概率论与数理统计 第六次作业 张锦程 2018012082

(1) 证明: (a) $\forall \varepsilon>0$, $P(|Y_n-0|\ge\varepsilon) = 1-P(-\varepsilon \le \frac{X_n}{n} \le \varepsilon)$

取 $n=[\frac{1}{\varepsilon}]+1$ 有 $-\varepsilon \le -\frac{1}{n} \le \frac{X_n}{n} \le \frac{1}{n} \le \varepsilon$ $\therefore P(|Y_n-0|\ge\varepsilon)=0$ #

[取整符]

(b) $P(Y_n=-1)=0$, $P(Y_n=1)=0$ $\therefore \forall\varepsilon>0$ $P(|Y_n-0|\ge\varepsilon)=1-P(-\varepsilon\le X_n^n\le\varepsilon)=1-P(|X_n|^n\le\varepsilon)$

取 $n=[\log_{|X_n|}\varepsilon]+1$, 有 $|X_n|^n<\varepsilon$, $P(|Y_n-0|\ge\varepsilon)=0$, #

(c) 需证明 $\forall\varepsilon>0$, $\delta>0$, $\exists n$ 使 $P(|Y_n-1|\ge\varepsilon) < \delta$

$|Y_n-1| = 1-Y_n \ge \varepsilon \Rightarrow Y_n \le 1-\varepsilon \Rightarrow X_1\le 1-\varepsilon,\ X_2\le 1-\varepsilon,\ \dots\ X_n\le 1-\varepsilon$

$\therefore P(|Y_n-1|\ge\varepsilon) = \left(\frac{2-\varepsilon}{2}\right)^n = (1-\frac{\varepsilon}{2})^n$

取 $n=[\log_{(1-\frac{\varepsilon}{2})}(\delta)]+1 > \log_{(1-\frac{\varepsilon}{2})}(\delta)$ 有 $(1-\frac{\varepsilon}{2})^n<\delta$ #

(2) (a) $\varphi_1(t)=\int_{-\infty}^{+\infty} e^{itx}\left(\frac{a}{2}e^{-a|x|}\right)dx = -\frac{ita}{t^2+a^2}$

$\therefore E_1(X)=\frac{1}{i}\varphi_1'(0) = \cancel{\dots} = \frac{-ia(3t^2+a^2)}{(t^2+a^2)^2}\Big|_{t=0} = -\frac{1}{a}$

$E_1(X^2)=\frac{1}{i^2}\varphi_1''(0) = \frac{2}{a^2}$

$\therefore Var_1(X) = \frac{2}{a^2}-(-\frac{1}{a})^2=\frac{1}{a^2}$

(b) $\varphi_2(t)=\int_{-\infty}^{+\infty} e^{itx}\frac{a}{\pi}\frac{1}{x^2+a^2}dx = \frac{a}{\pi}\left[\int_{-\infty}^{+\infty}\frac{i\sin tx}{x^2+a^2}dx+\int_{-\infty}^{+\infty}\frac{\cos tx}{x^2+a^2}dx\right]$

$= \cancel{\dots} = e^{-at}$

$\therefore E_2(X)=\frac{1}{i}\varphi_2'(0) = -\frac{a}{i} = ai$

$E_2(X^2)=\frac{1}{i^2}\varphi_2''(0)=\frac{a^2}{i^2} = -a^2<0$, 不合理, 则方差不存在

$\cancel{\therefore Var_2(X) = +a^2-(ai)^2=2a^2}$

(3) (a) $\lim_{n\to\infty}E(Y_n) = \lim_{n\to\infty}\frac{1}{n}\cdot n^2+(1-\frac{1}{n})\cdot 0 = \lim_{n\to\infty} n = +\infty$

(b) 它依概率收敛, 极限为0

$\forall\varepsilon>0$. $P(|Y_n-0|\ge\varepsilon)\le\frac{1}{n}$ $n\to\infty$, $\frac{1}{n}\to 0$

(4) $\cancel{Ga(\alpha,\lambda) \text{ 分布函数 } F_1 = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^x t^{\alpha-1}e^{-\lambda t}dt}$

$\cancel{Y=\frac{\lambda X-\alpha}{\sqrt{\alpha}} \text{ 分布函数 } F_2 = \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^Y\left(\frac{\sqrt{\alpha}Y+\alpha}{\lambda}\right)^{\alpha-1}e^{-(\sqrt{\alpha}Y+\alpha)}\frac{\sqrt{\alpha}}{\lambda}dY}$

$Y=\frac{\lambda X}{\sqrt{\alpha}} \sim Ga(\alpha,\sqrt{\alpha}) = \frac{(\sqrt{\alpha})^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-\sqrt{\alpha}y}$ $\cancel{\frac{\lambda}{\Gamma(\alpha)}\int_0^Y(\sqrt{\alpha}Y+\alpha)^{\alpha-1}e^{-\frac{\alpha}{\lambda}Y-\frac{\alpha\sqrt{\alpha}}{\lambda}}dY}$

$Z\sim N(\sqrt{\alpha},1)$

Y 特征函数为 $(1-\frac{it}{\sqrt{\alpha}})^{-\alpha}$ Z 特征函数为 $\exp(i\sqrt{\alpha}t-\frac{t^2}{2})$ 令 $\beta=\frac{1}{\alpha}$, 在 $\beta=0$处泰勒展开.

$\exp(i\sqrt{\alpha}t-\frac{t^2}{2}) = \cancel{1+i\sqrt{\alpha}x+(-0.5\alpha-0.5)x^2+o(x^2)} = 1+\frac{i}{\sqrt{\beta}}t+o(\beta) = 1+i\sqrt{\alpha}t+o(\frac{1}{\alpha})$

$(1-\frac{it}{\sqrt{\alpha}})^{-\alpha} = 1+\frac{i}{\sqrt{\beta}}t+o(\beta) = 1+i\sqrt{\alpha}t+o(\frac{1}{\alpha})$

$\therefore$ 当 $\alpha\to+\infty$ 时 $Y-Z\to 0$. 同时作变量代换 $Y'=Y-\sqrt{\alpha}$, $Z'=Z-\sqrt{\alpha}$

则 $Y'=\frac{\lambda X-\alpha}{\sqrt{\alpha}}$, $Z'\sim N(0,1)$ $\therefore Y'$ 依分布收敛到 $N(0,1)$

15) (a) $\varphi(t) = \int_{-\infty}^{+\infty} e^{itx} \frac{1}{\pi} \frac{\lambda}{\lambda^2+(x-\mu)^2} dx = \frac{1}{\pi} e^{it\mu} \int_{-\infty}^{+\infty} e^{i(\lambda t)z} \frac{1}{z^2+1} dz \quad (z = \frac{x-\mu}{\lambda})$

$= e^{it\mu} e^{-|\lambda t|}$ (由 Cauchy 分布的特征函数可得)

(b) $p(x) = p(y) = \frac{1}{\pi} \cdot \frac{1}{x^2+1}$

$\varphi_{X+Y}(t) = E(e^{it(X+Y)}) = \int_{-\infty}^{+\infty} e^{it(X+Y)} \frac{1}{\pi} \frac{1}{x^2+1} = \int_{-\infty}^{+\infty} e^{i(2t)x} \frac{1}{\pi} \frac{1}{x^2+1}$

$= e^{-2|t|} = e^{-|t|} \cdot e^{-|t|}$，但 $X=Y$，故 X 与 Y 不独立

(c) $\varphi_{\frac{1}{n}(X_1+\cdots+X_n)}(t) = \varphi_{\frac{X_1}{n}}(t) \cdot \varphi_{\frac{X_2}{n}}(t) \cdots \varphi_{\frac{X_n}{n}}(t) = e^{-\frac{1}{n}|t|} \cdots e^{-\frac{1}{n}|t|} = e^{-|t|}$

$\therefore \frac{1}{n}(X_1+\cdots+X_n)$ 的密度函数为 $p(y) = \frac{1}{\pi(1+x^2)}$

(6) X_n 不依概率收敛，因为 $n\to\infty$ 时 ~~$|X_n - X|$~~ $\exists\, n$，使得 $|X_n - X| \geq |X|$

则取 $\varepsilon \leq |X|$，$P(|X_n - X| \geq \varepsilon) \nrightarrow 0$

X_n 依分布收敛，因为 $X_n = X$ 或 $X_n = -X$，

$\lim_{n\to\infty} F_n(x) = F(x) = F(-x)$ 均为 $N(0,1)$ 的分布函数

(7) 不能，可以取 $X_n \sim N(0,1)$，$Y_n \sim N(0,1)$，$X = -X_n$，$Y = +X_n$

则 X_n，Y_n 均依分布收敛到 X, Y

但 $X_n + Y_n$ 不依分布收敛到 $X+Y = 0$

若 X_n 与 Y_n 相互独立，则 $X_n + Y_n$ 依分布收敛到 $X+Y$；因为此时 $\lim_{n\to\infty} F_{Z_n}(z) = \lim_{n\to\infty} \int_0^{z_n}\int_0^{u} p_{X_n}(u-t) p_{Y_n}(t)\, dt\, du$ $(Z = X+Y)\ (Z_n = X_n + Y_n)$

$= F_Z(z) = \int_0^{z}\int_0^{u} p_X(u-t) p_Y(t)\, dt\, du$

(8) $n\to\infty$ 时，Z_n 依概率收敛到 $E(Z_n) = N$

证明如下：设 $E(Y^2) = E(Y) = N \leq 1$

$Var(Y) = E(Y^2) - E^2(Y) = N - N^2$

$E(Z_n) = E\left(\frac{Y_1 + \cdots + Y_n}{n}\right) = E(Y) = N$

$Var(Z_n) = Var\left(\frac{Y_1+\cdots+Y_n}{n}\right) = \frac{1}{n} Var(Y) = \frac{1}{n}(N - N^2)$

由切比雪夫不等式：$P(|Z_n - EZ_n| \geq \varepsilon) = P(|Z_n - N| \geq \varepsilon) \leq \frac{Var(Z_n)}{\varepsilon^2} = \frac{\frac{1}{n}(N-N^2)}{\varepsilon^2} \leq \frac{\frac{1}{n}}{\varepsilon^2}$

$\forall \varepsilon > 0$，$n\to\infty$ 时，$P(|Z_n - E(Z_n)| \geq \varepsilon) \to 0$. #