Theorem 1. *Let $X_1, X_2, \ldots$ be i.i.d. with $E|X_1| = \infty$ and let $S_n = X_1 + \cdots + X_n$. Let a_n be a sequence of positive numbers with a_n/n increasing. Then* $\limsup_{n\to\infty} |S_n|/a_n = 0$ *or ∞ according as $\sum_n P(|X_1| \geq a_n) < \infty$ or $= \infty$*

Proof. Since $a_n/n \uparrow$, $a_{kn} \geq k a_n$ for any integer k. Using this and $a_n \uparrow$,

$$\sum_{n=1}^{\infty} P(|X_1| \geq k a_n) \geq \sum_{k=1}^{\infty} P(|X_1| \geq a_{kn}) \geq \frac{1}{k} \sum_{m=k}^{\infty} P(|X_1| \geq a_m)$$

The last observation shows that if the sum is infinite,$\limsup_{n\to\infty} |X_n|/a_n = \infty$. Since $max\{|S_{n-1}|,|S_n|\} \geq |X_n|/2$, it follows that $\limsup_{n\to\infty} |S_n|/a_n = \infty$.

To prove the other half, we begin with the identity

$$(*) \qquad \sum_{m=1}^{\infty} m P(a_{m-1} \leq |X_i| < a_m) = \sum_{n=1}^{\infty} P(|X_i| \geq a_{n-1})$$

To see this, write $m = \sum_{n=1}^{m} 1$ and then use Fubini's theorem. We now let $Y_n = X_n 1_{(|X_n| < a_n)}$, and $T_n = Y_1 + \ldots + Y_n$. When the sum is finite, $P(Y_n \neq X_n\ i.o.) = 0$ and it suffices to investigate the behavior of the T_n. To do this, we let $a_0 = 0$ and compute

$$\begin{aligned}
\sum_{n=1}^{\infty} var(Y_n/a_n) &\leq \sum_{n=1}^{\infty} E{Y_n}^2/{a_n}^2 \\
&= \sum_{n=1}^{\infty} {a_n}^{-2} \sum_{m=1}^{n} \int_{[a_{m-1}, a_m)} y^2 dF(y) \\
&= \sum_{m=1}^{\infty} \int_{[a_{m-1}, a_m)} y^2 dF(y) \sum_{n=m}^{\infty} {a_n}^{-2}
\end{aligned}$$

Since $a_n \geq n a_m/m$, we have $\sum_{n=m}^{\infty} {a_n}^{-2} \leq (m^2/{a_m}^2) \sum_{n=m}^{\infty} n^{-2} \leq C m {a_m}^{-2}$, so

$$\leq C \sum_{m=1}^{\infty} m \int_{[a_{m-1}, a_m)} dF(y)$$

Using $(*)$ now, we conclude $\sum_{n=1}^{\infty} var(Y_n/a_n) < \infty$.

The last step is to show $ET_n/a_n \to 0$. To begin, we note that if $E|X_i| = \infty$, $\sum_{n=1}^{\infty} P(|X_i| > a_n) < \infty$, and $a_n/n \uparrow$ we must have $a_n/n \uparrow \infty$. To estimate ET_n/a_n now, we observe that

$$\begin{aligned}
\left| {a_n}^{-1} \sum_{m=1}^{n} EY_m \right| &\leq {a_n}^{-1} n \sum_{m=1}^{n} E(|X_m|\,; |X_m| < a_m) \\
&\leq \frac{n a_N}{a_n} + \frac{n}{a_n} E(|X_i|\,; a_N \leq |X_i| < a_n)
\end{aligned}$$

where the last inequality holds for any fixed N. Since $a_n/n \to \infty$, the first term converges to 0. Since $m/a_m \downarrow$, the second is

$$\begin{aligned}
&\leq \sum_{m=N+1}^{n} \frac{m}{a_m} E(|X_i|\,; a_{m-1} \leq |X_i| < a_m) \\
&\leq \sum_{m=N+1}^{n} m P(a_{m-1} \leq |X_i| < a_m)
\end{aligned}$$

($*$) shows that the sum is finite, so it is small if N is large and the desired result follows. $\square$