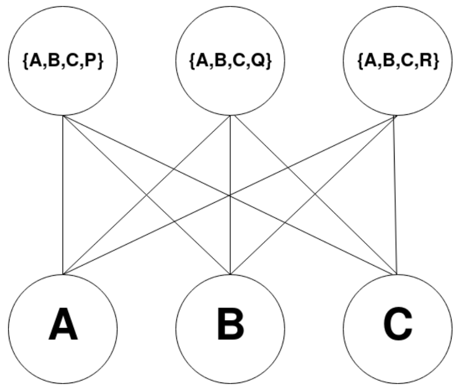

Figure 2: Hasse Diagram for $R_{\subset}$

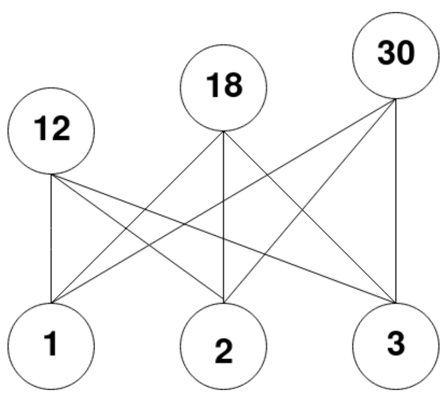

Figure 3: Hasse Diagram for $R_{|}$