欧氏平面上平方距离最小的直线拟合（续）

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1 旧有结论

按旧文推导，对点集{(x,y)}满足欧氏距离误差最小的最优直线拟合为：

$$x\cos\alpha + y\sin\alpha - r = 0 \tag{1}$$

其中

$$\begin{aligned}\alpha &= \frac{1}{2}\arctan\frac{2\mathrm{cov}(x,y)}{\mathrm{cov}(x,x)-\mathrm{cov}(y,y)}+\frac{k\pi}{2}\\ r &= \bar{x}\cos\alpha+\bar{y}\sin\alpha\end{aligned} \tag{2}$$

2 Total Least Squares

按(2)式，令$x' = x - \bar{x}, y' = y - \bar{y}$，直线方程可改写为线性形式：

$$x'\cos\alpha + y'\sin\alpha = 0 \tag{3}$$

优化目标为

$$E(X) = (XA)^T(XA) = A^T X^T X A, \quad X = \begin{bmatrix} \vdots & \vdots \\ x_i' & y_i' \\ \vdots & \vdots \end{bmatrix}_{m\times 2}, A = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \tag{4}$$

求$A = \mathrm{Argmin}_A\, E(X)$。对X作SVD分解：$X = U_{m\times 2}\Sigma_{2\times 2}V_{2\times 2}^T$

$$E(X) = A^T V \Sigma^T U^T U \Sigma V^T A = A^T V \Sigma^T \Sigma V^T A \tag{5}$$

设$V = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right), \Sigma = \left(\begin{smallmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{smallmatrix}\right), \sigma_1 \geqslant \sigma_2 \geqslant 0$，可直接求出

$$E(X) = \frac{1}{2}(\sigma_1^2 + \sigma_2^2 + (\sigma_1^2 - \sigma_2^2)\cos(2\alpha - 2\theta)) \tag{6}$$

当$2(\alpha-\theta) = \pi + 2k\pi$即$\alpha = \theta + \frac{\pi}{2} + k\pi$时取最小值。考虑到$\alpha + \pi$仅仅是$\alpha$的180°反方向，故只取一个解$\alpha = \theta + \frac{\pi}{2}$即可。此时

$$A = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = V[\![2]\!] \tag{7}$$

另一方面，按Total Least Squares完全最小二乘法求解，也能有：

$$A = \mathrm{Argmin}_A \|\tilde{X}\|_F, \quad (X + \tilde{X})A = 0 \tag{8}$$

SVD分解

$$
\begin{aligned}
X &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T \\
X + \tilde{X} &= \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T = U_1 \sigma_1 V_1^T \\
\text{两式相减} \quad \tilde{X} &= -U_2 \sigma_2 V_2^T
\end{aligned} \tag{9}
$$

故在$X + \tilde{X}$的补空间中取$A = kV_2$，可使$(X + \tilde{X})A = U_1 \sigma_1 V_1^T \cdot kV_2 = kU_1 \sigma_1 (V_1^T V_2) = 0$. Q.E.D.