量子与统计　第五周作业　2018011082　材84　张锦程

1. 取 $\alpha = x$. $[\hat L_x, \hat L^2] = [\hat L_x, \hat L_x^2 + \hat L_y^2 + \hat L_z^2] = [\hat L_x, \hat L_x]\hat L_x + [\hat L_x, \hat L_y]\hat L_y + [\hat L_x, \hat L_z]\hat L_z + \hat L_y[\hat L_x, \hat L_y] + \hat L_z[\hat L_x, \hat L_z]$

$$\hat L \times \hat L = \begin{vmatrix} \vec x & \vec y & \vec z \\ L_x & L_y & L_z \\ L_x & L_y & L_z \end{vmatrix} = (\hat L_z \hat L_y - \hat L_y \hat L_z)\vec x + (\hat L_z \hat L_x - \hat L_x \hat L_z)\vec y + (\hat L_x \hat L_y - \hat L_y \hat L_x)\vec z$$

$$= i\hbar(\hat L_x \vec x + \hat L_y \vec y + \hat L_z \vec z) = i\hbar \hat L$$

$\Rightarrow [\hat L_z, \hat L_y] = i\hbar \hat L_x \qquad [\hat L_x, \hat L_y] = i\hbar \hat L_z, \quad [\hat L_z, \hat L_x] = i\hbar \hat L_y$

$\therefore [\hat L_x, \hat L^2] = i\hbar \cancel{\hat L_z} \hat L_y + (-i\hbar \hat L_y)\cancel{\hat L_z} + \hat L_y \cancel{(i\hbar \hat L_z)} + \hat L_z(-i\hbar \hat L_y)$

$= 0$ #

2. 取非 Hermite 算符 F

令 $A = \frac{F + F^*}{2} \qquad B = \frac{F - F^*}{2i}$

由 $(F^*)^* = F$ 可得 $A^* = \frac{F^* + F}{2} = A \qquad B^* = \frac{F^* - F}{-2i} = B$

则 A 与 B 均为厄密算符

因可以 $F = A + iB$，故：$A = \frac{F+F^*}{2}$，$B = \frac{F-F^*}{2i}$

3. $\bar x = \int_{-\infty}^{+\infty} \psi^*(x)\, x\, \psi(x)\, dx = \int_{-\infty}^{+\infty} \frac{1}{2\pi\gamma^2}\, x \cdot \exp\left(-\frac{x^2}{2\gamma^2}\right) dx = -\frac{1}{2\pi} \exp\left(-\frac{x^2}{2\gamma^2}\right)\Big|_{-\infty}^{+\infty} = 0$

$\overline{x^2} = \int_{-\infty}^{+\infty} \psi^*(x)\, x^2 \cdot \psi(x)\, dx = \int_{-\infty}^{+\infty} \frac{1}{2\pi\gamma^2} \cdot x^2 \cdot \exp\left(-\frac{x^2}{2\gamma^2}\right) dx = \int_{-\infty}^{+\infty} \frac{\sqrt{2}\gamma}{\pi} t^2 \exp(-t^2)\, dt =$

$\frac{\sqrt{2}\gamma}{\pi} \cdot \frac{\sqrt{\pi}}{4} \cdot \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-t^2} dt = \cancel{\frac{\sqrt{2}}{2\sqrt{\pi}}}\, \gamma^2$

$\therefore (\Delta x)^2 = \overline{x^2} - \bar x^2 = \cancel{\frac{\sqrt{2}}{2\sqrt{\pi}}}\, \gamma^2$

$\bar p = -i\hbar \int_{-\infty}^{+\infty} \psi^*(x) \frac{d}{dx} \psi(x)\, dx = \frac{-i\hbar}{2\pi\gamma^2} \int_{-\infty}^{+\infty} \left(\frac{i}{\hbar} p_0 - \frac{x}{2\gamma^2}\right) \exp\left(-\frac{x^2}{2\gamma^2}\right) dx$

$= \frac{+i\hbar}{2\pi\gamma^2} \cdot \frac{i p_0}{\hbar} \sqrt{2\pi}\, \gamma = \cancel{\frac{p_0\sqrt{2\pi}}{2\pi\gamma}}\; p_0$

$\overline{p^2} = -i\hbar \int_{-\infty}^{+\infty} \psi^*(x) \left(\frac{d}{dx}\right)^2 \psi(x)\, dx = \frac{-i\hbar}{2\pi\gamma^2} \int_{-\infty}^{+\infty} \left[\frac{x^2}{4\gamma^4} - \frac{i}{\hbar} p_0 \frac{x}{2\gamma^2} - \frac{p_0^2}{\hbar^2} - \frac{1}{2\gamma^2}\right] \exp\left(-\frac{x^2}{2\gamma^2}\right) dx$

$= \frac{-i\hbar}{2\pi\gamma^2} \left[\frac{\gamma}{4}\sqrt{\pi} - \left(\frac{p_0^2}{\hbar^2} + \frac{1}{2\gamma^2}\right)\sqrt{2}\gamma\sqrt{\pi}\right] = \frac{\hbar^2}{4\gamma^2} + p_0^2$

$(\Delta p)^2 = \overline{p^2} - \bar p^2 = \frac{\hbar^2}{4\gamma^2}$

$\therefore (\Delta x)^2 \cdot (\Delta p)^2 = \frac{\hbar^2}{4\gamma^2} \cdot \gamma^2 = \frac{\hbar^2}{4}$

4. $\frac{dp_x}{dt} = \frac{1}{i\hbar}[p_x, H] = \frac{1}{i\hbar}[p_x, \omega L_z] = \frac{\omega}{i\hbar}[p_x, x p_y - y p_x] = \omega p_y$

$\frac{dp_y}{dt} = \frac{1}{i\hbar}[p_y, H] = \frac{1}{i\hbar}[p_y, \omega L_z] = \frac{\omega}{i\hbar}[p_y, x p_y - y p_x] = -\frac{\omega}{i\hbar}[p_y, y p_x] = -\frac{\omega}{i\hbar}[p_y, y]p_x = -\omega p_x$

$\frac{dp_z}{dt} = \frac{1}{i\hbar}[p_z, H] = \frac{1}{i\hbar}[p_z, \omega L_z] = \frac{\omega}{i\hbar}[p_z, x p_y - y p_x] = 0$

$\therefore \frac{d^2 p_x}{dt^2} = -\omega^2 p_x, \quad \frac{d^2 p_y}{dt^2} = -\omega^2 p_y, \quad \frac{dp_z}{dt} = 0$

$\therefore \bar p_x(t) = p_0 \cos(\omega t), \quad \bar p_y(t) = -p_0 \sin(\omega t), \quad \bar p_z(t) = 0$

因此沿磁场方向（z轴方向）动量守恒

由 $\hat L \times \hat L = i\hbar \hat L$，$[\hat L, \hat p^2] = 0$，$[\hat L, \hat p^2] = 0$，可知：$[\hat L_z, \hat H] = 0$，沿磁场方向的角动量守恒

又 H 不显含 t，则能量守恒

5. $K=LM$. $[L,M]=LM-ML=1$

$K\psi_n=\lambda_n\psi_n$

$K(L\psi_n)=LML\psi_n=\cancel{L(1+ML)\psi_n=\psi_n+ML\psi_n}$

$=L(K-1)\psi_n=LK\psi_n-L\psi_n=L\lambda_n\psi_n-L\psi_n=(\lambda_n-1)L\psi_n$ #

$K(M\psi_n)=LMM\psi_n=(1+ML)M\psi_n=M\psi_n+MLM\psi_n=M\psi_n+MK\psi_n=M\psi_n+\lambda_nM\psi_n$
$=(\lambda_n+1)M\psi_n$ #

则 $L\psi_n$, $M\psi_n$ 为 K 的本征函数, 本征值分别为 (λ_n-1) 与 (λ_n+1)

6. 证明: $\overline{P_x}=\int\psi_n^*(r)P_x\psi_n(r)d\tau=\frac{1}{E_n}\int\psi_n^*(r)P_xH\psi_n(r)d\tau$

$=\int\left(\frac{1}{E_n}H\psi_n(r)\right)^*P_x\psi_n(r)d\tau=\frac{1}{E_n}\int\psi_n^*(r)HP_x\psi_n(r)d\tau$

$\therefore\int\psi_n^*(r)(P_xH-HP_x)\psi_n(r)d\tau=0$, 对任意 $\psi_n(r)$ 均成立.

$\therefore$ 有 $\frac{dP_x}{dt}=[P_x,H]=P_xH-HP_x=0$

$\therefore\frac{d}{dt}\overline{P_x}(t)=0$, $\overline{P_x}(t)=C$, 又由于粒子处于束缚态. $\therefore\overline{P_x}(t)=0$

同理可得 $\overline{P_y}(t)=\overline{P_z}(t)=0$