概统第九次作业　张锦程　2018012082　材84

(1) 似然函数 $f(x_1,\cdots,x_n;\theta,H) = \frac{1}{\theta^n}\exp\left(-\frac{\sum x_i - nH}{\theta}\right)I_{x_{(1)}} = \frac{1}{\theta^n}\exp\left(-\frac{n\bar{x}-nH}{\theta}\right)I_{x_{(1)}} = g(\bar{x}, x_{(1)}, \theta, H)\cdot h(x_1,\cdots,x_n)$

其中 $h(x_1,\cdots,x_n)=1$. 由因子分解定理得 $(\bar{x}, x_{(1)})$ 为 (θ,H) 的充分统计量

$$I_{x_{(1)}} = \begin{cases} 1 & x_{(1)} > H \\ 0 & x_{(1)} \le H \end{cases}$$

(2) $f(x_1,\cdots,x_n) = \frac{\lambda^{\alpha n}}{[\Gamma(\alpha)]^n}\prod_{i=1}^n x_i^{\alpha-1}\exp\left(-\lambda\sum_{i=1}^n x_i\right)$

设 $T=(t_1,t_2)$ $\quad \boxed{t_1=\prod_{i=1}^n x_i \quad t_2=\sum_{i=1}^n x_i}$

则 $f(x_1,\cdots,x_n) = \frac{\lambda^{\alpha n}}{[\Gamma(\alpha)]^n}t_1^{\alpha-1}\exp(-\lambda t_2) = g(t_1,t_2,\alpha,\lambda)\,h(x_1,\cdots,x_n)$

其中 $h(x_1,\cdots,x_n)=1$. 则由因子分解定理得 (t_1,t_2) 为 (α,λ) 的充分统计量

$Ga(1,\lambda) = Exp(\lambda)$

(3) (a) $E(\bar{x}) = \frac{1}{\lambda} = \frac{1}{\lambda}$, 故 $\bar{x}$ 为 $\frac{1}{\lambda}$ 的无偏估计

(b) $E(\frac{1}{\bar{x}}) = E\left(\frac{n}{\sum_{i=1}^n x_i}\right)$ 由伽玛分布的可加性 ($Z=X+Y\sim Ga(\alpha_1+\alpha_2,\lambda)$):

$\therefore \sum_{i=1}^n\left(\frac{x_i}{n}\right)\sim Ga(1,\lambda) = Exp(\lambda) = \begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & x<0\end{cases}$

$\therefore E\left(\frac{n}{\sum x_i}\right) = \int_0^{+\infty}\frac{1}{x}\lambda e^{-\lambda x}\,dx = \infty \ne \frac{1}{\lambda}$. 故 $\frac{1}{\bar{x}}$ 不是 λ 的无偏估计

(c) 构造 $T=\frac{1}{2}(x_{(1)}+x_{(n)})$. $(x_{(1)}, x_{(n)})$ 的联合分布密度函数为 $n(n-1)\lambda^2(e^{-\lambda y}-e^{-\lambda z})^{n-2}e^{-\lambda y-\lambda z}$

故: $E(x_{(1)}+x_{(n)}) = \iint (y+z)\,n(n-1)\lambda^2(e^{-\lambda y}-e^{-\lambda z})^{n-2}e^{-\lambda y-\lambda z}\,dz\,dy$

$= \int_0^{+\infty} -\frac{\lambda e^{-2z}}{4n}\left(4(z+1)-e^{z}(2z+1)\right)dz$

(???)

$=2\lambda \qquad E\left(\frac{x_{(1)}+x_{(n)}}{2}\right)=\lambda$

(4) (a) $P(x)=1.\quad F(x)=x-\theta+\frac{1}{2}$

$\therefore P_{1n}(y,z) = \frac{n!}{0!\,(n-2)!\,(n-n)!}(y-\theta+\frac{1}{2})^0\,(z-y)^{n-1-1}\,(\frac{1}{2}+\theta-z)^0\times1\times1 = n(n-1)(z-y)^{n-2}$

$\therefore E\left(\frac{x_{(1)}+x_{(n)}}{2}\right) = \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}\frac{1}{2}(y+z)\,n(n-1)(z-y)^{n-2}\,dz\,dy = \left[-\frac{n}{2}(y+z)(z-y)^{n-1}-\frac{1}{2}(z-y)^n\right]\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}$

$E(x) = E(U(\theta-\frac{1}{2},\theta+\frac{1}{2}))=\theta$. 故 $\bar{x}$ 与 $\frac{1}{2}(x_{(1)}+x_{(n)})$ 均为参数为 θ 的无偏估计

(b) $Var(\bar{x}) = \frac{\sigma^2}{n} = \frac{1}{12n}$

$Var\left(\frac{x_{(1)}+x_{(n)}}{2}\right) = \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}\int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}\frac{1}{4}(y+z)^2(n-1)n(z-y)^{n-2}\,dz\,dy - E^2\left(\frac{x_{(1)}+x_{(n)}}{2}\right)$

$= \int_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}\left[-\frac{n}{4}(y+z)^2(z-y)^{n-1}-\frac{1}{2}(y+z)(z-y)^n-\frac{1}{2(n+1)}(z-y)^{n+1}\right]\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}dz - \theta^2$

$= \left[-\frac{(z-y)^n\left(n^2(y+z)^2+n(y^2+6yz+5z^2)+8z^2\right)}{4(n+1)(n+2)} + \frac{(z-y)^{n+1}\left((n+1)y+(n+3)z\right)}{2(n+1)(n+2)} + \frac{(z-y)^{n+2}}{2(n+1)(n+2)}\right]\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}}\Big|_{\theta-\frac{1}{2}}^{\theta+\frac{1}{2}} - \theta^2$

$= \frac{(4n^2+12n+8)\left((2\theta+1)^2-(2\theta-1)^2\right)}{-4(n+1)(n+2)} + \frac{(2n+4)\left((\theta+\frac{1}{2})-(\theta-\frac{1}{2})\right)}{2(n+1)(n+2)} + \frac{0}{2(n+1)(n+2)} - \theta^2$

(???)

$= -\cancel{\ }-4\theta+\frac{1}{n+1}-\theta^2$

$\bar{x}$ 比 $\frac{x_{(1)}+x_{(n)}}{2}$ 更有效, 它综合了所有样本的信息, 方差更小

(5)(a) $E(Z)=E(aS_1^2+bS_2^2)=aE(S_1^2)+bE(S_2^2)=(a+b)\sigma^2=\sigma^2$ $\therefore aS_1^2+bS_2^2$ 为 σ^2 的无偏估计

(b) S_1^2 与 S_2^2 相互独立，$\therefore Var(Z)=a^2Var(S_1^2)+b^2Var(S_2^2)$

~~S_1^2 为 σ^2 的相合估计，S_2^2 为 σ^2 的相合估计~~

$\therefore$ ~~$a^2S_1^2+b$ $aS_1^2+bS_2^2$ 为~~

$\frac{(n_1-1)S_1^2}{\sigma^2}\sim\chi^2(n_1-1)$ $\qquad$ $\frac{(n_2-1)S_2^2}{\sigma^2}\sim\chi^2(n_2-1)$

$\therefore Var(S_1^2)=\frac{2n_1}{(n_1-1)^2}\sigma^4$ $\qquad$ $Var(S_2^2)=\frac{2n_2}{(n_2-1)^2}\sigma^4$

$\therefore Var(Z)=2\sigma^4\left(\frac{n_1}{(n_1-1)^2}a^2+\frac{n_2}{(n_2-1)^2}b^2\right)=2\sigma^4(Aa^2+Bb^2)$

又 $a+b=1$，故当 $\begin{cases} a=\frac{B}{A+B}=\frac{n_2(n_1-1)^2}{n_1(n_2-1)^2+n_2(n_1-1)^2} \\ b=\frac{n_1(n_2-1)^2}{n_1(n_2-1)^2+n_2(n_1-1)^2}\end{cases}$ 时，Z方差最小

(6)、对于该样本：$E(X)=mp$ $\qquad$ $Var(X)=mp(1-p)$

$\Rightarrow m=\frac{E(X)-Var(X)}{E(X)}$，$p=\frac{E^2(X)}{E(X)-Var(X)}$

$\Rightarrow$ 矩估计 $\hat{m}=\frac{\bar{x}-S^2}{\bar{x}}$，$\hat{p}=\frac{\bar{x}^2}{\bar{x}-S^2}$

(7)(a) $E(X)=\int_0^\theta\frac{2x}{\theta^2}(\theta-x)d\theta=\frac{\theta}{3}$ $\qquad$ $E(X^2)$ ~~$Var(X)$~~ $=\int_0^\theta\frac{2x^2}{\theta^2}(\theta-x)d\theta=\frac{2}{3}\theta^2-\frac{1}{2}\theta^2=\frac{1}{6}\theta^2$

$\therefore Var(X)=E(X^2)-E^2(X)=\frac{1}{18}\theta^2$

$\Rightarrow\hat{\theta}=3\bar{x}$ 或 $\hat{\theta}=$ ~~$18Var$~~ ~~$\sqrt{18}S$~~ $3\sqrt{2}S$

(b) $E(X)=\int_u^\infty\frac{x}{\theta}\exp\left(-\frac{x-u}{\theta}\right)d\theta=\frac{\theta(\theta+u)}{\theta}=\theta+u$，$E(X^2)=\int_u^\infty\frac{x^2}{\theta}\exp\left(-\frac{x-u}{\theta}\right)d\theta=2\theta^2+u^2+2\theta u$

$\therefore Var(X)=E(X^2)-E^2(X)=2\theta^2+u^2+2\theta u-(\theta+u)^2=\theta^2$

$\therefore\hat{\theta}=S$，$\hat{u}=\bar{x}-S$

(8) $Var\left(\frac{2jX_j}{n(n+1)}\right)=\frac{4j^2Var(X)}{n^4(n+1)^2}$ $\qquad$ $\therefore\frac{1}{n^2}Var\left(\frac{2}{(n+1)n}\sum_{j=1}^n jX_j\right)=\frac{\cancel{4}\,4}{n^3\,\cancel{n}(n+1)^2}Var(X)\frac{\cancel{n}(n+1)(2n+1)}{6}$

$=\frac{4(2n+1)}{n^3(n+1)}Var(X)\to0\quad(n\to\infty)$

由马尔可夫大数定律 $\hat{u}$ 为 u 的相合统计量

$E(\hat{u})=E\left(\frac{2}{n(n+1)}\sum_{j=1}^n j\right)\cdot E(X_j)=u$

$\frac{1^2+2^2+3^2}{}$

$\frac{j(j+1)(2j+1)}{6}$