Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme

Mathematics – 2 (303191151)

Unit 1: Higher order ordinary differential equations:

(Lecture Notes)

Higher Order Linear Differential equation:

Consider n^{th} order linear differential equation,

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = r(x),$$

The linear homogeneous differential equation of the n^{th} order if $r(x) = 0$

$$y^{(n)}(x) + a_1 y^{(n-1)}(x) + \cdots + a_{n-1} y'(x) + a_n y(x) = 0,$$

where $a_1, a_2, \ldots, a_n$ are constants which may be real or complex or functions of x.

Using the linear differential operator L(D), this equation can be represented as

$$L(D)y(x) = 0,$$

Where, $D = \frac{d}{dx}$ and $L(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n$

For each differential operator with constant coefficients, we can introduce the *characteristic polynomial.*

$$L(m) = m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n.$$

The algebraic equation

$$L(m) = m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n.$$

is called the *characteristic equation* ***(Auxiliary equation)*** of the differential equation.

According to the fundamental theorem of algebra, a polynomial of degree n has exactly n roots, counting multiplicity. Let us consider in more detail the different cases of the roots of the characteristic equation and the corresponding formulas for the general solution of differential equations.

CASE 1. ALL ROOTS OF THE CHARACTERISTIC EQUATION ARE REAL AND DISTINCT

Assume that the characteristic equation $L(m) = 0$ has n roots $m_1, m_2, \ldots, m_n$. In this case the general solution of the differential equation is written in a simple form:

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x},$$

where $c_1, c_2, \dots, c_n$ are constants depending on initial conditions.

CASE 2. THE ROOTS OF THE CHARACTERISTIC EQUATION ARE REAL AND MULTIPLE

Let the characteristic equation $L(m) = 0$ of degree n have m roots $m_1, m_2, \dots, m_n$., the multiplicity of which, respectively, is equal to $m_1, m_2, \dots, m_n$. It is clear that the following condition holds: $m_1, m_2, \dots, m_n$ Then the general solution of the homogeneous differential equations with constant coefficients has the form

$$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_2 x} + \dots + c_{k1} x^{k_1 - 1} e^{m_1 x} + \dots + c_n x^{km-1} e^{m_m x}.$$

It is seen that the formula of the general solution has exactly ki terms corresponding to each root $m_i$of multiplicity k_i.

CASE 3. THE ROOTS OF THE CHARACTERISTIC EQUATION ARE COMPLEX AND DISTINCT

If the coefficients of the differential equation are real numbers, the complex roots of the characteristic equation will be presented in the form of conjugate pairs of complex numbers:

$$m_{1,2} = \alpha \pm i\beta, m_{3,4,} = \gamma \pm i\delta, \dots$$

In this case the general solution is written as

$$y(x) = e^{\alpha x}(c_1 cos\beta x + c_2 sin\beta x) + e^{\gamma x}(c_3 cos\delta x + c_4 sin\delta x) + \cdots$$

Homogeneous Linear Equations of the Second Order:

Linear Differential Equation of the Second Order

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x)$$

or in alternative notation,

$y'' + p(x)y' + q(x)y = r(x)$ ____________ (1) , if $r(x) = 0$ in equation (1) then it is called homogeneous linear differential equation of second order.

[Example]

1.	$y'' + 2y' + y = 0$	Linear Homogeneous differential equation with constant coefficients
2.	$(1-x^2)y'' - 2xy' + 6y = 0$ $y'' - \frac{2x}{1-x^2}y' + \frac{6}{1-x^2}y = 0$	Linear Homogeneous differential equation with variable coefficients
3.	$y'' + 5y' + 6y = e^x$	Linear non homogeneous differential equation with constant coefficients
4.	$x^2y'' + xy' + y = sin\, x$	Linear non homogeneous differential equation with variable coefficients
5.	$y''y + y' = 0$	Non linear

Homogeneous Equations with Constant Coefficients :

Consider an equation

$$y'' + a\,y' + b\,y = 0$$

where a and b are real constants, this type of equation is called Homogeneous Equations with Constant Coefficients .

Auxiliary equation of homogeneous differential equation is given by

$$m^2 + a\,m + b = 0 \qquad \text{--}\textbf{Auxiliary equation}$$

Since the Auxiliary equation is quadratic, we have two roots:

$$m_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}$$

$$m_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

Thus, there are three possible situations for the roots of m_1 and m_2 of the characteristic equation:

Case I	$a^2 - 4b > 0$	m_1 and m_2 are distinct real roots
Case II	$a^2 - 4b = 0$	$m_1 = m_2$, a real double root
Case III	$a^2 - 4b < 0$	m_1 and m_2 are two complex conjugate roots

We have following cases:

Case	Roots of m	General Solution
I	Distinct & real m_1, m_2	$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}$
II	Real & double root $m_1 = m_2 = m$	$y(x) = (c_1 + c_2 x)\, e^{mx}$
III	Complex conjugate $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$	$y(x) = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

[Example 1] Solve: $y'' - y' - 6y = 0$

[Solution] The characteristic equation is

$$m^2 - m - 6 = (m+2)(m-3) = 0$$

we have two distinct roots

$$m_1 = -2 \quad ; \quad m_2 = 3$$

$$y(x) = c_1 e^{3x} + c_2 e^{-2x} \qquad \text{--general solution}$$

[Example 2] $y'' + 3y' - 10y = 0 \quad ; \quad y(0) = 1, \; y'(0) = 3.$

[Solution] The characteristic equation is

$$m^2 + 3m - 10 = (m-2)(m+5) = 0$$

we have two distinct roots

$$m_1 = 2 \quad ; \quad m_2 = -5$$

$$y(x) = c_1 e^{2x} + c_2 e^{-5x} \qquad \text{--general solution}$$

The initial conditions can be used to evaluate c_1 and c_2:

$$y(0) = c_1 + c_2 = 1$$

$$y'(0) = 2c_1 - 5c_2 = 3$$

$$c_1 = 8/7 \quad , \quad c_2 = -1/7$$

$$y(x) = \frac{1}{7}(8e^{2x} - e^{-5x}) \qquad \text{— particular solution}$$

[Example 3] Solve: $\boldsymbol{\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0 \; ; \quad y(1) = e^2 \, , y(1) = 3e^2.}$

[Solution] $(D^2 - 5D + 6)y = 0$

Auxiliary equation is $m^2 - 5m + 6 = 0$

$$\therefore (m-3)(m-2) = 0$$

$$\therefore m = 3 \, , \quad m = 2$$

The general solution is $y(x) = c_1 e^{3x} + c_2 e^{2x}$

$$\therefore y'(x) = 3c_1 e^{3x} + 2c_2 e^{2x}$$

$$y(1) = e^2 \rightarrow e^2 = c_1 e^3 + c_2 e^2$$

$$\therefore 1 = c_1 e + c_2$$

$$y'(1) = 3e^2 \rightarrow 3e^2 = 3c_1 e^{3x} + 2c_2 e^{2x}$$

$$\therefore 3 = 3c_1 e + 2c_2$$

$$\therefore c_1 = \frac{1}{e}, c_2 = 0$$

$$\therefore y(x) = \frac{1}{e} e^{3x} = e^{3x-1}$$

[Example 4] **Solve: y'' - 6 y' + 9 y = 0.**

[Solution] The characteristic equation is $m^2 - 6m + 9 = 0, m = 3,3$

Thus, the general solution is $y(x) = (c_1 + c_2 x)e^{3x}$.

[Example 5] **Solve y'' + 4y'+ 4y = 0, y(0) = 2, y'(0) = 1**

[Solution] The characteristic equation is $m^2 + 4m + 4 = 0, m = -2, -2$

Thus, the general solution is

$$Y(x) = (c_1 + c_2 x) e^{-2x}$$ --general solution

The initial conditions can be used to evaluate c_1 and c_2:

$$y(0) = c_1 = 2$$

$$y'(0) = -2 c_1 + c_2 = 1 \quad \therefore c_2 = 5$$

$$y(x) = (2 + 5 x) e^{-2x}$$ — particular solution

[Example 6] **Solve: y'' + 9 y = 0 .**

[Solution] The characteristic equation is

$$m^2 + 9$$

$$m = 3i, -3i$$

Thus, the general solution is $y = (c_1 \cos 3x + c_2 \sin x)$

[Example 7] **Solve: y'' + y' + y = 0 ; y(0) = 1, y'(0) = 3**

[Solution] The characteristic equation is $m^2 + m + 1 = 0$

$$m_1 = \frac{-1 + i\sqrt{3}}{2}, m_2 = \frac{-1 - i\sqrt{3}}{2}$$

Thus, the general solution is

$$y(x) = e^{-x/2}\left[A \cos\frac{\sqrt{3}}{2}x + B \sin\frac{\sqrt{3}}{2}x \right]$$

The constants A and B can be evaluated by considering the initial conditions:

$$y(0) = 1 \Rightarrow A = 1$$

$$y'(0) = 3 \Rightarrow \frac{\sqrt{3}}{2}B - \frac{1}{2}A = 3$$

$$A = 1, B = -\frac{7}{\sqrt{3}}$$

Thus $$y(x) = e^{-x/2}\left[\cos\frac{\sqrt{3}}{2}x + \frac{7}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}x \right]$$

Higher Order Constant Coefficient Differential Equations :

[Example 8] Solve: $\boldsymbol{\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0.}$

[Solution] Auxiliary equation is $\therefore A.E. is\ m^3 - 4\,m^2 + m + 6 = 0$

Since sum of coefficient of odd power terms = 1 + 1=2

And sum of coefficient of even power terms = -4 +6= 2

$\therefore m + 1\ is\ a\ factor.$

-1	1	-4	1	6
	0	-1	5	-6
	1	-5	6	0

$m^3 - 4\,m^2 + m + 6 = 0$

$(m + 1)(m - 2)(m - 3) = 0$

$m = -1\,, m = 2\,, m = 3$

$y(x) = c_1e^{-x} + c_2e^{2x} + c_3e^{3x}$

[Example 9] Solve y^{iv} -5 y'' + 4y = 0;

[Solution] Auxiliary equation is

$$\lambda^4 - 5\lambda^2 + 4 = 0$$

$$(\lambda - 1)(\lambda + 1)(\lambda - 2)(\lambda + 2) = 0$$

$$\lambda = \pm 1, \pm 2$$

$$y = c_1 e^{x} + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-2x}$$

[Example 10] Solve $(D^2 + 2D + 2)^2 y = 0 \; where \; D = \frac{d}{dx}$.

[Solution] **A.E =** $(m^2 + 2m + 2)^2 = 0$

Roots of equation are $m = -1 \pm i$ (Repeated 2 times),

G.S=$y(x) = e^{-x}(c_1 + c_2 x)cosx + (c_3 + c_4 x)sinx$

Practice examples:

Find general solution of following,

1. $4y^{(iv)} - 8y^{(\prime\prime\prime)} - 7y^{(\prime\prime)} + 11y' + 6y = 0$
2. $y''' + y = 0$
3. find particular sol of : $y''' - y'' + 100y' - 100y = 0$, $y'(0) = 11, y''(0) = -299, y(0) = 4$.
4. $4y'' - 12y' + 5y = 0$.
5. $(D^2 + 8D + 25)y = 0$; $y(0) = 2, y\left(\frac{\pi}{6}\right) = 5$.

Non-homogeneous Linear Differential equations:

General Concepts

A general solution of the non-homogeneous equation

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = r(x) \quad \text{where } r(x) \neq 0.$$

on some interval I is a solution of the form

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x) = c_1 y_1(x) + \dots + c_n y_n(x)$ is a solution of the homogeneous equation

$$y^{(n)} + p_{n-1}(x)\, y^{(n-1)} + \dots + p_1(x)\, y' + p_0(x)\, y = 0$$

and $y_p(x)$ is a particular solution of the non-homogeneous equation.

$y'' + p(x)y' + q(x)y = r(x)$ The general solution of the homogeneous equation can be obtained by the method discussed in the above sections. On the other hand, there are two methods to obtain the particular solution $y_p(x)$: Method of Undetermined Coefficients and Method of Variation of Parameters.

Method of Undetermined Coefficients :

Solution of non –homogeneous equation is determined by trial solution or initial guessing based on the form of *r(x)*.

In summary, *for* a constant coefficient nonhomogeneous linear differential equation of the form

$$y^{(n)} + a\,y^{(n-1)} + \dots + f\,y' + g\,y = r(x)$$

we have the following rules for the method of undetermined coefficients:

(A) Basic Rule: If r(x) in the nonhomogeneous differential equation is one of the functions in the first column in the following table, choose the corresponding function y_p in the second column and determine its undetermined coefficients by substituting y_p and its derivatives into the nonhomogeneous equation.

$f(x)$	Initial guess of y_p
$f(x) = k$	$y_p = A$
$f(x) = ke^{ax}$	$y_p = Ae^{ax}$
$f(x) = a_0 + a_1x + \dots + a_nx^n$	$y_p = A_0 + A_1x + \dots + A_nx^n$
$f(x) = \cos bx / \sin bx$	$y_p = A_1 \cos bx + A_2 \sin bx$
$f(x) = e^{ax} cosbx / e^{ax} sinbx$	$y_p = e^{ax}[A_1 \cos bx + A_2 \sin bx]$
$f(x) = e^{ax}(a_0 + a_1x + \cdots a_{nx^n})$	$y_p = e^{ax}[A_0 + A_1x + \cdots . + A_nx^n]$

(B) Modification Rule: If any term of the suggested solution $y_p(x)$ is the solution of the corresponding homogeneous equation, multiply y_p by x repeatedly until no term of the product $x^k y_p$ is a solution of the homogeneous equation. Then use the product $x^k y_p$ to solve the nonhomogeneous equation.

(C) Sum Rule: If r(x) is sum of functions listed in several lines of the first column of the following table, then choose for y_p the sum of the functions in the corresponding lines of the second column.

In summary, a P.S. of an equation of the form $y''+py'+qy = k\,e^{ax}$ can be obtained as follows :

Step 1. Start with an initial guess (from table)

Step 2. Determine if the initial guess is a solution of the C.F. $y''+py'+qy = 0$.

Step 3. If the initial guess is not a solution of the C.F. , then inital guess is the correct form of a particular solution ($y_p(x)$).

Step 4. If the initial guess is a solution of the C.F., then multiply it by the smallest positive integer power of *x* required to produce a function that is not a solution of the C.F. .

for example, if initial guess $y_p = Ae^{ax}$ is the solution then multiply x with given equation $y_p = Axe^{ax}$ if this is also solution of given equation then again multiply initial guessing with x^2 so new guessing will be $y_p = Ax^2e^{ax}$.

Let us illustrate this method by some examples:

[Example 1] **$y'' + 4y = 12$.**

[Solution] The general solution of $y'' + 4y = 0$ is

$$y_c(x) = c_1\cos 2x + c_2 \sin 2x$$

If we assume the particular solution

$$y_p(x) = A$$

then we have $y_p'' = 0$, and

$$4A = 12 \qquad \text{or} \qquad A = 3$$

Thus, the general solution of the nonhomogeneous equation is

$$y(x) = c_1\cos 2x + c_2 \sin 2x + 3$$

[Example 2] **$y'' + 4y = 8x^2$.**

[Solution] If we now assume the particular solution is of the form

$$y_p(x) = A x^2 + B x + C$$

then $\quad y_p' = 2Ax + B$

$$y_p'' = 2A$$

thus $\quad 2A + 4(Ax^2 + Bx + C) = 8x^2$

or $\quad 4Ax^2 + 4Bx + (2A + 4C) = 8x^2$

or $\quad 4A=8$

$$4B=0$$

$$2A+4C=0$$

or $\quad A = 2 \qquad B = 0 \qquad C = -1$

$$y_p(x) = 2x^2 - 1$$

and $\quad y(x) = c_1\cos 2x + c_2 \sin 2x + 2x^2 - 1$

[Example 3] **$y'' - 4y' + 3y = 10e^{-2x}$ OR $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 10e^{-2x}$.**

[Solution] The general solution of the homogeneous equation

$$y'' - 4y' + 3y = 0 \quad \text{is } y_c(x) = c_1 e^{x} + c_2 e^{3x}$$

If we assume a particular solution of the nonhomogeneous equation is of the form

$$y_p(x) = A e^{-2x}$$

then $y_p' = -2 A e^{-2x}$ $\qquad$ $y_p'' = 4 A e^{-2x}$

and $4 A e^{-2x} - 4 (-2 A e^{-2x}) + 3 (A e^{-2x}) = 10 e^{-2x}$

or $15 A e^{-2x} = 10 e^{-2x}$

or $A = 2/3$

Thus $y(x) = c_1 e^{x} + c_2 e^{3x} + \frac{2}{3} e^{-2x}$

[Example 4] $\mathbf{y'' + y = x e^{2x}}$.

[Solution] The general solution to the homogeneous equation is

$$y_c = c_1 \sin x + c_2 \cos x$$

Since the nonhomogeneous term is of the form

$$x e^{2x}$$

If we assume the particular solution be

$$y_p = e^{2x} (A + B x)$$

we will have

$$y_p = \frac{e^{2x}}{25} (5 x - 4)$$

Therefore, the general solution is

$$y(x) = c_1 \sin x + c_2 \cos x + \frac{e^{2x}}{25} (5 x - 4)$$

[Example 5] $\mathbf{y'' + 4 y' + 3 y = 5 \sin 2x}$.

[Solution] The general solution of the homogeneous equation is

$$y_c = c_1 e^{-x} + c_2 e^{-3x}$$

We assume

$$y_p = A \sin 2x + B \cos 2x$$

and substitute y_p, y_p' and y_p'' into the nonhomogeneous equation, we have

$$A = -\frac{1}{13} \qquad \text{and} \qquad B = -\frac{8}{13}$$

Thus
$$y = c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{13}(\sin 2x + 8\cos 2x)$$

[Example 6] $y'' - 3y' + 2y = e^x \sin x$.

[Solution] The general solution to the homogeneous equation is

$$y_c = c_1 e^x + c_2 e^{2x}$$

Since the $r(x) = e^x \sin x$, we assume the particular solution of the form

$$y_p = A e^x \sin x + B e^x \cos x$$

Substituting the above equation into the differential equation and equating the coefficients of sin x and cos x, we have

$$y_p = \frac{e^x}{2}(\cos x - \sin x)$$

and
$$y(x) = c_1 e^x + c_2 e^{2x} + \frac{e^x}{2}(\cos x - \sin x)$$

[Example 7] $y'' + 2y' + 5y = 16 e^x + \sin 2x$.

[Solution] The general solution of the homogeneous equation is

$$y_c = e^{-x}(c_1 \sin 2x + c_2 \cos 2x)$$

Since the nonhomogeneous term r(x) contains terms of e^x and sin 2x, we can assume the particular solution of the form $y_p = A e^x + B \sin 2x + C \cos 2x$

After substitution the above y_p into the nonhomogeneous equation, we arrive

$$y_p = 2e^x - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

Thus
$$y(x) = e^{-x}(c_1 \sin 2x + c_2 \cos 2x) + 2e^x - \frac{4}{17}\cos 2x + \frac{1}{17}\sin 2x$$

[Example 8] $y'' - 3y' + 2y = e^x$.

[Solution] The general solution of the homogeneous equation is

$$y_c(x) = c_1 e^x + c_2 e^{2x}$$

Assume

$$y_p = A\,x\,e^x$$

then $\quad y_p' = A\,(e^x + x\,e^x) \qquad y_p'' = A\,(2\,e^x + x\,e^x)$

and $\quad A\,(2\,e^x + x\,e^x) - 3\,A\,(e^x + x\,e^x) + 2\,A\,x\,e^x = e^x$

or $\quad -A = 1 \qquad$ or $\qquad A = -1$

Thus, $\quad y(x) = c_1\,e^x + c_2\,e^{2x} x\,e^x$

[Example 9] $\mathbf{y'' - 2\,y' + y = e^x}$**.**

[Solution] The general solution of the homogeneous equation is

$$y_c = (c_1 + c_2\,x)\,e^x$$

If we assume $\quad y_p = A\,x^2\,e^x$

then we have $\quad A = \dfrac{1}{2}$

thus $\quad y(x) = (c_1 + c_2\,x)\,e^x + \dfrac{1}{2}\,x^2\,e^x$

[Example 10] $\mathbf{y'' - 4\,y' + 4\,y = 6\,x\,e^{2x}}$**.**

[Solution] $\quad y_c = c_1 e^{2x} + c_2\,x\,e^{2x}$

$$y_p = x^2\,(a + b\,x)\,e^{2x}$$

[Example 11] **Solve** $\mathbf{y'' - 2\,y' + y = e^x + x}$**.**

[Solution] $\quad y_c = (c_1 + c_2\,x)\,e^x$

$$y_p = A + B\,x + C\,x^2\,e^x$$

$$\Rightarrow y_p = 2 + x + \frac{1}{2}\,x^2\,e^x$$

Method of Variation of Parameters

In this section, we shall consider a procedure for finding a particular solution of *any* nonhomogeneous second order linear differential equation

$$y'' + p(x)\,y' + q(x)\,y = r(x)$$

here p(x), q(x) and r(x) are continuous on an open interval I. The general solution of the corresponding homogeneous equation

$$y'' + p(x)\, y' + q(x)\, y \;=\; 0$$

is given, $\quad y_h \;=\; c_1 y_1 + c_2 y_2$

where c_1 and c_2 are arbitrary constants.

Thus, the particular solution y_p is

$$y_p(x) \;=\; -\,y_1\int \frac{y_2\, r}{W}\,dx \;+\; y_2\int \frac{y_1\, r}{W}\,dx$$

$$W(y_1, y_2, ,,, y_n) = \begin{vmatrix} y_1 & .. & y_n \\ .. & .. & .. \\ y_n^n & .. & y_n^n \end{vmatrix} \neq 0$$

[Example 1] $\quad$ **$y'' - y \;=\; e^{2x}$.**

[Solution] The general solution to the homogeneous equation is

$$y_c \;=\; c_1 e^{-x} + c_2 e^{x}$$

i.e., $\quad y_1 \;=\; e^{-x} \qquad y_2 \;=\; e^{x}$

The Wronskian of y_1 and y_2 is $\quad W \;=\; \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 2$

$$\text{thus, } y_p = -y_1\int \frac{y_2\, r}{W}\,dx + y_2\int \frac{y_1\, r}{W}\,dx$$

$$y_p = -e^{-x}\int \frac{e^{x}\, e^{2x}}{2}\,dx + e^{x}\int \frac{e^{-x}\, e^{2x}}{2}\,dx$$

$$= -\frac{e^{3x}}{6}e^{-x} + \frac{e^{x}}{2}e^{x} = \frac{e^{2x}}{3}$$

and the general solution is $\qquad y(x) \;= y_h + y_p = \; c_1 e^{-x} + c_2 e^{x} + \dfrac{e^{2x}}{3}$

[Example 2] $\quad$ **$y'' + y \;=\; \tan x$.**

[Solution] The general solution to the homogeneous equation is

$$y_c = c_1\cos x + c_2 \sin x$$

thus, $\quad y_1 = \cos x \qquad y_2 = \sin x$

Also $\quad W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$

$$y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$$

$$= -\cos x \int simx \tan x\, dx + \sin x \int \cos x \tan x \, dx$$

$$= -\cos x \int \frac{1-\cos^2 x}{\cos x} dx + \sin x \int \sin x\, dx$$

$$= -\cos x \int (\sec x - \cos x) dx - \sin x \cos x$$

$$= -\cos x (\ln(\sec x + \tan x) - \sin x) - \sin x \cos x$$

$$= -\cos x \ln(\sec x + \tan x)$$

[Example 3] **Solve $(D^2 + 1)y = cosec\ x.$**

[Solution] A.E. is $m^2 + 1 = 0$

$\therefore m = \pm i$

$\therefore\ y_c = c_1 cos\ x + c_2 sin\ x$

Here, $y_1 = cos\ x\ and\ y_2 = sin\ x$ form two independent solutions to the corresponding homogeneous equation

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} cos\ x & sin\ x \\ -\ sinx & cos\ x \end{vmatrix} = 1 \neq 0$$

$$\therefore\ y_p = -y_1 \int \frac{y_2 F(x)}{W} dx + y_2 \int \frac{y_1 F(x)}{W} dx$$

$$= -\cos \int sinx\ cosecx\ dx + \sin x \int \cos x\ cosec\ x\ dx$$

$$= -\cos x \int 1\, dx + \sin x \int \cot x\, dx$$

$$= -x \cos x + \sin x \log(\sin x)$$

$$= \boldsymbol{y} = y_c + y_p = c_1 cos\, x + c_2 sin\, x - x\cos x + \sin x \log(\sin x)$$

Practice Examples:

Find the General solution of following using variation of parameter

1. **Solve:** $\frac{d^2y}{dx^2} + a^2 y = sec\, ax$
2. **Solve:** $y'' + 2y' + y = e^{-x} lnx.$
3. **Solve :** $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$
4. **Solve:** $\frac{d^2y}{dx^2} + 4y = tan2x$

Euler Equations (Linear 2nd-order ODE with variable coefficients):

For most linear second order equations with variable coefficients, it is necessary to use techniques such as the power series method to obtain information about solutions. However, there is one class of such equations for which closed form solutions can be obtained the *Euler equation*:

$$x^2 y'' + a x y' + b y = 0, \qquad x \neq 0$$

Let $\log x = z$ then, $x = e^z \Rightarrow \frac{dz}{dx} = \frac{1}{x}$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

$$= e^{-z}\frac{dy}{dz} = \frac{1}{x}\frac{dy}{dz}$$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz}$$

$$\Rightarrow xy' = Dy$$

Similarly $\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx}$

$$= \frac{d}{dz}\left(e^{-z}\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \left(-e^{-z}\frac{dy}{dz} + e^{-z}\frac{d^2y}{dz^2}\right)e^{-z} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\therefore x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$\Rightarrow x^2 y'' = D^2 y - Dy = D(D-1)y$ -

Substitute in given equation, we get

$$\left(D(D-1)+aD+b\right)=0$$

Therefore, Auxiliary equation is

$$m(m-1)+am+b = 0$$

or $$m^2+(a-1)m+b = 0$$ **Auxiliary Equation**

which is constant coefficient equation with independent variable z. which is discussed in previous section.

Case	Roots of λ	General Solution
I	*Distinct & real* m_1, m_2	$y=\left(c_1e^{m_1z}+c_2e^{m_2z}\right)$ $=\left(c_1x^{m_1}+c_2x^{m_2}\right)$
II	*Real & double root* $\lambda_1=\lambda_2=m$	$y=(c_1+c_2z)e^{mz}$ $=(c_1+c_2\log x)\,x^m$
III	*Complex conjugate* $\lambda_1 = \alpha+i\beta$ $\lambda_2 = \alpha-i\beta$	$y=e^{\alpha z}(c_1\cos\omega z+c_2\sin\omega z)$ $x^{\alpha}(c_1\cos\omega\log x+c_2\sin\omega\log x)$

[Example 1] $\mathbf{x^2y''+2xy'-12y = 0.}$

[Solution] Let $x = e^z \to z = ln\,x$

$$\therefore xy = Dy; where\ D = \frac{d}{dz}$$

$$\therefore x^2y'' = D(D-1)y$$

Given Equation becomes,

$$(D(D-1)+2D-12)y = 0$$

The characteristic equation is

$$m(m-1)+2m-12 = 0$$

with roots $m = -4$ and 3

Thus, the general solution is

$$Y(x) = c_1 x^{-4} + c_2 x^3$$

[Example 2] $x^2 y'' - 3xy' + 4y = 0.$

[Solution] Let $x = e^z \to z = \ln x$

$$\therefore xy = Dy; \text{where } D = \frac{d}{dz}$$

$$\therefore x^2 y'' = D(D-1)y$$

Given Equation becomes,

$$(D(D-1) - 3D + 4)y = 0$$

The characteristic equation is

$$m(m-1) - 3m + 4 = 0$$

$$m = 2, 2 \text{ (double roots)}$$

Thus, the general solution is

$$y = x^2(c_1 + c_2 \log|x|)$$

[Example 3] $x^2 y'' + 5xy' + 13y = 0.$

[Solution] Let $x = e^z \to z = \ln x$

$$\therefore xy = Dy; \text{where } D = \frac{d}{dz}$$

$$\therefore x^2 y'' = D(D-1)y$$

Given Equation becomes,

$$(D(D-1) + 5D + 13)y = 0$$

The characteristic equation is $m(m-1) + 5m + 13 = 0$

or $m = -2 + 3i$ and $-2 - 3i$

Thus, the general solution is

$$y = x^{-2}[c_1 \cos(3\log|x|) + c_2 \sin(3\log|x|)]$$

[Example 4] $x^2 y'' - 5xy' + 8y = 2\log x, \qquad x > 0$

[Solution] Let $x = e^z \to z = \ln x$

$$\therefore xy = Dy; where\ D = \frac{d}{dz}$$

$$\therefore x^2y'' = D(D-1)y$$

Given Equation becomes,

$$(D(D-1) - 5D + 8)y = 2z$$

$$y_c = c_1 e^{4z} + c_2 e^{2z}$$

and $\quad y_p = c\,z + d = \dfrac{1}{4} z + \dfrac{3}{16}$

$$y(z) = c_1 e^{4z} + c_2 e^{2z} + \frac{1}{4} z + \frac{3}{16}$$

$$y(x) = c_1 x^4 + c_2 x^2 + \frac{1}{4} \log x + \frac{3}{16}$$

[Example 5] $\mathbf{x^2 y'' + 2 x y' - 12 y = \sqrt{x}}$ **.**

[Solution] This is an Euler equation. The general solution to the homogeneous equation is

$$y_h = c_1 x^{-4} + c_2 x^3$$

or $\quad y_1 = x^{-4} \qquad y_2 = x^3$

and $\quad W = \begin{vmatrix} x^{-4} & x^3 \\ -4x^{-5} & 3x^2 \end{vmatrix} = 7 x^{-2}$

or $\quad \dfrac{1}{W} = \dfrac{x^2}{7}$

In order to use the method of variation of parameters, we must write the differential equation in the standard form in order to obtain the correct r(x), i.e.,

$$y'' + \frac{2}{x} y' - \frac{12}{x^2} y = x^{-\frac{3}{2}}$$

or $\quad r(x) = x^{-3/2}$

Thus, $\quad u' = -\dfrac{y_2\, r}{W} = -\, x^3 x^{-3/2} \dfrac{x^2}{7} = -\dfrac{x^{7/2}}{7}$

and $\quad v' = \frac{y_1 r}{W} = x^{-4} x^{-3/2} \frac{x^2}{7} = \frac{x^{-7/2}}{7}$

Hence $\quad u = -\frac{1}{7}\frac{2}{9} x^{9/2} \qquad v = -\frac{1}{7}\frac{2}{5} x^{-5/2}$

so that $\quad y_p = u\, y_1 + v\, y_2$

$$= -\frac{2}{63} x^{9/2} x^{-4} - \frac{2}{35} x^{-5/2} x^3$$

$$= -\frac{4}{45} x^{1/2}$$

Thus, the general solution is given by

$$y(x) = c_1 x^{-4} + c_2 x^3 - \frac{4}{45} x^{1/2}$$

Practice Examples:

1. **Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$.**
2. **Solve $x^2y'' - 4xy' + 6y = 21x^{-4}$.**
3. **Solve $x^2y'' - 5xy' + 8y = 0$; $y(1) = 5, y'(1) = 18$ using variation parameter method.**
4. **Solve $x^3y''' - 3x^2y'' + 6xy' - 6y = x^4 \log x$.**

Applications:

- **Buckling of a Thin Vertical Column**

In the eighteenth-century Leonhard Euler was one of the first mathematicians to study an eigenvalue problem in analyzing how a thin elastic column buckles under a compressive axial force. Consider a long slender vertical column of uniform cross section and length L. Let y(x) denote the deflection of the column when a constant vertical compressive force, or load, P is applied to its top, as shown in **FIGURE** By comparing bending moments at any point along the column we obtain

$$EI\frac{d^2y}{dx^2} = -Py$$

Therefore; $EI\frac{d^2y}{dx^2} + Py = 0$

where E is Young's modulus of elasticity and I is the moment of inertia of a cross section about a vertical line through its centroid.

SOLUTION:

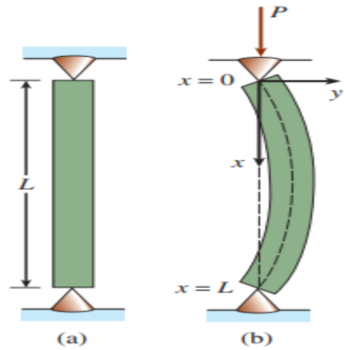
$\frac{d^2y}{dx^2} + \frac{P}{EI}y = 0$

$\frac{d^2y}{dx^2} + \lambda y = 0$ where $\lambda = \frac{P}{EI}$

A.E: $m^2 + \lambda = 0$

$m = \pm\sqrt{\lambda}$

$y(x) = c_1 cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$, where $\lambda = \frac{P}{EI}$



Young's modulus (E) is a measure of a solid's stiffness or resistance to elastic deformation under load. It relates stress (force per unit area) to strain (proportional deformation) along an axis or line. The basic principle is that a material undergoes elastic deformation when it is compressed or extended, returning to its original shape when the load is removed. More deformation occurs in a flexible material compared to that of a stiff material

- **Free Oscillation**

Consider a spring OA suspended vertically from a fixed support at O. let a body of mass being large in comparison with mass of the spring, that the later may be neglected. Let $e = (= AB)$ be the elongation produced by the mass hanging in equilibrium, and then B is called the position of static equilibrium.

Let k be the restoring force per unit stretch of the spring due to elasticity.

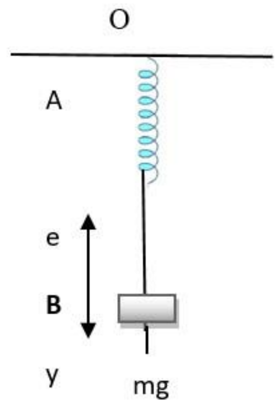
For the equilibrium at B $mg = ke$

Let the mass be displaced through a further distance y from equilibrium position.

The acceleration of the mass m at this position is $\frac{d^2y}{dx^2}$ and the forces acting upon it are weight mg down wards and restoring force $k\ (e + y)$ upward.

$\therefore By\ Newton's second\ law\ of\ motion$

$$m\frac{d^2y}{dx^2} = mg - k(e+y)$$
$$= ke - ky$$
$$= -ky$$
$$\therefore m\frac{d^2y}{dx^2} + ky = 0$$
$$\therefore \frac{d^2y}{dx^2} + wy = 0 \quad where\ w = \frac{k}{m}$$



Ex. Find the steady state oscillation of the mass spring system governed by the equation $y''+3y'+2y = 20\cos 2t$.

Solution: The given equation can be written as

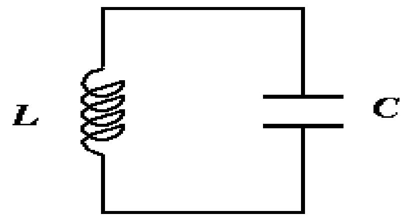
$m^2 + 3m + 2 = 0$
$m = -1, -2$
$y_c = c_1e^{-t} + c_1e^{-2t}$
$y_p = A\cos 2t + B\sin 2t$
$y' = -2A\sin 2t + 2B\cos 2t$
$y'' = -4A\cos 2t - 4B\sin 2t$
substitute in given equation
$-4A\cos 2t - 4B\sin 2t + 3(-2A\sin 2t + 2B\cos 2t) + 2(A\cos 2t + B\sin 2t) = 20\cos 2t$
$(-2A + 6B)\cos 2t + (-2B - 6A)\sin 2t = 20\cos 2t$
compare both side
$-2B - 6A = 0$ and $-2A + 6B = 20$
$B = -3A$ $\quad\quad\therefore -2A - 18A = 20$
$\quad\quad A = -1 \quad B = 3$
$y_p = 3\sin 2t - \cos 2t$
$y = c_1e^{-t} + c_1e^{-2t} + 3\sin 2t - \cos 2t$

- **Electric Circuit**

1. **L – C Circuit :** Consider an electrical circuit containing an inductance L and capacitance C. Let q be the electric charge on condenser plate and i be the currebt in the circuit at any time t. The voltage drop across L and C being L $\frac{di}{dt}$ and $\frac{q}{c}$ respectively and since there is no EMF in the circuit.

$\therefore$ *by Kirchhoff's law*

$$L\frac{di}{dt}+\frac{q}{C}=0$$



Since, $\quad i=\frac{dq}{dt}$

$$L\frac{d^2q}{dt^2}+\frac{q}{C}=0$$

$$\frac{d^2q}{dt^2}+\frac{q}{C}=0$$

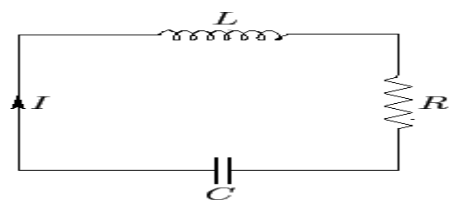
$$\frac{d^2q}{dt^2}+w^2q=0 \qquad Where, w^2=\frac{1}{LC}$$

2. **L –C – R Circuit:**

   Consider the discharge of the condenser C through an inductance L and the resistance R. Let q be the charge and i the current in the circuit at any time t. The voltage drops across L, C and R are respectively.

$$L\,\frac{d^2q}{dt^2}+R\,\frac{dq}{dt}+\,\frac{q}{C}=0$$

$$\therefore \frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\,\frac{1}{LC}q=0$$

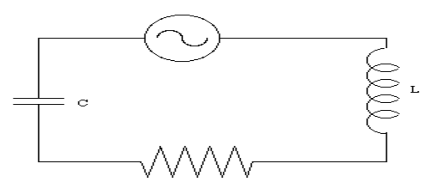


3. **L- C – R Circuit with EMF**

An emf of $E(t)$ is connected with inductance L, Resistance R, Capacitor C in series

By Kirchhof's Law

$$L\,\frac{d^2Q}{dt^2}+R\,\frac{dQ}{dt}+\,\frac{Q}{C}=E(t)$$



Example: Find the charge q on the capacitor in LCR circuit when L=0.25 henry ,R is 10 ohms and C is 0.001 farad, no external voltage is applied on the circuit .

Solution : No external voltage is applied therefore $E(x)=0$,

Here $L=0.25=\frac{1}{4}\,H$, $\quad R=10\,ohm,\quad C=0.001=\frac{1}{1000}\,F$

From equation

$$L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{Q}{C} = E(t),$$

$$\frac{1}{4}\frac{d^2q}{dt^2} + 10\,\frac{dq}{dt} + 1000q = 0$$

$there\ fore\quad q'' + 40q' + 4000 = 0.$

A.E: $m^2 + 40m + 4000 = 0.$

$$m = \frac{-40 \pm \sqrt{1600 - 16000}}{2},\ \text{so},\ m = -20 \pm 60i$$

$$q(t) = c_1 cos60t + c_2 sin60t$$

PRIVIOUS YEARS PAPERS'S QUESTION:-

1. **Solve.** $\frac{dy}{dx} + 2y\ tanx = sinx$; $y\left(\frac{\pi}{3}\right) = 0.$ **(Nov 2022).**
2. **Solve.** $\frac{dy}{dx} + \frac{y}{x} = x^2y^6.$ **(Nov 2022).**
3. **Solve** $y'' - 5y' + 6y = 3e^{-2x}$ **.** **(April 2022).**
4. **Solve.** $y^2\frac{dy}{dx} + xy = x^2y^3.$ **(Oct 2021).**
5. **Solve.** $\frac{d^2y}{dx^2} + y = cosecx$ **by variation parameter method. (May 2023).**