A Century of Generalized Momentum;
From Flexible Asset Allocations (FAA) to Elastic Asset Allocation (EAA)

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Abstract

This paper follows Keller (2012), which introduced the Flexible Asset Allocation (FAA) concept. FAA is based on a weighted ranking score of historical asset returns (R), volatilities (V), and correlations to an equal weighted index (C). We call this "generalized momentum" since we assume persistence in the short-term, not only for R, but also for V and C. Portfolios were formed monthly from a specified quantile of assets with the highest combined score.

In this paper we generalize FAA, starting from a tactical version of Modern Portfolio Theory (MPT) proposed in Keller (2013). Instead of choosing assets in the portfolio by a weighted ordinal rank on R, V, and C as in FAA, our new methodology – called *Elastic Asset Allocation* (EAA) – uses a geometrical weighted average of the historical returns, volatilities and correlations, using elasticities as weights.

In order to avoid datasnooping (or curvefitting), we optimize the EAA model exclusively during a 50-year in-sample period (IS) from 1914 and apply these optimal IS parameters to test the model during an out-of-sample (OS) period from 1964-2014. The EAA model demonstrates impressive risk-adjusted and absolute OS performance over an equal weighted index for a variety of global asset universes.

Keywords: Tactical Asset Allocation, Momentum, Elasticities, Markowitz, MPT, minimum variance, maximum diversification, Sharpe, EW, smart beta

JEL Classification: C00, C10, G00, G11

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1. Introduction

Keller (2012) introduced a Flexible Asset Allocation (FAA) model, where an equally weighted portfolio is constructed monthly from a universe of global asset classes based on relative rankings on rolling estimates of return, volatility and correlation. Subsequently, Keller (2013, 2014) introduced a Modern Asset Allocation (MAA) model, which was consistent with the tactical nature of FAA, but also incorporated the Single Index Model (SIM) of Elton (1976) to approximate the mean-variance efficiency prescribed by Modern Portfolio Theory (MPT, see Markowitz, 1952).

The common ingredient for both FAA and MAA was the concept of *generalized momentum*, which loosely describes the persistence in returns (R), but also in volatilities (V) and correlations (C), across assets over short horizons up to one year. This stands in contrast to the typical implementation of MPT, which uses parameters estimated from long-term asset returns for optimization (see eg. DeMiguel, 2007).

Keller (2012, 2013, 2014) demonstrated that a "tactical" approach was successful in terms of risk/return in several backtests for various global, multi-asset universes consisting of ETFs and/or index funds. In each case, tactical decisions were primarily informed by the factor of relative price momentum (see Jegadeesh, 1993, Blitz, 2007, Faber, 2010, Asness, 2012), which we have also extended to account for the observed persistence of volatility and correlation over intermediate horizons. Both FAA and MAA also incorporated a Crash Protection filter based on absolute (or time-series) momentum (see Moskowitz, 2011, and Antonacci, 2011 and 2013), and assumed long-only allocations.

Testing for FAA and MAA was conducted using daily data, which facilitated daily level granularity for observing the risk character of the strategy. Unfortunately, daily data for most indexes is only available for recent decades which, while providing several notable cyclical regimes and volatility clusters, were dominated by just one, steadily declining interest rate and inflation regime.

In this paper, we employ a longer dataset of monthly returns, spanning over a century, and therefore covering several structural regimes of growth and inflation. Consistent with best practices in quantitative system development, we investigated the EAA model in two steps. First, we observed the performance of the model exclusively during a 50-year in-sample period (IS) from 1914-1964, in order to discover the parameter combination that optimized performance. Next we applied the optimal parameters from the IS period to test the performance of the model during an out-of-sample (OS) period from 1964-2014. As a result, we were able to observe the performance of the model during several cyclical and secular equity bull and bear market periods, as well as periods of rising and falling economic growth, interest rates, and inflation.

This paper also generalizes the FAA model by moving from ordinal ranking to cardinal elasticities. The result is a simple geometric analog of FAA, which includes some basic MAA models (like Naïve Risk Parity and Equal Weight) as special cases and others (like Maximum Diversification) by proxy. We arrive at elasticities by using a geometrical form of the optimal momentum score as a function of return, volatility, and index correlation. As index we use the equal weighted universe.

As with many Global Tactical Asset Allocation (GTAA) methods, including FAA and MAA, EAA provides for restricting the optimal portfolio to a fraction of the assets in the universe with the best scores. Further, while many GTAA strategies allocate to top assets in equal weight, our proposed EAA allocates asset weights in proportion to cardinal asset scores. A long-only assumption provides the necessary weight normalization.

Consistent with FAA and MAA, EAA also utilizes so-called Crash Protection (CP) based on the concept of absolute momentum . In EAA we replace a portion of the optimal portfolio with cash depending on the fraction of assets in the universe with non-positive returns.

2. From FAA to EAA: the Elastic Asset Allocation (EAA) model

The traditional mean-variance literature asserts that, all things equal, investors will prefer the maximum amount of return for the minimum amount of risk. In a mean-variance context, risk is measured as the volatility of the portfolio. However, Markowitz (1952) demonstrated that the volatility of a portfolio is a function of assets' individual volatilities, as well as their correlations with the portfolio itself.

We can therefore state that, all things equal, investors will prefer to own assets in a portfolio which deliver higher returns, with lower volatility, and lower correlations with the portfolio. FAA incorporated these preferences by ranking assets on each of these three dimensions and weighting the ranks to emphasize returns while de-emphasizing volatility and correlations, using a linear weighting scheme.

The motivation for EAA is to generalize the FAA model by replacing ordinal ranking scores on each of the three portfolio estimates, returns ($r_{i,}$) volatilities (v_i), and correlations (c_i), with cardinal scores based on non-negative elasticities wR, wV and wC. Instead of computing a final score for each asset as a weighted average of ordinal ranks, EAA portfolio weights are derived by computing a geometrically weighted score z_i for each asset i, on the same attributes. We will call this $z_{i,}$ the (generalized) *momentum score* for asset i. The main formula for z_i, for asset i=1,..,N in the universe, is:

$$\textbf{(1)} \quad \mathbf{w_i \sim z_i = (r_i)^{wR} \,.\, (1\text{-}c_i)^{wC} \;/\; (v_i)^{wV}}, \qquad \text{for } r_i > 0\text{, else } w_i = z_i = 0\text{, for } i=1..N$$

where "~"stands for "proportional to", and

- z_i is the generalized momentum score for asset i
- w_i is the optimal budget share for asset i, with w_i>=0 and $\sum w_j$=1 (long only)
- r_i is the return for asset i
- v_i is the volatility for asset i
- c_i is the correlation to the index (EW universe) for asset i

So, the non-negative *elasticities* **wR, wV, wC** appear as exponents[2] for r_i, v_i and (1-c_i), such that the share w_i of asset i (=1,..,N) is larger, the larger is the return r_i and the smaller is the volatility v_i and/or correlation to the index c_i. For all assets with non-positive returns (r_i<=0) the momentum score z_i and therefore the weight w_i is zero.

The optimal (non-negative) weights w_i are proportional to the momentum score z_i. We can normalize w_i by using the long-only restriction so that

$$\textbf{(2)} \quad \mathbf{\sum w_j = 1, \quad for\ j=1,.., N}$$

[2] See en.wikipedia.org/wiki/Exponentiation for some simple rules like $x^a.x^b = x^{a+b}$ and $(x^a)^b = x^{ab}$, $(xy)^b = x^by^b$, $x^{1/2}$=sqrt(x), etc.

Notice that the proportionality with z_i implies that the weights w_i are not necessarily equal, as was the case for FAA. Note also that, as a function of our absolute momentum filter (r_i>0), and our transform of the correlation metric c_i to (1- c_i), all attribute scores r_i, v_i and (1-c_i) will be non-negative by construction. This is critical, as the fractional exponent of a negative value is undefined.

The geometric weights wR, wV, and wC are called *elasticities* since they reflect the relative change in w_i when r_i, v_i and (1-c_i) change by a small relative amount, say 1%. For example, an elasticity wR=50% implies that w_i changes approximately 5% when r_i changes by 10%. These elasticities are in fact the geometric weights for these attributes in eq. (1) and are parameters to be specified in advance through optimization. We investigate the optimal settings for these elasticities in Section 5, using IS data, and then demonstrate how the optimal settings hold in the OS period in Section 6.

As with FAA, we assume that only a specified top quantile (NTop) of the N assets in the universe are included in the optimal portfolio. These are the best scoring assets on the generalized momentum score z_i>0. NTop is a fraction of N, the size of the universe, so NTop = TopX*N. Where TopX applied to N results in a non-integer, we will round down. For example, if the universe has 11 assets and TopX=50% we will hold only 5 assets in the portfolio with the best z_i>0. As TopX is a parameter to be specified, we will estimate the optimal TopX rule in-sample, and test it out-of-sample, just like the elasticities. The fraction TopX might also be a function of N, such that the fraction TopX decreases for larger N.

Lastly, as noted above, EAA imposes a form of Crash Protection (CP) in the form of an absolute momentum filter. To apply the filter, we replace a fraction w_{CP} of the optimal portfolio with cash, where w_{CP} is equal to the share of all assets N with a non-positive return r_i. When the CP and TopX rules are both active, eq. (2) becomes

$$\textbf{(3)} \qquad \sum\nolimits^{NTop} \mathbf{w_j = (1 - w_{CP})},$$

where the summation runs from j=1 to NTop and w_{CP} is the CP fraction of assets with non-positive returns r_i.

An example should crystallize the concept: when we have N=11 assets in our universe, TopX=50%, and 7 out of 11 assets have r_i<=0, then 64% (=w_{CP}=7/11) of the final portfolio will be allocated to cash or a cash proxy fund. The remaining 36% will be allocated to the best scoring NTop=4 (<= TopX*N=5) assets in proportion to their momentum score, z_i>0. Notice that only 4 (instead of 5) assets with ri>0 were available for allocation of capital due to the TopX (50%) rule.

3. EAA: some special cases

We can make the EAA model congruent with the FAA model by rewriting eq. (1) without loss of generality as**:**

$$\textbf{(4)} \quad \mathbf{w_i \sim z_i = \{ (r_i)^{wR} . (1\text{-}c_i)^{wC} / (v_i)^{wV} \}^{wS}}, \quad \text{for } r_i > 0, \text{ else } w_i = z_i = 0,$$

where we use wR=1 for normalization, such that wV and wC are relative to wR=1. A similar normalization was applied to the FAA model (see Keller, 2012).

The general exponent wS is a (non-negative) "scaling" elasticity which takes over the role of wR (which is therefore fixed at one: wR=1). When this non-negative parameter wS goes to zero, the EAA allocation of eq. (3) goes to equal weights (EW). Then EAA will produce an equally weighted portfolio of the TopX assets with the highest geometrical mean score on RVC in zi. Computationally, we arrive at this solution by choosing wS=0 and adding a small positive epsilon (eg. e=1E-6) to wS in eq. (4)[3]. Subsequently, we will refer to this FAA-like variant of EAA by wS=0.

Notice that wS is not only interesting when wS=0. When wS becomes very large (wS goes to plus infinity), only the best scoring asset on zi (the geometrical mean of RVC in eq. 4) will get a positive asset weight w_i and all other weights (apart from cash) will tend to zero. So wS also determines the degree of concentration of our optimal portfolio. In the empirical models below we will always use eq. (4) for the EAA model and therefore use wS as scaling weight instead of wR, which we fix at 1.

Notice that the EAA formula in eq. (4) (and eq. 1) is invariant for changes in units for both r_i and v_i, since constants are absorbed by the normalization rule of eq. (2). So it does not matter for w_i if we express r_i and v_i as eg. annual or monthly returns or volatilities. This does not hold for the correlation parameter c_i, which is unit-free and bounded by $1<=c_i<=1$.

Besides its congruence with FAA when wS=0, from eqs. (1) and (4) we also show that EAA can be used to express some special cases of MPT exactly or by proxy, such as **EW, Risk Parity and Maximum Diversification (**see Choueifaty et. al., 2011 and Keller, 2014, and Appendix A). The most trivial is the **Equal Weight** (EW) model

(5) $\mathbf{w_i = 1/N}$ for i=1,..N,

when wR=wV=wC=0 in eq. (1) or wS=0 in eq. (4). This 1/N (or EW) solution can be combined (or not) with the CP ($r_i<=0$) and TopX (best z_i) rules to limit the number of assets in the EW portfolio, as in FAA (see above). This holds also for all subsequent special cases. In fact, both the CP rule and the TopX rule can be applied to any asset allocation model, such as eg. the Risk Parity (RP) model discussed below.

When wV=1 (and wR=wC=0 in eq. (1)), we arrive at

(6) $\mathbf{w_i \sim 1/v_i}$

which is consistent with the (naïve) **Risk Parity model** where asset weights w_i are inversely proportional to their **volatilities** $\mathbf{v_i}$. This is also a special case of MAA (and therefore of MPT) when all Sharpe ratios r_i/v_i are assumed to be equal and the market (index) volatility v (see Appendix A) is assumed to be zero (see Keller, 2014). All assets are included. Of course, we can limit the number of assets by applying the CP and/or TopX rules.

When wV=2 (and wR=wC=0 in eq. (1)) we arrive for the EAA model at a similar model as naïve Risk Parity, but where assets are weighted as a function of the inverse of their **variances** (v_i^2) instead of volatilities:

(7) $\mathbf{w_i \sim 1/v_i^2}$

[3] By using this very small epsilon we accomplish almost identical z_i scores (and therefore an EW allocation), while we still are able to distinguish between the scores z_i for TopX selection.

This model also corresponds to a special case of the MAA (and therefore of MPT) model when we assume that the market volatility v is zero and all returns r_i are equal (see Keller, 2014).

When wR=1, wV=2 and wC=0 in eq. (1), we arrive at a more general EAA model with **return momentum** (and asset variance):

$$\textbf{(8)} \quad w_i \sim r_i / v_i^2$$

This model also corresponds to a special case of the MAA (and therefore of MPT) model when we only assume that the market volatility v is zero (zie Keller, 2014). All assets are included in eqs. (7) and (8) but we can limit these again by applying our CP/TopX rules.

We can show (see Appendix A) that the more general EAA model with wC>0 is a proxy for the MAA (and therefore MPT) model where the market volatility is assumed to be positive (v>0). In particular, the EAA model of eq. (1) with wR=0, and wV=1

$$\textbf{(9)} \quad w_i \sim (1 - c_i)^{wC} \; / \; v_i$$

turns out to be a (good) proxy for the MAA (and therefore MPT) model for **Maximum Diversification** when we assume constant Sharpe ratio's (see also Appendix A). Again all assets are included unless our CP/TopX rules are applied.

From the above examples it is clear that the EAA model is able to encompass all kinds of special and not so-special cases of the MAA (and therefore MPT) model exactly or by proxy. This gives a justification for the chosen form of eq. (1).

4. Testing the EAA model in- and out-of-sample

4.1 The In-sample (IS) and Out-of-sample (OS) method

In this section we will test the EAA model (as expressed in eq. 4), using monthly data for over a century (Apr 1914-Aug 2014). We have adopted best practices by dividing historical data into two independent sample periods: an in-sample (IS) period ranging from (end of) April 1914 through March 1964; and an out-of-sample (OS) period ranging from April 1964 to Aug 2014. See also Bailey (2013) for a discussion of this IS/OS practice.

Many financial models are both specified and tested on the same sample data. However, this introduces a material probability of over-specifying the model to fit historical data, where the parameters do not generalize well out of sample. In constructing our testing using two independent sample periods, we are able to optimize model parameters on one sample period, and apply them without bias to the second sample period.

We would note that, due to the availability of more than a century of historical data on a variety of indexes, we can feel confident that both the IS and OS periods cover a wide range of cyclical and secular equity bull and bear market periods, as well as periods of rising and falling economic growth, interest rates and inflation. We might also have chosen the last 50 years as IS (and the first as OS), but in view of the dominant bond regime (with decreasing interest rates for most of the time) in the recent decades we prefer the more heterogenous period of 1914-1964 (with high and low, rising and falling rates and the heaviest stockmarket crash over a century), as our in-sample (or learning) period.

In addition, because of the large sample size, and heterogeneous nature of the IS periods, we have sufficient data density to optimize across a wide range of parameter combinations, without making use of OS data. As such, the probability of 'datasnooping' or 'curvefitting' of parameters is remote.

4.2 Universe and data

For testing purposes, we have assembled several global, multi-asset universes, ranging in size from 7 to 38 asset classes. The asset classes represent a mixture of US, international and emerging market stocks, bonds (government and corporate) and alternative (REIT and commodities) indexes. Index data is drawn from various sources until the onset of tradeable index mutual funds and Exchange Traded Funds in recent decades. All indexes are of monthly frequency based on month-end total return values, including geometrically linked dividends.

The three universes represent a small global, multi-asset N=7 universe (SP500, EAFE, EEM, US Tech, Japan Topix, and two bonds: US Gov10y, and US HighYield); a N=15 US sector universe (10 Fama/French US sectors plus five US bonds: Gov10y, Gov30y, US Muni, US Corp, and US HighYield), and; a large global multi-asset N=38 universe, which includes all of the above plus Small-cap US equities, GSCI, Gold, Foreign bonds, US TIPS, US Composite REITs, US Mortgage REITs, FTSE US 1000/US 1500/Global ex US/Developed/EM, JapanGov10y, Dow Util/Transport/Industry, FX-1x/2x, and Timber.

For the N=7 and N=15 universes, most data was available from 1913, facilitating testing from Apr 1914 to accommodate a one year lookback period. For the large universe (N=38) about half of the assets are available from 1970 or later. The first two smaller universes were chosen out of the large N=38 universe as a global (N=7) and a US (N=15) multi-asset universe with as much available data as possible.

The monthly data for these three universes were from various sources: we used monthly TR index data mainly from MSCI, Fama/French, Ibbotson, Barclays and Global Financial Data, extended with closest analog index and ETF data from Yahoo and Bloomberg for the more recent years.

4.3 Model specification

We propose that the EAA model (as the FAA and MAA model) is effective for *tactical* asset allocation. Therefore, we will rebalance the portfolio at the end of each month using a short-term rolling historical window of observed values for returns, volatilities and correlations. In other words, the model asserts that returns, volatility and correlation exhibit short-term positive autocorrelation, such that the most recent observed value is a good estimate for the next month.

Many studies of MPT optimality rely on lookback windows of 3 to 5 years or even longer for statistical significance in the estimation of parameters. However, the literature on momentum demonstrates that future returns are a positive function of historical returns only over periods of about 12 months or less. Over lookback windows greater than 12 months, and especially over lookback horizons of 3 to 5 years, historical returns negatively predict future returns (see Asness, 2012). That is, asset returns tend to mean-revert rather than persist when observed over longer horizons.

In view of the well-established optimal range of momentum for estimation, we will investigate combinations of horizons from 1, 3, 6, 9 and/or 12 months for monthly portfolio formation. This is consistent with both Faber (2010) and Hurst (2012). By examining return momentum over multiple horizons, we are less likely to optimize on a spurious periodicity in the historical data, and more likely to capture price persistence which exists at multiple fractal scales. We will optimize these lookback combinations for historical returns in the IS period, and test these optimality conditions in the OS period.

For v_i we will simply use the rolling standard deviations v_i of returns. Correlation parameters c_i for each asset are calculated against the returns of the EW universe index over the past 12 months. Notice that these vi and ci estimates are based on just 12 return observations (e.g. ultimo Jan 2012 through ultimo Dec 2012) for these second-order statistics, which means the estimator is much sparser than what is possible with daily data, which yields about 21 observations per month. There is a tradeoff however, between data density and the time decay of information, and our testing confirms that 12 monthly observations are enough to generate meaningful estimates for our purposes.

4.4 The optimization metric

The EAA model has many parameters to specify, including:

- The size and asset compositions of the testing universes
- The TopX rule for the fraction of assets to be included in the optimal monthly portfolio
- The choice of the 'safe' asset for Crash Protection
- The lookback weights for the last twelve months of R, V and C
- The use of excess returns (or not) calculating R
- The normalized EAA elasticities wV, wC
- The scaling weight wS (see eq. 4)
- Transaction costs and maximum leverage

The optimization process generally requires that we examine a wide variety of parameter combinations in terms of how they impact portfolio performance over the test horizon. But how should we quantify 'performance'? Some investors might be risk agnostic, and prefer the parameters which generate the highest average returns over the test horizon. Others may be especially risk conscious, and prefer specifications that produce low volatility or drawdowns. More thoughtful and balanced investors know that over finite investment horizons, risk and return are inextricably linked; these investors would prefer performance measures that normalize returns for the amount of risk taken to generate those returns.

The Sharpe ratio, which captures the excess portfolio returns (in excess of cash) per unit of portfolio volatility is, perhaps, the most well-known risk-adjusted performance metric. We submit that this is a legitimate and unbiased objective function for optimization. However, the Sharpe ratio is vulnerable to misguided assumptions about the return distribution; namely that returns are identically and independently distributed.

In the context of very long horizons of several decades, the volatility (V) adjustment applied in the calculation of the Sharpe ratio may not be a robust estimator for risk. Given the very long horizons we bring to bear with over a century of data, we are more interested in the thick tails of the return distribution, as observed during risky asset crashes such as the Great Depression in 1929, and the Global Financial Crisis of 2008. In this context, we decided to use Maximum Drawdown (D) to capture the risk dimension for optimization.

Notice that the impact of D over time is much more severe than the impact of V. When we have a crash of D=40%, such as was observed in the S&P 500 in 2008, we need a 40%/(1-40%)= 66% rise after the crash to get back to previous highs. Given that mean returns for the S&P 500 are a small fraction of 66%, such a drop in portfolio value has a disproportionately large impact on long-term returns. That said, D does not stabilize over time, as V does; in fact, the chance of a big drawdown increases over time (see also MagdonIsmail, 2004). On the other hand, any multi-asset portfolio with a maximum drawdown of around 10% over 100 years (as the N=38 defensive portfolio in the next sections) is clearly exceptional.

4.5 The Calmar ratio and its target return

With D as risk, we will use the so-called Calmar ratio of return R above a threshold (or target) return t divided by the max drawdown D. So our optimization metric is CRt:

$$CRt = (R-t)/D \qquad \textbf{(10)}$$

We take a target return of t=5% as default, which we will call CR5. A 5% threshold was chosen because it comes close to the preferences of an investor with an average risk aversion (gamma=3, see Ang 2012). A more offensive investor might require a threshold of 10%, so we will provide results for both CR5 and CR10 in testing, but focus on the more neutral CR5 as our preferred optimization metric. In recognition of the merits of the traditional Sharpe ratio, we also present the risk-free (target=rf) SR, SR5 (target=5%), and SR10 (target=10%) in most test results. Readers should also find comfort in the fact that parameters which optimize on CR also provide for high SRs.

We have optimized all above parameters exclusively on IS for our three universes (N=7, 15 and 38), over all possible parameter values within certain dimensions. We will focus here mostly on the best parameters when optimizing for these three universes over IS using CR5 as our preferred optimization metric.

4.6 The EAA recipe

Now we will discuss the outcome of our estimation of the EAA parameters for our three universes in the form of a simple EAA recipe. Equation (4) gives us the monthly scores z_i (i=1..N) and therefore the shares w_i for the optimal portfolio, given the elasticities wS, wV and wC. We will call z_i in eq. (4) the "generalized momentum score", since it summarizes the combined momentum of r_i (returns), v_i (volatilities) and c_i (correlation to the EW index). These momentums are computed over a maximum lookback of one year per asset i (=1,..N) as follows:

- r_i is computed as the average total *excess* return over the last 1, 3, 6 and 12 months (not 9 months) where excess return is defined relative to the (13 week) TBill yield.
- v_i is computed as the standard deviation of the last 12 monthly total nominal (not excess) returns
- c_i is computed as the correlation between the last 12 monthly total nominal returns and the index
- the index is computed as the equal weighted (EW:1/N) monthly total nominal return over all N assets

This recipe is motivated by a 'rough' optimization of CR5 over IS, and by a desire to keep the process as simple as possible. It turns out to be marginally optimal to use excess returns in r_i

but not in v_i and c_i. We also skipped the 9-month lookback horizon in calculating r_i as it was not helpful IS. Also notice that we only use twelve observations (one year) for vi and ci, consistent with the shorter-term nature of generalized momentum. This estimation horizon turns out to be long enough for reliable estimates. Finally, we added a very small positive number e (=1E-6) to wS in eq. (4). This implies that when wS=0 we arrive at an EW portfolio including the best TopX assets on z_i (the generalized momentum score as a function of wV and wC only), much like FAA (see also Section 3).

We also found that the TopX fraction of the N assets in each universe to be included in the EAA allocation to be a decreasing function on N (from N=7, 15, to 38) for the three universes over IS. In order to limit the number of optimization dimensions, we set the number of assets to be included, NTop, as a simple function[4] of Sqrt(N) with a maximum TopX of 50%, such that NTop=3 when N=7, NTop=5 when N=15 and NTop=8 when N=38. Notice that when N changes over time due to data (un)availability (as is the case with the large N=38 universe), the NTop number changes with it. So our universes can expand or contract over time with the best NTop number of portfolio assets following N over time.

We used a (symbolic, one-way) transaction tax of 0.1% and no leverage. This cost is realistic only for the most recent OS years. Finally, for our 'safe asset' proxy, we tested the US 3-month Treasury Bill, as well as US bonds like Gov10y, Gov30y, Muni, Corp, etc. For all universes, US Gov10y was a clear winner, so we will use that as the proxy for cash in all tests. Notice that our IS period included periods with increasing and decreasing interest rates, periods with near zero or very high interest rates, several inflationary regimes, stock crashes, etc. To reiterate, when a fraction x of the universe has non-positive excess returns r_i, the portfolio will hold a fraction x of the optimal portfolio to the 'safe asset' per our Crash Protection (CP) rule. See also eq. (3).

4.7 The optimal elasticities

With respect for the elasticities wV and wC, we found over IS that the volatility (V) term in eq. (4) had little impact on strategy performance, given wS and wC. This is probably related to the fact that ci and vi are collinear, such that most bonds have low vi and low ci. Therefore, and for the sake of simplicity, we set wV=0 for all tests.

This leaves wS and wC to be optimized over IS for each of the three universes. To find the best elasticities wS and wC over IS, we optimized CR5 per universe with these elasticities ranging from 0%, 25%, 50%, 75%, 100%, 150%, and 200%. So we made "sweeps" over 7x7=49 parameter combinations on wS and wC in order to find the combination which maximized CR5.

wS / wC	0%	25%	50%	75%	100%	150%	200%
0%	Scen 1	Scen 2	Scen 3	Scen 4	Scen 5	Scen 6	Scen 7
25%	Scen 8	Scen 9	Scen 10	Scen 11	Scen 12	Scen 13	Scen 14
50%	Scen 15	Scen 16	Scen 17	Scen 18	Scen 19	Scen 20	Scen 21
75%	Scen 22	Scen 23	Scen 24	Scen 25	Scen 26	Scen 27	Scen 28
100%	Scen 29	Scen 30	Scen 31	Scen 32	Scen 33	Scen 34	Scen 35
150%	Scen 36	Scen 37	Scen 38	Scen 39	Scen 40	Scen 41	Scen 42
200%	Scen 43	Scen 44	Scen 45	Scen 46	Scen 47	Scen 48	Scen 49

[4] The actual formula used is NTop = Min (1+Roundup (Sqrt(N)) , Rounddown (N/2))

Recall again that wS=0 gives an equal weight (EW) portfolio (net of our 'safety asset' where CP has been triggered) while wS=2 produces a more concentrated portfolio. With wS=2, the assets with the highest score on zi receive a majority of the portfolio weight budget. So while wC governs the "hedging"[5] by emphasizing lowly-correlated assets, wS governs the degree of concentration of the portfolio weights (emphasize on the assets with the biggest general momentum score zi).

So we will make 7x7 sweeps over wS and wC to study all 49 scenarios. We will present the performance of some of these 49 scenarios using several performance metrics over IS (1914-1964), including:

- R (CAGR net of costs),
- D (Maximum Drawdown),
- V (Annualized Volatility),
- CR5 and CR10 (Calmar Ratio with target 5% and 10% resp.),
- SR2, SR5 and SR10 (Sharpe Ratio with target 2%, 5% and 10% resp.)
- 3y: the proportion of rolling 3 year periods where R of EAA > R of the Equal Weight index of all assets, rebalanced monthly.

Notice that the annual target rate of 2% is consistent with the TBill rate over IS, so SR2 is the traditional Sharpe ratio over IS.

As discussed, the most important performance measurements, in our opinion are R and D, and CR5 in particular for a risk-adjusted performance metric. To summarize these statistics visually, we present a "**Calmar Scatter**" of these 49 scenarios. This scatter is very similar to the traditional R-V scatter but now with Maximum Drawdown (D) on the horizontal axis instead of volatility V, but preserving returns (R) on the vertical axis. Each dot represents the outcome of one of the 49 EAA backtests as described above.

Next we found the points in the Calmar Scatter representing highest R for a given D, or lowest D for a given R. These are the dots on the top-left portion, or the so-called Pareto envelope, of the Calmar Scatter. We can also find these dots by maximizing CRx over many targets x. The result of this procedure is the **"Calmar Frontier"** (CF). This is very similar to the traditional "Efficient Frontier" (see eg. Ang, 2012), but with D replacing V for risk.

Besides D, R and CRx, we also focus on the proportion of 3 year rolling periods where the return on the strategy is greater than the return on the equal weight index because this statistic quantifies the proportion of times when the EAA model deviates negatively vis-à-vis the index for meaningful periods.

Note again that our benchmark (or index) is always the equal weighted portfolio of the full universe (with all N assets, all with weights 1/N), rebalanced monthly. We also show comparable statistics for the commonly cited US S&P 500 total return index.

[5] See eg. http://en.wikipedia.org/wiki/Hedge_(finance)

5. The three universes on IS

5.1 Introduction

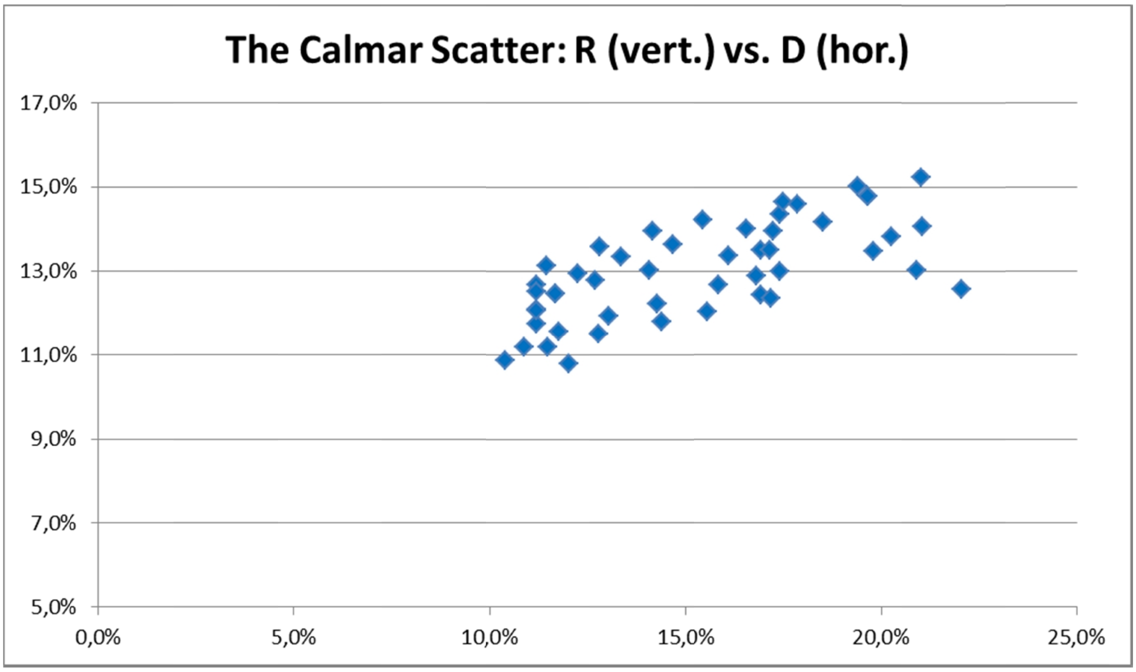
In this section we will optimize the model on IS over wS and wC for each of the three universes, mainly focusing on CR5 as the optimization metric, but with an eye on other criteria like CR10 and various Sharpe Ratio's SRx. We will also examine the Calmar Frontiers in order to find for each universe the most efficient offensive (max R) and defensive (min D) model.

We will conclude this section with an effort to find two pairs of elasticities wS/wC (one pair defensive and one pair offensive) which optimize a model for *all* three universes at the same time. We will call these the *Golden* elasticities and the corresponding EAA models the *Golden* EAA models. These golden models will be then used for the OS tests in the next section.

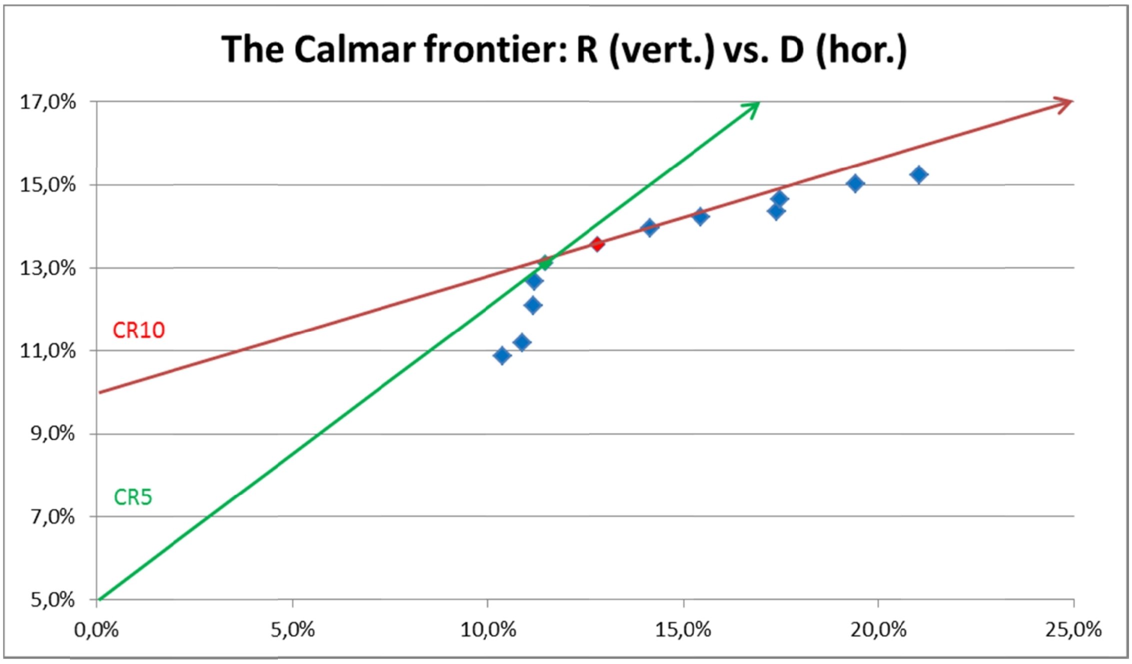
5.2 The small global, multi-asset universe (N=7) on IS

The N=7 universe is a small global, multi-asset universe (SP500, EAFE, EEM, US Tech,Japan Topix, and two bonds: US Gov10y, and US HighYield).

We present all 7x7=49 scenarios on IS by the Calmar Scatter (CS) for our N=7 universe:



On the below diagram the Calmar Frontier (CF) is depicted for the N=7 universe on IS as a subset of the CS dots (12 out of the 49 scenarios on the Pareto envelop), including the CR5 and CR10 line and the two best CR5 (green dot) and CR10 (red dot) scenarios:



We can also display all the 12 (of 49) best dots on the Calmar Frontier for the N=7 universe on IS (with all the performance metrics) in the following table (plus the two "golden scenarios 19 and 45 to be discussed later):

wS	wC	Scen	R	V	D	CR5	CR10	SR2	SR5	SR10	3y>EV
0,00	2,00	7	10,9%	7,2%	10,4%	56,4%	8,3%	122,5%	81,1%	12,0%	67,8%
0,25	2,00	14	11,2%	7,6%	10,9%	56,8%	11,0%	120,3%	81,0%	15,6%	69,3%
0,25	1,00	12	12,1%	8,2%	11,2%	63,5%	18,8%	122,5%	86,2%	25,5%	79,2%
0,00	0,50	3	12,7%	8,1%	11,2%	68,6%	23,9%	131,3%	94,4%	32,9%	83,3%
0,25	0,50	10	13,1%	8,8%	11,5%	70,8%	27,3%	126,6%	92,5%	35,6%	89,5%
0,50	0,50	17	13,6%	9,6%	12,8%	66,9%	27,9%	120,7%	89,4%	37,3%	91,7%
0,75	0,50	24	13,9%	10,3%	14,2%	63,1%	27,8%	115,4%	86,4%	38,1%	92,3%
1,00	0,50	31	14,2%	11,0%	15,5%	59,7%	27,4%	111,4%	84,1%	38,5%	92,3%
0,75	0,25	23	14,4%	11,0%	17,4%	53,7%	25,0%	111,9%	84,8%	39,4%	92,8%
1,00	0,25	30	14,6%	11,6%	17,5%	55,1%	26,5%	108,8%	83,0%	39,9%	94,0%
1,50	0,25	37	15,0%	12,4%	19,4%	51,5%	25,8%	104,9%	80,7%	40,4%	95,3%
2,00	0,25	44	15,2%	12,9%	21,1%	48,6%	24,8%	102,9%	79,5%	40,6%	95,7%
0,50	1,00	19	12,5%	9,0%	11,7%	64,0%	21,1%	116,7%	83,3%	27,5%	79,7%
2,00	0,50	45	14,8%	12,3%	19,7%	49,7%	24,3%	103,8%	79,4%	38,8%	92,2%

As you can see, CR5 is best (CR5=70,8%) on IS for scenario 10 (the green dot in the CF graph) with wS= 25% and wC=50%, resulting in a neutral R=13,1% and D=11,5% on IS. The best CR10 gives the slightly more offensive scenario 17 (the red dot on the CF) with CR10=27,9% and wS=50% and wC=50%, resulting in an R=13,6% and D=12,8%.

Notice that the most offensive scenario 44 (with the highest R=15,2% and D=21,1%) gives also the best 3y (of 95,7%) of EAA over the EW index. This scenario refers to a rather concentrated, minimally hedged portfolio in view of the high value of wS (200%), and low value of wC (25%). The most defensive is scenario 7 (with the lowest D=10,4% and R=10,9%), which is EW (since wS=0%) and very diversified (wC=200%); it also presents the lowest 3y (67,8%). Notice also that there are no scenarios on the CF envelop with the elasticity wC=0. This shows that the "hedging" effect of wC is effective.

It's important to also note that the 'golden' scenarios at the bottom both lie in the top quintile of CR5 for this universe. We will discuss this more below.

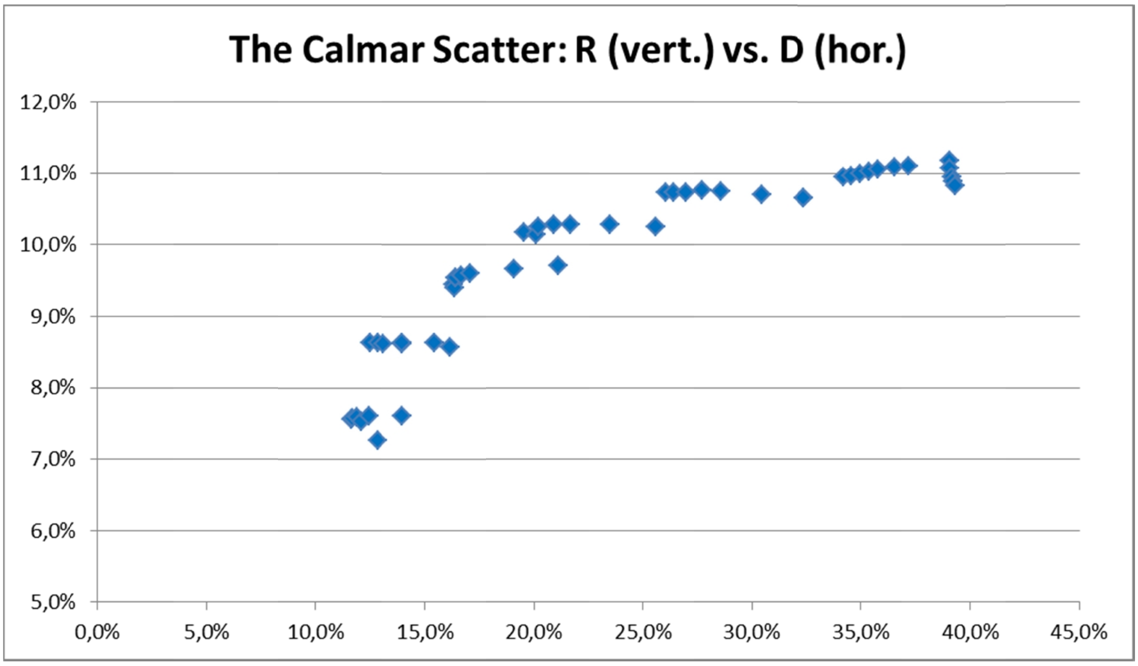
Finally, we will compare the best IS scenario on CR5 and CR10, scenario 10 (wS=25%, wC=50%) and 17 (wS=50%, wC=50%), with the index or benchmark (EW universe) over IS (and SP500).

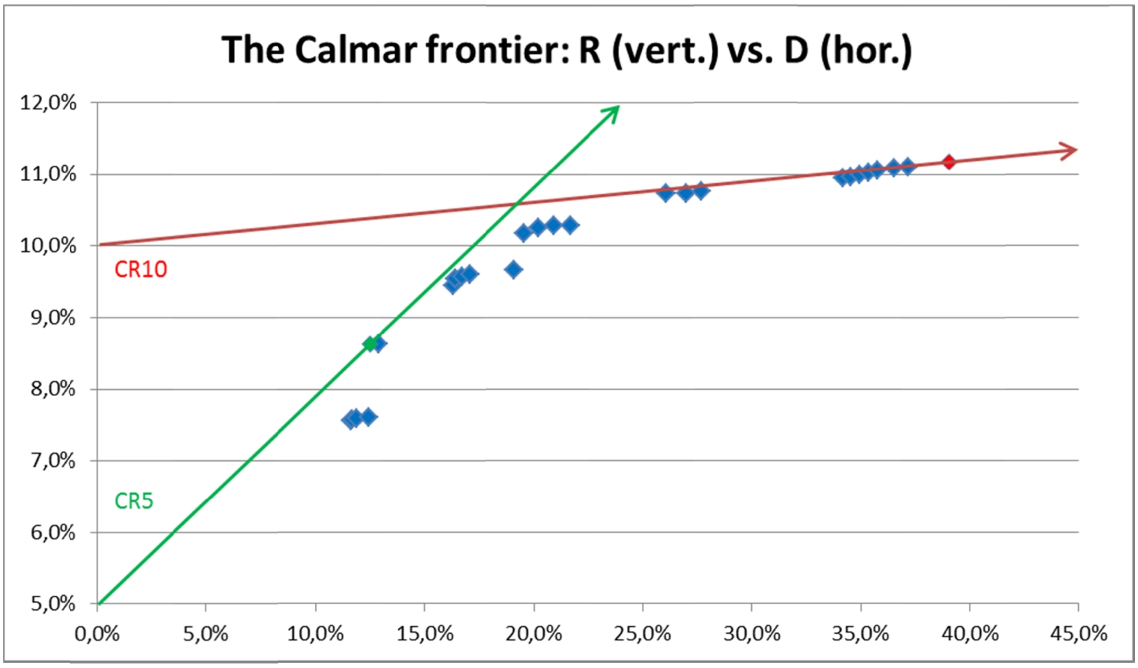
(N=7, IS)	EAA Sc. 10	EAA Sc. 17	EW	SP500
R (CAGR)	13,1%	13,6%	8,9%	10,1%
V (An. Volatility)	8,8%	9,6%	10,1%	21,4%
D (Max DDown)	11,5%	12,8%	54,9%	83,5%
CR5 (Calmar 5%)	70,8%	66,9%	7,1%	6,2%
SR (Sharpe Rf)	126,6%	120,6%	68,3%	38,0%
3y (Roll >=EW)	89,5%	91,7%	50,0%	?
Costs (Annual)	0,4%	0,5%	0,0%	0,0%

Notice that both EAA scenarios compare quite favorably with EW, especially on D, but also on R and V. For comparison we also give the statistics on IS for the SP500, which compares even less favourably with both scenarios on R, V, and D (and various return/risk statistics).

5.3 The US-sector universe (N=15) on IS

The US sector universe consists of the 10 Fama/French US sectors (from 1926) plus five US bond sectors (Gov10y, Gov30y, Muni, Corp, and HighYield). Most bonds are available from 1920, some from 1900. So this is an expanding universe: we start in 1914 with N=2 (Gov10y and Muni) and through N=4 in 1921 grow to N=15 from 1927 through 1964. We first present the Calmar Scatter (49 dots) and Calmar Frontier for N=15 on IS (1914-1964).





Next we show the CF scenario table, which contains all 26 scenarios on the Calmar Frontier (plus the two golden scenarios). The best CR5 is scenario 20, the best CR10 is scenario 43, which is also the most offensive (max R). The most defensive (min D) is scenario 35. The highest 3y is found with scenarios 44 (3y=63,0%) and 43(3y=62,3%).

We observe that the most offensive scenario 43 is maximally concentrated (wS=200%) and has no-hedging (wC=0) while the opposite is true for the most defensive portfolio (wC=200%). This is intuitive.

wS	wC	Scen	R	V	D	CR5	CR10	SR2	SR5	SR10	3y>EV
1,00	2,00	35	7,6%	6,6%	11,6%	22,0%	-21,0%	83,6%	38,5%	-36,74%	40,7%
0,75	2,00	28	7,6%	6,5%	11,7%	22,0%	-20,8%	85,9%	39,6%	-37,48%	40,3%
0,50	2,00	21	7,6%	6,4%	11,9%	21,7%	-20,3%	87,6%	40,5%	-37,89%	31,7%
0,25	2,00	14	7,6%	6,4%	12,5%	20,9%	-19,2%	88,3%	41,1%	-37,61%	32,0%
0,50	1,50	20	8,6%	7,1%	12,5%	29,0%	-11,0%	93,4%	51,1%	-19,33%	37,7%
0,25	1,50	13	8,6%	7,0%	12,9%	28,2%	-10,6%	94,4%	51,7%	-19,43%	39,8%
0,25	1,00	12	9,4%	8,4%	16,3%	27,3%	-3,4%	88,9%	53,1%	-6,57%	51,0%
0,50	1,00	19	9,5%	8,5%	16,4%	27,6%	-2,8%	89,1%	53,7%	-5,48%	52,3%
0,75	1,00	26	9,6%	8,6%	16,7%	27,4%	-2,5%	88,5%	53,4%	-4,95%	52,8%
1,00	1,00	33	9,6%	8,7%	17,1%	27,0%	-2,3%	87,5%	53,0%	-4,52%	52,3%
1,50	1,00	40	9,7%	9,0%	19,1%	24,4%	-1,7%	85,3%	51,9%	-3,70%	62,0%
0,25	0,75	11	10,2%	9,0%	19,6%	26,5%	0,9%	91,2%	57,8%	2,05%	59,7%
0,50	0,75	18	10,3%	9,1%	20,2%	26,0%	1,3%	90,9%	57,9%	2,82%	60,2%
0,75	0,75	25	10,3%	9,2%	20,9%	25,2%	1,3%	89,9%	57,3%	3,02%	61,0%
1,00	0,75	32	10,3%	9,4%	21,7%	24,4%	1,3%	88,5%	56,5%	3,06%	61,2%
0,00	0,50	3	10,7%	9,9%	26,1%	22,0%	2,8%	88,0%	57,7%	7,36%	60,3%
0,50	0,50	17	10,7%	10,2%	27,0%	21,3%	2,7%	86,1%	56,5%	7,29%	60,2%
0,75	0,50	24	10,8%	10,3%	27,7%	20,8%	2,7%	85,1%	56,0%	7,40%	60,7%
0,00	0,25	2	11,0%	10,9%	34,2%	17,4%	2,8%	81,9%	54,5%	8,75%	61,2%
0,25	0,25	9	11,0%	11,0%	34,6%	17,3%	2,8%	81,3%	54,1%	8,79%	60,8%
0,50	0,25	16	11,0%	11,2%	35,0%	17,1%	2,8%	80,6%	53,7%	8,91%	61,0%
0,75	0,25	23	11,0%	11,3%	35,4%	17,0%	2,9%	80,0%	53,4%	9,09%	61,8%
1,00	0,25	30	11,1%	11,4%	35,8%	17,0%	3,0%	79,4%	53,2%	9,34%	62,2%
1,50	0,25	37	11,1%	11,7%	36,5%	16,7%	3,0%	77,8%	52,1%	9,36%	62,2%
2,00	0,25	44	11,1%	12,0%	37,2%	16,4%	3,0%	76,0%	51,0%	9,18%	63,0%
2,00	0,00	43	11,2%	12,2%	39,1%	15,8%	3,0%	75,1%	50,5%	9,64%	62,3%
0,50	1,00	19	9,5%	8,5%	16,4%	27,6%	-2,8%	89,1%	53,7%	-5,48%	52,3%
2,00	0,50	45	10,7%	11,2%	32,4%	17,4%	2,0%	77,6%	50,7%	5,85%	70,7%

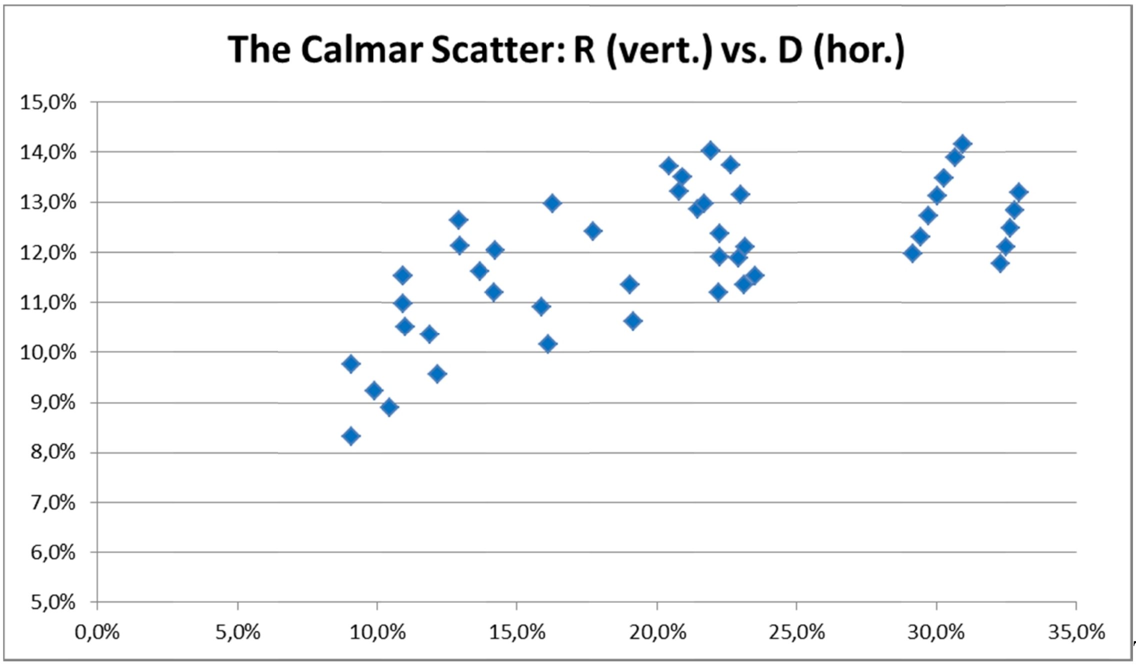
Below we also show the comparison for the best (CR5 an CR10) scenarios (20 and 43 respectively) with EW on IS. Notice that nearly all statistics (R, V, D, CR5, and SR2) are favorable for EAA, except the 3y for CR5, which is somewhat disappointing. In general, we observe that the more defensive the model, the lower the 3y statistic. So we will find higher 3y stats when considering more offensive models, like the offensive scenario 43 on CR10 (with 3y =62,3%).

(N=15, IS)	EAA Sc. 20	EAA Sc. 43	EW	SP500
R (CAGR)	8,6%	11,2%	8,0%	10,1%
V (An. Volatility)	7,1%	12,2%	13,6%	21,4%
D (Max DDown)	12,5%	39,1%	67,6%	83,5%
CR5 (Calmar 5%)	29,0%	15,8%	4,4%	6,2%
SR (Sharpe Rf)	93,3%	75,0%	43,8%	38,0%
3y (Roll >=EW)	37,7%	62,3%	50,0%	?
Costs (Annual)	0,4%	0,5%	0,0%	0,0%

5.4 The large universe (N=38) on IS

The large global multi-asset universe, includes 38 assets (N=38): SP500, EAFE, EEM, Japan Topix, US Gov10y, US HighYield, US Gov10y, US Gov30y, US Muni, US TIPS, US Corp, all 10 Fama/French US sectors plus US SmallCaps, GSCI, Gold, Foreign bonds, REIT, NAREIT, 5x FTSE (US 1000/US 1500/Global ex US/Developed/EM), JapanGov10y, 3xDow (Util/Transport/Industry), FX-1x, FX-2x, and Timber.

This is also an expanding universe: we start in 1914 with N=4 (SPY500, EAFE, 10y, and Muni) and grow through N=9 in 1921 to N=26 in 1927 through N=28 in 1964. All 38 assets were available in OS from 1994 to mid 2011, after which the universe contracted to 13 assets in 2014. So this is also a test how well the EAA model can cope with expanding and contracting universes. The Calmar Scatter (49 dots) and Frontier (8 dots) on IS are:.



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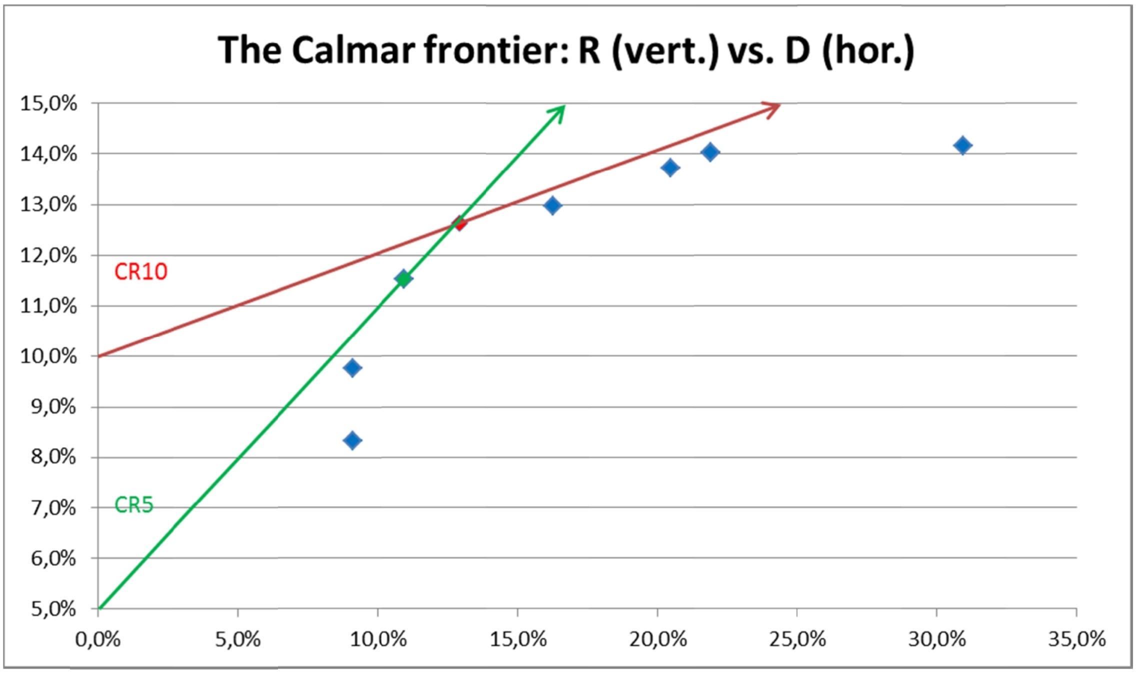


The table with the 8 CF scenarios (on IS) plus the two golden scenarios (see below), is:

wS	wC	Scen	R	V	D	CR5	CR10	SR2	SR5	SR10	3y>EV
0,00	2,00	7	8,3%	5,6%	9,1%	36,48%	-18,5%	113,6%	59,7%	-30,2%	45,8%
0,25	1,50	13	9,8%	6,4%	9,1%	52,44%	-2,5%	121,0%	74,3%	-3,58%	59,7%
0,50	1,00	19	11,5%	7,8%	10,9%	59,70%	14,0%	121,5%	83,3%	19,50%	78,7%
0,75	0,75	25	12,6%	9,0%	12,9%	59,12%	20,4%	118,1%	84,8%	29,25%	87,0%
1,00	0,75	32	13,0%	9,8%	16,3%	48,99%	18,3%	111,9%	81,3%	30,30%	86,5%
1,50	0,50	38	13,7%	11,4%	20,5%	42,68%	18,2%	102,5%	76,3%	32,6%	87,5%
2,00	0,50	45	14,0%	12,0%	21,9%	41,19%	18,4%	100,3%	75,3%	33,6%	87,7%
2,00	0,25	44	14,2%	12,9%	30,9%	29,60%	13,4%	94,3%	71,0%	32,24%	89,7%
0,50	1,00	19	11,5%	7,8%	10,9%	59,70%	14,0%	121,5%	83,3%	19,50%	78,7%
2,00	0,50	45	14,0%	12,0%	21,9%	41,19%	18,4%	100,3%	75,3%	33,61%	87,7%

The best CR5 scenario is 19 (D=10,9%, R=11,5%), while the best CR10 scenario is 45 (D=21,9%, R=14,0%). The most offensive (max R) scenario is 44 (D=30,9% and R=14,2%), which also maximizes 3y (89,7%). The most defensive is scenario 7, with D=9,1% and R=8,3%. Again we see wS trending up and wC trending down, from most defensive to most offensive.

The comparison of the best CR5 and CR10 scenarios 19 and 45 with the EW index on IS is below. Again, all statistics (R, V, D, CR%, SR2, 3y) on IS are in favor of the EAA scenarios relative to EW and SP500. Again the offensive scenario 45 gives a higher 3y (88%).

(N=38, IS)	EAA Sc. 19	EAA Sc. 45	EW	SP500
R (CAGR)	11,5%	14,0%	8,0%	10,1%
V (An. Volatility)	7,8%	12,0%	12,5%	21,4%
D (Max DDown)	10,9%	21,9%	65,1%	83,5%
CR5 (Calmar 5%)	59,7%	41,2%	4,5%	6,2%
SR (Sharpe Rf)	121,4%	100,2%	47,7%	38,0%
3y (Roll >=EW)	78,7%	87,7%	50,0%	?
Costs (Annual)	0,5%	0,5%	0,0%	0,0%

5.5 The Golden models on IS

IS/OS tests often take the optimized parameters from IS and test them on OS. In the present case this means we would test, for example, the pairs wS/wC=25/50% (defensive) and wS/wC=50/50% (offensive) for the N=7 universe. However, since we will often have much shorter IS periods for arbitrary assets (outside our present database), we might wonder if there are more generally effective pairs of elasticities which, on average, generate strong results (on CR5 and CR10 on IS) for *all three universes*. This is our “golden” combination of wS/wC elasticities which work well on all three (N=7, 15, 38) universes. In general it’s clear that these are not optimal on any one universe, but are nearly optimal in all cases, and generalize well.

After some experimentation on all three universes, exclusively on IS, we arrived at wS=50% and wC=100% for CR5, and wS=200% and wC=50% for CR10. Both “golden” sets are also the optimal CR5 and CR10 elasticities (Top1 out of the 49 CS scenarios) for the broad N=38 universe, which we feel is our richest testing sandbox. Both parameter setups (defensive and offensive) also score within the Top5% and Top25% (of the 49 scores on CR5 and CR10) for the N=15 universe, respectively, with a better D (32% instead of 39%) than the optimized CR10. And for the N=7 universe both golden setups score in the Top10% and Top 15% (of the 49 scores on CR5 and CR10, respectively).

For each universe (N=7, 15, 38) we have also provided the statistics (R, V, D, etc) for these two Golden models in above sections below the table with the Calmar Frontier scenarios. From these performance figures it is clear that our golden models also score well on metrics like CR5 and CR10 for N=7 and N=15, and generate the best CR5 and CR10 scenario for N=38.

More validating still, we observe that the golden setups generate Sharpe Ratio performance that is quite near to the performance of the best scenarios. For example, the worst scoring (Top25%) offensive set for the N=15 universe still generates a Sharpe Ratio (SR2) of 78% compared to 93% for the best and 44% for EW.

We will call these combinations the **Golden Defensive EAA model** (with wS=50%, wC=100%), and the **Golden Offensive EAA model (**wS=200%, wC=50%).

Notice that these golden scenarios were completely determined on IS, without looking at OS. Below we will examine the OS performance for all three universes (N=7, 15, 38) on these golden models.

Notice that with these golden scenarios we can rewrite eq. (4), given wS and wC (and wR=1 and wV=0), simply as

(11) $\mathbf{w_i \sim z_i = sqrt\{r_i\,(1-c_i)\}}$, for $r_i > 0$, else $w_i = z_i = 0$ **(Golden Defensive EAA)**

(12) $\mathbf{w_i \sim z_i = (1-c_i)\, r_i^2}$, for $r_i > 0$, else $w_i = z_i = 0$ **(Golden Offensive EAA)**

We will call these the two **Golden EAA** models. Together with the (sqrt) TopX rule and our Crash Protection (CP) rule these rules give two very simple EAA models, to be applied on monthly data with a maximum of one year (12 months) lookback. Below we will test both Golden scenarios on all three universes on OS (1964-2014).

6. The Out-of-sample (OS) tests

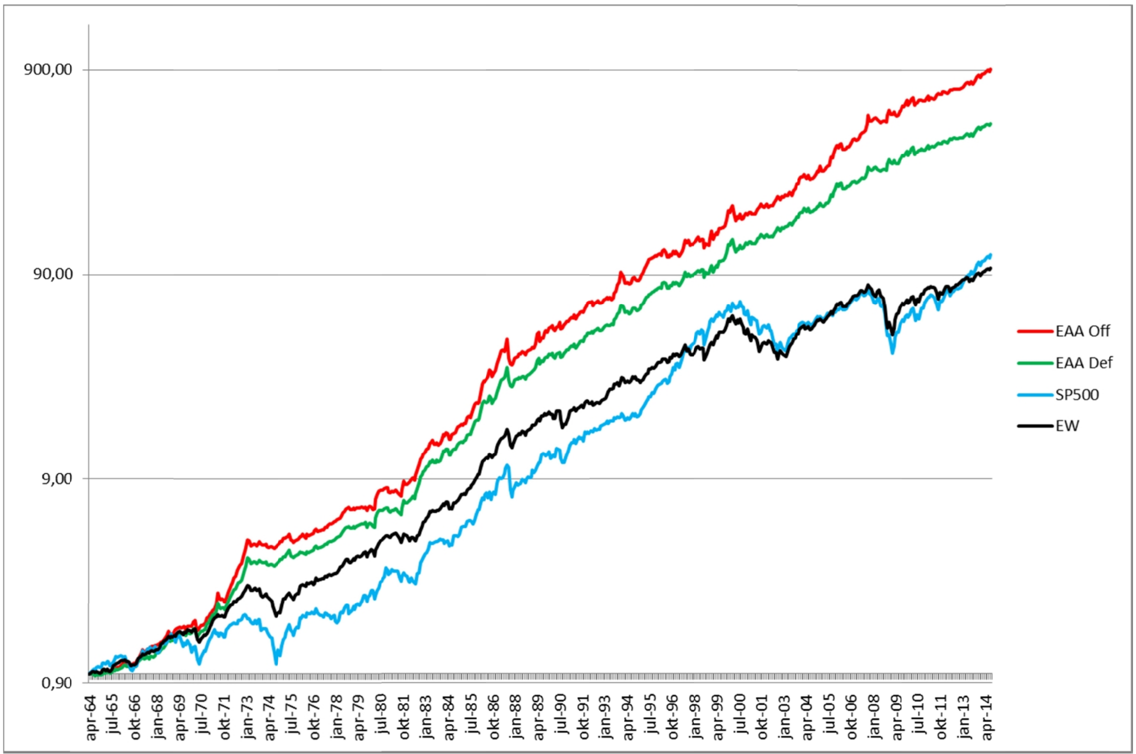
6.1 Golden models on OS: the small universe (N=7)

In this Section we will present the Out-of-Sample tests of the Golden EAA (defensive and offensive) elasticities, on all the three universes. Since all optimizations were performed on the first 50 year In-Sample (IS) period (1914-1964), there could be no datasnooping when applying these optimal parameters to the Out-of-Sample (OS) period from April 1964 through August 2014.

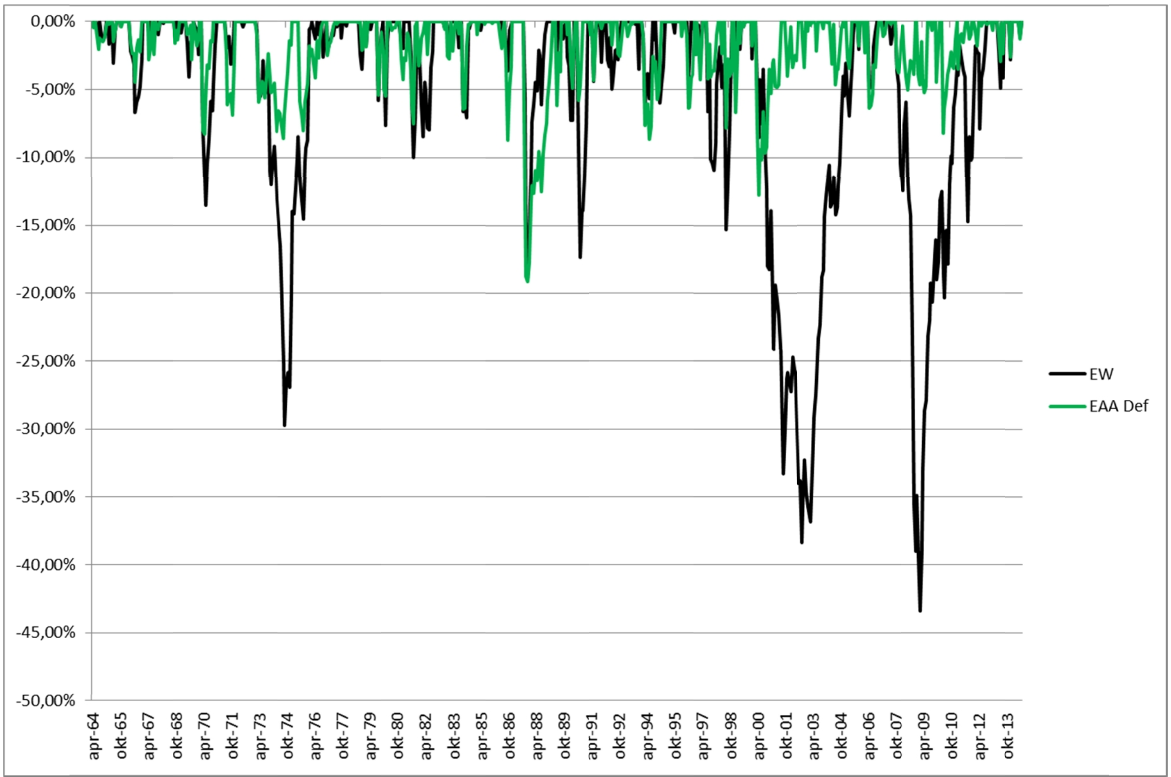
Both Golden EAA models (defensive and offensive) applied to the N=7 universe generates the following OS performance comparison:

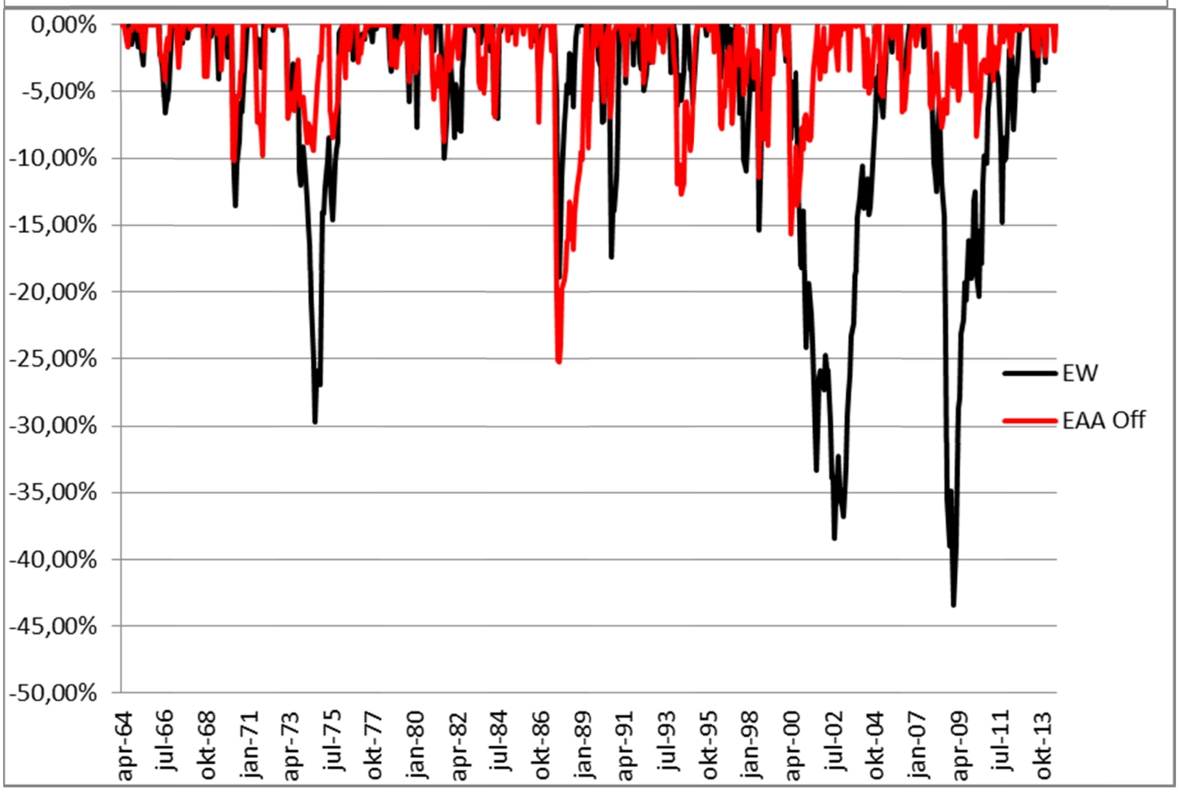
(N=7, OS, Golden)	EAA Def.	EAA Off.	EW	SP500
R (CAGR)	13,1%	14,5%	9,5%	9,9%
V (An. Volatility)	9,7%	11,4%	11,3%	15,0%
D (Max DDown)	19,1%	25,3%	43,4%	50,8%
CR5 (Calmar 5%)	42,4%	37,6%	10,4%	9,6%
SR (Sharpe Rf)	83,3%	82,6%	39,7%	32,1%
3y (Roll > EW)	68,5%	79,0%	0,0%	?
Costs (Annual)	0,5%	0,6%	0,0%	0,0%

It is clear that both the golden EAA models have superior risk-adjusted and absolute performance compared to the EW index (and the SP500). And below we show the corresponding equity curves (in green and red) for OS with also the EW index (in black) and the SP500 (blue).



And here are the drawdown curves over OS for the Golden Defensive (green) and Offensive (red) models, respectively and for the EW index (black). All drawdowns (over the 50 year OS period) for the defensive model are less than 10%, except the large drawdown after Black Monday (19%, Oct 1987), and the internet crash (13%, May 2000). For the offensive EAA model the Oct 1987 and May 2000 drawdowns are 25% and 15% respectively; while the balance are generally also 10% or less.



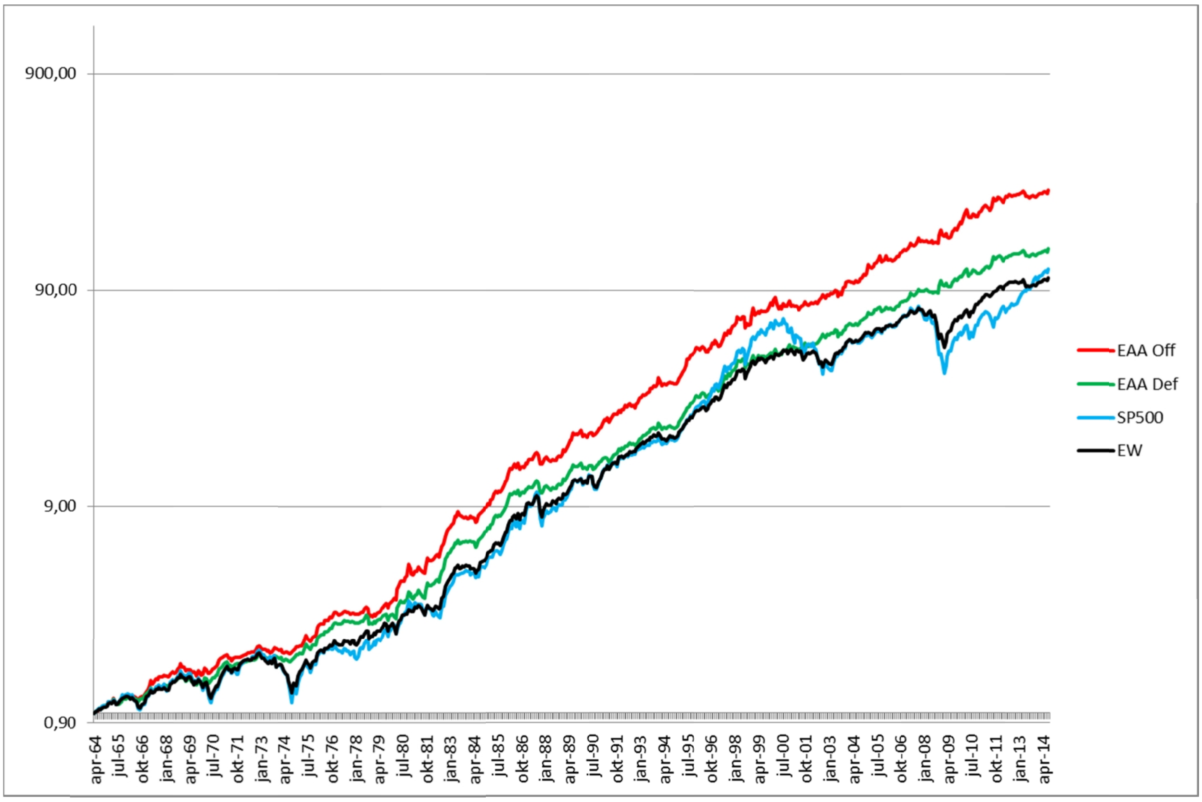


6.2 Golden models on OS: the US sector universe (N=15)

Both golden EAA models (defensive and offensive) applied to the N=15 universe generates the following comparison:

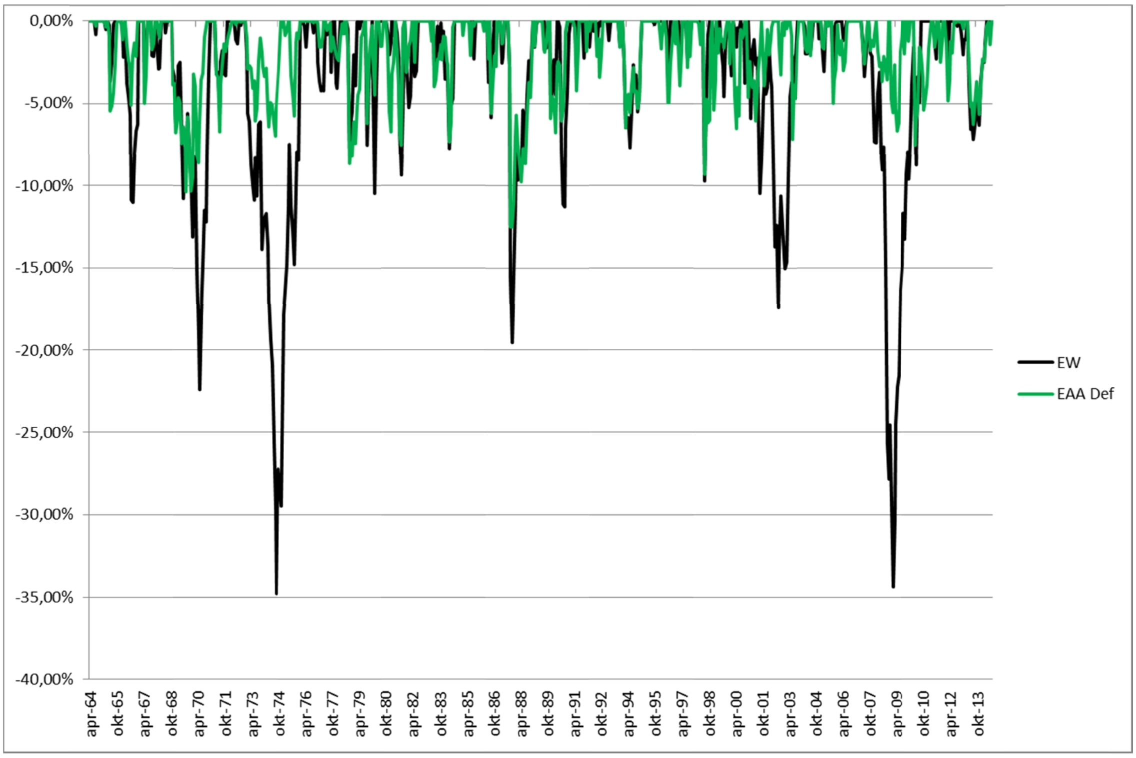
(N=15, OS, Golden	EAA Def.	EAA Off.	EW	SP500
R (CAGR)	10,3%	11,7%	9,6%	9,9%
V (An. Volatility)	8,8%	10,1%	10,7%	15,0%
D (Max DDown)	12,5%	12,5%	34,8%	50,8%
CR5 (Calmar 5%)	42,6%	53,4%	13,4%	9,6%
SR⁻ (Sharpe Rf)	59,9%	65,6%	43,1%	32,1%
3y (Roll > EW)	50,5%	64,4%	0,0%	?
Costs (Annual)	0,6%	0,7%	0,0%	0,0%

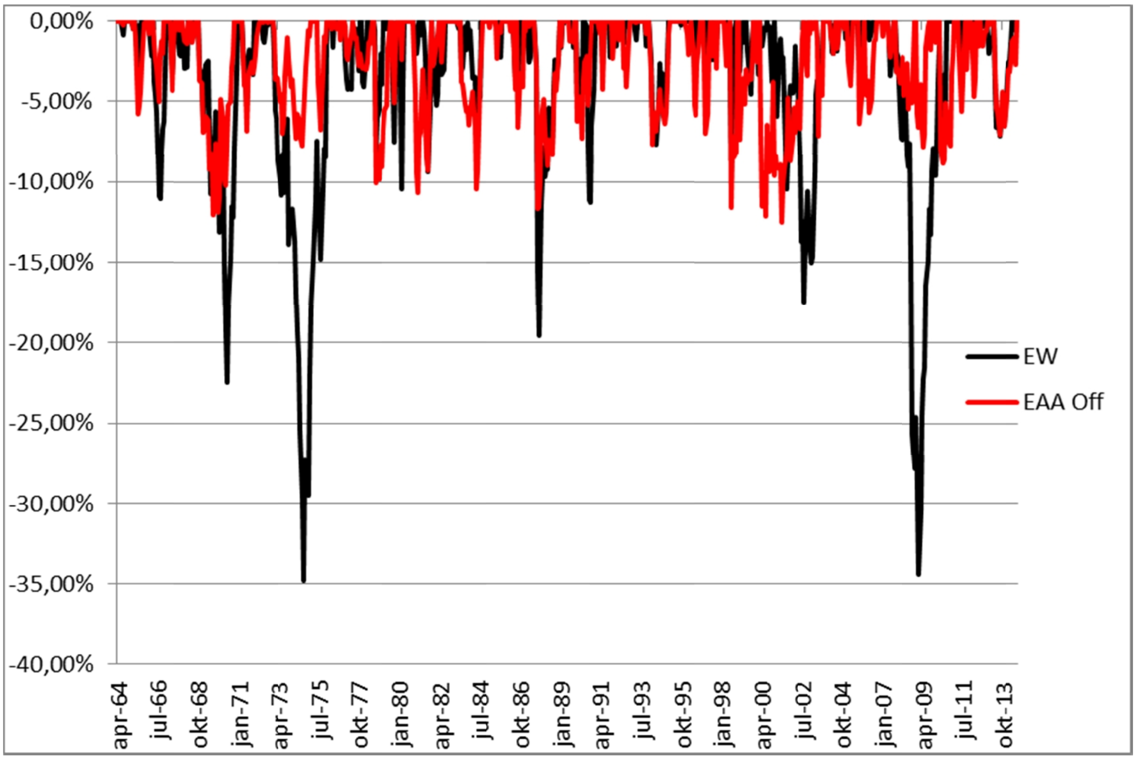
Both the Golden Defensive EAA model as the Offensive EAA model beats the EW index and SP500 on R, V, and D and various risk/return statistics. And below we show the corresponding equity curve (Defensive in green, Offensive in red) for OS with the EW index (in black) and the SP500 (blue).



Below are the drawdown curves for the Golden Defensive (green) and Offensive (red) EAA models, respectively. Observe that all drawdowns (over the 50 year OS period) on the defensive model are approximate 10% or less, except the drawdown on Black Monday (Oct

1987) which is 12,5%. All drawdowns on the offensive model over the OS period are also bounded at 12,5%.



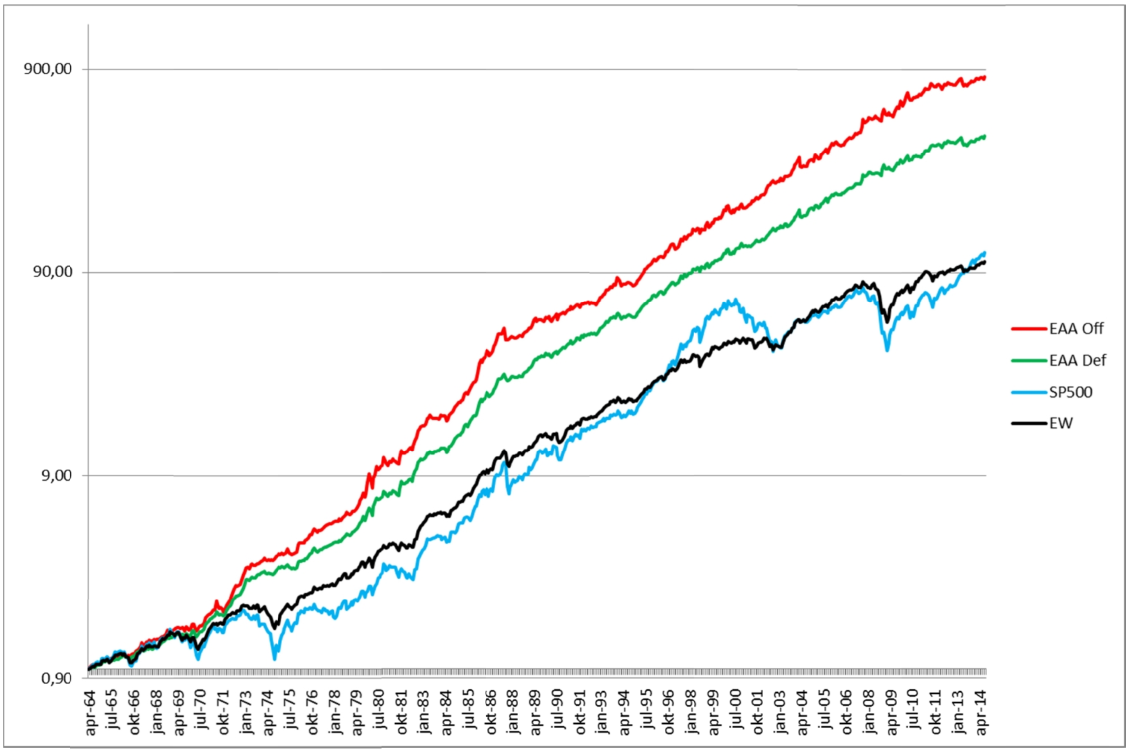


6.3 Golden models on OS: the large universe (N=38)

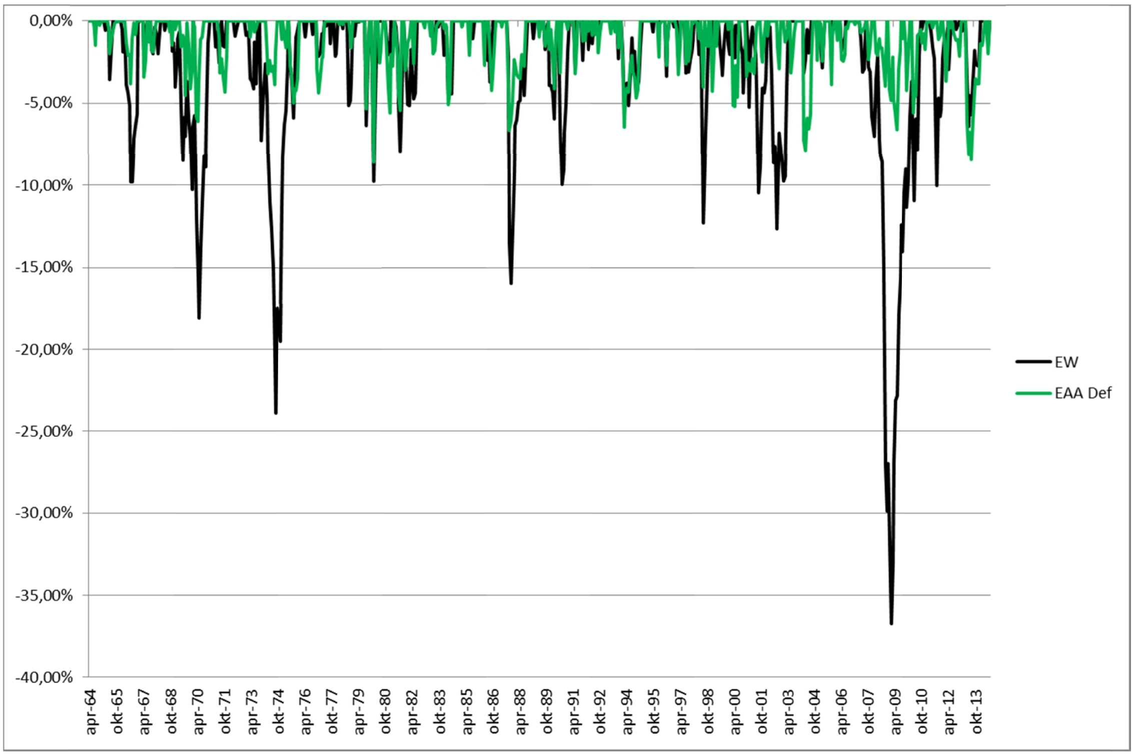
Both golden models applied to the large N=38 universe provide the following favourable comparisons with EW (and SP500). In particular the Golden Defensive EAA model is impressive in terms of its maximum drawdown (8,6% over 50 years OS).

(N=38, OS, Golden	EAA Def.	EAA Off.	EW	SP500
R (CAGR)	12,8%	14,3%	9,6%	9,9%
V (An. Volatility)	7,9%	10,1%	9,6%	15,0%
D (Max DDown)	8,6%	14,1%	36,7%	50,8%
CR5 (Calmar 5%)	90,2%	65,7%	12,6%	9,6%
SR (Sharpe Rf)	97,4%	91,0%	47,8%	32,1%
3y (Roll > EW)	76,0%	79,3%	0,0%	?
Costs (Annual)	0,6%	0,6%	0,0%	0,0%

And below is the corresponding equity curve (Defensive in green and Offensive in red) for OS, with the EW index (in black) and the SP500 (blue). The OS is based on the Golden ratios above.



Finally, examine the drawdown curves below for the Golden Defensive (green) and Offensive (red) EAA models resp. with the EW index (in black) for the OS period (1964-2014). All drawdowns (over the 50 year OS period) for this large universe (N=38) are less than 10% in the case of the defensive model, including the drawdown over Black Monday (Oct 1987) which is 9%. All drawdowns for the offensive model are less than 15%.



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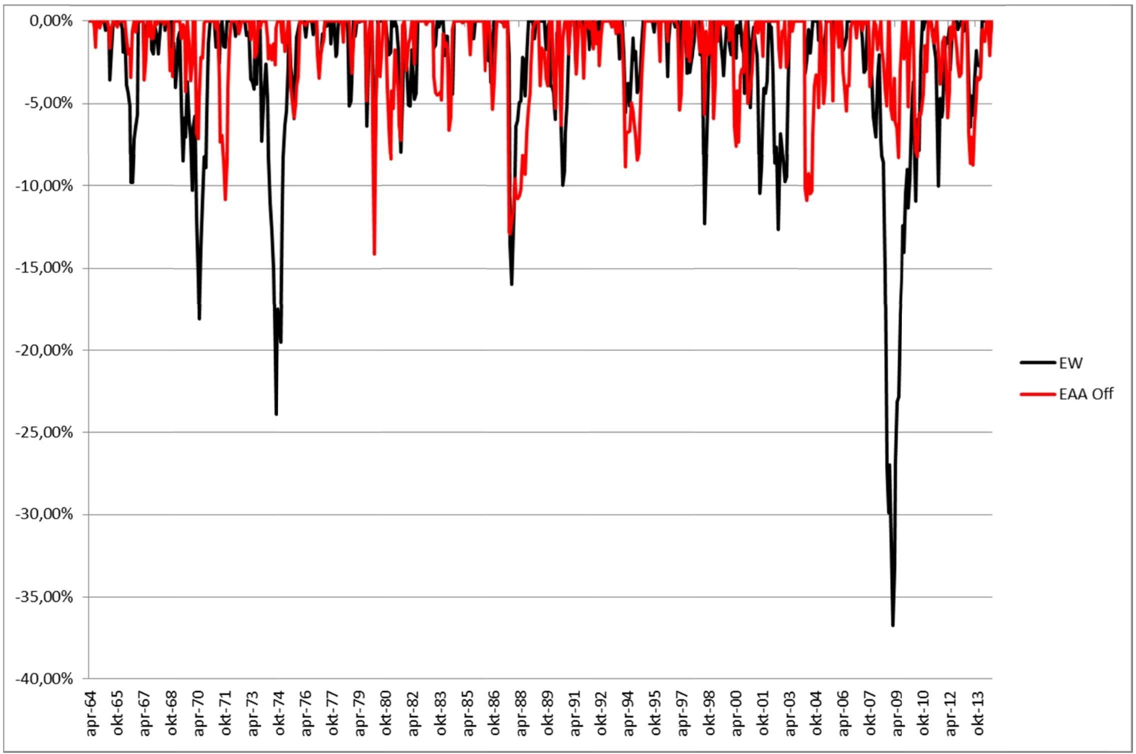
EW
EAA Def



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apr-06
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apr-09
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EW
EAA Off

7. Conclusions

We will summarize our main objectives and results:

1. We have shown that **MPT** and its proxies (like our Elastic Asset Allocation, EAA) show good performance when we apply a **tactical** instead of strategic approach. In particular, we have demonstrated that two Golden models generalize to strong performance, relative to an equally weighted benchmark, on three heterogeneous universes. Both golden models were optimized In-Sample (IS) and tested Out-of-Sample (OS) against the corresponding benchmark (the EW universe). This conclusion is in contrast to the seminal article of DeMiguel (2007), which states that equally weighted portfolios compare favourably versus optimized portfolios.

2. We have shown that complex MPT optimizations can be **proxied** by a much simpler form such as our EAA model (and the FAA and MAA models of Keller, 2012, 2013, 2014). Here, instead of using a **two-dimensional** covariance matrix of size **NxN** per MPT, we apply **one-dimensional** vectors of length N for both volatilities and index correlations, where the index is the EW (1/N) index of the N assets in the universe.

3. We have shown that in a tactical context our concept of "**generalized momentum**" also applies in the context of our EAA model, as was demonstrated for FAA and MAA. This implies not only persistence of short-term (momentum) returns R, but also persistence of short-term volatilities V and index correlations C. We have also shown that only the last twelve observations (of monthly returns) were enough to estimate tactical volatility and correlation effects.

4. We have shown that our EAA model (as MAA and MPT) can be adapted to proxy many well-known "**smart-beta**" variations, such as Risk Parity, Minimum Variance, Maximum Diversification and Equal Weight, as a special cases. In our IS optimizations, return momentum and low beta (more precisely: low index correlation) turns out to be the dominant drivers in our empirical EAA models.

5. We have shown that we can use monthly index **data over a century** (1914-2014) in order to optimize (IS: 1914-1964) and test (OS: 1964-2014) EAA over many economic regimes. We prefer to use the first 50 years as learning (IS) period since it contains a wider variety of economic and financial regimes than the most recent 50 years, which are largely dominated by a decreasing interest-rate regime.

6. We have shown that we can successfully use **geometrical weights (or elasticities**) for RVC in EAA instead of the traditional RVC shrinkages (as in MAA, and in traditional MPT). Since EAA also proxies MAA, we think that our EAA formula with elasticities for R, V and C is a (tactical) **MPT proxy** under relaxed assumptions. Notice that MAA under the single index model assumption (SIM, see Elton 1976) is an exact analog of MPT (See Keller, 2013, 2014).

7. We have shown that RVC elasticities imply many degrees of freedom and therefore offer material **datasnooping** opportunities. In anticipation of such a pitfall, we estimated "golden" EAA elasticities (and other rules) by optimizing our model exclusively on our first 50 years of IS data. We found two very simple **Golden EAA models** for the best defensive and offensive models $(\mathbf{w_i \sim sqrt(r_i\ (1\text{-}c_i))})$ respectively $\mathbf{w_i \sim (1\text{-}c_i)\ r_i^2)}$ on IS, to be tested on OS.

9. We have shown that we can easily estimate parameters other than elasticities on the 50 years of the IS period (1914-1964). For example, for return lookback we settled on lookbacks of (1,3,6 and 12 months) for return momentum. We also chose to use *excess returns* for the calculation of return momentum, while using nominal returns to calculate volatility and correlation parameters. Furthermore, we set an optimal TopX rule (sqrt(N) or N/2 for small universes). We also performed a cursory analysis to specify universe selection. **All of these specifications were decided based exclusively on our 50 years of IS**.

10. We have shown that these parameters plus the "golden" EAA elasticities for defensive and offensive allocations, optimized over the IS period (1914-1964), **show preferential risk-adjusted performance** over the EW index for small (N=7), medium (N=15) and large (N=38) heterogeneous universes, on a **50 year OS period (1964-2014)**. Since we optimized exclusively over IS and compared/tested over OS, there is no datasnooping involved.

Cautions regarding our findings include:

1. Focusing on **maximum drawdown** effectively reduces sample size, such that fitting to minimize a relatively few large drawdowns amplifies bias. Optimization on Sharpe ratio may be informative and more comfortable for many in the target audience.

In addition, the use of **old (1914-) index data** conveys several cautions:

2. Could the data have been compiled in a timely manner in the past? For example, would it have taken a week or a month to compile an **index level** in the IS period, or even in the early part of the OS period? Such a delay would have retarded market feedback from investors and may have substantially disrupted the strategy. A robustness test with **one-month delay** may be informative. We leave this for subsequent research.

3. What would have been the frictions associated with maintaining tracking funds in the early part of the sample? Frictions may have been very high (much higher than the assumed 0,1% one-way cost), thereby making fund returns much lower than index returns[6]. If tracking funds had existed, what would market feedback have done to prices?

[6] We did a quick and preliminary test with the offensive model for the N=38 universe with larger one-way transaction costs than the assumed 0.1%. As long as the costs where less than 0.7% the return R was still better than EW, as was the V and D. So there seems some room to allow for larger costs.

A. Technical Appendix: EAA vs. MAA/MPT

In this section we will compare EAA to MAA (see Keller 2013, 2014) and to the Modern Portfolio Theory (MPT) of Markowitz (1952). We restricted MPT to the tactical or short-term use (instead of strategic, or long-term use as seen in most MPT studies) in order to take advantage of (generalized) momentum, like we did in the MAA model.

The MAA model is based on the Single Index Model (SIM) of Elton (1976). The core of the SIM is the distinction between the systematic (or market) and the non-systematic (or idiosyncratic) drivers of returns. These effects separate the return of an asset into two components, namely the systematic component, which captures the sensitivity of an asset to the returns on the market index (in our case, the equal weighted portfolio), and; the non-systematic component, which is the residual return not explained by the market index. The MAA solution is congruent with the MPT solution when we assume that the single-index model (SIM) of Elton (1976) captures the systematic (or beta) component of returns. In this Appendix, we will show that EAA equals or proxies special cases of MAA, and therefore MPT.

Under SIM assumptions, the optimal MPT long-only asset allocation, which maximizes the Sharpe ratio, can be expressed as an elegant analytical formula. This is the general MAA model:

(A.1) $\mathbf{w_i \sim (1-t/t_i)\, r_i / s_i}$ for $t_i > t$, else $w_i = 0$, for i=1..N, with sumj wj =1 for normalization

where

- ri is return of asset i
- t_i is the Treynor ratio of asset i (with ti = ri/bi),
- t is the long-only Treynor threshold, so that $w_i = 0$ for all assets with $t_i < t$
- b_i is the beta of asset i wrt. the market index (the EW universe), with $b_i = c_i * v_i / v$
- s_i is the idiosyncratic variance of the returns of asset i, with $s_i = v_i$^2-v^2*b_i^2
- v is the market volatility

As before, c_i equals the correlation of asset i=1,..,N to the index, for which we use the EW return series of the universe of all N assets. Given the SIM assumption, eq. (A.1) is also the MPT solution.

As shown in eq. (A.1), there is a threshold t for the Treynor ratio ti=ri/bi such that only assets with large Treynor ratios ti>t are allowed in the optimal portfolio. The formula for this Teynor threshold t is rather complex (see the Appendix in Keller, 2014). In our EAA model, we use a simplified TopX rule of the assets with the best momentum score zi as a proxy, since the computation of the threshold t in MAA is rather daunting.

When we assume that Sharpe Ratios ri/vi are constant, we arrive at the Maximum Diversification variant of MAA (the MAA-MD model) with wi ~ (1-ci/c)/(vi*(1-ci^2) or by (Taylor) approximation

(A.2) $\mathbf{wi \sim (1-ci)^\wedge(1/c) \ / \ vi}$ for ci<c, else wi=0, i=1,..,N, with sum wj =1 for normalization,

with c the correlation threshold. This approximation turns out to be rather precise for values of ci (the index correlation of asset i) smaller than the threshold c, and larger than about -0.5, which holds for nearly all asset classes. For example, with a threshold c=0.8, the approximation is near perfect for ci around zero and good for a range of 0.8 > ci > -0.3.

In eq. (A.2) we recognize the geometrical expression (1-ci)^wC (with wC=1/c) used in EAA for the correlation ci, as well as the (inverse) term for the volatility vi^wV with weight wV=1. So the EAA model with wR=0, wC=1/c and wV=1 corresponds (by close approximation) to the MAA-MD model.

When ci is also constant for all i (and all Sharpe ratio's are equal), the optimal share wi of eq. (A.2) becomes :

(A.3) **wi ~ 1/vi,** with with sumj wj =1 for normalization

which corresponds exactly to the so-called **Naïve Risk Parity** (RP) model, where all assets are held in the portfolio in proportion to the inverse of their respective volatility. This corresponds to the EAA model with wR=0, wC=0, wV=1. When we also assume that all volatilities vi are equal, we arrive at the **Equal Weight** (EW) solution where wi =1/N. Now for EAA, all elasticities are zero (wR=wV=wC=0).

When we assume all returns to be equal (ri=r), MAA degenerates to the Minimium Variance variant (MAA-MV) with

wi ~ (1-bi/b) /si , for bi <b, else wi = 0, for i=1..N, with sumj wj =1 for normalization

where b is the beta threshold such that only assets with beta's bi<b below this threshold are included. When we assume the systematic component to be zero (v=0), this degenerates to

(A.4) **wi ~ 1/vi^2,** so wi>0, for i=1..N, with sumj wj =1 for normalization

so in this special MV case all asset weights are inversely proportional to their variances vi^2 (and therefore positive: all assets are included). This corresponds to the EAA model with wR=wC=0 and wV=2.

Finally, if we assume return momentum but no systematic (or market) effect (so v=0 : market volatility equal to zero), we arrive at the "non-systematic" MAA variant with return momentum which corresponds to weighting each asset in proportion to its Sharpe ratio:

(A.5) **wi ~ ri / vi^2,** for ri>0, else wi=0, with with sumj wj =1 for normalization

Now correlations ci don't play a role anymore. This simple MAA formula corresponds to the EAA formula with wR=1, wC=0 and wV=2.

So we have shown that EAA can encompass various non-systematic MAA/MPT solutions exactly and is a good approximation (with the term (1-ci)) for the systematic MAA/MPT models.

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