量子力学作业四 张锦程 2018012082 材84

1. $\psi(x,0)=\frac{1}{\sqrt{2}}[\psi_1(x)+\psi_2(x)]$ $\therefore \psi(x,t)=\frac{1}{\sqrt{2}}[\psi_1(x)e^{-\frac{iE_1t}{\hbar}}+\psi_2(x)e^{-\frac{iE_2t}{\hbar}}]=$

$E_1=\frac{1}{2m}(\frac{\pi\hbar}{2a})^2$. $E_2=\frac{1}{2m}(\frac{\pi\hbar}{a})^2$ $\quad \frac{1}{\sqrt{2a}}[\cos\frac{\pi x}{2a}e^{-\frac{iE_1t}{\hbar}}+\sin\frac{\pi x}{a}e^{-\frac{iE_2t}{\hbar}}]$

$\bar{E}=\int_{-\infty}^{\infty}\psi^*(x,t)\hat{H}\psi(x,t)dx=\frac{1}{2}E_1+\frac{1}{2}E_2$ $\quad (\bar{E}=|c_1|^2E_1+|c_2|^2E_2=\frac{5}{16m}(\frac{\pi\hbar}{a})^2)$

$\bar{E^2}=\int_{-\infty}^{\infty}\psi^*(x,t)\hat{H}^2\psi(x,t)dx=\frac{1}{2}E_1^2+\frac{1}{2}E_2^2$ $\quad (\bar{E^2}=|c_1|^2E_1^2+|c_2|^2E_2^2=\frac{17}{128m^2}(\frac{\pi\hbar}{a})^4)$

$\bar{x}(t)=\int\psi^*(x,t)\,x\,\psi(x,t)dx=\frac{1}{2}[(\psi_1(x,t),x\psi_1(x,t))+(\psi_2(x,t),x\psi_2(x,t))+(\psi_1(x,t),x\psi_2(x,t))+(\psi_2(x,t),x\psi_1(x,t))]$

$=\frac{1}{2}[(\psi_1(x),x\psi_1(x))+(\psi_2(x),x\psi_2(x))+e^{-\frac{i(E_2-E_1)t}{\hbar}}(\psi_1(x),x\psi_2(x))+e^{-\frac{i(E_1-E_2)t}{\hbar}}(\psi_2(x),x\psi_1(x))]$

$=\frac{1}{2}[(\psi_1(x),x\psi_1(x))+(\psi_2(x),x\psi_2(x))]+\exp(-\frac{i(E_2-E_1)t}{\hbar})\int_{-a}^{a}\frac{1}{a}x\cos\frac{\pi}{2a}x\sin\frac{\pi}{a}x\,dx+\exp(-\frac{i(E_1-E_2)t}{\hbar})\int_{-a}^{a}\frac{1}{a}x\cos\frac{\pi}{2a}x\sin\frac{\pi}{a}x\,dx$

$=\frac{1}{2}[\int_{-a}^{a}x\cos^2(\frac{\pi x}{2a})dx+\int_{-a}^{a}x\sin^2(\frac{\pi x}{a})dx+\exp(\ldots)]$

$=\frac{32a}{9\pi^2}\cos[(\frac{E_2-E_1}{\hbar})t]$

2. $\forall\psi$与χ，有 $\int\chi^*\hat{L}_z\psi d\tau=\int\psi(\hat{L}_z\chi)^*d\tau$

如取$\chi=\chi(r,\theta)$，与φ无关，即 $\hat{L}_z\chi=0 \Rightarrow \int\chi^*\hat{L}_z\psi d\tau=0$

利用 $d\tau=r^2\sin\theta drd\theta d\varphi$，可证 ~~$\psi(r,\theta,\varphi)$~~ $\quad \iiint\chi(r,\theta)\hat{L}_z\psi(r,\theta,\varphi)r^2\sin\theta drd\theta d\varphi=0$

$=\iint\chi(r,\theta)(-i\hbar)r^2\sin\theta(\psi(r,\theta,\varphi)|_0^{2\pi})drd\theta=0$

$\therefore \psi(r,\theta,\varphi)|_0^{2\pi}=0\Rightarrow\psi(r,\theta,\varphi)=\psi(r,\theta,\varphi+2\pi)$

3. $\forall$波函数ψ_i与ψ_j：

有 $(\psi_i,p\psi_j)=-i\hbar\int_{-\infty}^{+\infty}\psi_i^*\frac{d}{dx}(\psi_j)dx=-i\hbar\int_{-\infty}^{+\infty}(\frac{d}{dx}\psi_j)^*\psi_i dx$

$=-i\hbar[\psi_i^*\psi_j|_{-\infty}^{+\infty}-\int_{-\infty}^{+\infty}(\frac{d}{dx}\psi_i^*)\psi_j dx]$

$=i\hbar[\int_{-\infty}^{+\infty}(\frac{d}{dx}\psi_i)\psi_j^*]dx \quad -i\hbar[\int_{-\infty}^{+\infty}((\frac{d}{dx})^*\psi_i)\psi_j^*dx$

$\because (\frac{d}{dx})=(\frac{d}{dx})^* \quad \therefore p^*=p$

4. 由平方可积性，当$x\to\pm\infty$时 $|\psi|^2\to0$，即$u(x)\to0$

$\bar{p}=\int_{-\infty}^{+\infty}\psi^*(x)p\psi(x)dx=-i\hbar\int_{-\infty}^{+\infty}\psi^*(x)\frac{d\psi(x)}{dx}dx=-i\hbar\psi^*(x)\psi(x)+i\hbar\int_{-\infty}^{+\infty}\psi(x)\frac{d\psi^*(x)}{dx}dx$

$=-i\hbar u^2(x)+i\hbar\int_{-\infty}^{+\infty}[u(x)\frac{du(x)}{dx}-iku^2(x)]dx=i\hbar[u^2(x)|_{-\infty}^{+\infty}-ik\int_{-\infty}^{+\infty}u^2(x)dx]$

$=\hbar k$

5. $E_2 = \frac{1}{2m}\left(\frac{\pi\hbar}{a}\right)^2 \quad \psi_2(x) = \frac{1}{\sqrt{a}}\sin\left(\frac{\pi x}{a}\right)$

扩展后，粒子能量的可能测值为 $2a$ 阱宽下的能量本征值，即

$$E_n = \frac{1}{2m}\left(\frac{n\pi\hbar}{4a}\right)^2, \ n = 1,2,3,4,5,\cdots$$

$$\cancel{E_n = \frac{1}{2m}\left(\frac{n\pi\hbar}{4a}\right)^2, \ n = \ldots}$$

$u = \frac{\pi x}{4a}$

新势阱基态的波函数为 $\psi_1'(x) = \frac{1}{\sqrt{2a}}\cos\frac{\pi x}{4a}$

∴ 处于基态的系数 $C_1 = \int_{-2a}^{2a}\psi_1'^{*}\psi_2 dx = \frac{1}{\sqrt{2}a}\int_{-2a}^{2a}\cos\left(\frac{\pi x}{4a}\right)\sin\left(\frac{\pi x}{a}\right)dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos u \sin 4u \left(\frac{4a}{\pi}\right)du$

$= \frac{4a}{\pi}\left(-\frac{1}{10}\cos 5u - \frac{1}{6}\cos 3u\right)\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$

$$P = |C_1|^2 = 0$$

6. $\overline{xp_x} = \int_{-\infty}^{+\infty}\psi^{*} x p_x \psi dx = \int_{-\infty}^{+\infty}\psi x p_x \psi^{*} dx = -i\hbar\psi^{*} x\psi\Big|_{-\infty}^{+\infty} + i\hbar\int_{-\infty}^{+\infty}\psi\frac{d}{dx}(\psi^{*}x)dx$

$= i\hbar\int_{-\infty}^{+\infty}\psi\left(\frac{d\psi^{*}}{dx}x + \psi^{*}\right)dx = 0 - \left(-i\hbar\int_{-\infty}^{+\infty}\psi x\frac{d}{dx}\psi^{*}dx\right) + i\hbar$

$= -\overline{xp_x} + i\hbar$

而 $\overline{xp_x} = \int_{-\infty}^{+\infty}\psi^{*} x p_x \psi dx = \int_{-\infty}^{+\infty}(\psi x p_x)\psi^{*}dx = \int_{-\infty}^{+\infty} x(p_x x)^{*}\psi^{*}dx = \int_{-\infty}^{+\infty}\psi^{*}(p_x x)^{*}\psi dx = \overline{p_x x}^{*}$

∴ $\overline{xp_x} = \frac{i\hbar}{2}$ ∴ $\overline{p_x \cdot x} = \overline{xp_x}^{*} = -\frac{i\hbar}{2}$

若取定态波函数 $\psi(x) = \frac{1}{\sqrt{a}}\cos\frac{\pi x}{2a}$，可以得到

$\overline{xp_x} = -\frac{i\hbar}{a}\int_{-a}^{a}\cos\frac{\pi x}{2a}\, x\left(\frac{d}{dx}\right)\cos\frac{\pi x}{2a}dx = +\frac{i\hbar}{a}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t\left(\frac{2a}{\pi}t\right)\frac{\pi}{2a}\sin t\,\frac{2a}{\pi}dt$

$= \frac{2i\hbar}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t \cdot t \cdot \sin t\, dt$

$= \frac{i\hbar}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t\sin 2t\, dt = \frac{i\hbar}{2}$