Towards High-Performance Quantum Fourier Transform

The Quantum Fourier Transform (QFT) is the quantum analog of the Discrete Fourier Transform (DFT), widely used in signal processing, physics, and computer science. The QFT transforms quantum states from the computational basis to the Fourier basis, enabling applications in quantum algorithms like Shor's algorithm and phase estimation.

The QFT acts on a quantum state $|x\rangle$ as follows:

$$|x\rangle = |x_{n-1}x_{n-2}\dots x_0\rangle \quad \rightarrow \quad \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi i x k/N}|k\rangle,$$

where $N = 2^n$ is the dimension of the Hilbert space.

For a classical vector $\{a_x\}$, the DFT computes the coefficients:

$$b_k = \sum_{x=0}^{N-1} a_x e^{-2\pi i x k/N}, \quad k = 0, 1, \dots, N-1.$$

The QFT produces a quantum superposition with coefficients $e^{2\pi i x k/N}$, while the CFT outputs deterministic values $\{b_k\}$.

Steps in the QFT Algorithm

The QFT can be implemented using a sequence of quantum gates, acting on an n-qubit quantum register. The key steps are as follows:

1. Initial State Preparation The input state is a computational basis state $|x\rangle = |x_{n-1}x_{n-2}\dots x_0\rangle$, representing an n-bit integer x.

2. Apply the Hadamard Gate to the Most Significant Qubit The Hadamard gate H is applied to the first qubit x_{n-1}:

$$H|x_{n-1}\rangle \rightarrow \frac{1}{\sqrt{2}}\left(|0\rangle + e^{2\pi i(0.x_{n-1})}|1\rangle\right).$$

3. Introduce Controlled Phase Rotations For each remaining qubit x_j, controlled rotation gates R_k apply conditional phase shifts:

$$R_k = \begin{bmatrix} 1 & 0 \\ 0 & e^{2\pi i/2^k} \end{bmatrix}.$$

These gates introduce phase factors based on the binary fraction $0.x_j x_{j-1}\dots$.

4. Repeat for All Qubits The Hadamard and controlled rotations are applied sequentially to each qubit in decreasing significance, $x_{n-2}, x_{n-3}, \dots, x_0$.

After applying the gates, the final quantum state is:

$$\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1} e^{2\pi i x k/N}|k\rangle.$$

Note that this is different from the results obtained from DFT where the vector contains coefficients to the Fourier basis. Here, the qubits themselves form a complete Fourier basis and the amplitude for each basis element is encoded in the quantum register. The only operation we can obtain is to sample the state after which the entire state vector collapses onto a single basis element.