概率论与数理统计 第十一次作业 张锦程 2018012082

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1. $X \sim b(80, p)$，$\bar{x} \approx N(p, \frac{p(1-p)}{80})$ [中心极限定理] $n=80$，$\alpha = 0.1$

$u = \frac{\bar{x}-p}{\sqrt{p(1-p)/n}} \sim N(0,1)$；$P(|u| \le u_{1-\alpha/2}) \approx 1-\alpha$

令 $(\bar{x}-p)^2 = u^2_{1-\frac{\alpha}{2}}\, p(1-p)/n$，得 $p = \frac{1}{1+u^2_{1-\alpha}/n}\left(\bar{x} + \frac{u^2_{1-\alpha/2}}{2n} \pm \sqrt{\frac{\bar{x}(1-\bar{x})}{n}u^2_{1-\alpha/2} + \frac{u^4_{1-\alpha/2}}{4n^2}}\right)$

$\approx \bar{x} \pm u_{1-\alpha/2}\sqrt{\frac{\bar{x}(1-\bar{x})}{n}} = \frac{11}{80} \pm 0.8289\sqrt{\frac{\frac{11}{80}(1-\frac{11}{80})}{80}}$

$= 0.1375 \pm 0.0319$

0.9 置信区间为 $[0.1056, 0.1694]$

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2. μ 未知时 σ 的置信区间为 $\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}(n-1)}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}(n-1)}}\right]$

$n=9$，$s = 11\,m/s$，$\alpha = 0.05$

$\chi^2_{0.975}(8) = 17.5345$，$\chi^2_{0.025}(8) = 2.1797$

∴ 置信区间为 $[7.430, 21.074]$

$\sigma_1^2 = 64$，$\sigma_2^2 = 49$

~~$s_x^2 = 58$，$s_y^2 = 49$~~

$\bar{x} = 72$，$\bar{y} = 70$

$n = 7$，$m = 12$

3. (a). σ_1 与 σ_2 已知，0.95 置信区间为 $\bar{x} - \bar{y} \pm u_{1-\frac{\alpha}{2}}\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}$，$\alpha = 0.05$

$= 72 - 70 \pm 0.8352\sqrt{\frac{64}{7} + \frac{49}{12}}$

$= 2 \pm 3.037$

即：$[-1.037, 5.037]$

(b) $\sigma_1 = \sigma_2 = \sigma$，$\sigma$ 未知，0.95 置信区间为 $\bar{x} - \bar{y} \pm \sqrt{\frac{m+n}{mn}}\, s_w\, t_{1-\alpha/2}(m+n-2)$

$s_w = \sqrt{\frac{(n-1)s_x^2 + (m-1)s_y^2}{m+n-2}} = \sqrt{\frac{6\times 58 + 11 \times 49}{12+7-2}} = 1.7519$

$t_{1-\alpha/2}(m+n-2) = 2.1098$

∴ 0.95 置信区间为 $2 \pm \sqrt{\frac{7+12}{7\times 12}} \times 1.7519 \times 2.1098 = 2 \pm 1.7579$

即：$[0.2421, 3.7579]$

(c) 此时近似置信区间为 $\bar{x} - \bar{y} \pm s_0\, t_{1-\alpha/2}(l)$

$s_0^2 = \frac{s_x^2}{n} + \frac{s_y^2}{m} = \frac{58}{7} + \frac{49}{12} = 12.369$

$l = s_0^4 / \left(\frac{s_x^4}{n^2(n-1)} + \frac{s_y^4}{m^2(m-1)}\right) = 12.369^2 / \left(\frac{58^2}{7^2(7-1)} + \frac{49^2}{12^2(12-1)}\right) = 11.8068$

$t_{0.975}(11.8068) \approx 2.185$

∴ 区间为 $2 \pm \sqrt{12.369} \times 2.185$，即 $[-5.6845, 9.6845]$

(d) $\frac{\sigma_1^2}{\sigma_2^2}$ 的 0.95 置信区间为 $\left[\frac{s_x^2}{s_y^2}\frac{1}{F_{1-\alpha/2}(n-1, m-1)}, \frac{s_x^2}{s_y^2}\frac{1}{F_{\alpha/2}(n-1, m-1)}\right]$

即 $F_{0.975}(6, 11) = 3.90$ $F_{0.025}(6, 11) = \frac{1}{F_{0.975}(6, 11)} = 0.2564$

故区间为 $[0.3035, 4.616]$

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4. (a) $\frac{u_{1-\alpha/2}}{2\sqrt{n}} \le d \Rightarrow n \ge \left(\frac{u_{1-\alpha/2}}{2d}\right)^2 = 1743.8936$，故至少访问 1744 名顾客

(b) 可以，此时 $\bar{x}(1-\bar{x}) \le 0.2(1-0.2) = 0.16$ ∴ $\frac{u_{1-\alpha/2}}{\sqrt{n}} \times 0.4 \le d$

∴ $n \ge \left(\frac{0.4 \times u_{1-\alpha/2}}{d}\right)^2 = 1116.09$，故至少访问 1117 名顾客，小于1744名

5. n较大时: $m_{0.5} \sim N(x_{0.5}, \frac{1}{4n\cdot p^2(x_{0.5})})$

不难看出θ为总体的分位数. $\therefore m_{0.5} \sim N(\theta, \frac{\pi^2}{4n})$

$\therefore$ θ的置信区间为 $[\bar{x} - u_{1-\alpha/2}(n-1)\frac{s}{\sqrt{n}}, \bar{x} + u_{1-\alpha/2}(n-1)\frac{s}{\sqrt{n}}]$

即 $[\bar{x} - u_{1-\alpha/2}\frac{(n-1)\pi}{2n}, \bar{x} + u_{1-\alpha/2}\frac{(n-1)\pi}{2n}]$

6.(a) $x_{(1)}$ 的密度函数 $p_1(x) = n(1-F(x))^{n-1}p(x) = n\cdot e^{(n-1)(-x+\theta)}\cdot e^{-x+\theta} = n\cdot e^{n(-x+\theta)}$

$(F(x) = \int_0^x e^{-x_1+\theta}dx_1 \cdot I_{[x>\theta]} = (1-e^{-x+\theta})I_{[x>\theta]})$

$\therefore$ $x_{(1)}-\theta$ 的密度函数为 $n e^{-nx}$

(b) $x_{(1)}-\theta \sim Exp(n)$ $\therefore E(x_{(1)}-\theta) = \frac{1}{n} \to 0$

Exp(n) 的分布函数为 $F(x)=\int_0^x n e^{-nx_1}dx_1 = 1-e^{-nx}$

令 $F(x)=\frac{\alpha}{2}$, 得 $x = -\frac{1}{n}\ln(1-\frac{\alpha}{2})$

令 $F(x)=1-\frac{\alpha}{2}$, 得 $x = -\frac{1}{n}\ln\frac{\alpha}{2}$

$P(x_{(1)}-\theta \in [-\frac{1}{n}\ln(1-\frac{\alpha}{2}), -\frac{1}{n}\ln\frac{\alpha}{2}]) = 1-\frac{\alpha}{2}-\frac{\alpha}{2} = 1-\alpha$

$= P(\theta \in [x_{(1)}+\frac{1}{n}\ln\frac{\alpha}{2}, x_{(1)}+\frac{1}{n}\ln(1-\frac{\alpha}{2})])$

$\therefore$ θ的一个 1-α 置信区间为 $[x_{(1)}+\frac{1}{n}\ln\frac{\alpha}{2}, x_{(1)}+\frac{1}{n}\ln(1-\frac{\alpha}{2})]$

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7. $x_{(1)}$ 的密度函数为 $p_1(x) = n(1-F(x))^{n-1}p(x)$

其中 $p(x)=\theta x^{-2}$, $F(x)=\int_\theta^x p(x_1)dx_1 = 1-\theta x^{-1}$

$\therefore p_1(x) = n\cdot\theta^n\cdot x^{-1-n}$; $F_1(x) = 1-\theta^n\cdot x^{-n}$

$\therefore E(x_{(1)}) = \int_\theta^\infty n\theta^n\cdot x^{-n} = \frac{n}{n-1}\theta$

令 $F_1(x)=\frac{\alpha}{2}$, 得 $x = (1-\frac{\alpha}{2})^{-\frac{1}{n}}\theta$

令 $F_1(x)=1-\frac{\alpha}{2}$, 得 $x = (\frac{\alpha}{2})^{-\frac{1}{n}}\cdot\theta$

$P(\frac{x_{(1)}}{\theta} \in [(1-\frac{\alpha}{2})^{-\frac{1}{n}}, (\frac{\alpha}{2})^{-\frac{1}{n}}]) = 1-\frac{\alpha}{2}-\frac{\alpha}{2} = 1-\alpha$

$= P(\theta \in [x_{(1)}(\frac{\alpha}{2})^{\frac{1}{n}}, x_{(1)}(1-\frac{\alpha}{2})^{\frac{1}{n}}])$

$\therefore$ θ的一个 1-α 置信区间为 $[x_{(1)}(\frac{\alpha}{2})^{\frac{1}{n}}, x_{(1)}(1-\frac{\alpha}{2})^{\frac{1}{n}}]$

8.(a) 此时区间长度为 $2\cancel{u_{1-\alpha/2}\sqrt{\frac{\bar{x}(1-\bar{x})}{n}}} \cancel{= \frac{u_{1-\alpha/2}}{\sqrt{n}}} \cancel{= \frac{u_{0.975}}{\sqrt{n}}}$

$2u_{1-\alpha/2}\frac{s}{\sqrt{n}} = 2\times0.8352\times\frac{1}{\sqrt{n}}\times6 \le \frac{6}{4}$ $\Rightarrow$ 解 $n \ge (2\times0.8352\times4)^2 = 44.6438$

n应取45及以上

(b) 区间长度为 $2t_{1-\alpha/2}(n-1)\frac{s}{\sqrt{n}} = 2\times t_{0.975}(n-1)\frac{\sqrt{\chi^2(n-1)}}{\sqrt{(n-1)n}}\cdot6 \le \frac{6}{4}$, 成立的概率为0.9

$\cancel{\chi^2_{0.9}(n-1)}$ $\Rightarrow \frac{(n-1)n}{64}\frac{1}{t^2_{0.975}(n-1)} \ge \chi^2_{0.9}(n-1)$

经计算机数值计算后得, n=276时此不等式成立

n应取276及以上