PARUL UNIVERSITY
Faculty Of Engineering & Technology
Department of Applied Sciences & Humanities
1st year B.Tech Programme (All branches)
Mathematics-II (Subject Code :303191151)

UNIT-5 Multiple Integration (Lecture Note)

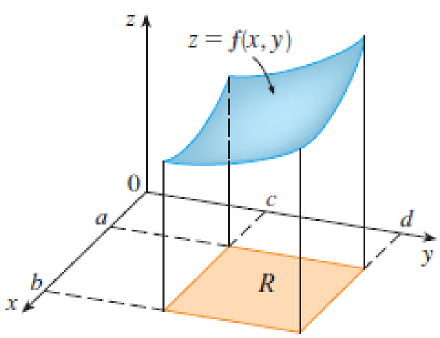
After Completion of this Chapter student is able to

1. Evaluate the integration, Change the limits of integration

2. Find the area, Volume using Double Integration

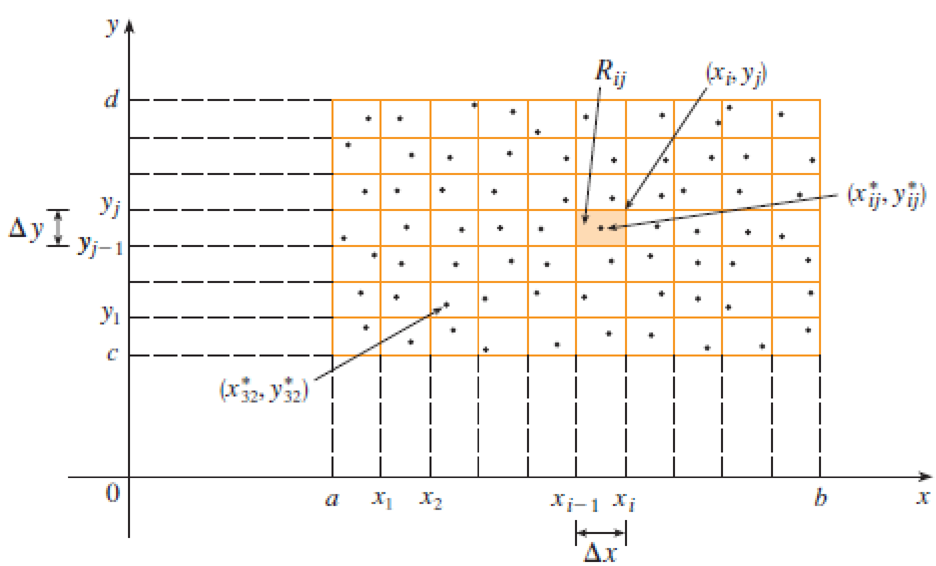
Introduction:
In this chapter we extend the idea of a definite integral to double and triple integrals of functions of two or three variables. These ideas are then used to compute Area, volume.

Volume of Solid:



Consider, $z = f(x, y)$ defined on a closed rectangle $R = \{(x, y) / a \le x \le b, c \le y \le d\}$ and the graph of f is shown in the figure.

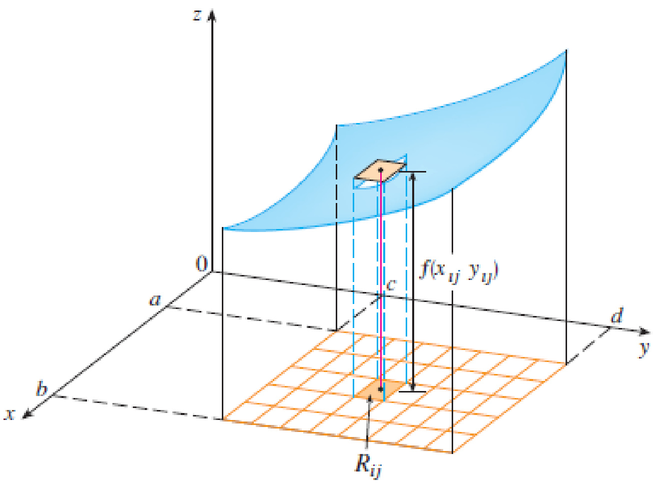
The first step is to divide the rectangle into sub rectangles. We accomplish this by dividing the rectangle by drawing lines parallel to co-ordinate axis. Number each rectangle 1,2,………
Let the area of ith rectangle R_{ij} is $dx_{ij} \times dy_{ij}$.

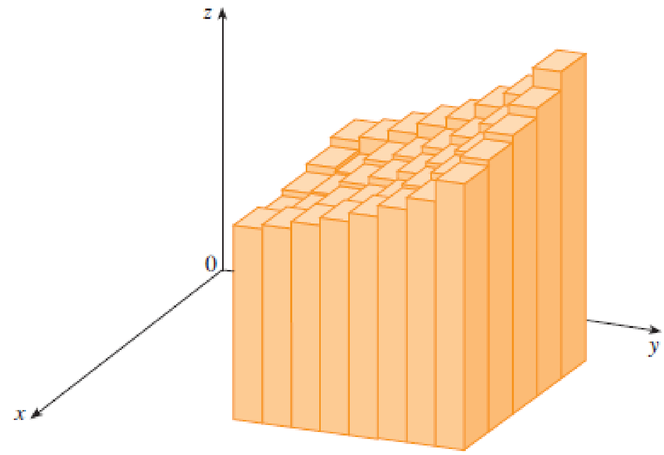


then we can approximate the part of S that lies above each R_{ij} by thin rectangular box (column) with base R_{ij} and height $f(x_{ij}, y_{ij})$. The volume this thin box is

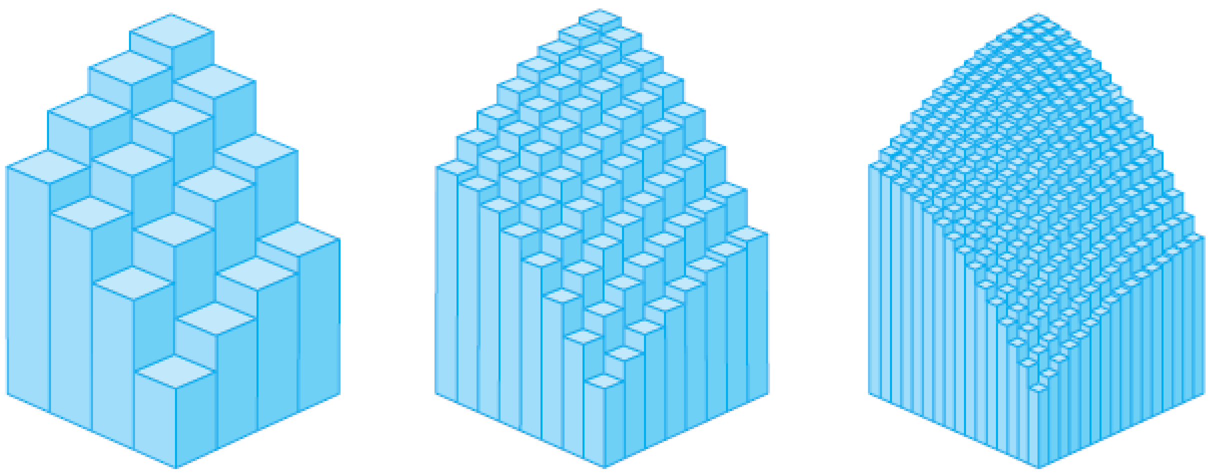
$$f(x_{ij}, y_{ij})dx_{ij}\, dy_{ij}\,.$$

If we follow this procedure for all rectangles and add volumes of corresponding boxes. We get an approximate volume of S.





as number of rectangular boxes increases, accuracy of volume increases (as shown in figure).



Hence, Volume is

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_{ij}, y_{ij}\right) dx_{ij}\;\; dy_{ij} = \iint_R f(x, y)\, dx\, dy$$

How to find the limits of integration:

Limits of Integral over Rectangle R:	

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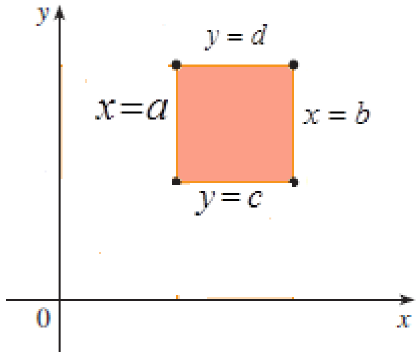
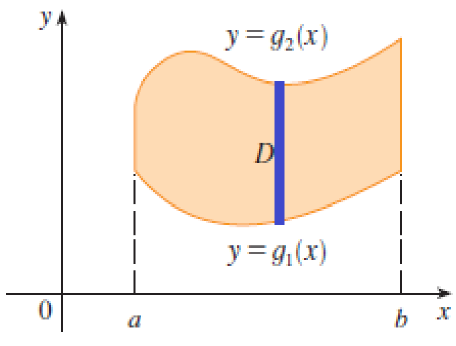
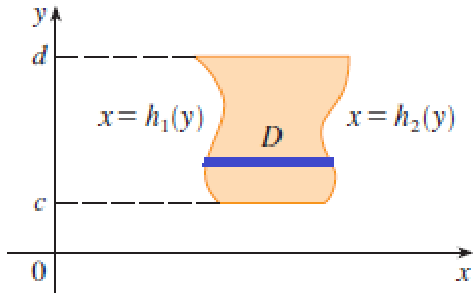
<tr><td>
</td><td>Rectangle R whose sides are $x=a$, $x=b$ $y=c$ and $y=d$
$\int_a^b \int_c^d f(x,y)\,dydx$
OR
$\int_c^d \int_a^b f(x,y)\,dx\,dy$</td></tr>
<tr><td colspan="2">Double Integral over general region:</td></tr>
<tr><td>
</td><td>For $\int \int f(x,y)\,dy\,dx$
Draw vertical strip in region, lower end of strip touches curve $y=g_1(x)$ and upper end of strip touches curve $y=g_2(x)$, hence limit of y : From $y=g_1(x)$ to $y=g_2(x)$.
This strip moves from $x=a$ to $x=b$.
Hence, limit of x: From $x=a$ to $x=b$.
$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$</td></tr>
<tr><td>
</td><td>For $\int \int f(x,y)\,dx\,dy$
Draw horizontal strip in region, strip touches curve $x=h_1(y)$ and $x=h_2(y)$, hence limit of y : From $x=h_1(y)$ and $x=h_2(y)$.
This strip moves from $y=c$ to $y=d$.
Hence, limit of x: From $y=c$ to $y=d$.
$\int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$</td></tr>
</table>

Double Integration as Iterated Integration:
Double integration is evaluated as iterated integration,

e.g. (i) $\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$,

We first integrate it with respect to y and then we integrate with respect to x.

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx = \int_{x=a}^{x=b} \left(\int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy \right) dx$$

(ii) $\int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$

We first integrate it with respect to x and then with respect to y

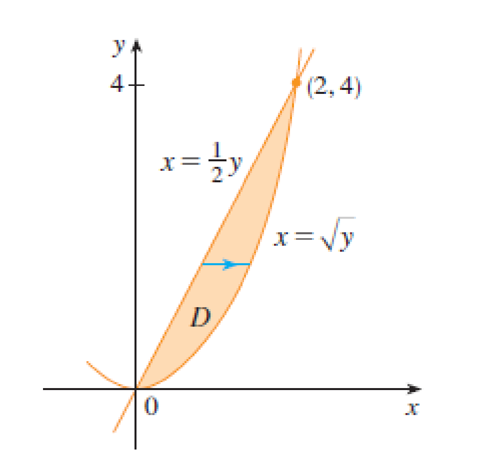
$$\int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left(\int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dy \right) dx$$

Examples:

1) $\int_1^2 \int_3^4 (xy + e^y)\,dy\,dx$
2) $\int_0^1 \int_0^{x^2} (e^{y/x})\,dy\,dx$
3) $\int_0^1 \int_y^{y^2+1} x^2 y\,dx\,dy$

Example 1: Evaluate the $\iint_R (x^2+y^2)\,dA$, where R is the region bounded by the line $y=2x$ and the parabola $y=x^2$.

Ans:



$$I=\iint_R (x^2+y^2)\,dA$$

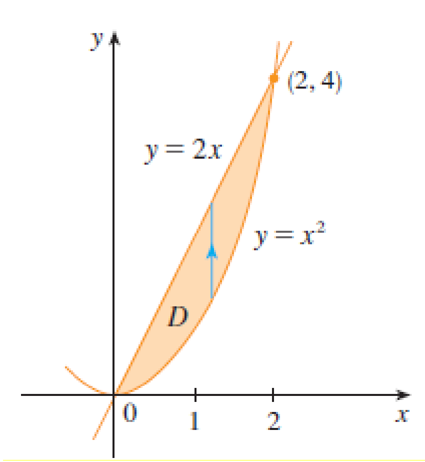
$$=\int_0^4 \int_{\frac{y}{2}}^{\sqrt{y}} (x^2+y^2)\,dx\,dy$$

$$=\int_0^4 \left(\frac{x^3}{3}+xy^2\right)_{\frac{y}{2}}^{\sqrt{y}} dy$$

$$=\int_0^4 \left(\frac{y^{\frac{3}{2}}}{3}+y^{\frac{5}{2}}-\frac{13y^3}{24}\right) dy$$

$$=\left(\frac{2y^{\frac{5}{2}}}{15}+\frac{2y^{\frac{7}{2}}}{7}-\frac{13y^4}{96}\right)_0^4=\frac{216}{35}$$

OR



$$\iint_R (x^2+y^2)\,dA$$

$$=\int_0^2 \int_{2x}^{x^2} (x^2+y^2)\,dy\,dx$$

$$=\int_0^2 \left(x^2y+\frac{y^3}{3}\right)_{2x}^{x^2} dx$$

$$=\int_0^2 \left(x^4+\frac{x^6}{3}-\frac{14x^3}{3}\right) dx$$

$$=\left(\frac{x^5}{5}+\frac{x^7}{21}-\frac{7x^4}{6}\right)_0^2=\frac{216}{35}$$

Example 2: Evaluate $\iint_R \sin(y^2)\,dA$, where R is the region bounded by the lines $y=x, y=\pi, x=0$.

$$\int_0^{\pi}\int_0^{y} \sin\left(y^2\right) dx\,dy$$

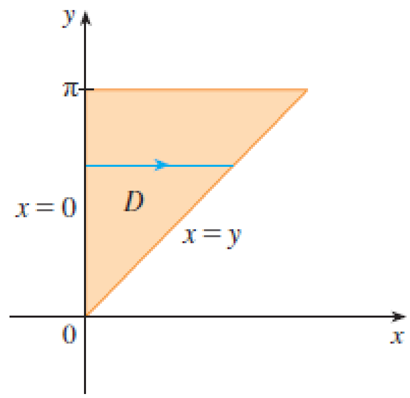
$$= \int_0^{\pi} y\sin\left(y^2\right) dy$$

Suppose $y^2 = t$

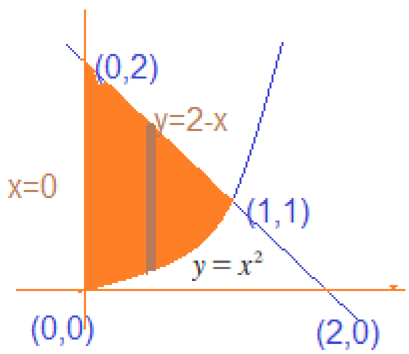
$$\therefore\ ydy = \frac{dt}{2}$$

$$= \int_0^{\pi} \sin t\,\frac{dt}{2}$$

$$= \left(-\frac{\cos t}{2}\right)_0^{\pi} = 1$$



Example 3: Compute the double integral of the function $f(x,y) = 6 - x + 2y$ over the region bounded by the curves $x = y^2$ and $y = 2 - x$ in the x-y plane.



$$\int_0^{1}\int_{x^2}^{2-x} (6 - x + 2y)\,dy\,dx$$

$$= \int_0^{1} \left(6y - xy + y^2\right)_{x^2}^{2-x} dx$$

$$= \int_0^{1} \left(16 - 10x - 4x^2 + x^3 - x^4\right) dx$$

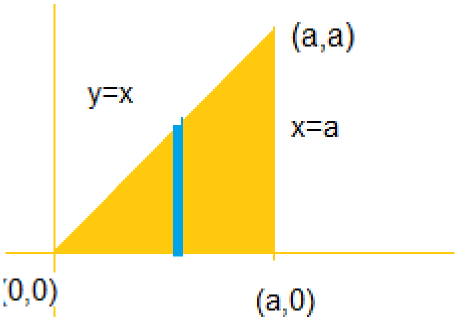
$$= \frac{-187}{660}$$

Change of Order:

The change of order often makes the evaluation of double integration easier e.g. in example 2 in previous section, $\int_0^{\pi}\int_x^{\pi} \sin\left(y^2\right) dy\,dx$ will be evaluated on reversing the order of integration.

In double integral with constant limits, the order of integration is immaterial provided the limits of integration are changed accordingly. But in case of double integral with variable limits, the limits of the integration changes with the change of order of integration. The new limits are obtained by sketching the region of integration. Sometime in changing the order of integration, it is required to split up the region of integration, and the given integral is expressed as the sum of number of double integrals with the changed limits.

Example 1 Change the order of integration in $\int_0^{a}\int_y^{a} \frac{x\,dx\,dy}{x^2 + y^2}$, and evaluate the same



From the limits of integration, it is clear that the region of integration is bounded by y=x, x=a and y= a. Thus, the region of integration as shown in figure. Draw vertical strip. So new limits of integration are

Limits of x: from y=0 to y=x

Limits of y: from x=0 to x=a

$$\int_0^a\int_0^x \frac{x\,dy\,dx}{x^2+y^2}$$

$$=\int_0^a x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)_0^x dx$$

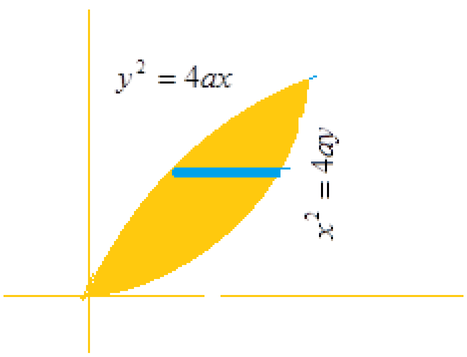
$$=\int_0^a \frac{\pi}{4}dx$$

$$=\frac{\pi}{4}(x)_0^a$$

$$=\frac{\pi a}{4}$$

Example 2:

Change the order of integration in the following integral and evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{az}} dy\,dx$



After changing the order, new limits are

From $x=\frac{y^2}{4a}$ to $x=2\sqrt{ax}$

From $y=0$ to $y=4a$

$$\int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} dx\,dy$$

$$=\int_0^{4a}(x)_{\frac{y^2}{4a}}^{2\sqrt{ay}}\,dy$$

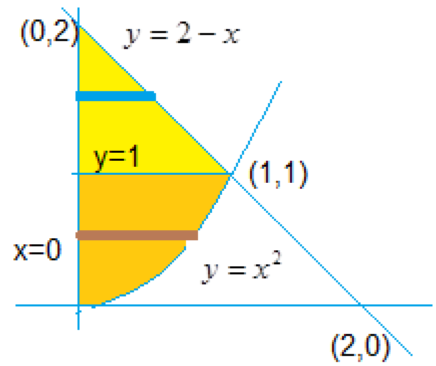
$$=\int_0^{4a}\left(2\sqrt{ay}-\frac{y^2}{4a}\right)dy$$

$$=\left(\frac{4\sqrt{a}\,y^{\frac{3}{2}}}{3}-\frac{y^3}{12a}\right)_0^{4a}$$

$$=\frac{16a^2}{3}$$

Example 3: Change the order of integration $\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$

$\cancel{y^2 = x\ y}$ Region 1 : $x = 0$ to $x = \sqrt{y}$
y=0 to y= 1
Region 2: $x = 0$ to $x = 2 - y$
y=1 to y=2



$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_0^1 \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} y\,dy = \int_0^1 \frac{y^2}{2}dy = \left(\frac{y^3}{6}\right)_0^1 = \frac{1}{6}$$

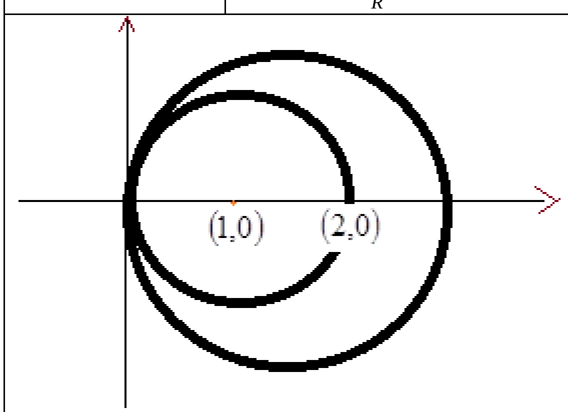
$$I_2 = \int_1^2 \int_0^{2-y} xy\,dx\,dy \qquad = \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} y\,dy \qquad = \int_0^1 \frac{(2-y)^2 y}{2} dy$$

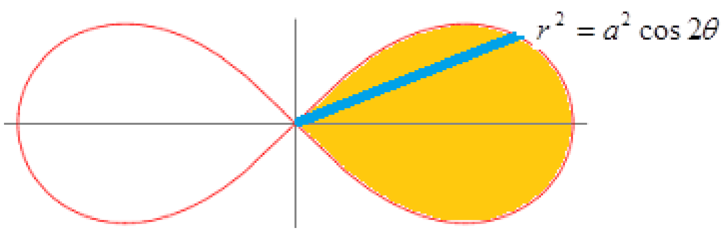
$$= \int_1^2 \frac{\left(4y - 4y^2 + y^3\right)}{2} dy = \left(y^2 - \frac{2}{3}y^3 + \frac{y^4}{8}\right)_1^2 = \frac{5}{24}$$

$$I = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

Evaluation of Double integral in Polar Co-ordinates (Applications)

To evaluate the double integral $\iint f(r,\theta)dr\,d\theta$ over the region R bounded by the curves $r = a$, $r = b$ and the straight lines $\theta = \alpha$ and $\theta = \beta$. First integrate with respect to r and θ.

Example 1:	Evaluate $\iint_R r^3 dr\,d\theta$, over the area bounded between the circles $r = 2\cos\theta$ and $r = 4\cos\theta$.
	$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3 dr d\theta$ $= 2\int_0^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} r^3\,dr\,d\theta$ $= 2\int_0^{\frac{\pi}{2}} \left(\frac{r^4}{4}\right)_{2\cos\theta}^{4\cos\theta} d\theta$ $= 120\int_0^{\frac{\pi}{2}} \cos^4\theta\,d\theta$ $= 120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$ $= \frac{45\pi}{2}$
Example 2:	Evaluate $\iint_R \frac{rdr\,d\theta}{\sqrt{a^2 + r^2}}$ over one loop of lemniscate $r^2 = a^2\cos 2\theta$.



$$\iint_R \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$$= \int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$$= 2\int_0^{\frac{\pi}{4}} \int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{a^2+r^2}}$$

$$= 2\int_0^{\frac{\pi}{4}} \left(2\sqrt{a^2+r^2}\right)_0^{a\sqrt{\cos 2\theta}}$$

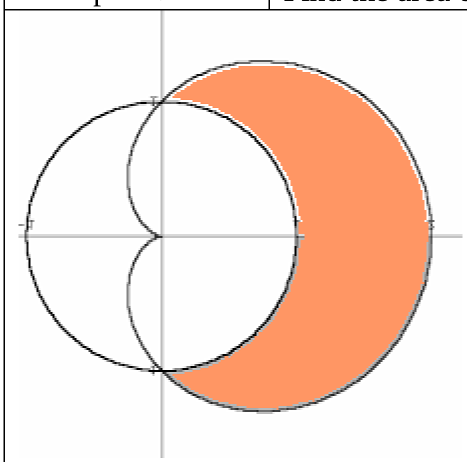
$$= 4\int_0^{\frac{\pi}{4}} \left(\sqrt{a^2+a^2\cos 2\theta}-a\right)d\theta$$

$$= 4a\int_0^{\frac{\pi}{4}} (2\cos\theta-1)\,d\theta$$

$$= 4a(2\sin\theta-\theta)_0^{\frac{\pi}{4}}$$

$$= 4a\left(\sqrt{2}-\frac{\pi}{4}\right)$$

Example 3: Find the area of the region that lies inside the cardioid $r = 1+\cos\theta$ and the circle $r = 1$.



$$\iint_R r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \int_1^{1+\cos\theta} r\,dr\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_1^{1+\cos\theta} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[(1+\cos\theta)^2-1\right]d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(1+2\cos\theta+\cos^2\theta-1\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(2\cos\theta+\cos^2\theta\right)d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \left(2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)_0^{\frac{\pi}{2}}$$

	$=\left(2+\dfrac{\pi}{4}\right)$

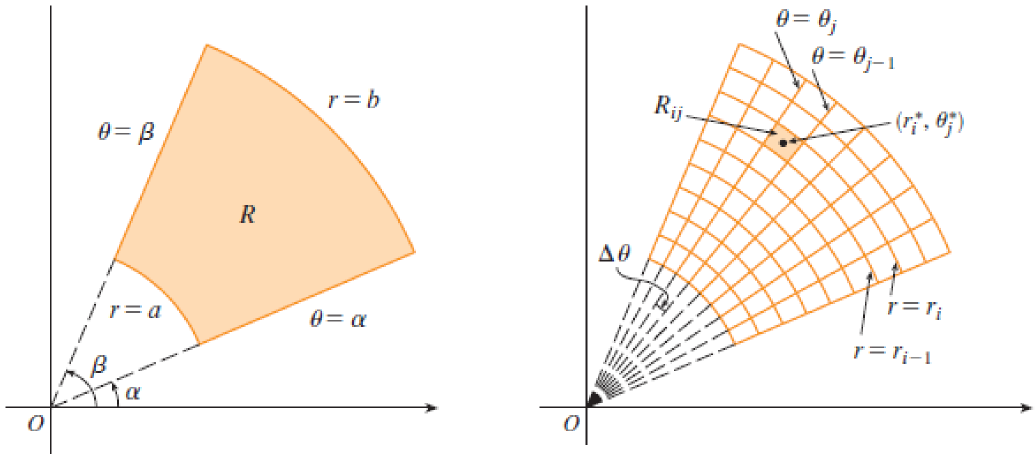
Change into Polar Form:

Integration is often transformed into polar form to make easier.

Suppose that we want to evaluate a double integral $\iint_R f(x,y)dA$, where R is either circle or semicircle.

In either case the description of in terms of rectangular coordinates is rather complicated but is easily described using polar coordinates.

In order to evaluate the double integral $\iint_R f(x,y)dA$, where $R=\{(r,\theta): a\le r\le b;\ \alpha\le\theta\le\beta\}$, we divide the interval $[a,b]$ into m subintervals $[r_{i-1},r_i]$ of equal width $\Delta r=\dfrac{b-a}{m}$ and we divide the interval $[\alpha,\beta]$ into n subintervals $[\theta_{j-1},\theta_j]$of the equal width $\Delta\theta=\dfrac{\beta-\alpha}{n}$.Then the circle will be divided into small polar rectangles as shown in figure



(a) (b)

The center of the Shaded polar sub rectangle $\{(r,\theta): r_{i-1}\le r\le r_i;\theta_{j-1}\le\theta\le\theta_j\}$ has polar coordinates

$$r_i^*=\frac{r_i+r_{i-1}}{2},\ \theta_j^*=\frac{\theta_j+\theta_{j-1}}{2}$$

$\therefore$ Area of polar rectangle $R_{i,j}=\dfrac{1}{2}r_i^2\Delta\theta_j-\dfrac{1}{2}r_{i-1}^2\Delta\theta_j$

$$=\frac{1}{2}\left(r_i^2-r_{i-1}^2\right)\Delta\theta_j$$

$$=\frac{1}{2}(r_i-r_{i-1})(r_i+r_{i-1})\Delta\theta_j$$

$$=r_i^*\Delta r_i\Delta\theta_j$$

Although we have defined the double integral $\iint_R f(x,y)dA$ in terms of ordinary rectangle, it can be shown that, for continuous functions f, we always obtain the same answer using polar rectangles, The rectangular coordinates of the centre of $R_{i,j}$ are $(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*,)$, so a typical Riemann sum is

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f\left(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*\right)\Delta A_{i,j}=\sum_{j=1}^{n}\sum_{i=1}^{m} f\left(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*\right)\Delta r_i\Delta\theta_j$$

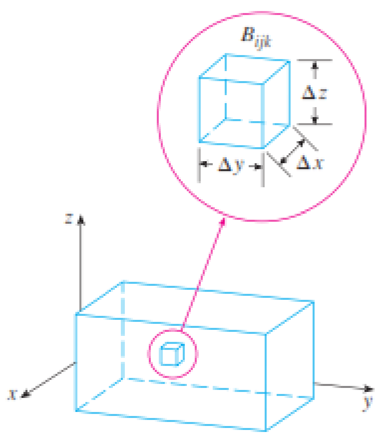
$$= \int_{\alpha}^{\beta} \int_{a}^{b} f(r\cos\theta, r\sin\theta) dr\, d\theta$$

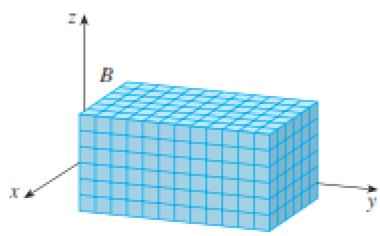
$$= \int_{\alpha}^{\beta} \int_{a}^{b} f(r, \theta)\, r\, dr\, d\theta$$

Example:

1) Evaluate $\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} x^2 dydx$
2) Evaluate $\int_0^{2a} \int_0^{\sqrt{2x - x^2}} \frac{x}{\sqrt{x^2+y^2}} dydx$
3) Evaluate $\int_0^{1} \int_0^{\sqrt{1 - y^2}} (x^2 + y^2) dxdy$

<u>Triple Integrations</u>





Just as we defined single integrals for functions of one variable and double integrals for functions of two variables , so we can define triple integrals for functions of three variable. Lets first deal with the simplest case where f is defined on a rectangular box:

The first step is to divide B into sub boxes. We do this by dividing the interval[a,b] in to l sub intervals $[x_{i-1}, x_i]$ of equal width Δx, dividing (c,d) into m subintervals of width Δz. The planes through the endpoints of these subintervals purallel to the coordinate planes divide the box B into lmn sub-boxes

$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ which are shown in figure 1 each sub box has volume ΔV =Δx Δy Δz

Then we form triple riemann sum.

Defination : the triple integral f over the box B is

$$\iiint_b f(x,y,z)dv = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}, y_{ijk}, z_{ijk})\Delta V$$

If this limit exists>

Again the triple integral always exists if f is continuous> we can choose sample points to be any point in the sub box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler looking expression for the triple integral.

$$\iiint_b f(x,y,z)dv = \lim_{l,m,n\to\infty} \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_i, y_j, z_k)\Delta V$$

Just as for double integrals , the practice methods for evaluating triple is to express them as iterated integrals as follows.

Now we define the triple integral over a general bounded region E in three dimentional spce by much the same procedure that we used for double intyegrals . we enclose E in a box B for the type given by equation . then we define a function F so that it agrees with f on E but is 0 for points in b that are outside E. by defination.

$$\iiint_B f(x,y,z)dv = \iiint_E f(x,y,z)dv$$

The integral exists if f is continuous and the boundary of E is “reasanobaly smooth”. The triple integrals has essentially the same properties as the double integrals .

We restricts our attention to continuous function f and to certain simple types of regions. A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y, that is,

E= { (x,y,z)| (x,y)∈ D, u_1(x,y)<= z <= u_2(x,y)}

Where D is the projection of e onto the xy-plane. Notice that the upper boundry of the solid E is the surface with equation z= u_2(x,y), while the lower bdry is the surface z= u_1(x,y).

just as for double integrals , the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

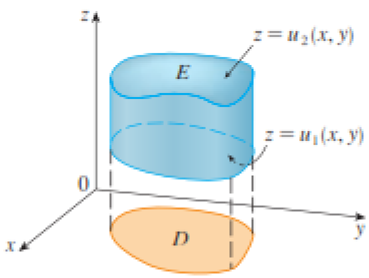


FIGURE 2
A type 1 solid region

By the same sort of argument that led to formula .it can be shown that if E is a type 1 region given by equation 5, then

$$\iiint_E f(x,y,z)dv = \iint_D \left[\int_{u2(x,y)}^{u1(x,y)} f(x,y,z)dz \right] dA$$

The meaning of the inner integral on the right side of equation is that x and y are held fixed, and therefore u1(x,y) and u2(x,y) are regarded as constants, while f(x,y,z) is integrated with respect to z.

In particular, if the projection D of E onto the type xy plane is a type 1 region then,

E= { (x,y,z)| a<= x<=b, g1(x)<=y<=g2(x), u_1(x,y)<= z <= u_2(x)}

And expression becomes,

$$\iiint_E f(x,y,z)dv = \int_a^b \int_{g1(x)}^{g2(x)} \int_{u2(x,y)}^{u1(x,y)} f(x,y,z)dxdydz$$

If on other hand D is type 2 region then

E= { (x,y,z)| c<= x<=d, h1(x)<=y<=h2(x), u_1(x,y)<= z <= u_2(x)}

And expression becomes,

$$\iiint_E f(x,y,z)dv = \int_c^d \int_{h1(x)}^{h2(x)} \int_{u2(x,y)}^{u1(x,y)} f(x,y,z)dxdydz$$

Evaluate the following triple integrations

1. $\int_1^2 \int_2^3 \int_0^1 xyzdxdydz = \int_1^2 \left[\int_2^3 \left[\int_0^1 xyzdz \right] dx \right] dy$

$$= \int_1^2 \left[\int_2^3 xy \left(\frac{z^2}{2} \right)_0^1 dx \right] dy$$

$$=\frac{1}{2}\int_1^2\left[\int_2^3 xydx\right]dy$$

$$=\frac{1}{2}\int_1^2 y\left(\frac{x^2}{2}\right)_2^3 dy \quad =\frac{5}{4}\int_1^2 ydy = \frac{5}{4}\left(\frac{y^2}{2}\right)_1^2 = \frac{15}{8}$$

2. $\int_0^1\int_0^{2-x}\int_0^{x-y} dzdydx$

$$=\int_0^1\int_1^{2-x}[z]_0^{x-y}dydx \quad =\int_0^1\int_1^{2-x}[x-y]dydx \quad =\int_0^1\left[xy-\frac{y^2}{2}\right]_0^{2-x}dx \quad =\int_0^1\left[x(2-x)-\frac{(2-x)^2}{2}\right]dx$$

$$=-\frac{1}{2}\int_0^1\left[3x^2-8x+4\right]dx=-\frac{1}{2}\left[x^3-4x^2+4x\right]_0^1=-\frac{1}{2}$$

3. Evaluation of triple integrals yields a volume instead of an area.

Parallelepipeds (boxes) are summed rather than rectangles

$$\int_1^2\int_x^{2x}\int_0^{y-x} \mathrm{dz}\,\mathrm{dy}\,\mathrm{dx} = \int_1^2\int_x^{2x} \mathrm{z}\Big|_0^{y-x}\mathrm{dy}\,\mathrm{dx} = \int_1^2\int_x^{2x}(\mathrm{y}-\mathrm{x})\mathrm{dy}\,\mathrm{dx} = \int_1^2\left[\frac{\mathrm{y}^2}{2}-\mathrm{xy}\right]\Big|_x^{2x}\mathrm{dx}$$

$$=\int_1^2\left[\left(\frac{(2\mathrm{x})^2}{2}-\mathrm{x}\cdot 2\mathrm{x}\right)-\left(\frac{\mathrm{x}^2}{2}-\mathrm{x}\cdot\mathrm{x}\right)\right]\mathrm{dx} = \int_1^2\frac{\mathrm{x}^2}{2}\mathrm{dx}$$

$$=\frac{\mathrm{x}^3}{6}\Big|_1^2 = \frac{8}{6}-\frac{1}{6}=\frac{7}{6}$$

Application of triple integration

Recall that if f(x) > 0. then the single integral. $\int_a^b$ f(x) (it represents the area under the Curve y = f (x) from a to b. and if f (x. y)> 0. then the double integral f (x,y) (M rep-resents the volume under the surface : = f (x. y) and above D. The corresponding interpretation of a triple integral is not very useful because it would be the "hyper volume" of a four-dimensional object and. of course. That is very difficult to visualize. (Remember that E is just the domain of the function f z the graph of f lies in four-dimensional space.) Nonetheless. The triple integral $\iiint f(x,y,z)dv$ can be interpreted in different ways in different physical situations depending on the physical interpretations of x, y, z and f (x. y. z).

$$V(E)=\iiint f(x,y,z)dV=\iiint dV$$

Let's begin with the special case where f (.x. y. z) = l for all points in E. Then the triple integral does represent the volume of E.

$$\iiint 1dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} 1\, dZ \right] dA = \iint_D u_2(x,y) - u_1(x,y)\; dA$$

For example. You can see this in the case of a type I region by putting f (x. y. z) = I in

formula. We know this represents the volume that lies between the surfaces

z= u_1(.x. y) and z = u_2(.x. y).

Examples.

$$1 \int_0^1 \int_0^2 \int_0^{x+2} 6xzdxdydz$$

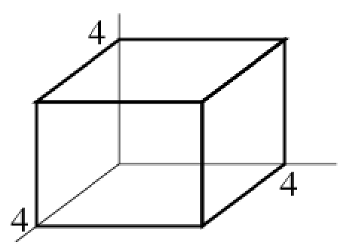
$$2 \int_0^1 \int_0^{2x} \int_0^{y} 2xyzdxdydz$$

$$3 \int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y dxdydz$$

$$4 \int_0^1 \int_0^z \int_0^y ze^{-y^2} dxdydz$$

$$5 \int_0^1 \int_0^y \int_0^x \cos(x+y+z) dxdydz$$

Example : A cube has sides of length 4. Let one corner be at the origin and the adjacent corners be on the positive x, y, and z axes.



If the cube's density is proportional to the distance from the xy-plane, find its mass.

Solution: The density of the cube is f(x,y,z)=kz for some constant k.

If W is the cube, the mass is the triple integral.

$$\iiint_W kzdV = \int_0^4 \int_0^4 \int_0^4 kzdxdydz = 128k.$$

If distance is in cm and k=1 gram per cubic cm per cm, then the mass of the cube is 128 grams.

Suppose we change from the Cartesian coordinates (x1, x2, x3) to the curvilinear coordinates, which we denote ui , each of which are functions of the xi :- u1 = u1(x1, x2, x3) u2 = u2(x1, x2, x3) u3 = u3(x1, x2, x3) The ui should be single-valued, except possibly at certain points, so the reverse transformation, xi = xi (u1, u2, u3) can be made. A point may be referred to by its Cartesian coordinates xi , or by its curvilinear coordinates ui . For example, in 2-D, we might have:- Now consider coordinate surfaces defined by keeping one coordinate constant. • The Cartesian coordinate surfaces 'xi = constant' are planes, with constant unit normal vectors ei (or e1 , e2 and e3), intersecting at right angles. • The surfaces 'ui = constant' do not, in general, have constant unit normal vectors, nor in general do they intersect at right angles. 89 Example: Spherical polar co-ordinates r = q x 2 + y 2 + z 2 ; θ = cos−1 (z √ x 2 + y 2 + z 2) ; φ = tan−1 y x . The surfaces of constant r, θ, and φ are:- r = constant =⇒ spheres centred at the origin unit normal er θ = constant =⇒ cones of semi-angle θ and axis along the z-axis unit normal e θ φ = constant =⇒ planes passing through the z-axis unit normal eφ These surfaces are not all planes, but they do intersect at right angles. If the coordinate surfaces intersect at right angles (i.e. the unit normals intersect at right angles), as in the example of spherical polars, the curvilinear coordinates are said to be orthogonal. 23. 1. Orthogonal Curvilinear Coordinates Unit Vectors and Scale Factors Suppose the point P has position r = r(u1, u2, u3). If we change u1 by a small amount, du1, then r moves to position (r + dr), where dr = ∂r ∂u1 du1 ≡ h1 e1 du1 where we have defined the unit vector e1 and the scale factor h1 by h1 = ∂r ∂u1 and e1 = 1 h1 ∂r ∂u1 . • The vector e1 is a unit vector in the direction of increasing u1. • The scale factor h1 gives the magnitude of dr when we make the change u1 → u1+du1. Thus for an infinitessimal change of u1 |dr| = h1 du1 90 Similarly, we can define hi and ei for i = 2 and 3. • The unit vectors ei are not constant vectors. In general they are nonCartesian basis vectors, they depend on the position vector r, i.e. their directions change as the ui are varied. • If ei · ej = δij, then the ui are orthogonal curvilinear coordinates.