概统第十次作业　孙锦程　2018012082

(1) (a) 联合密度函数 $L(\theta,X)=\frac{1}{(2\theta)^n}\exp\left(-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|\right) = \cancel{\exp(\ldots)}\ \exp\left(\sum_{i=1}^{n}|x_i|\right)\frac{1}{(2\theta)^n}\exp\left(-\frac{1}{\theta}\right)$

令 $x=\frac{1}{2\theta}$，当 $\frac{1}{2\theta}=\frac{n}{2}$ 时 有 $\frac{d(x^n\cdot\exp(-2x))}{dx}=0$，此时 $\hat{\theta}=\frac{1}{n}$ 为 MLE

(b) $L(\theta_1,\theta_2,x)=\frac{1}{\theta_2-\theta_1}I_{(\theta_1,\theta_2,x)}$　要让 $L(\theta_1,\theta_2,x)\to\max$ 则 $\theta_2-\theta_1\downarrow$，$\theta_2\downarrow$，$\theta_1\uparrow$，同时保证 $p(x,\theta_1,\theta_2)\le 1$　$\theta_1<x<\theta_2$

得 (θ_1,θ_2) 的一个MLE为 $(x-\frac{1}{2},x+\frac{1}{2})$

(c) $L(p,X)=\frac{\prod_i^n\binom{2}{x_i}p^{x_i}(1-p)^{2-x_i}}{2p-p^2}=\prod_i^n\binom{2}{x_i}\left(\frac{p^{\sum_i^n x_i}(1-p)^{\sum_i^n(2-x_i)}}{2p-p^2}\right)\to f(p)$

$\ln f(p)=\sum_i^n x_i\ln p+\sum_i^n(2-x_i)\ln(1-p)-\ln p-\ln(2-p)$

当 $\frac{d\ln f(p)}{dp}=\frac{m}{p}+\frac{n}{1-p}+\frac{1}{2-p}=0$ 时，$p=\frac{2m}{n+2}=\frac{2\ n+2}{n+2}=\frac{2m}{m+1}=2-\frac{2}{\sum_i^n x_i}$

$m=\sum_i^n x_i-1$，$n=\sum_i^n(2-x_i)$　$m=n+1$

(2) (a) $p(\lambda,X)=\frac{1}{\lambda}e^{-\frac{1}{\lambda}x}$ 则 $E(x)=\frac{1}{\frac{1}{\lambda}}=\lambda$

$\therefore L(\lambda,X)=\frac{1}{\lambda^n}\exp\left(-\frac{1}{\lambda}\sum_i^n x_i\right)=\frac{1}{\lambda^n}\exp\left(-\frac{n}{\lambda}\bar{x}\right)$　$\frac{dL(\lambda,x)}{d\lambda}=\lambda^{-n-2}n\exp\left(-\frac{n}{\lambda}\bar{x}\right)(\bar{x}-\lambda)$

令 $\frac{dL}{d\lambda}=0$，得 $\hat{\lambda}=\bar{x}$ 为 MLE．而样本均值 $\bar{x}$ 为总体均值 λ 的相合估计．$E(\bar{x})=E(X)=\lambda$ 为无偏估计

(b) 令 $a=\frac{n}{n+1}$，此时均方误差 $MSE(\hat{\lambda})=\frac{\lambda^2}{n+1}<\bar{x}$ 的均方误差 $MSE(\bar{x})=Var(\bar{x})+[E(\bar{x})-\lambda]^2=\frac{\lambda^2}{n}+(\lambda-\lambda)^2=\frac{\lambda^2}{n}$

$=\left(\frac{n+1}{n}a^2-2a+\frac{n}{n+1}+\frac{1}{n+1}\right)\lambda^2$

(3) 联合密度函数为 $(2\pi)^{-\frac{m+n}{2}}\sigma^{-m-n}\exp\left(-\sum_i^m\frac{(x_i-\mu_1)^2}{2\sigma^2}-\sum_i^n\frac{(y_i-\mu_2)^2}{2\sigma^2}\right)=L(\theta)$

$\ln L(\theta)\propto(-m-n)\ln\sigma-\frac{1}{2\sigma^2}\left[\sum_i^m(x_i-\mu_1)^2+\sum_i^n(y_i-\mu_2)^2\right]$

$L(\theta)\to\max$　则 $\ln L(\theta)\to\max$，则 $\sum_i^m(x_i-\mu_1)^2+\sum_i^n(y_i-\mu_2)^2\to\min$．则 $\sum_i^m|x_i-\mu_1|+\sum_i^n|y_i-\mu_2|\to\min$

$\therefore\mu_1=\bar{x}$，$\mu_2=\bar{y}$ 为 x,y 的样本均值．

此时 $\ln L(\theta)=-(m+n)\ln\sigma-\frac{1}{2\sigma^2}[mS_m^2+nS_n^2]$．　S_m^2、S_n^2 分别为 X、Y 的样本

令 $\frac{\partial\ln L(\theta)}{\partial\sigma}=0$．得 $-\frac{m+n}{\sigma}+\frac{1}{\sigma^3}[mS_m^2+nS_n^2]=0$

$\therefore\sigma=\sqrt{\frac{mS_m^2+nS_n^2}{m+n}}=\sqrt{\frac{\sum_i^m(x_i-\bar{x})^2+\sum_i^n(y_i-\bar{y})}{m+n}}$

(4) (a) 先求θ的 MLE：$L(\theta)=\theta^n\prod_i^n x_i^{\theta-1}$

$\ln L(\theta)=n\ln\theta+(\theta-1)\sum_i^n\ln x_i$ $=\cancel{n\ln\theta+(\theta-1)\ldots}$

当 $\ln L(\theta)$ 取max时，$\theta=-\frac{n}{\sum\ln x_i}$．则 $\frac{1}{\theta}$ 的MLE为 $-\frac{1}{n}\sum\ln x_i$ (不变性)

$E(\hat{g})=-\frac{1}{n}E\left(\sum\ln X_i\right)=\frac{1}{\theta}=g(\theta)$　$\therefore\hat{g}$ 为 $g(\theta)$ 的无偏估计

$f(x)=\int_0^1\theta x^\theta dx=\frac{\theta}{\theta+1}$

(b) $I(\theta)=E\left[\frac{\partial}{\partial\theta}\ln p(x;\theta)\right]^2=\cancel{\ldots}=E\left(\frac{1}{\theta}+\ln x\right)^2=\frac{1}{\theta^2}+\frac{2}{\theta}E(\ln x)+E(\ln^2 x)$
$=\frac{1}{\theta^2}+\frac{2}{\theta}\cdot\left(-\frac{1}{\theta}\right)+\frac{2}{\theta^2}=\frac{1}{\theta^2}$
$=Var(\ln X)$

(c) 首先 $g(\theta)$ 的 MLE $\hat{g}$ 为无偏估计，则 $Var(\hat{g})=\frac{1}{n^2}\sum_{i=1}^n Var(\ln X_i)=\frac{1}{n^2}\cdot n\cdot\frac{1}{\theta^2}=\frac{1}{n\theta^2}$

$\frac{\partial}{\partial\theta}\ln p(x,\theta)=\frac{1}{\theta}+\ln x$，$\frac{\partial^2}{\partial\theta^2}\ln p(x,\theta)=-\frac{1}{\theta^2}$，$I(\theta)=\frac{1}{\theta^2}$

$\therefore$ C-R下界为 $\frac{[g'(\theta)]^2}{nI(\theta)}=\frac{(-\frac{1}{\theta^2})^2}{n\frac{1}{\theta^2}}=\frac{1}{n\theta^2}=Var(\hat{g})$，故 $\hat{g}=-\frac{1}{n}\sum_{i=1}^n\ln X_i$ 为 $g(\theta)=\frac{1}{\theta}$ 的有效估计

(5) $p(x,\lambda)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}I_{x>0}$，当 $x>0$ 时，$\cancel{\ln f(x,\lambda)}$ $\ln p(x,\lambda)=\alpha\ln\lambda-\ln\Gamma(\alpha)+(\alpha-1)\ln x-\lambda x$；$\frac{\partial}{\partial\lambda}\ln p(x,\lambda)=\frac{\alpha}{\lambda}-x$

$\frac{\partial^2}{\partial\lambda^2}\ln p(x,\lambda)=-\frac{\alpha}{\lambda^2}$，$I(\lambda)=-E\left[\frac{\partial^2}{\partial\lambda^2}\ln p(X;\lambda)\right]=\frac{\alpha}{\lambda^2}$

~~(5)(c) $E[\frac{\partial}{\partial\lambda}\ln p(x,\lambda)]=E\left(x^{\alpha-1}\exp(-\lambda\sum_i^n x_i)(n\alpha-\lambda\sum_i^n x_i)\frac{\lambda^{n\alpha-n}}{\Gamma^n(\alpha)}\right)$~~

(b) $E\left(\frac{\bar{x}}{\alpha}\right)=\frac{1}{\alpha}E(\bar{x})=\frac{1}{\alpha}E(x)=\frac{1}{\alpha}\cdot\frac{\alpha}{\lambda}=\frac{1}{\lambda}=g(\lambda)$．$\therefore\frac{\bar{x}}{\alpha}$ 为 $g(\lambda)=\frac{1}{\lambda}$ 的无偏估计

(c) $Var\left(\frac{\bar{x}}{\alpha}\right)=\frac{1}{\alpha^2}Var(\bar{x})=\frac{1}{\alpha^2}\cdot\frac{1}{n}Var(x)=\frac{1}{n\alpha\lambda^2}$

$g(\lambda)=\frac{1}{\lambda}$ 的C-R下界为 $\frac{[g'(\lambda)]^2}{nI(\lambda)}=\frac{(-\frac{1}{\lambda^2})^2}{n\cdot\frac{\alpha}{\lambda^2}}=\frac{1}{n\alpha\lambda^2}=Var\left(\frac{\bar{x}}{\alpha}\right)$

$\therefore\frac{\bar{x}}{\alpha}$ 为 $g(\lambda)=\frac{1}{\lambda}$ 的有效估计．一定为UMVUE

~~贝叶斯估计 θ ... 的后验分布为 ...~~

~~X与θ的联合分布 (X, θ) =~~

$y_1 = \frac{n}{2}(S_n^2 - \bar{x})$

$y_2 = n - nS_n^2$

$y_3 = \frac{n}{2}(S_n^2 + \bar{x})$

(6) (a) $L(\theta, X) \propto \left(\frac{1-\theta}{2}\right)^{y_1}\left(\frac{1}{2}\right)^{y_2}\left(\frac{\theta}{2}\right)^{y_3}$ $\quad \ln L(\theta, x) \propto y_1 \ln\left(\frac{1-\theta}{2}\right) + y_2 \ln\frac{1}{2} + y_3 \ln\left(\frac{\theta}{2}\right)$

$\frac{d \ln L(\theta, x)}{d\theta} = \frac{y_1}{\theta - 1} + y_3 \cdot \frac{1}{\theta}$ $\quad$ 令 $\frac{d \ln L(\theta, x)}{d\theta} = 0$，得 $\hat{\theta} = \frac{y_3}{y_1 + y_3}$ $\quad$ (y_1, y_2, y_3 分别为 $X = -1, 0, 1$ 次数)

注意到 $E(X) = 0$，~~$Var(X)$~~ $S_n^2 = \frac{1}{n}(y_1 + y_3)$

样本均值 $\bar{x} = \frac{1}{n}(y_3 - y_1)$

~~$P(X = -1) \ldots P(X = 0) \ldots P(X = 1) = \frac{\theta}{2} \ldots \theta = \ldots$~~ $\quad \therefore \hat{\theta} = \frac{S_n^2 + \bar{x}}{2S_n^2}$

$E(X) = \frac{\theta - 1}{2} + 0 \times \frac{1}{2} + \frac{\theta}{2} = \theta - \frac{1}{2}$ $\quad \therefore$ 矩估计 $\hat{\theta} = \bar{x} + \frac{1}{2}$

(b) ~~$I(\theta) = E\left(\frac{\partial}{\partial\theta}\ln p\right)^2 = E\left(\frac{y_1}{\theta-1} + \frac{y_3}{\theta}\right)^2$~~ =

$p(x, \theta) = \frac{1}{2}(1-\theta)^{\frac{x^2 - x}{2}} \theta^{\frac{x^2 + x}{2}}$, $x = -1, 0, 1$ $\quad \ln p(x, \theta) = \frac{x^2 - x}{2}\ln(1-\theta) + \frac{x^2 + x}{2}\ln\theta - \ln 2$

$\frac{\partial}{\partial\theta}\ln p(x;\theta) = \frac{x^2 - x}{2}\cdot\frac{-1}{1-\theta} + \frac{x^2 + x}{2}\cdot\frac{1}{\theta}$ $\quad \frac{\partial^2}{\partial\theta^2}\ln p(x, \theta) = \frac{x^2 - x}{2}\cdot\frac{-1}{(1-\theta)^2} - \frac{x^2 + x}{2}\cdot\frac{1}{\theta^2} = -\frac{[(1-\theta)^2 + \theta^2]x^2 + [(1-\theta)^2 - \theta^2]x}{2\theta^2(1-\theta)^2}$

$\therefore I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2}\ln p(\cdot, \theta)\right] = \frac{[(1-\theta)^2 + \theta^2]E(X^2) + [(1-\theta)^2 - \theta^2]E(X)}{2\theta^2(1-\theta)^2}$

而 $E(X) = \theta - \frac{1}{2}$, $E(X^2) = 1 \times \frac{1-\theta}{2} + 0 + 1 \times \frac{\theta}{2} = \frac{1}{2}$ $\quad \therefore I(\theta) = \frac{1}{2\theta(1-\theta)}$

$\therefore$ C-R 下界为 $\frac{1}{nI(\theta)} = \frac{2\theta(1-\theta)}{n}$

(c) 由 MLE 的渐近正态性，$\hat{\theta}_n \sim AN\left(\theta, \frac{1}{nI(\theta)}\right) = AN\left(\theta, \frac{2\theta(1-\theta)}{n}\right) \to P(\hat{\theta} = \theta) = 1$

(7) $E(\theta \mid x) = \int_{\max[x_1, \ldots, x_n, \theta_0]}^{+\infty} \theta \cdot \pi(\theta \mid X)\, d\theta = \int_{\max[x_1, \ldots, \theta_0]}^{+\infty} \frac{(n+\beta)[\max[x_1, \ldots, x_n, \theta_0]]^{n+\beta}}{\theta^{n+\beta}}\, d\theta$

$= (n+\beta)[\max[x_1, \ldots, x_n, \theta_0]]^{n+\beta} \frac{[\max[x_1, \ldots, x_n, \theta_0]]^{-(n+\beta)+1}}{n+\beta-1} = \frac{n+\beta}{n+\beta-1}\max[x_1, \ldots, x_n, \theta_0]$

$\therefore \theta$ 的贝叶斯估计为 $\hat{\theta}_B = \frac{n+\beta}{n+\beta-1}\max[X_1, \ldots, X_n, \theta_0]$

(8) $\theta \sim Exp(\lambda)$, $\quad \pi(\theta) = e^{-\lambda\theta} I_{\theta > 0}$

$p(x_1, \ldots, x_n \mid \theta) = \prod_{i=1}^{n} \theta x_i^{\theta - 1} I_{0 < x_i < 1} = \theta^n (x_1 \cdots x_n)^{\theta - 1} I_{0 < x_1, \ldots, x_n < 1} = \theta^n e^{(\theta - 1)\ln(x_1 \cdots x_n)} I_{0 < x_1, \ldots, x_n < 1}$

联合分布 $h(x_1, \ldots, x_n, \theta) = \frac{1}{x_1 \cdots x_n}\theta^n \exp[(-\lambda + \ln(x_1 \cdots x_n))\theta] I_{0 < x_1 \cdots x_n < 1, \theta > 0}$

$X_1, \ldots X_n$ 的边际分布 $m(x_1, \ldots, x_n) = \int_0^{+\infty} \frac{1}{x_1 \cdots x_n}\theta^n \exp[(-\lambda + \ln(x_1 \cdots x_n))\theta] I_{0 < x_1, \ldots, x_n < 1}\, d\theta$

$= \frac{1}{x_1 \cdots x_n}\frac{\Gamma(n+1)}{[\lambda - \ln(x_1 \cdots x_n)]^{n+1}} I_{0 < x_1, \ldots, x_n < 1}$

即 θ 的后验分布(贝叶斯分布):

$\pi(\theta \mid x_1, \ldots, x_n) = \frac{h(x_1, \ldots, x_n, \theta)}{m(x_1, \ldots, x_n)} = \frac{[\lambda - \ln(x_1 \cdots x_n)]^{n+1}}{\Gamma(n+1)}\theta^n \exp[(-\lambda + \ln(x_1 \cdots x_n))\theta] I_{\theta > 0}$