G54DMA - Lab 3: Modelling in R

Question Sheet

This question sheet presents a series of exercises designed to help you create and classify models as well as implement them in R. You will be applying the concepts you learned during the first two labs (vectors, matrices, data frames, control structures, functions, etc.) to define, create and test models.

It is recommended that you work on the models with pen and paper first until you have defined them. Once you have defined and classified them, try implementing them in R.

In this lab session, you will learn to:

- **Create models by hand**
- **Implement and solve models in R using functions**
- **Find the equilibrium values for models**
- **Classify models**
- **Analyse behavior of models using plots and functions.**

To begin, start your R editor (RStudio or your editor of choice). **This lab session does not have an Instruction Sheet, instead revise Lectures 2 and 3, and the Disease Model solved in class, along with your notes**. Once you have finished revising them and are familiar with the concepts of models, mathematical models and classification of models, start working on these exercises.

A. Model 1: Variation in price

An economist is interested in the variation of the price of a single product. It is observed that a high price for the product in the market attracts more suppliers. However, increasing the quantity of the product supplied tends to drive the price down. Over time, there is an interaction between price and supply. The economist has proposed the following model, where P_n represents the price of the product at year *n*, and Q_n the quantity:

$$P_{n+1} = P_n - 0.1\,(Q_n - 500)$$
$$Q_{n+1} = Q_n + 0.2\,(P_n - 100)$$

1. Does the model make sense intuitively? What is the significance of the constants 100 and 500? Explain the significance of the sign of the constants -0.1 and 0.2.
2. Write a function in R that solves this model.
3. Test your function with the initial conditions present in Table 1 and predict the long-term behavior for the next 25 years at intervals of 1 year:

Table 1. Relationship between Price and Quantity.

	Price	Quantity
Case 1	100	500
Case 2	200	500
Case 3	100	600
Case 4	100	400

B. Model 2: The D'Hondt Method

The D'Hondt Method, widely used in Europe, is a method for allocating election seats with proportional representation.

It is calculated as follows: the total votes cast for each party in the electoral district are divided, first by 1, then by 2, then 3, up to the total number of seats to be allocated for the district.

Say there are *p* parties and *s* seats. Then a matrix of numbers can be created, with *p* rows and *s* columns, where the entry in the *i*th row and *j*th column is the number of votes won by the *i*th party, divided by *j*.

The *s* winning entries are the *s* highest numbers in the whole grid; each party is given as many seats as there are winning entries in its row.

1. Write a function in R that models the D'Hondt Method. The prototype of the function should be:

   dhondt_model(vote_results, seats)

   Where:

   - *vote_results* is a dataframe with two fields: *party* which has the name of each party, and *votes*, which has an integer with the number of votes that that party has received.
   - *seats*, an integer with the number of seats up for the election.

   It should then **return and print** a dataframe with the name of each party and the number of seats assigned to it.

   (**Hint:** R's in-built functions *sweep* and *order* might be of use).

2. Test your model with the two cases shown in Table 2:

Table 2. Test scenarios

(a) Scenario 1. Total number of seats: 5.

	Votes
Party 1	17,920
Party 2	11,490
Party 3	11,170
Party 4	4,420

(b)Scenario 2. Total number of seats: 12.

	Votes
Party A	250,351
Party B	431,935
Party C	125,679
Party D	64,873
Party E	125,828

C. Model 3: Wolf and Moose Population

In the remote area of Isle Royale, wolves' primary food source is a single prey: moose. You have been hired to predict the population levels of wolves and moose in the region. Let W_n represent the wolf population in year *n*, and M_n represent the moose population in year *n*. The ecologist has suggested the following model:

$$M_{n+1} = 1.2M_n - 0.004\, W_nM_n$$
$$W_{n+1} = 0.7W_n + 0.005\, W_nM_n$$

The ecologist wants to know whether these two species can coexist in the habitat.

1. Compare the signs of the coefficients in the previous model. Do they make sense?

2. Find the equilibrium values for this population. The equilibrium values of a model are those values such as, given a population *P* and a time *n*:

$$P_{n+1}= P_n =....= P$$

Or, that is, the values of the population remain constant in time.

3. Write a function in R that calculates and plots the behaviour of both species.

4. Test your model's behaviour in the next 10 years with the initial populations shown in Table 3:

Table 3. Initial Wolf and Moose populations.

	Wolves	Moose
Case 1	50	50
Case 2	50	60
Case 3	100	200
Case 4	151	204

D. Model 4: Car Rental Company

A car rental company has distributorships in Nottingham and Leicester. Travelers may rent a car in one city and drop the car off in the second city. Because cars are dropped off in both cities, the company wants to know the following: will a sufficient number of cars end up in each city to satisfy the demand for cars in that city? If not, how many cars must the company transport from Nottingham to Leicester or viceversa? The answers to these questions will help the company figure its expected cost.

Analyzing the historical records, it is determined that 60% of the cars rented in Nottingham are returning to Nottingham, whereas 40% end up in Leicester. Of the cars rented from Leicester office, 70% are returned to Leicester, whereas 30% end up in Nottingham.

1. Develop a model of the system for the behavior of the number of cars on each city at the end of day *n*.

2. Classify all the elements of this model.

3. Write a function in R to model and solve the system. A possible prototype is:

   car_dealer_model = function (*nott*, *leic*, *days*)

   This function must also plot the behavior of both distributorships during a number of days *days*.

4. How many cars should the offices at Nottingham and Leicester need to have to ensure that the company does not have to transport cars between cities at the end of the day to meet demand? (**Note:** This scenario is called “in equilibrium” in the literature.)

5. Study the behavior of each distributorship in these four different cases as shown in Table 4. Considering the behavior of the distributorships what can be said about the system's stability?

Table 4. Initial cars in each dealership populations in four different cases.

	Nottingham	Leicester
Case 1	7000	0
Case 2	3000	4000
Case 3	5000	2000
Case 4	2000	5000
Case 5	0	7000

E. Model 5: Human vs Aliens

The year is 2500. Twenty five years ago, the Earth was invaded by aliens. A war, in which humans and aliens exterminated each other systematically, ensued. Right now, only two small colonies remain: one alien colony and one human colony.

Both humans and aliens are competing for survival and control of the planet and the leaders of both colonies need to decide if they should start the second wave of the war.

Suppose that in the absence of war against each other, each specie exhibits unconstrained growth. In this case, the change in population during an interval of time (for example, one month) is proportional to the population size at the beginning of the interval.

1. Create a model for the case in which no war is taking place between species. Assume constant and unknown growth rates for humans (k_1) and aliens (k_2) respectively.
    a. Classify all the elements of the model.
    b. Create a function in R that solves this model.
    c. If the growth rate for humans was $k_1= 0.2$ and for aliens was $k_2= 0.3$:
        i. Plot the changes of both populations in the same figure for the first 3 months if the human colony currently has 150 humans and the alien colony has 70.
        ii. What is the situation after 11 months?
        iii. What is the situation after 25 months?
2. Imagine that there is to be another war. Write the model for how both colonies will change. Assume constant and unknown killing rates for a human being killed by an alien when there is an encounter (k_3) and for an alien being killed by a human when there is an encounter (k_4).
    a. Classify all elements of this model.
    b. Create a function in R that solves it.
    c. If the growth rates for and aliens remains that same that in 1., and if the killing rates that were observed in the first war are: **0.1%** for humans being killed by aliens and **0.2%** in the opposite case, which

population values would ensure that both populations remain constant, with the colonies remaining at war forever?

3. Study the behavior of the colonies in these three different cases as shown in Table 5.

Table 5. Initial Human and Alien populations in three different cases.

	Humans	Aliens
Case 1	151	199
Case 2	149	201
Case 3	10	10

   a. Write a function that calculates and plots the behavior of the populations in each of the cases in the first 25 months at intervals of 1 month.
   b. What is the situation in all of those 3 cases by month 30?
   c. Which scenario would the human leader prefer?
   d. Which scenario would the alien leader prefer?

F. Model Specification

For the following exercises, identify problems worth studying and list the variables that affect the behavior you have identified. Which variables are necessary? Which variables can be neglected completely? Which constants would you need? Identify data you would need to collect.

1. A botanist is interested in studying the shapes of leaves and the forces that mold them. She clips some leaves from the bottom of a white oak tree and finds the leaves to be rather broad and not very deeply indented. When shoe goes to the top of the tree, she finds very deeply indented leaves with hardly any broad expanse of blade.

2. How can you improve your ability to sign up for the best classes each term?

3. A transportation company is considering transporting people between skyscrapers in New York City via helicopter. You are hired as a consultant to determine the number of helicopters needed.

4. When should a person replace their vehicle?