Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme

Mathematics – II (303191151)

Unit 3: Laplace Transform:(Lecture Notes)

Laplace Transform:

Let $f(t)$be a function of $t \geq 0$, then the Laplace Transformation of $f(t)$is defined as

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$$

Provided that integral exists. s is a parameter which may be real or complex number.

Laplace transform of elementary function:

***Ex*: Find the Laplace transform of 1, where $s > 0$.**

***Sol*. We know that,**

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$$

$$\Rightarrow L\{1\} = \int_0^\infty 1.e^{-St}dt = \left[\frac{e^{-St}}{-s}\right]_0^\infty = \left[\frac{e^{-S(\infty)}}{-s} - \frac{e^{-S(0)}}{-s}\right] = \left[0 - \frac{1}{-s}\right] = \frac{1}{s}(\because \quad e^{-\infty} = 0)$$

$$\therefore \quad L\{1\} = \frac{1}{s}$$

***Ex*: Find the Laplace transform of e^{-at}, where, $s > -a$.**

***Sol*. We know that,**

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$$

$$\Rightarrow L\{e^{-at}\} = \int_0^\infty e^{-at}.e^{-St}dt = \left[\frac{e^{-(S+a)t}}{-(S+a)}\right]_0^\infty = \left[\frac{e^{-(S+a)(\infty)}}{-(S+a)} - \frac{e^{-(S+a)(0)}}{-(S+a)}\right]$$

$$= \left[0 - \frac{1}{-(S+a)}\right] (\because \ e^{-\infty} = 0) = \frac{1}{(S+a)}$$

$$\therefore \quad L\{e^{-at}\} = \frac{1}{(S+a)}$$

Also

$$\therefore \quad L\{e^{at}\} = \frac{1}{(S-a)}, s > a$$

***Ex*: Show that** $\mathrm{L}\{\mathrm{t}^{\mathrm{n}}\} = \frac{\Gamma_{\mathrm{n}+1}}{\mathrm{s}^{\mathrm{n}+1}}$ for n > -1

$= \frac{\mathrm{n}!}{\mathrm{s}^{\mathrm{n}+1}}$ for n is a positive Integer and $s > 0$

Sol. We know that,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

Here $f(t) = \mathrm{t}^{\mathrm{n}}$

$$\Rightarrow L\{\mathrm{t}^{\mathrm{n}}\} = \int_0^\infty \mathrm{t}^{\mathrm{n}}.e^{-St}dt$$

Putting $st = x \Rightarrow dt = \frac{1}{s}dx$

Also, when $t = 0, x = 0$ and $t = \infty, x = \infty$

$$L\{\mathrm{t}^{\mathrm{n}}\} = \int_0^\infty \mathrm{t}^{\mathrm{n}}.e^{-St}dt = \int_0^\infty \left(\frac{x}{s}\right)^n.e^{-x}\frac{dx}{s} = \frac{1}{\mathrm{s}^{\mathrm{n}+1}}\int_0^\infty x^n.e^{-x}dx = \frac{1}{\mathrm{s}^{\mathrm{n}+1}}\int_0^\infty x^{(n+1)-1}.e^{-x}dx$$

We know that

$$\Gamma_n = \int_0^\infty x^{n-1}.e^{-x}dx$$

Therefore,

$$L\{t^n\} = \frac{\Gamma_{n+1}}{s^{n+1}}$$

Also, if n is a positive integer, then $\Gamma_{n+1} = n!$

So,

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

***Ex*: Show that $\mathbf{L\{\sin at\} = \frac{a}{s^2+a^2}}$**

Sol. Here $f(t) = \ sin\ at = \frac{e^{iat}-e^{-iat}}{2i}$

We know that,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

$$L\ (sin\ at) = L\ \left\{\frac{e^{iat}-e^{-iat}}{2i}\right\}$$

$$= \frac{1}{2i}[L(e^{iat}) - L(e^{-iat})]$$

$$= \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{s+ia-s+ia}{(s-ia)(s+ia)}\right]$$

$$= \frac{1}{2}\left[\frac{2ia}{s^2-i^2a^2}\right] = \frac{a}{s^2+a^2}$$

$$\therefore L\ (sin\ hat) = \frac{a}{s^2+a^2}$$

Exercise 1: Showthat $L\{cos\ at\} = \frac{\boldsymbol{s}}{\boldsymbol{s^2+a^2}}$(Hint:$cos\ at = \frac{e^{iat}+e^{-iat}}{2}$)

Exercise 2: Prove that $L\{sin\ hat\} = \frac{a}{s^2-a^2}$(Hint: $sin\ hat = \frac{e^{at}-e^{-at}}{2}$)

Exercise 3: Find $L\{cos\ hat\}$ (Hint:$cos\ hat = \frac{e^{at}+e^{-at}}{2}$)**Ans**: $\frac{s}{s^2-a^2}$

Linearity of Laplace Transform:

If $L\{f(t)\} = F(s)$and$\{g(t)\} = G(s)$, then for any constants a and b,

$$L\{af(t) + b\ g(t)\} = a\ L\{f(t)\} + bL\{g(t)\} = aF(s) + bG(s)$$

Proof: We know that,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

$$L\{af(t) + b\ g(t)\} = \int_0^\infty \{af(t) + b\ g(t)\}e^{-St}dt = \int_0^\infty af(t)e^{-St}dt + \int_0^\infty b\ g(t)e^{-St}dt$$

$$= a\int_0^\infty f(t)e^{-St}dt + b\int_0^\infty g(t)e^{-St}dt = a\ L\{f(t)\} + bL\{g(t)\} = aF(s) + bG(s)$$

Ex:Find the Laplace transform of $\boldsymbol{f(t) = t^2 + sin3t - 2e^{-t}}$.

Sol: Given that $f(t) = t^2 + sin3t - 2e^{-t}$

By using formula and linearity property,

$$L\{f(t)\} = L\{t^2 + sin3t - 2e^{-t}\} = L\{t^2\} + L\{sin3t\} - L\{2e^{-t}\}$$
$$= \frac{2!}{s^3} + \frac{3}{s^2 + 3^2} - \frac{2}{s+1}$$

Ex: Evaluate $\boldsymbol{L\{cos^2 t\}}$ $= L\left(\frac{1+cos\ 2t}{2}\right)$

Sol: $L\{cos^2 t\} = L\left(\frac{1+cos\ 2t}{2}\right) = L\left(\frac{1}{2}\right) + \frac{1}{2}L\ (cos\ 2t) = \frac{1}{2S} + \frac{1}{2}\frac{S}{S^2+4}$

Ex: Find $\boldsymbol{L\{cos^3 2t\}}$

Sol: we know that $\cos^3\theta = \frac{1}{4}(3\cos\theta + \cos 3\theta)$

$$L\{\cos^3 2t\} = \frac{1}{4}L\,(3\cos 2t + \cos 6t) = \frac{1}{4}\left(3\frac{S}{S^2+4} + \frac{S}{S^2+36}\right)$$

Ex:Evaluate $L\{\boldsymbol{sin2tcos2t}\}$**.**

Sol: $L\{sin2tcos2t\} = L\left\{\frac{1}{2}(2sin2tcos2t)\right\} = \frac{1}{2}L\{sin4t\} = \frac{1}{2}\frac{4}{s^2+16} = \frac{2}{s^2+16}$

Ex:Find the Laplace transformation of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{-1} & \boldsymbol{0 < t \le 4} \\ \boldsymbol{1} & \boldsymbol{t > 4} \end{cases}$**.**

Sol: We have,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

Here $f(t) = \begin{cases} -1 & 0 < t \le 4 \\ 1 & t > 4 \end{cases}$

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = \int_0^4 (-1)e^{-St}dt + \int_4^\infty (1)e^{-St}dt = -\left[\frac{e^{-St}}{-s}\right]_0^4 + \left[\frac{e^{-St}}{-s}\right]_4^\infty$$

$$= -\left(\frac{e^{-4S}}{-s} - \frac{e^0}{-s}\right) + \left(\frac{e^{-\infty}}{-s} - \frac{e^{-4S}}{-s}\right) = 2\frac{e^{-4S}}{s} - \frac{1}{s} = \frac{2e^{-4S}-1}{s}$$

Exercise: Find the Laplace transformation $\boldsymbol{f(t) = 2cos\,5t\,sin2t}$**, Ans:** $\frac{4s^2-84}{(s^2+49)(s^2+9)}$

Exercise: Find $\boldsymbol{L\{cos^2 2t\}}$**, Ans:**$\frac{s^2+8}{s\,(s^2+16)}$

Exercise: Find $\boldsymbol{L\{5t^3 + 3t^2 - 6t + 3e^{-5t}\}}$

Exercise: Find $\boldsymbol{L\{e^{2t}cos\,h^2 2t\}}$

- **First Shifting Theorem:**

If $\mathrm{L\{f(t)\} = F(s)}$, then $\mathrm{L\{e^{at}f(t)\} = F(s-a) = [L\{f(t)\}]_{s \to s-a}}$

Proof: By, definition of L.T.,

$$L\left(e^{at}f(t)\right) = \int_0^\infty e^{-st}e^{at}f(t)dt$$

$$= \int_0^\infty e^{-st+at}f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t}f(t)dt$$

$$L\left(e^{at}f(t)\right) = F(s-a)$$

$$(\because \int_0^\infty e^{-st}f(t)dt = F(s))$$

Some important formula of First shifting theorem:

$$\mathrm{L}\{e^{at}.1\} = \frac{1}{(s-a)}$$

$$\mathrm{L}\{e^{at}.t^n\} = \frac{\Gamma_{n+1}}{(s-a)^{n+1}}\left(n = \frac{p}{q}\right) = \frac{n!}{(s-a)^{n+1}}(n = \text{Integer})(n > -1)$$

$$\mathrm{L}\{e^{at}.\sin bt\} = \frac{b}{(s-a)^2+b^2}$$

$$\mathrm{L}\{e^{at}.\cos bt\} = \frac{(s-a)}{(s-a)^2+b^2}$$

$$\mathrm{L}\{e^{at}.\sin h\, bt\} = \frac{b}{(s-a)^2-b^2}$$

$$\mathrm{L}\{e^{at}.\cos h\, bt\} = \frac{(s-a)}{(s-a)^2-b^2}$$

$$\mathrm{L}\{\sin h\, at\, f(t)\} = \frac{1}{2}\left[\bar{f}(s-a) - \bar{f}(s+a)\right]$$

and

$$\mathrm{L}\{\cos h\, at\, f(t)\} = \frac{1}{2}\left[\bar{f}(s-a) + \bar{f}(s+a)\right]$$

Examples:

Ex: Evaluate $\boldsymbol{L\{e^{-3t}(\cos 4t + 3\sin\ 4t\}}$

Sol: $L\{e^{-3t}(\cos 4t + 3\sin\ 4t\} = L\left(e^{-3t}\cos\ 4t\right) + L\left(e^{-3t}\, 3\sin 4t\right)$

First, $L\ (e^{-3t}\cos\ 4t)$

$$L\ (\cos 4t) = \frac{s}{s^2+16}$$

$$L\ (e^{-3t}\cos\ 4t) = \frac{s+3}{(s+3)^2+16}$$

Second, $L\ (e^{-3t}\, 3\sin 4t)$

$$L\ (\sin 4t) = \frac{4}{s^2+16}$$

$$L\ (e^{-3t}\cos\ 4t) = \frac{4}{(s+3)^2+16}$$

$$\therefore L\{e^{-3t}(\cos 4t + 3\sin\ 4t\} = \frac{s+3}{(s+3)^2+16} + \frac{3(4)}{(s+3)^2+16} = \frac{s+15}{(s+3)^2+16}$$

Ex: Find $\boldsymbol{L\ (e^{-3t}sin^2t)}$

***Sol*:**

$$L\ (sin^2t) = L\left\{\frac{(1-\cos 2t\ )}{2}\right\} = \frac{1}{2s} - \frac{s}{2\ (S^2+4)}$$

$$L\ (e^{-3t}sin^2t) = \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right]$$

Exercise: Find $\boldsymbol{L\{e^{2t}sin^2 3t\}}$Ans: $\frac{18}{(s-2)(s^2-4s+40)}$

Exercise: Evaluate $\boldsymbol{L\{(t+1)^3e^t\}}$Ans: $\frac{s\ (s^2-2s+7)}{(s-1)^4}$

Exercise: Find the Laplace transformation $\boldsymbol{g(t) = (t+1)^2e^t}$

Ans: $\frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$

Exercise: Find $\boldsymbol{L\{1+2\sqrt{t}+3/\sqrt{t}\}}$Ans: $\frac{1}{s} + \frac{\sqrt{\pi}}{s} + \sqrt[3]{\frac{\pi}{s}}$

Exercise: Find $\boldsymbol{L\{coshat\ sinat\}}$Ans: $\frac{a(s^2+2a^2)}{(s^4+4a^4)}$

Exercise: Find $L\{e^{-t}sin^2t\}$

Exercise: Find $L\{e^{t}sin2t\ sin2t\}$

Differentiation and Integration of Transform:

If $L\{\mathrm{f(t)}\} = \mathrm{F(s)}$, then $\mathrm{L\{t^n f(t)\}} = (-1)^{\mathrm{n}} \frac{\mathrm{d^n}}{\mathrm{ds^n}}\{\mathrm{F(s)}\} = (-1)^{\mathrm{n}} \frac{\mathrm{d^n}}{\mathrm{ds^n}}\{\mathrm{L\{f(t)\}}\}$.

Proof: Given $L\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)dt$

Differentiating both sides w.r.t. s , we get

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^{\infty} e^{-st} f(t)dt\right]$$

$$= \int_0^{\infty} \frac{\partial}{\partial s} e^{-st} f(t)dt$$

$$= \int_0^{\infty} (-t)e^{-st} f(t)dt$$

$$= -\int_0^{\infty} e^{-st} (t\, f(t))dt$$

$$L\{t\, f(t)\} = (-1)^n \frac{d}{ds}[F(s)]$$

Similarly, differentiating both the side, w.r.t. s(times), one can find

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Examples:

Ex: Find the Laplace Transform of $h(t) = t^2 \sin \pi t$

Sol: We know that, $L\,(sin\, \pi t) = \frac{\pi}{s^2+\pi^2}$

$$L\,(t^2 \sin \pi\, t) = (-1)^2 \frac{d^2}{ds^2}[F(s)]$$

$$= \frac{d^2}{ds^2}\left(\frac{\pi}{s^2+\pi^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s\pi}{(s^2+\pi^2)^2}\right)$$

$$= -2\,\pi\ \frac{d}{ds}\left(\frac{s}{(s^2+\pi^2)^2}\right)$$

Using $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,u' - u\,v\prime}{v^2}$

$$\frac{d}{ds}\left(\frac{s}{(s^2+\pi^2)^2}\right) = \frac{(s^2+\pi^2)^2 - 2s(s^2+\pi^2)(2s)}{(s^2+\pi^2)^4}$$

$$= \frac{(s^2+\pi^2)\left[s^2+\pi^2-4s^2\right]}{(s^2+\pi^2)^4}$$

$$= \frac{\pi^2-3s^2}{(s^2+\pi^2)^3}$$

$$= -2\,\pi\ \left[\frac{\pi^2-3s^2}{(s^2+\pi^2)^3}\right]$$

$$= \frac{(3s^2-\pi^2)2\,\pi}{(s^2+\pi^2)^3}$$

Ex: Evaluate $\boldsymbol{L}\ \{\boldsymbol{t\ e^{-2t} sin\ t}\}$

Sol: We know that $L\ (sin\ t) = \frac{1}{s^2+1}$

$$L\ \{t\ sin\ t\} = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{1}{(s^2+1)^2}(2s) = \frac{2s}{(s^2+1)^2}$$

$$\therefore L\ \{t\ e^{-2t} sin\ t\} = \frac{2(s+2)}{((s^2+1)^2+1)^2}$$

Exercise: Find $\boldsymbol{L}\ [\boldsymbol{t\ e^{-t}\ cos\ t}]$Ans: $\frac{s\ (s+2)}{(s^2+2s+2)^2}$

Exercise: Evaluate$\boldsymbol{L}\ [\boldsymbol{t\ sin\ \omega t}]$Ans:$\frac{2\omega s}{(s^2+1)^2}$

Exercise: Evaluate$\boldsymbol{L}\ [\boldsymbol{t\ e^{-4t} sin3t}]$Ans:$\frac{6(s+4)}{(s^2+8s+25)^2}$

Exercise: Evaluate$\boldsymbol{L}\ [\boldsymbol{t^2 sin\pi t}]$Ans:$\frac{2\pi(3s^2-\pi^2)}{(s^2+\pi^2)^3}$

Change of scale Property:

If $L\{f(t)\} = F(s)$then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}\left[\ L\{f(t)\}\right]_{s\to\frac{s}{a}}$.

Division by t Property:

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty L\{f(t)\}\,ds$ provided the integral exists.

Proof: We have $L\,\{f(t)\} = F(s) = \int_0^\infty e^{-st}\,f(t)dt$

Integrating both sides,w.r.t.s to ∞,

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st}\,f(t)dt\right] ds = \int_0^\infty \left[\int_s^\infty e^{-st}ds\right] f(t)dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)dt$$

$$= \int_0^\infty \frac{e^{-st}}{-t} f(t)dt = L\left\{\frac{f\,(t)}{t}\right\}$$

$$\therefore L\left\{\frac{f\,(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

Examples:

Ex: Evaluate $L\left\{\frac{t-\sin h\,5t}{t}\right\}$

Sol: We know that,

$$L\left\{\frac{f\,(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

By comparison we get,

$f(t) = (t - \sin h\,5t)$

$$L\{f(t)\} = L(t - \sin h\,5t) = \frac{1}{s^2} - \frac{5}{s^2 - 25}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \left(\frac{1}{s^2} - \frac{5}{s^2-25}\right) ds = \left[-\frac{1}{s} - \frac{5}{10}\log\left(\frac{s-5}{s+5}\right)_s^\infty\right]$$

$$\therefore L\left\{\frac{f(t)}{t}\right\} = \frac{1}{s} - \frac{1}{2}\log\left(\frac{s-5}{s+5}\right)$$

Ex: Find $L\left\{\frac{1-\cos t}{t}\right\}$

Sol: We know that,

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

$$L\{f(t)\} = L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty \frac{1}{s} - \frac{s}{s^2+1}\, ds = \int_s^\infty \frac{1}{s} - \frac{1}{2}\left(\frac{2s}{s^2+1}\right) ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+1}}\right)\right]_s^\infty = -\log\left(\frac{s}{\sqrt{s^2+1}}\right)$$

Exercise: Find $L\left[\frac{e^{-3t}\sin 2t}{t}\right]$Ans:$cot^{-1}\frac{s+3}{2}$

Exercise: Evaluate $L\left[\frac{sin^3 t}{t}\right]$Ans:$\frac{3}{4}\left[cot^{-1}s - \frac{1}{3}cot^{-1}\frac{s}{3}\right]$

Exercise: Evaluate $L\left[\frac{cos2t - cos3t}{t}\right]$Ans:$\frac{1}{2}log\left[\frac{s^2+9}{s^2+4}\right]$

- **Transform of Derivatives and Integrals:**

- **Laplace transform of the derivative:**

$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$,

$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$, Similarly, in general

$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - f^{(n-1)}(0).$

- **Laplace transform of the integral of a function:**

If $L\{f(t)\} = F(s)$, then $L\left[\int_0^t f(u)du\right] = \frac{1}{s}F(s)$

Proof: The function $f(t)$ should be integrable in such a way that

$$g(t) = \int_0^t f(u)du$$

is of exponentialorder. Then $g(0) = 0$and$g'(t) = f(t)$. Therefore,

$L\{g'(t)\} = s\,L\{g(t)\} - g(0) = s\,L\{g(t)\}$

and so

$$L\left[\int_0^t f(u)du\right] = L\{g(t)\} = \frac{L\{g'(t)\}}{s} = \frac{L\{f(t)\}}{s} = \frac{1}{s}F(s)$$

Ex: Find the Laplace Transform of $\int_0^t e^{-t}\ dt$

Sol:$L\ \{e^{-t}\} = \frac{1}{s+1}$

$L\left\{\int_0^t e^{-t}\ dt\right\} = \frac{1}{s}L\ \{e^{-t}\} = \frac{1}{s(s+1)}$

Ex: Find $L\left\{\int_0^t e^{-t}\cos t\ dt\right\}$

Sol: We know that $L\ \{cos\ t\} = \frac{s}{s^2+1}$

$$\therefore L\ \{e^{-t}cos\ t\} = \frac{(s+1)}{(s+1)^2+1} = \frac{(s+1)}{s^2+2s+2}$$

$$\therefore L\left\{\int_0^t e^{-t}\cos t\ dt\right\} = \frac{1}{s}\left(\frac{s+1}{s^2+2s+2}\right)$$

Exercise: Evaluate $L\left\{\int_0^t te^{-4t}sin3t\ dt\right\}$

Ans: $\frac{1}{s}\left(\frac{6(s+4)}{(s^2+8s+25)^2}\right)$

➢ Evaluation of integrals by Laplace transform:

Examples:

Ex: Evaluate $\int_0^\infty e^{-3t}t^5\ dt$.

Sol: $\int_0^\infty e^{-st}t^5\ dt = L\{t^5\} = \frac{\mathbf{5!}}{\boldsymbol{s^6}}$

Putting $s = 3$, we have

$$\int_0^\infty e^{-3t}t^5\ dt = \frac{120}{3^6} = \frac{40}{243}$$

Ex: Evaluate $\boldsymbol{\int_0^\infty e^{-2t}sin^3t\ dt}$ **.**

Sol: $\int_0^\infty \mathrm{e^{-st}sin^3t\ dt = L\{sin^3t\}}$

$= \mathrm{L}\{\frac{3\mathrm{sint}-\mathrm{sin3t}}{4}\}$

$=\frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$

$=\frac{3}{4}[\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}]$

$=\frac{6}{(s^2+1)(s^2+9)}$…(1)

Putting $\mathrm{s} = 2$ in eq (1)

$$\int_0^\infty \mathrm{e^{-2t}sin^3t\ dt} = \frac{6}{(4+1)(4+9)} = \frac{6}{65}$$

Ex: Evaluate $\int_0^\infty te^{-2t}cost\,dt$.

Sol: $\int_0^\infty te^{-st}cost\,dt = L\{tcost\}$

$=-\frac{\boldsymbol{d}}{\boldsymbol{ds}}\ L\{cost\}$

$=-\frac{\boldsymbol{d}}{\boldsymbol{ds}}\ (\frac{s}{s^2+1})$

$=\frac{s^2-1}{(s^2+1)^2}$…(1)

Putting s=2 in equation 1, we have

$$\int_0^\infty te^{-2t}cost\,dt = \frac{4-1}{(\mathbf{4}+\mathbf{1})^2} = \frac{3}{25}$$

Exercise: Evaluate $\int_0^\infty t^2 e^{-2t} sin3t\, dt$.   Ans $=\frac{18}{2197}$

Show that $\int_0^\infty e^{-2t} \frac{sinht}{t} dt = \frac{1}{2}$ log3

- **Laplace Transform of periodic function:**

  Examples:

If $\boldsymbol{f(t)}$ **is a periodic function with period** $\boldsymbol{p}$**, i.e.**$\boldsymbol{f(t+p) = f(t), then}$

$$\boldsymbol{L\left(f(t)\right) = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t) dt}$$

Examples:

Ex: Find Laplace Transform of f(t) if

$$\boldsymbol{f(t) = \begin{cases} 3t \;\; if\; 0 < t < 2 \\ 6 \;\; if \;\; 2 < t < 4 \end{cases}}$$

Where,$\boldsymbol{f\ (t+\ 4) = f(t)}$

Sol: The given function is a periodic function with period 4

$$L\{f(t)\} = \frac{1}{1-e^{-4s}} \int_0^4 f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-4s}} \left[\int_0^2 e^{-st}\, 3t\, dt + \int_2^4 e^{-st} 6\, dt \right]$$

$$= \frac{1}{1-e^{-4s}} \left[3t\left(\frac{e^{-st}}{-s}\right) - 3\left(\frac{e^{-st}}{s^2}\right) \right]_0^2 + \frac{1}{1-e^{-4s}} \left[6\left(\frac{e^{-st}}{-s}\right) \right]_2^4$$

$$= \frac{1}{1-e^{-4s}} \left[-\frac{6}{s} e^{-2s} - \frac{3e^{-2s}}{s^2} + \frac{3}{s^2} - \frac{6}{s} e^{-4s} + \frac{6}{s} e^{-2s} \right]$$

$$= \frac{1}{s^2(1-e^{-4s})} [3 - 3e^{-2s} - 6se^{-4s}]$$

Ex: Find the Laplace transform of the function $\boldsymbol{f(t) = |sin\ \omega t|; t \geq 0}$

Sol. The given function is a periodic function of period $\frac{\pi}{\omega}$.

$$L\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st}\, f(t)dt \;=\; \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}|sin\,\omega t|dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s\sin\omega t - \;\omega\cos\omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$(\because \int e^{ax}\, sinbxdx = \frac{e^{ax}}{a^2+b^2}(asinbx - bcosbx))$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left[\frac{\omega e^{-\frac{\pi s}{\omega}}}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right]$$

Exercise: Find the Laplace transform of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{k} & \boldsymbol{0} < t < a \\ -\boldsymbol{k} & \boldsymbol{a} < t < 2\boldsymbol{a} \end{cases}$ **and**

$\boldsymbol{f\,(t+2a) = f(t).}$

Ans: $\frac{k}{s}\; tan\,h\,(\frac{as}{2})$

Inverse Laplace Transform:

If $L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$ then $L^{-1}\{F(s)\} = f(t)$

Some basic Inverse Laplace Transform Formula:

$$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}}\right\} = 1$$

$$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^2}\right\} = \mathrm{t} \;\text{ and }\; \mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^{\mathrm{n}+1}}\right\} = \frac{\mathrm{t}^{\mathrm{n}}}{\mathrm{n}!}$$

$$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}-\mathrm{a}}\right\} = \mathrm{e}^{\mathrm{at}} \quad \text{and} \quad \mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}+\mathrm{a}}\right\} = \mathrm{e}^{-\mathrm{at}}$$

$$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^2+\mathrm{a}^2}\right\} = \frac{\mathrm{sinat}}{\mathrm{a}}$$

$$\mathrm{L}^{-1}\left\{\frac{\mathrm{s}}{\mathrm{s}^2+\mathrm{a}^2}\right\} = \cos\mathrm{at}$$

$$L^{-1}\left\{\frac{1}{s^2-a^2}\right\}=\frac{\sin h\ at}{a}$$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cos h\ at$$

$$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\}=\frac{e^{at}\sin bt}{b}$$

$$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\}=e^{at}\cos bt$$

$$L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\}=\frac{e^{at}\sin h\ bt}{b}$$

$$L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\}=e^{at}\cos h\ bt$$

Partial Fractions:

Mainly four types of partial fraction are use repeatedly.

Case-1: If the denominator has non – repeated linear factors $(s-a),(s-b),(s-c)$ then $Partial\ fraction=\frac{A}{(s-a)}+\frac{B}{(s-b)}+\frac{C}{(s-c)}$.

Case-2: If the denominator has repeated linear factors $(s-a)(n\text{ times})$, then $Partial\ fraction=\frac{A_1}{(s-a)}+\frac{A_2}{(s-a)^2}+\frac{A_3}{(s-a)^3}+\cdots+\frac{A_n}{(s-a)^n}$.

Case-3: If the denominator has non-repeated quadratic factors

$(s^2+as+b),(s^2+cs+d)$ then

$$Partial\ fraction=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+cs+d)}.$$

Case-4: If the denominator has repeated quadratic factors $(s^2+as+b)(n\text{ times})$ then

$$Partial\ fraction=\frac{As+B}{(s^2+as+b)}+\frac{Cs+D}{(s^2+as+b)^2}+\cdots(n\ times).$$

Examples

Ex: $\boldsymbol{L^{-1}\left(\frac{1}{s-7}\right) = e^{7t}}$

Ex: $\boldsymbol{L^{-1}\left(\frac{4}{s^2-121}\right) = \frac{4}{11}\ sin\,h\ 11t}$

Ex: Find $\boldsymbol{L^{-1}\left\{\frac{6s-8}{s^2-s-6}\right\}}$

Sol: we have

$$\frac{6s-8}{s^2-s-6} = \frac{6s-8}{(s+2)(s-3)} = \frac{A}{(s+2)} + \frac{B}{(s-3)} = \frac{A(s-3)+B(s+2)}{(s+2)(s-3)}$$

$\therefore 6s-8 = A(s-3)+B(s+2)\ \dots(1)$

Put $s=3$ in above equation, we can find

$18-8 = A(0)+B(5)$

$10 = 5B$

$\therefore \boldsymbol{B = 2}$

Put $s=-2$ in equation (1), one can get

$6(-2)-8 = A\,(-2-3)+B(0)$

$-12-8 = A(-5)$

$-20 = -5A$

$\therefore \boldsymbol{A = 4}$

By using Partial Fraction,

$$L^{-1}\left\{\frac{6s-8}{s^2-s-6}\right\} = L^{-1}\left(\frac{4}{(s+2)} + \frac{2}{(s+3)}\right) = 4\,e^{-2t} + 3e^{3t}$$

Ex: Evaluate $\boldsymbol{L^{-1}\left\{\log\frac{1}{s}\right\}}$

Sol: Here

$$F(s) = \log\frac{1}{s} = \log 1 - \log s = -\log s$$

Differentiating with respect to s

$$-\frac{d}{ds}\,F(s) = -\frac{d}{ds}\,(-\log s) = \frac{1}{s}$$

We know that,

$$L^{-1}\left\{-\frac{d}{ds}\,F(s)\right\} = t\,f(t)$$

$$\Rightarrow 1 = L^{-1}\left\{\frac{1}{s}\right\} = t\,f(t)$$

$$\Rightarrow f(t) = \frac{1}{t}$$

Exercise: Show that $L^{-1}\left(\frac{s}{s^2+4}\right) = \boldsymbol{\cos 2t}$

Exercise: Prove that $L^{-1}\left(\frac{3}{s^2+6s+18}\right) = \boldsymbol{e^{-3t}\sin 3t}$

Exercise: Show that $L^{-1}\left(\frac{5s+3}{(s^2+2s+5)(s-1)}\right) = \boldsymbol{-e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t + e^{t}}$

Exercise: Prove that $\mathbf{L^{-1}\left\{\log\left(\frac{s+1}{s}\right)\right\} = \frac{1-e^{-t}}{t}}$

Exercise: Show that $\mathbf{L^{-1}}\left\{\int_s^\infty \cot^{-1}(s+1)ds\right\} = \frac{-e^t \sin t}{t^2}$

Definition of convolution:

The $(f * g)$ is known as convolution and it is defined as

$(f * g)(t) = \int_0^t f(u)\,g(t-u)\,du$. Also $f * g = g * f$

Convolution Theorem: If $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$ then

$$L^{-1}\{F(s).G(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\} = f(t) * g(t) = \int_0^t f(u)\,g(t-u)\,du$$

Examples:

Ex: Find $\mathbf{1 * 1}$

Sol: Let $f(t) = 1$ and $g(t) = 1$

Then by definition of convolution,

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du = \int_0^t 1 \cdot 1\, du = \int_0^t 1\, du = [u]_1^t = t$$

$$\therefore 1 * 1 = t$$

Ex: Using convolution theorem, evaluate $\boldsymbol{L^{-1}\left\{\frac{1}{s\,(s^2+4)}\right\}}$

Sol: $f(s) = \frac{1}{s}$ and $g(s) = \frac{1}{(s^2+4)}$

We know that

$$g(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$$

and

$$f(t) = L^{-1}\left\{\frac{1}{(s^2+4)}\right\} = \frac{1}{2}\sin 2t$$

Therefore, we get

$$g(t) = 1 \quad ; \qquad f(t) = \frac{1}{2}\sin 2t$$

By convolution theorem,

$$L^{-1}\left\{\frac{1}{s\,(s^2+4)}\right\} = \int_0^t f(u)g(t-u)du = \int_0^t \frac{1}{2}\sin 2u \cdot (1)du = \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t$$

$$= \frac{1}{4}[-\cos 2t + \cos 0] = \frac{1}{4}[1 - \cos 2t]$$

Exercise: Find $\boldsymbol{\cos \omega t * \sin \omega t}$Ans:$\frac{t \sin \omega t}{2}$

Exercise:Apply Convolution Theorem to find $L^{-1}\left\{\frac{s^2}{s^4-a^4}\right\}$ **Ans:** $\frac{\sin at}{2a}+\frac{e^{at}-e^{at}}{2a}$

Exercise: Find the convolution of e^t and e^{-t} Ans: $\sin ht$

- **Unit Step Function(or Heaviside's unit function):**

Find the Laplace transform of unit step function

Or

Show that $L\{u(t-a)\}=\frac{e^{-as}}{s}$ **where** $u(t-a)=\begin{cases}0\,, t<a\\ 1\,, t\geq a\end{cases}$ *(This function is known as unit step function).*

Proof:

$$L\left(u\,(t-a)\right)=\int_0^{\infty} e^{-st}\,u(t-a)dt=\int_0^{a} e^{-st}(0)dt+\int_a^{\infty} e^{-st}(1)dt=\left[\frac{e^{-st}}{-s}\right]_a^{\infty}$$

$$=\frac{1}{-s}\left[e^{-\infty}-\,e^{-as}\right]=\frac{e^{-as}}{s}$$

$$\therefore L\left(u\,(t-a)\right)=\frac{e^{-as}}{s}$$

Dirac's Delta Function:

Definition: Consider the *Dirac Delta Function* $f_\varepsilon(where\ \varepsilon>0)$ which is defined by,

$$f_\varepsilon(t)=\delta\,(t-\varepsilon)=\begin{cases}\frac{1}{\varepsilon}\,, a\leq t\leq a+\varepsilon\\ 0\,,\qquad t>\varepsilon\end{cases}.$$

Exercise: Find the Laplace transform of Dirac – delta function $\delta\,(t-a)$

Ans: $L\,\{\delta\,(t-a)\}=e^{-as}$

Second Shifting Theorem:

If $L\{f(t)\}=F(s)$, then $L\{f(t-a)\,.\,u(t-a)\}=e^{-as}F(s)=e^{-as}L\{f(t)\}$

Proof: We know that $u(t-a)=\begin{cases}0 & for\ t<a\\ 1 & t\geq a\end{cases}$

$L\{f(t-a)u\,(t-a)\}$

$$= \int_0^\infty e^{st} f(t-a)u\,(t-a)dt$$

$$= \int_0^a e^{st}(0)dt + \int_a^\infty e^{-st} f\,(\,t-a)dt$$

$$= \int_a^\infty e^{-st} f\,(\,t-a)dt$$

Let $u = t - a \Rightarrow du = dt$

Also $if\ t \to a\ then\ u \to 0$ and $t \to \infty\ \ then\ u \to \infty$

$$\int_a^\infty e^{-st} f\,(\,t-a)dt = \int_0^\infty e^{-s\,(u+a)} f(u)du = \int_0^\infty e^{-su-sa} f(u)du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)du = e^{-as} F(s)$$

$$\therefore L\,\{f\,(t-a)u\,(t-a)\} = e^{-as} F(s)$$

Note: If a function $f(t)$ is defined as follows

$f(t) = \begin{cases} f_1(t); 0 < t < a \\ f_2(t); a < t < b \\ f_3(t); \qquad t > b \end{cases}$ Then $f(t)$ can be written as,

$$f(t) = [f_1(t)u(t) - f_1(t)u\,(t-a)] + [f_2(t)u(t-a) - f_2(t)u\,(t-b)] + f_3(t)u\,(t-b)$$

Corollary 1: $L\,\{f\,(t)u\,(t-a)\} = e^{-as}\,L\{f\,(t+a)\}$

Corollary 2: $L\,\{u\,(t-a)\} = L\,\{1.u\,(t-a)\} = \frac{e^{-as}}{s}$

Corollary 3: $L\,\{u\,(t-a) - u\,(t-b)\} = \frac{e^{-as}-e^{-bs}}{s}$

Corollary 4: $L\,\{f(t)[u\,(t-a) - u\,(t-b)]\} = [e^{-as}\,L\{f\,(t+a)\} - e^{-bs}\,L\{f\,(t+b)\}]$

Ex: Evaluate $\boldsymbol{L\,\{(t-3)^2 u\,(t-3)\}}$

Sol. Here $f(t-a) = (t-3)^2$

$f(t+a) = (t)^2$

$\therefore L\{(t-3)^2 u(t-3)\} = e^{-3s} L\{f(t+a)\} = e^{-3s} L\{(t)^2\} = \dfrac{2e^{-3s}}{s^3}$

Ex: Find $L\{(t)^2 u(t-3)\}$

Sol. Here, $f(t) = t^2$

$L\{f(t)u(t-a)\} = e^{-as} L\{f(t+a)\}$

$L\{(t)^2 u(t-3)\} = e^{-3s} L\{f(t+3)\} = e^{-3s} L\{(t+3)^2\} = e^{-3s} L\{t^2+6t+9)\}$

$= e^{-3s}[L(t^2) + 6L(t) + 9L(1)]$

$\therefore L\{(t)^2 u(t-3)\} = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$

Ex: Evaluate $L\{e^t u(t-2)\}$

Sol: $L\{e^t u(t-2)\} = e^{-2s} L\{e^{t+2}\} = e^{-2s} e^2 L\{e^t\} = \frac{e^{-2(s-1)}}{s-1}$

Exercise: Find$L\{e^{-3t} u(t-2)\}$ **Ans:** $\frac{e^{-2s-6}}{s+3}$

Exercise: Evaluate $L\left\{Sin\, t\, u(t-\frac{\pi}{2})\right\}$ **Ans:** $e^{-\frac{\pi s}{2}} \frac{s}{s^2+1}$

Inverse Laplace by using second shifting theorem:

$$L^{-1}\{e^{-as} F(s)\} = f(t-a)u(t-a)$$

Ex: Find $L^{-1}\left\{\frac{2e^{-s}}{s^3}\right\}$

Sol: Here $L^{-1}\left\{\frac{2e^{-s}}{s^3}\right\} = L^{-1}\left\{e^{-s}\frac{2}{s^3}\right\}$

By comparison, we get $a = 1$

We know that,

$$L^{-1}\left\{\frac{2}{s^3}\right\} = 2\,\frac{t^2}{2!} = t^2$$

$$\therefore L^{-1}\left\{e^{-s}\frac{2}{s^3}\right\} = (t-1)^2 u\,(t-1)$$

Examples:

1**.** Find $L^{-1}\left\{\frac{se^{-as}}{s^2-w^2}\right\}$ Ans: cosh$w$(t-a). u(t-a).

2.Find $L^{-1}\left\{\frac{2se^{-3s}}{s^2-16}\right\}$ Ans: 2cosh4 (t-3). u(t-3)

Application of Laplace Transform:

Solution of Ordinary Differential Equations:

$L\{f'(t)\} = s\,L\{f(t)\} - f(0),$

$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$,

Similarly, in general Laplace transform of the n^{th} derivatives of $f\ (t)$

$L\left(f^n(t)\right) = s^n L\left(f(t)\right) - s^{n-1} f(0) - s^{n-2} f'(0) \ldots - f^{\,n-1}(0)$

Ex: Solve the differential equation

$$\boldsymbol{y'' + 6y = 1}\,, \qquad \boldsymbol{y(0) = 2}, \qquad \boldsymbol{y'(0) = 0}$$

Sol: Here $\quad y'' + 6y = 1$

Taking Laplace Transform on both sides, we get

$L(y'' + 6y\,) = L(1)$

$L(y'') + 6\,L(y) = \dfrac{1}{s}$

$s^2 L\left(y(t)\right) - sy(0) + 6L\left(y(t)\right) = \dfrac{1}{s}$

Here, given conditions are $y(0) = 2,\ y'(0) = 0$

$$s^2L\left(y(t)\right) - 2s + 6L\left(y(t)\right) = \frac{1}{s}$$

$$(s^2+6)L\left(y(t)\right) = \frac{1}{s} + 2s = \frac{1+2s^2}{s}$$

$$L\left(y(t)\right) = \frac{1+2s^2}{s\,(s^2+6)}$$

Apply Inverse Laplace Transformation on both sides, we get

$$y(t) = L^{-1}\left(\frac{1+2s^2}{s\,(s^2+6)}\right)$$

By using Partial fraction,

$$\frac{1+2s^2}{s\,(s^2+6)} = \frac{A}{s} + \frac{Bs+c}{(s^2+6)} \Rightarrow 1+2s^2 = A(s^2+6) + (Bs+c)s$$

$$\Rightarrow 1+2s^2 = A(s^2+6) + Bs^2 + cs$$

Put $s = 0, 2(0) + 1 = A(0+6) \Rightarrow 1 = 6A \Rightarrow \boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{6}}$

Substitute $s = 1, 2(1) + 1 = A(1+6) + B + C \Rightarrow 3 = 7\left(\frac{1}{6}\right) + B + C \Rightarrow 3 = \frac{7+6B+6c}{6}$

$$\Rightarrow 18 = 7 + 6B + 6C$$

$$\Rightarrow 6B + 6C = 11 \qquad (1)$$

Put $S = -1, 2(1) + 1 = A(1+6) + B - C \Rightarrow 3 = 7A + B - C \Rightarrow 3 = 7\left(\frac{1}{6}\right) + B - C$

$$\Rightarrow 3 = \frac{7+6B-6c}{6} \Rightarrow 18 = 7 + 6B - 6c$$

$$\Rightarrow 6B - 6c = 11 \qquad (2)$$

Solving equation (1) and (2), we get $\boldsymbol{B} = \frac{\mathbf{11}}{\mathbf{6}}$ and $\boldsymbol{C} = \mathbf{0}$

$$\therefore L^{-1}\left\{\frac{1+2s^2}{s\,(s^2+6)}\right\} = L^{-1}\left\{\frac{1}{6}\left(\frac{1}{s}\right) + \frac{11}{6}\left(\frac{s}{s^2+6}\right)\right\} = \frac{1}{6} + \frac{11}{6}\cos\sqrt{6}\,t$$

Therefore, the required solution of given differential equation is,

$$y(t) = \frac{1}{6} + \frac{11}{6}\cos\sqrt{6}\,t$$

Ex: Solve the differential equation

$$\boldsymbol{y'' - 4y = 24cos2t, y(0) = 3, y'(0) = 4}$$

Sol: Apply Laplace Transformation on both sides, we get

$$L\,(y'' - 4y) = L(24cos2t)$$
$$L\,(y'') - 4L(y) = \frac{24s}{s^2+4}$$
$$s^2L\big(y(t)\big) - sy(0) - y'(0) - 4L\big(y(t)\big) = \frac{24s}{s^2+4}$$

Now substitute boundary conditions Immediately before solving in above equation, we find

$$s^2L\big(y(t)\big) - s(3) - 4 - 4\,L\big(y(t)\big) = \frac{24s}{s^2+4}$$

$$(s^2-4)L\big(y(t)\big) - 3s - 4 = \frac{24s}{s^2+4}$$

$$L\big(y(t)\big) = 3s + 4 + \frac{24s}{s^2+4}$$

$$L\big(y(t)\big) = \frac{3s^3+4s^2+36s+16}{(s+2)(s-2)(s^2+4)}$$

$$\big(y(t)\big) = L^{-1}\left\{\frac{3s^3+4s^2+36s+16}{(s+2)(s-2)(s^2+4)}\right\}$$

Taking Laplace Transform on both sides, we get

$$y(t) = L^{-1}\left\{\frac{4}{s-2} + \frac{2}{s+2} - \frac{3s}{s^2+4}\right\}$$

$$\therefore\ y(t) = 4e^{2t} + 2e^{-2t} - 3cos2t$$

Therefore, the required solution of given differential equation is

$$y(t) = 4e^{2t} + 2e^{-2t} - 3cos2t$$

Exercise: Using Laplace transformation solve the initial value Problem

$$\boldsymbol{y'' + y = \sin 2t\ ;\ \ y(0) = 2; y'(0) = 1}$$

Ans: $y(t) = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + 2\cos t$

Exercise: Using the method of Laplace transform, solve the IVP

$$\boldsymbol{x'' - 10x' + 9x = 5t, x(0) = -1\ , x'(0) = -2}$$

Ans:$x(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}e^{9t} - 2e^{t}$

Exercise: Using the method of Laplace transform, solve the differential equation

$$\boldsymbol{y'' + y' - 6y = 1; \qquad y(0) = 0\ ; \quad y'(0) = 1}$$

Ans:$y(t) = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$

Exercise: Solve $\boldsymbol{y'' + 5y' + 4y = 3\delta(t-2),\ y(0) = 2\ ,\ y'(0) = -2}$

Ans:$y(t) = e^{-(t-2)} - e^{-4(t-2)}\,u(t-2) + 2\,e^{t}$