概率论与数理统计：部分参考解答

第六题：对应的密度函数为 $p(x) = 2xe^{-x^2}, x > 0$. 则数学期望为

$$E(X) = \int_0^\infty 2x^2 e^{-x^2} dx = \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}.$$

(高斯积分的计算，可以这样考虑 $(\int_{-\infty}^{\infty} e^{-x^2})^2 = \int_{\mathbb{R}^2} e^{-x^2-y^2} dxdy$, 接着通过极坐标变换可以计算)

$$E(X^2) = \int_0^\infty 2x^3 e^{-x^2} dx = \int_0^\infty x^2 e^{-x^2} d(x^2) = \int_0^\infty y e^{-y} dy = \int_0^\infty e^{-y} dy = 1.$$

所以方差为 $Var(X) = E(X^2) - (E(X))^2 = 1 - \frac{1}{4}\pi$.

第九题 (b) 考虑答题顺序

$$L = (i_1, i_2, \ldots, i_{k-1}, i_k, i_{k+1}, \ldots, i_n),$$
$$L' = (i_1, i_2, \ldots, i_{k-1}, i_{k+1}, i_k, \cdots, in_n).$$

注意到答题顺序 L 和 L' 的差别只在于对调了第 k 题和 $k+1$ 题的顺序，其他没有变化。下面我们计算总奖金的期望，

$$\begin{aligned} E(L) &= p_{i_1} v_{i_1} + p_{i_1} p_{i_2} v_{i_2} + \cdots + p_{i_1} p_{i_2} \cdots p_{i_n} v_{i_n}, \\ E(L') &= p_{i_1} v_{i_1} + \cdots p_{i_1} \ldots p_{i_{k-1}} v_{i_{k-1}} \\ &\quad + p_{i_1} \ldots p_{i_{k-1}} p_{i_{k+1}} v_{i_{k+1}} + p_{i_1} \ldots p_{i_{k-1}} p_{i_{k+1}} p_{i_k} v_{i_k} \\ &\quad + p_{i_1} \cdots p_{i_{k+2}} v_{i_{k+2}} + \cdots + p_{i_1} \cdots p_{i_n} v_{i_n}. \end{aligned}$$

所以

$$\begin{aligned} &E(L') - E(L) \\ &= p_{i_1} \cdots p_{i_{k-1}} (p_{i_{k+1}} v_{i_{k+1}} - p_{i_k} v_{i_k} + p_{i_k} p_{i_{k+1}} v_{i_k} - p_{i_k} p_{i_{k+1}} v_{i_{k+1}}) \\ &= p_{i_1} \cdots p_{i_{k-1}} (1 - p_{i_k})(1 - p_{i_{k+1}}) \left(\frac{p_{i_{k+1}} v_{i_{k+1}}}{1 - p_{i_{k+1}}} - \frac{p_{i_k} v_{i_k}}{1 - p_{i_k}}\right). \end{aligned}$$

如果 $\frac{p_{i_{k+1}} v_{i_{k+1}}}{1-p_{i_{k+1}}} - \frac{p_{i_k} v_{i_k}}{1-p_{i_k}} > 0$, 可以通过交换第 k 题和 $k+1$ 的顺序让增大。所以在期望意义下的最佳答题顺序 $(i_1, \ldots, i_n)$ 应该满足 $r(i_1) \geqslant r(i_2) \geqslant .... \geqslant r(i_n)$, 其中 $r(i_k) = \frac{p_{i_k} v_{i_k}}{1-p_{i_k}}$.