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Problems

1. (80 pt. in total)

Assume that we have n data points, $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$. Let the degree of polynomial be d. Then, we want to find $w_0, w_1, w_2, \cdots, w_d$ of the polynomial such that

$$
\begin{aligned}
\hat{f}(x_1) &= w_0 + w_1 x_1 + w_2 x_1^2 + \cdots + w_d x_1^d = y_1,\\
\hat{f}(x_2) &= w_0 + w_1 x_2 + w_2 x_2^2 + \cdots + w_d x_2^d = y_2,\\
\hat{f}(x_3) &= w_0 + w_1 x_3 + w_2 x_3^2 + \cdots + w_d x_3^d = y_3,\\
\hat{f}(x_4) &= w_0 + w_1 x_4 + w_2 x_4^2 + \cdots + w_d x_4^d = y_4,\\
\hat{f}(x_5) &= w_0 + w_1 x_5 + w_2 x_5^2 + \cdots + w_d x_5^d = y_5,\\
&\vdots\\
\hat{f}(x_n) &= w_0 + w_1 x_n + w_2 x_n^2 + \cdots + w_d x_n^d = y_n.
\end{aligned}
$$

Now, we reformulate the equations into the vector and matrix form. First, let $\mathbf{w} = [w_0, w_1, \cdots, w_d]^T$ and $\mathbf{y} = [y_1, y_2, \cdots, y_n]^T$. Then, the above equations can be rewritten as

$$
\hat{f}(x_1) = [1, x_1, x_1^2, x_1^3, \cdots, x_1^d] \cdot \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ \vdots \\ w_d \end{bmatrix} = [1, x_1, x_1^2, x_1^3, \cdots, x_1^d]\mathbf{w} = y_1
$$

Similarly, we have,

$$
\begin{aligned}
[1, x_2, x_2^2, x_2^3, \cdots, x_2^d]\mathbf{w} &= y_2,\\
[1, x_3, x_3^2, x_3^3, \cdots, x_3^d]\mathbf{w} &= y_3,\\
[1, x_4, x_4^2, x_4^3, \cdots, x_4^d]\mathbf{w} &= y_4,\\
[1, x_5, x_5^2, x_5^3, \cdots, x_5^d]\mathbf{w} &= y_5,\\
&\vdots\\
[1, x_n, x_n^2, x_n^3, \cdots, x_n^d]\mathbf{w} &= y_n.
\end{aligned}
$$

Then, all equations can be written as the form of linear equation,

$$
A\mathbf{w} = \mathbf{y},
$$

where A is the stack of $[1, x_i, x_i^2, x_i^3, \cdots, x_i^d]$ for $i = 1, \cdots, n$. Under this setting, answer the following questions.

1-(a) What is the size of vector w and y? (10pt)

$w = \sqrt{w_0^2 + w_1^2 + w_2^2 + \cdots + w_d^2}$

$y = \sqrt{y_1^2 + y_2^2 + \cdots + y_n^2}$

1-(b) What is the size of matrix A? Write A. (10pt)

$n \times (d+1)$

$$A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^d \\ 1 & x_2 & x_2^2 & \cdots & x_2^d \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^d \end{bmatrix}$$

1-(c) Let ~~$d = n$~~ $d+1=n$, then, A becomes a square matrix. Compute the determinant of A. (40pt in total, Derivation: 30pt, Answer: 10pt, Hint: Vandermonde Matrix.)

$d=n-1$, $A = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$

위의 A 행렬에서 j열을 i열로 바꾸자 (단 $i<j$)

$V(x_1, x_2, \cdots, x_i, \cdots, x_j, \cdots x_n) = 0$ 이므로

$V(x_1, \cdots x_n)$의 인수 $x_j - x_i$가 존재한다

그러면 차수를 생각하면

$V(x_1, x_2, \cdots, x_n) = Q \prod_{i<j} (x_j - x_i)$ 인 상수 Q가 존재한다

$$\det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \cdots & x_3^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

차수 $0 \quad 1 \quad 2 \quad \cdots \quad n-1$

$(x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \cdots (x_n - x_1)$ $n-1$개

$(x_3 - x_2)(x_4 - x_2) \cdots (x_n - x_2)$ $n-2$개

$(x_4 - x_3) \cdots (x_n - x_3)$ $\vdots$

$\vdots$

$(x_n - x_{n-1})$ 1

위 식에서 $x_2, x_3^2, x_4^3 \cdots x_n^{n-1}$의 계수를 비교하면 $Q=1$ 이된다.

$\therefore \det A = \prod_{1 \le i < j} (x_j - x_i)$

1-(d) What is the condition that makes the determinant of A non-zero? (10pt)

$(x_j - x_i) \neq 0 \quad (1 \le i < j \le n)$

1-(e) Assume that the determinant of A is non-zero, then, what is the solution of linear equation, $A\mathbf{w} = \mathbf{y}$, with respect to $\mathbf{w}$? (10pt)

$w = A^{-1} y$

2. (20pt)

Suppose that $n > d$. Then, we cannot compute the inverse of A since A is not a square matrix. In this case, how can we solve the linear equation $A\mathbf{w} = \mathbf{y}$? (Hint: Pseudo Inverse)

$Aw = y$ ($\det |A^T \cdot A| \neq 0$ 가정)

$$(A^T \cdot A)^{-1} \cdot \underbrace{(A^T \cdot A)}_{I}\, w = (A^T \cdot A)^{-1} A^T y$$

$$w = \underbrace{(A^T \cdot A)^{-1} \cdot A^T}_{\text{Pseudo Inverse}} \cdot y$$