概率论与数理统计 2018012082 张锦程 第5周作业

(1) $Cov(X,Y) = E(XY) - E(X)E(Y) = E(aX + bX^2 + cX^3) - E(X)E(a + bX + cX^2)$
$= (0 + b + 0) - 0 = b$

$Var(X) = E(X^2) - E^2(X) = 1$ $\quad Var(Y) = E(c^2X^4 + 2bcX^3 + (2ac + b^2)X^2 + 2abX + a^2) - E^2(Y)$
$= (3c^2 + 0 + (2ac + b^2) + 0 + a^2) - (a+c)^2 = 2c^2 + b^2$

$\therefore \rho(X,Y) = \dfrac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = \dfrac{b}{\sqrt{2c^2 + b^2}}$

(2) 右式 $= E^2(Y)[E(Y^2) - E^2(Y)] + E^2(Y)[E(X^2) - E^2(X)] + [E(X^2) - E^2(X)][\cancel{E^2(X) E^2(Y)}\ E(Y^2) - E^2(Y)]$
$= E(X^2)E(Y^2) - 2E^2(Y)E^2(X) + E^2(X)E^2(Y) = E(X^2Y^2) - E^2(XY) = Var(XY) =$ 左式

(3) $p(XY) = p(Y|X)p(X) = \begin{cases} xe^{-xy}, & y \geq 0, \quad 1<x<2 \\ 0, & y<0, \quad 1<x<2 \end{cases}$

(4) 由随机个随机变量和的数学期望公式知: $E(盒重) = E(K)E(m) = \frac{H}{\lambda}$
$E(箱重) = E(M)E(盒重) = \frac{1}{p} \cdot \frac{H}{\lambda} = \frac{H}{p\lambda}$

$Var(盒重) = Var(K)[E^2(m)] + E(K)Var(m) = \frac{2H}{\lambda^2}$
$Var(箱重) = Var(M)E^2(盒重) + E(M)Var(盒重) = \frac{1-p}{p^2} \cdot \frac{H^2}{\lambda^2} + \frac{1}{p}\frac{2H}{\lambda^2}$

(5) (a) $N(0,1)$: $p(Y) = p(X|Z=-1)p(Z=-1) + p(X|Z=1)p(Z=1) = p(X) = N(0,1)$
(b) 它们不相互独立. 因为 $Y = X$ 与 $Y = -X$ 必取其一, 但 X与Y不相关. 因为可以计算得
$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$

(6) (a) $E(等待时间) = \int_9^{10} (H-9)\frac{1}{2} dH = \frac{1}{4}h$
(b) $E(持续时间) = \int_8^9 (3)\frac{1}{2} du + \int_9^{10} (12-H)\frac{1}{2} du = (\frac{3}{2} + \frac{5}{4})h = \frac{11}{8}h$
(c) ~~P(迟到)~~ ~~P(不迟到)~~ $p(迟到) = \frac{1}{2} = p(M)$, $p(迟到多于 45 min) = \frac{1}{8} = p(N)$

~~$p(MN) = \frac{1}{2} \times \frac{1}{8} = \frac{1}{16}$~~
~~分析 在他们的第k次约会上, 他们在下一次分手的概率为~~

假设他们约会次数的期望为Y; 有 $Y = \frac{1}{2}[\frac{?}{?} + \frac{1}{2}(Y+1)] + \frac{1}{2} \times \frac{1}{8}[\frac{3}{8} + \frac{1}{2}(Y+1)] + (\frac{1}{2} \times \frac{1}{8} \times \frac{1}{8})[\frac{?}{8} + \frac{1}{2}(Y+1)]$ → 错位相减法

$= \lim_{n\to\infty} \{ \frac{1}{2}Y + \frac{1}{4}[Y + \frac{?}{?}Y + (\frac{?}{?})^2 Y + \cdots + (\frac{?}{?})^n Y] + \frac{1}{16}\overbrace{[2 + 3 \times \frac{?}{?} + 4 \times (\frac{?}{?})^2 + \cdots + (n+1) \times (\frac{?}{?})^{n-1}]}^{z} \}$
$+ [\frac{1}{2} + \frac{1}{4}[2 + (\frac{?}{?})3 + (\frac{?}{?})^2 \times 4 + \cdots]$ → 同样错位相减

$= \frac{1}{2}Y + \frac{?}{?}Y + \frac{9}{25} + \frac{26}{25}$

得 $\frac{1}{?}Y = \frac{9}{5}$ $\quad \therefore Y = 18$

$\frac{1}{2}$ 迟 → $\frac{1}{8}$ 分手
$\frac{1}{8}$ 迟
$\frac{7}{8}$ 不迟
$\frac{1}{2}$ 不迟 $(Y+1)$

按如图分解, 当出现"不迟"时, 期望仍为Y. 当出现分手时, 要加次数
当迟到时, 对下一步进行分析

(7) (a) $\forall x \in X$ 有 $E(g(x)Y|x) = E(g(x)\cdot Y|X=x) = g(x)E(Y|X=x) = g(x)E(Y|x)$

则 $E(g(X)Y|X) = g(X)E(Y|X)$

(b) $E(XY) = \iint xy\, p(x,y)\,dx\,dy$

$E(XE(Y|X)) = \int x p(x)\left(\int y\, p(y|x)\,dy\right)dx = \iint xy\, p(x)p(y|x)\,dy\,dx = \iint xy\, p(x,y)\,dx\,dy = E(XY)$

(c) $\mathrm{Cov}(X, E(Y|X)) = E(XE(Y|X)) - E(X)E(E(Y|X)) = E(XY) - E(X)E(Y) = \mathrm{Cov}(X,Y)$

(8) $p(X|X+Y=z) = \dfrac{p(X)P(Y=z-X)}{P(X+Y=z)} = \dfrac{\lambda_1\lambda_2 e^{-\lambda_2 z + (\lambda_2-\lambda_1)x}}{\int_0^z \lambda_1\lambda_2 e^{-\lambda_2 z+(\lambda_2-\lambda_1)x}\,dx}$

$= (\lambda_2-\lambda_1)\dfrac{e^{-\lambda_2 z+(\lambda_2-\lambda_1)x}}{e^{-\lambda_1 z}-e^{-\lambda_2 z}}$

$\therefore E(X|X+Y=z) = \int_0^z x\,p(X|X+Y=z) = \dfrac{1}{e^{(\lambda_2-\lambda_1)z}-1}\left(x-\dfrac{1}{\lambda_2-\lambda_1}\right)e^{(\lambda_2-\lambda_1)x}\Big|_0^z$

$= \dfrac{1}{e^{(\lambda_2-\lambda_1)z}-1}\left[\left(z-\dfrac{1}{\lambda_2-\lambda_1}\right)e^{(\lambda_2-\lambda_1)z}+\dfrac{1}{\lambda_2-\lambda_1}\right]$

(9) $0<2p-1\le 1$，可知赌资永远不会赔光

设第i次的赌资为 x_i

由题: $x_{i+1} = \begin{cases} 2p\,x_i, & 计赢，概率\ p \\ (2-2p)x_i & 计输，1-p \end{cases}$

设 n次中赢的次数为 a 次，则输了 $n-a$ 次

有 $x_n = (2p)^a(2-2p)^{n-a}x$

$\ln x_n = a\ln 2p + (n-a)\ln(2-2p)$

由二项分布知 $E(a) = np$

$\therefore E(\ln x_n) = np\ln(2p) + (n-np)\ln(2-2p)$

$\therefore E(x_n) = (2p)^{np}\cdot(2-2p)^{n-np}$