CSE 586A Problem Set 1

Name: Jin Yang
UID: 467527

1. Euler's method

Question: For $\frac{dy}{dt} + 2y = 2 - e^{-4t}$, $y(0) = 1$, derive its closed-form solution.

Solution: $y = Ce^{-\int P(x)dx} + e^{-\int P(x)dx} \int Q(x)e^{\int P(x)dx}dx$

In this question, $P(x) = 2$ and $Q(x) = 2 - e^{-4t}$. Then I change variable x with variable t .

$y = Ce^{-\int 2dt} + e^{-\int 2dt} \int (2 - e^{-4t})e^{\int 2dt}dt$ The Property of derivative equation

$y = Ce^{-2t} + e^{-2t} \int (2 - e^{-4t})e^{2t}dt$ Calculation of integral

$y = Ce^{-2t} + e^{-2t} \int (2e^{2t} - e^{-2t})dt$ Mathematical Calculation

$y = Ce^{-2t} + e^{-2t}(e^{2t} + \frac{1}{2}e^{-2t})$ Calculation of integral

$y = Ce^{-2t} + 1 + \frac{1}{2}e^{-4t}$ Mathematical Calculation

When $t = 0$ and $y = 1$, I can calculate and get the value of parameter C, $C = -\frac{1}{2}$

$y = 1 - \frac{1}{2}e^{-2t} + \frac{1}{2}e^{-4t}$ Final solution

From $\frac{dy}{dt} + 2y = 2 - e^{-4t}$, I can get the equation $\frac{dy}{dt} = 2 - e^{-4t} - 2y$.

From the definition of Euler's method, $y_{n+1} = y_n + h * f(t_n, y_n)$

I set step size to 0.1, 0.05, 0.01, 0.005, 0.001 respectively, and then I find approximation of solution at t = {1, 2, 3, 4, 5}. Finally, I calculate the exact solution from closed-form solution and compare these exact solutions to approximate solutions. Calculation results are presented in the Table I.

Table I. Exact Solutions and Approximate Solutions

Step Size	0.1	0.05	0.01	0.005	0.001	Exact Solutions
1	0.9313	0.9365	0.9405	0.9410	0.9414	0.9415
2	0.9914	0.9911	0.9910	0.9910	0.9910	0.9910
3	0.9991	0.9989	0.9988	0.9988	0.9988	0.9988
4	0.9999	0.9999	0.9998	0.9998	0.9998	0.9998
5	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

Then I plot the approximated values in the graph. The plot is shown in Figure 1.

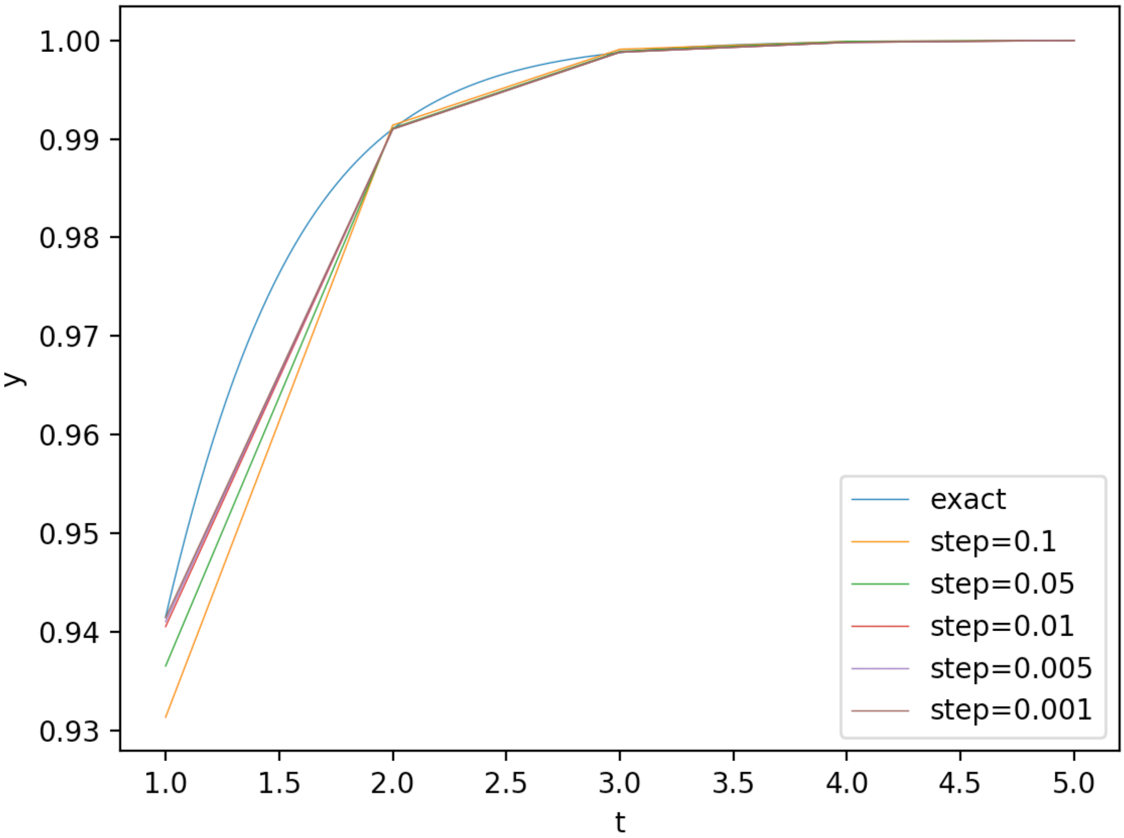

Figure 1. Approximated Values Plot

2. Geodesic Shooting

2.1 Problem Description

The goal of this question is to implement geodesic shooting by two strategies. After implementing geodesic shooting by these two methods, I need to compare the differences between two final transformations ϕ_1 at the time point t = 1. Then I can deform a given source image by using the transformations ϕ_1 obtained from these two methods.

2.2 Solution

I apply two strategies to implement geodesic shooting algorithm. The first one is presented below.

$$\frac{dv_t}{dt} = K[(Dv_t)^T v_t + div(v_t v_t^T)]$$
$$\frac{d\phi_t}{dt} = v_t \circ \phi_t$$

The second one is presented below.

$$\frac{dv_t}{dt} = -K[(Dv_t)^T v_t + div(v_t v_t^T)]$$
$$\frac{d\phi_t}{dt} = -D\phi_t \circ v_t$$

2.3 Experimental Details

As for the first method, before updating the velocity, I get the initial velocity from the file 'v0Spatial.mhd'. Then I calculate the derivative of velocity $\frac{dv_t}{dt}$ based on the first given strategy. After getting the value of $\frac{dv_t}{dt}$, I apply Euler's method to update the velocity. The step size is 0.01. The reason is that when I set the value of step size, I try different values, such as 1, 0.5, 0.1, 0.01, 0.001, and then I find that I can get the best results when I set the value of step size to 0.01.

$$v_{t+1} = v_t + stepsize * \frac{dv_t}{dt}$$

When I update velocity once, then I use the new value of v_t to calculate the derivative of transformations $\frac{d\phi_t}{dt}$. After acquiring the value of $\frac{d\phi_t}{dt}$, I apply Euler's method to update transformations. The step size is 0.01, which is the same with the step size that I use to update velocity. The reason is that when I set the value of step size, I try different values, such as 1, 0.5, 0.1, 0.01, 0.001, and then I find that I can get the best results when I set the value of step size to 0.01.

$$\phi_{t+1} = \phi_t + stepsize * \frac{d\phi_t}{dt}$$

As for the second methods, before updating the velocity, I get the initial velocity from the file ‘v0Spatial.mhd’. Then I calculate the derivative of velocity $\frac{dv_t}{dt}$ based on the second given strategies. After getting the value of $\frac{dv_t}{dt}$, I apply Euler’s method to update the velocity. The step size is 0.0007. The reason is that when I set the value of step size, I try different values, such as 0.1, 0.005, 0.001, 0.0005, 0.0001, and then I find that I can get the best results when I set the value of step size to 0.0007.

$$v_{t+1} = v_t + stepsize * \frac{dv_t}{dt}$$

When I update velocity once, then I use the new value of v_t to calculate the derivative of transformations $\frac{d\phi_t}{dt}$. After acquiring the value of $\frac{d\phi_t}{dt}$, I apply Euler’s method to update transformations. The step size is 0.0007, which is the same with the step size to update the velocity. The reason is that when I set the value of step size, I try different values, such as 0.1, 0.005, 0.001, 0.0005, 0.0001, and then I find that I can get the best results when I set the value of step size to 0.0007.

$$\phi_{t+1} = \phi_t + stepsize * \frac{d\phi_t}{dt}$$

The number of iterations is 100, which means that I apply two strategies to calculate the value of $\frac{dv_t}{dt}$ and the value of $\frac{d\phi_t}{dt}$, and then I update velocity and transformation 100 times through applying Euler’s method.

2.4 Experimental Results

I read in the source image from the file ‘source.mhd’. This image is used to inter-plot with transformation. The source image is shown in Figure 2.

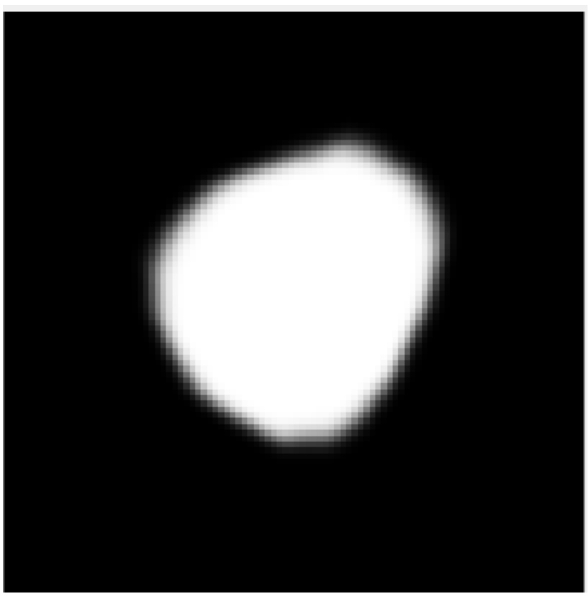

Figure 2. Source Image

After implementing the first geodesic shooting strategy, I plot the final image. The result is shown as Figure 3.

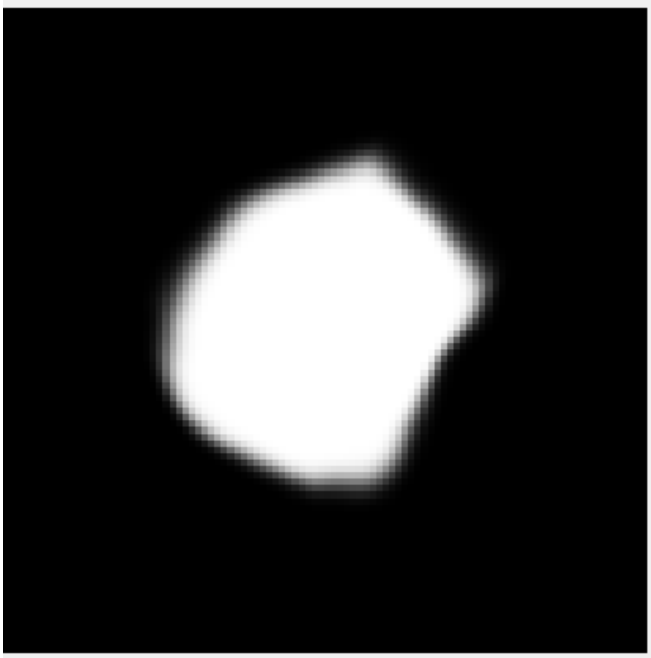

Figure 3. Result Image from the First Strategy

After implementing the second geodesic shooting strategy, I plot the final image. The result is shown as Figure 4.

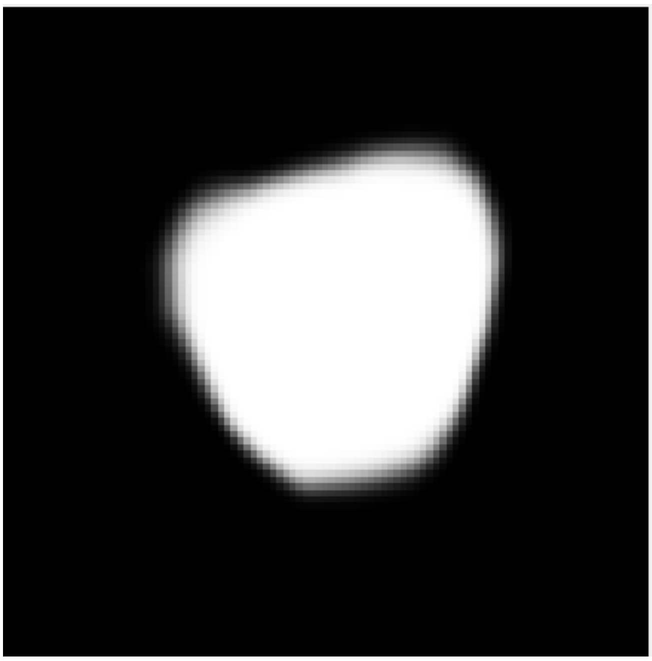

Figure 4. Result Image from the Second Strategy

2.5 Discussion

After I implement these two strategies, I use Euler's method to update velocity and transformation in both two strategies. Except the methods to calculate the $\frac{dv_t}{dt}$ and $\frac{d\phi_t}{dt}$, another difference between these two strategies is the value of step size that I use to update velocity and transformations. The value of step size in the first strategy is 0.01, and the value in the second strategy is 0.0007. The value of iteration is 100 in both two strategies.

First, I can assume that the distance between source images and result image is fixed. The number of iterations is the same in two methods, so I can update velocity and transformation both 100 times in two strategies. The value of step size in the second strategy is much smaller than that in the first strategy, so every time I update velocity and transformation, the changed values of the two variables in the second strategy is much smaller than these in the first strategy. Therefore, if I choose to implement the second strategy rather than the first one, I can get the best result more quickly.

Through comparing the approximated results and exact results in the first question, if we choose smaller step size, we can get better approximation for the true values. Therefore, I choose the smaller step size when I implement the second strategy rather than the first strategy, so the result deformed from the second strategy is more accurate than it from the first strategy.

In conclusion, the implementation of the second strategy is more efficient than that of the first strategy.