Example Prompt: Compute the Last Two Nonzero Digits of **80!** Using Modular Arithmetic

Instructions to the Model

You are solving factorial tail problems like a competition math expert. Follow this exact chain-of-logic structure. Do not jump steps. Whenever you divide under a modulus, you must replace it with a multiplication by the modular inverse. Connect all steps using symbolic logic and never rely on decimal approximations or estimates. Your goal is to compute:

Last two nonzero digits of 80!

Step-by-Step Structured Example (with Chain-of-Logic)

Step 1: Count the Number of Trailing Zeros

Trailing zeros in 80! come from factors of $10 = 2 \cdot 5$. So count the number of factors of 5:

$$v_5(80!) = \left\lfloor \frac{80}{5} \right\rfloor + \left\lfloor \frac{80}{25} \right\rfloor = 16 + 3 = 19$$

We remove $10^{19} = 2^{19} \cdot 5^{19}$ from 80!:

$$T = \frac{80!}{10^{19}} = \frac{80!}{2^{19} \cdot 5^{19}} = \frac{M}{2^{19}}, \quad \text{where } M = \frac{80!}{5^{19}}$$

We aim to compute $N = T \mod 100$ using CRT with:

- $N \mod 25$
- $N \mod 4$

Step 2: Evaluate $M \mod 25$

2.1: Non-5 Multiples

There are $80 - 16 = 64$ numbers not divisible by 5 from 1 to 80. These form 16 groups:

$$(5n+1)(5n+2)(5n+3)(5n+4) \equiv -1 \mod 25$$

So:

$$\prod_{\text{non-5 terms}} \equiv (-1)^{16} = 1 \mod 25$$

2.2: 5-Multiples

Original	Divided by	Residue
5	5	1
10	5	2
15	5	3
20	5	4
25	25	1
30	5	6
35	5	7
40	5	8
45	5	9
50	25	2
55	5	11
60	5	12
65	5	13
70	5	14
75	25	3
80	5	16

Groupable residues:

- Group A: $1 \cdot 2 \cdot 3 \cdot 4 \Rightarrow (-1)$
- Group B: $6 \cdot 7 \cdot 8 \cdot 9 \Rightarrow (-1)$
- Group C: $11 \cdot 12 \cdot 13 \cdot 14 \Rightarrow (-1)$

Unpaired: 16
Extra: $1 \cdot 2 \cdot 3$ from $25, 50, 75$

Thus:

$$M \equiv (-1)^3 \cdot 16 \cdot (1 \cdot 2 \cdot 3) = -96 \equiv 4 \mod 25$$

Step 3: Compute $2^{19} \mod 25$

$$2^{10} = 1024 \equiv -1 \mod 25 \Rightarrow 2^{20} \equiv 1 \Rightarrow 2^{19} \equiv 13 \mod 25$$

Step 4: Modular Division with Inverse

$$N \equiv \frac{M}{2^{19}} \equiv 4 \cdot 13^{-1} \mod 25$$

$$13^{-1} \equiv 2 \mod 25 \Rightarrow N \equiv 4 \cdot 2 = 8 \mod 25$$

Step 5: $N \mod 4$

80! has $v_2 = 78$ factors of 2. After removing 19, we still have 2^{59}:

$$2^{59} \equiv 0 \mod 4 \Rightarrow N \equiv 0 \mod 4$$

Step 6: Chinese Remainder Theorem (CRT)

$$\begin{cases} N \equiv 8 \mod 25 \\ N \equiv 0 \mod 4 \end{cases} \Rightarrow N = 25k + 8 \equiv 0 \mod 4 \Rightarrow k \equiv 0 \mod 4 \Rightarrow k = 4m$$

$$N = 100m + 8 \Rightarrow \boxed{N \equiv 08 \mod 100}$$

Final Answer

$$\boxed{\text{The last two nonzero digits of 80! are } \boxed{08}}$$