概率论与数理统计 第七次作业 张锦程 2018012082 材84

1. $E(X_n)$ 存在，为 2.5，$E(Y_n)$ 存在，为 $\frac{1}{2}$

$\therefore$ $\{X_n\}$ 与 $\{Y_n\}$ 均符合大数定律（依概率收敛）

$\therefore Z_n = \dfrac{\frac{1}{n}(X_1+\cdots+X_n)}{\frac{1}{n}(Y_1+\cdots+Y_n)}$ 也依概率收敛

2. ~~设 F_+：轮盘公正，F_-：轮盘不公正~~
~~G_+：认为轮盘公正，G_-：认为轮盘不公正~~

设 S_n 为奇数点的和，$p=0.5$，$n=100$

$Y_n^* = \dfrac{S_n - np}{\sqrt{npq}} = \dfrac{S_n - 50}{50}$

则认为 $P(S_n \le 55) \approx \lim\limits_{n\to\infty} P(Y_n^* \le \frac{1}{10}) = \Phi(\frac{1}{10}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{1}{10}} e^{-\frac{t^2}{2}}dt = 0.54$

错误概率 46%

3. $E(X_n) = 5$，$Var(X_n) = 9$

(a) $S_n = X_1 + X_2 + \cdots + X_n$ $\quad Y_n^* = \dfrac{S_n - n\mu}{6\sqrt{n}} = \dfrac{S_n - 500}{30}$

$P(S_n \ge 440) \approx \lim\limits_{n\to\infty} P(Y_n^* \ge -2) = 1 - \Phi(-2) = \Phi(2) = 0.9772$

(b) $P(S_n \ge 200 + 5n) \approx \lim\limits_{n\to\infty} P\left(Y_n^* \ge \dfrac{200}{3\sqrt{n}}\right) = \cancel{\Phi(\ldots)} = 1 - \Phi\left(\dfrac{200}{3\sqrt{n}}\right) \le 0.05$

$\Phi\left(\frac{200}{3\sqrt{n}}\right) \ge 0.95$

查表得：$\dfrac{200}{3\sqrt{n}} \ge 1.65$ $\quad\therefore$ $n \le 1632.48$

n 最大值为 1632

(c) ~~$P(S_N \ge 1000) \approx \lim\limits_{n\to\infty} P\left(Y_N^* \ge \frac{1000-5N}{3\sqrt{N}}\right) = 1 - \Phi\left(\frac{1000-5N}{3\sqrt{N}}\right)$~~

对立命题：$N = 219$ 时，$S_N > 1000$，$P(S_{219} > 1000) \approx \lim\limits_{n\to\infty} P\left(Y_{219}^* \ge \dfrac{1000 - 219\times 5}{3\times\sqrt{219}}\right) = \Phi(2.14) = 0.9838$

$\therefore$ 概率近似值为 1.62%

4. $Var(W) = \dfrac{Var(X)}{16} + \dfrac{Var(Y)}{16} = \frac{1}{16}\left(\frac{1}{12} + \frac{1}{12}\right) = \frac{1}{96}$ $\quad W_N = X_N - Y_N$，$E(W_N) = 0$，$Var(W_N) = \iint\limits_{0\,0}^{1\,1} (x-y)^2 dx dy = \frac{1}{6}$

~~$\therefore P(|W - E(W)| \ge 0.001) \le \frac{\ldots}{0.001^2}$~~

$\lim\limits_{n\to\infty} P\left(\frac{1}{B_N}\left(\sum\limits_{i=1}^{N} W_N - \sum\limits_{i=1}^{N} E(W_N)\right) \le \dfrac{0.001}{B_N}\right) = \dfrac{1}{\sqrt{2\pi}}\int_{-0.004\sqrt{6}}^{0.004\sqrt{6}} e^{-t^2/2}dt \approx 0.8\%$

$\therefore P(|W - E(W)| < 0.001) \approx 0.8\%$

5. $\frac{1}{n^2}Var\left(\sum\limits_{k=1}^{n} X_k\right) = \frac{1}{n^2}\left[\sum\limits_{k=1}^{n} Var(X_k) + 2\sum\limits_{i=1}^{n-1}\sum\limits_{j=1}^{j=i} Cov(X_i, X_j)\right]$

由题，$\forall \varepsilon > 0$，$\exists N > 0$，$|i - j| > N$ 时，$Cov(X_i, X_j) < \varepsilon$，取 $\varepsilon \to 0$

$\therefore \lim\limits_{n\to\infty}\frac{1}{n^2}Var\left(\sum\limits_{k=1}^{n} X_k\right) < \lim\limits_{n\to\infty}\left(\dfrac{n \cdot Var\,max}{n^2} + \frac{2}{n^2}\sum\limits_{i=1}^{n-1}\sum\limits_{i-j=-N}^{i-j=N} Cov(X_i, X_j) + \frac{2}{n^2}\cdot\varepsilon\cdot(n-N+1)^2\right)$

$= 0 + \lim\limits_{n\to\infty}\frac{2}{n^2}\cdot(2N\cdot n)\, Cov(X_i, X_j)_{max} + 0$

$= 0$

$\therefore \lim\limits_{n\to\infty}\frac{1}{n^2}Var\left(\sum\limits_{k=1}^{n} X_k\right) = 0$，符合马尔可夫大数定律

6. 服从 $Y_n = \sum\limits_{k=1}^{n} X_k \sim P(1 + \sqrt{2} + \cdots + \sqrt{n})$ $\quad Var(Y_n) = 1 + \sqrt{2} + \cdots + \sqrt{n}$

$\lim\limits_{n\to\infty}\frac{1}{n^2}Var\left(\sum\limits_{k=1}^{n} X_k\right) < \lim\limits_{n\to\infty}\dfrac{n\sqrt{n}}{n^2} = \lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ 符合马尔可夫条件

7. 设 $a < h(x) < b$, $a, b \in \mathbb{R}$

X_i 相互独立. 则 $h(X_i)$ 相互独立.

则 $Var(h(X_i)) \le E(h^2(X_i)) < \max[a^2, b^2] = const1$

$\therefore Var(Y_n) = Var(\frac{1}{n}\sum_{i=1}^{n} h(X_i)) < \frac{n}{n^2} \cdot \max[a^2, b^2] = const1$

$n \to \infty$ 时. $Var(Y_n) \to 0$, $E(Y_n) \to E(h(X_i)) = const2$

则 $\frac{1}{n}\sum_{i=1}^{n} h(X_i)$ 近似为一单点分布, 其期望值为 $E(h(X_i))$

8. $n \to \infty$ 时: $Y_n = E(X_n)$ (由于 $Var(X_n)$ 存在, 则 $E(X_n)$ 必存在) (???)

$Z_n = \frac{1}{n}\sum_{i=1}^{n} (X_i - E(X_n))^2 = E(X_i - E(X_i))^2 = Var(X_i)$

由于 $\{X_n\}$ 独立同分布, 则 Z_n 依概率收敛到 σ^2.