1.证：

下面证明$f(n)=\sum_{i=1}^{n} i^2=\frac{1}{6}n(n+1)(2n+1)$,

当$n=1$时, $f(1)=1$,

假设$n=k$时, $f(n)=\frac{1}{6}n(n+1)(2n+1)$成立,

当$n=k+1$时,

$$\begin{aligned} f(n+1) &= f(n)+(n+1)^2 \\ &= \frac{1}{6}n(n+1)(2n+1)+(n+1)^2 \\ &= \frac{1}{6}(n+1)(n+1+1)(2(n+1)+1) \end{aligned}$$

从而$f(n)=\frac{1}{6}n(n+1)(2n+1)$

故$f(n)=\Theta(n^3)$

2.解：

(1) $T(n)=O(n)$

即证$\exists c,n_0>0,s.t.$当$n>n_0,T(n)\le cn$

假设$n<k$时均成立,

当$n=k$时,

$$T(k)=T\left(\frac{k}{2}\right)+k\le c\frac{k}{2}+k=\left(\frac{c}{2}+1\right)k\le ck,\ c\ge 2,$$

即为$T(n)=O(n)$

$T(n)=\Omega(n)$

即证$\exists c,n_0>0,s.t.$当$n>n_0,T(n)\ge cn$

假设$n<k$时均成立,

当$n=k$时,

$$T(k)=T\left(\frac{k}{2}\right)+k\ge c\frac{k}{2}+k=\left(\frac{c}{2}+1\right)k\ge ck,\ c\le 2,$$

即为$T(n)=\Omega(n)$

(2) $T(n)=O(n\log^2 n)$

即证$\exists c,n_0>0,s.t.$当$n>n_0,T(n)\le cn\log^2 n$

假设$n<k$时均成立,

当$n=k$时,

$$
\begin{aligned}
T(k) = 2T\left(\frac{k}{2}\right) + k\log k &\le c\frac{k}{2}\log^2\frac{k}{2} + k\log k \\
&= ck\log^2 k + (1-2c\log 2)n\log n + cn\log^2 2 \\
&\le ck\log^2 k,\ c=1, n_0 \ge 10,
\end{aligned}
$$

即为$T(n) = O(n\log^2 n)$

$T(n) = \Omega(n\log^2 n)$

即证$\exists c, n_0 > 0, s.t.$当$n > n_0, T(n) \ge cn\log^2 n$

假设$n < k$时均成立,

当$n = k$时,

$$
\begin{aligned}
T(k) = 2T\left(\frac{k}{2}\right) + k\log^2 k &\ge c\frac{k}{2}\log^2\frac{k}{2} + k\log k \\
&= ck\log^2 k + (1-2c\log 2)n\log n + cn\log^2 2 \\
&\ge ck\log^2 k,\ c=\frac{1}{2}, n_0 \ge 5,
\end{aligned}
$$

即为$T(n) = \Omega(n\log^2 n)$

3.证:

对于$\forall x > -1, x > \ln(x+1)$,

从而

$$
\begin{aligned}
1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} &\ge \ln\left(\frac{2}{1}\right) + \ln\left(\frac{3}{2}\right) + ... + \ln\left(\frac{n+1}{n}\right) \\
&= \ln\left(\frac{2}{1}\cdot\frac{3}{2}\cdot ... \cdot\frac{n+1}{n}\right) \\
&= \ln(n+1)
\end{aligned}
$$

4.解:

$T(n) = O(n^2)$

即证$\exists c, n_0 > 0, s.t.$当$n > n_0, T(n) \le cn^2$

假设$n < k$时均成立,

当$n = k$时,

$$
T(k) = 2T\left(\frac{k}{2}\right) + k^2 \le c\left(\frac{k}{2}\right)^2 + k^2 = \left(\frac{c}{4}+1\right)k^2 \le ck^2,\ c \ge \frac{4}{3},
$$

即为$T(n) = O(n^2)$

$T(n) = \Omega(n^2)$

即证$\exists c, n_0 > 0, s.t.$当$n > n_0, T(n) \ge cn^2$

假设$n < k$时均成立,

当$n = k$时,

$$T(k)=2T\left(\frac{k}{2}\right)+k^2\geq c\left(\frac{k}{2}\right)^2+k^2=\left(\frac{c}{4}+1\right)k^2\geq ck^2,\ c\leq\frac{4}{3},$$

即为$T(n)=\Omega(n^2)$

5.解:

$T(n)=\Omega(n\log n)$

即证$\exists m,n_0>0,s.t.$当$n>n_0,T(n)\geq mn\log n$

假设$n<k$时均成立,

当$n=k$时,

$$\begin{aligned}T(k)&=T\left(\frac{k}{3}\right)+T\left(\frac{2k}{3}\right)+ck\\&\geq mc\frac{k}{3}\log\frac{k}{3}+mc\frac{2k}{3}\log\frac{2k}{3}+ck\\&=cmk\log k+\left(\frac{2}{3}\log 2-\log 3\right)cmk+ck\end{aligned}$$

当$c\leq 1$, 取$m<\frac{1}{3\log 3-2\log 2}$, 即有$T(k)\geq k\log k$,

当$c>1$, 取$m>\frac{1}{3\log 3-2\log 2}$, $n_0=10$, 即有$T(k)\geq k\log k$,

即为$T(n)=\Omega(n\log n)$