概统第十三次作业 张锦程 2018012082

(1) $L(p) = P(X_1=x_1,\cdots,X_n=x_n;p) = p^{\sum x_i}(1-p)^{n-\sum x_i}$ 记 X 有p机率取1，(1-p)机率取0

$\ln L(p) = \sum x_i \ln p + (n-\sum x_i)\ln(1-p)$

令 $\frac{d\ln L(p)}{dp} = \frac{\sum x_i}{p} + \frac{n-\sum x_i}{p-1} = 0$，得：$\hat{p} = \frac{\sum x_i}{n} = \bar{x}$

$\therefore \Lambda(x_1,\cdots,x_n) = \frac{p(x_1,\cdots,x_n;\hat{p})}{p(x_1,\cdots,x_n;\hat{p}_0)} = \left(\frac{\bar{x}}{p_0}\right)^{n\bar{x}}\left(\frac{1-\bar{x}}{p_0}\right)^{n-n\bar{x}}$

$\therefore$ 拒绝域 $W = \{\Lambda(x_1,\cdots,x_n) \geq c\}$，c满足 $P_p(\Lambda(x_1,\cdots,x_n)\geq c) \leq \alpha$，且 $p=p_0$

~~即有~~ 它的显著水平为α

(2) ①当 H_0 不时：Θ_0 上 μ,σ^2 的MLE分别为 μ_0，$\frac{1}{n}\sum_{i=1}^n(x_i-\mu_0)^2$；

Θ 上 μ,σ^2 的MLE分别为 $\bar{x}$，$\frac{1}{n}\sum_{i=1}^n(x_i-\bar{x})^2$

$\therefore \Lambda(x_1,\cdots,x_n) = \left(\frac{\sum(x_i-\bar{x})^2}{\sum(x_i-\mu_0)^2}\right)^{\frac{n}{2}} = \left(1+\frac{t^2}{n-1}\right)^{-\frac{n}{2}}$，$t = \frac{\sqrt{n}(\bar{x}-\mu_0)}{s}$

②当 H_0 不足时：Θ_0 上 μ,σ^2 的MLE分别为 $\bar{x}$，$\frac{1}{n}\sum_{i=1}^n(x_i-\bar{x})^2$；

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它们的拒绝域均如下：$[\Lambda(x_1,\cdots,x_n)\geq c]$

对①来说，相当于 $[|t|\leq d]$，$d = t_{1-\alpha/2}(n-1)$，$c = [1+\frac{\alpha^2}{n-1}]^{\frac{n}{2}}$，α为显著水平

对②来说，相当于 $[|t|\geq d]$，$d = t_{1-\alpha/2}(n-1)$，$c = [1+\frac{\alpha^2}{n-1}]^{\frac{n}{2}}$，α为显著水平

(3) (a) 样本联合密度函数为 $\frac{\theta^n v^{\theta n}}{(\prod x_i)^{\theta+1}} = L(\theta,v)$

$\ln L(\theta,v) = n\ln\theta + n\theta\ln v - (\theta+1)\sum\ln x_i$

$\frac{d\ln L(\theta,v)}{d\theta} = \frac{n}{\theta} + n\ln v - \sum\ln x_i = 0$；$\frac{d\ln L(\theta,v)}{dv} = \frac{n\theta}{v} > 0$

$\Rightarrow$ v尽量大，$\therefore \hat{v} = x_{(1)}$ $(x\geq v)$

$\Rightarrow \theta = \frac{n}{\sum\ln\frac{x_i}{v}} = \frac{n}{\sum\ln\frac{x_i}{x_{(1)}}} = \frac{n}{T(x)}$

(b) 由(a)：$L(\theta,v)_{max} = \frac{\theta^n v^{n\theta}}{(\prod x_i)^{\theta+1}} = \frac{\theta^n \cdot \frac{\prod x_i^\theta}{e^n}}{(\prod x_i)^{\theta+1}} = \frac{\left(\frac{\theta}{e}\right)^n}{\prod x_i} = \frac{n^n}{T(x)^n\cdot e^{n-T(x)}}$

$\therefore \Lambda(x_1,\cdots,x_n) = \frac{\frac{\theta^n v^{n\theta}}{(\prod x_i)^{\theta+1}}}{\frac{x_{(1)}^n}{(\prod x_i)^2}} = \frac{\theta^n x_{(1)}^{n(\theta-1)}}{(\prod x_i)^{\theta-1}} = \frac{\left(\frac{n}{T(x)}\right)^n}{[e^{T(x)}]^{\cancel{\theta}-1}}$

拒绝域为 $\Lambda(x_1,\cdots,x_n)\geq b$

$\frac{d\Lambda}{dT} = \frac{-n^n\ n^n}{T^{2n}e^{2n-2T}}\left[nT^{n-1}e^{n-T} - e^{n-T}\cdot T^n\right]$

$= \frac{-n^n\cdot n^n e^{n-T}T^{n-1}}{T^{2n}e^{2n-2T}}(n-T)$ T=n时有极小值

$\therefore$ 拒绝域的形式为 $T\leq c$ 或 $T\geq d$，

~~两~~ 满足 $\Lambda(T=c) = \Lambda(T=d) = b$

(4) 统计量 $\chi^2 = \sum_{i=1}^{6} \frac{(n_i - np_{i0})^2}{np_{i0}} = \frac{(10-10)^2}{10} + \frac{(9-10)^2}{10} + \frac{(12-10)^2}{10} + \frac{(7-10)^2}{10} + \frac{(13-10)^2}{10} + \frac{(9-10)^2}{10}$

$= \frac{24}{10} = 2.4$

$\chi^2_{1-\alpha}(r-1) = \chi^2_{0.95}(5) = 11.07$ $\quad \chi^2 < 11.07$ 不在拒绝域内

∴可以认为该硬币均匀

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(5) 记三类品种分别对应数值1, 2, 3

$\therefore \bar{x} = p^2 + 4p(1-p) + 3(1-p)^2 = 3-2p$ 易看出 p 的 MLE 为 $\frac{3-\bar{x}}{2}$

证明如下: $L(p) = (p^2)^{n_1} \times [2p(1-p)]^{n_2} \times (1-p)^{2n_3} = p^{2n_1+n_2}(1-p)^{n_2+2n_3} \cdot 2^{n_2}$, 其中 n_i 为第 i 种的数目

$\ln L(p) = (2n_1+n_2)\ln p + (n_2+2n_3)\ln(1-p) + n_2 \ln 2$

令 $\frac{d \ln L(p)}{dp} = \frac{(2n_1+n_2)}{p} + \frac{(n_2+2n_3)}{p-1} = 0$ 得:

$(2n_1+2n_2+2n_3)p = (2n_1+n_2)$

又 $\begin{cases} n_1+n_2+n_3 = n \\ n_1+2n_2+3n_3 = n\bar{x} \end{cases} \Rightarrow 2n_1+n_2 = n(-\bar{x}+3) \Rightarrow 2np = n(-\bar{x}+3) \Rightarrow \boxed{p = \frac{3-\bar{x}}{2}}$ 为MLE

$\bar{x} = \frac{10+53\times2+46\times3}{10+53+46} = \frac{254}{109} \approx 2.33$ $\quad \therefore p \approx \cancel{1.835}\ 0.33486$, $p_1 = \cancel{3.369}\ 0.112$, $p_2 = 0.446$, $p_3 = 0.442$

$\therefore \chi^2 = \frac{(10-12.208)^2}{12.208} + \frac{(53-48.614)^2}{48.614} + \frac{(46-48.178)^2}{48.178} = 0.39935 + 0.3957 + 0.0985 = 0.89355$

而 $\chi^2_{0.95}(2) = 5.9915$ $\quad \therefore \chi^2 < 5.9915$, 不在拒绝域内

∴数据与模型相符合.

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(6) $p_{11} = \frac{66}{150} \times \frac{38}{150} = 0.1115$

$p_{12} = \frac{66}{150} \times \frac{34}{150} = 0.0997$

$p_{13} = \frac{66}{150} \times \frac{78}{150} = 0.2288$

$p_{21} = \frac{84}{150} \times \frac{38}{150} = 0.14187$

$p_{22} = \frac{84}{150} \times \frac{34}{150} = 0.12693$

$p_{23} = \frac{84}{150} \times \frac{78}{150} = 0.2912$

$\therefore \chi^2 = \sum_{j=1}^{3}\sum_{i=1}^{2} \frac{(n_{ij} - n\hat{p}_{ij})^2}{n\hat{p}_{ij}}$

$= \frac{(18-16.72)^2}{16.72} + \frac{(15-14.96)^2}{14.96} + \frac{(33-34.32)^2}{34.32} + \frac{(20-21.28)^2}{21.28} + \frac{(19-19.04)^2}{19.04} + \frac{(45-43.68)^2}{43.68}$

$= 0.09799 + 0.00\ldots + 0.05077 + 0.07699 + 0.00 + 0.03989$

$= 0.26564$

$\chi^2_{0.95}(5) = 11.0705 > \chi^2$ $\quad$ ∴原假设, 臂部大小与智商不相关