Subject: Mathematics – II (303191151)

Semester: 2nd Sem. B.Tech Programme (All Branches)

Lecture Note : Unit – 2 Power Series

Series Solutions of Differential Equations

Introduction

In mathematics, the **Power Series Method** is used to seek a power series solution to certain differential equations. In general, such a solution assumes a power series with unknown coefficients, then on substituting that solution into the differential equation to find a recurrence relation for the coefficients. In this way a power series solution is obtained.

In this unit, we will learn two methods to obtain power series solution.

(i) *Power series method*

(ii)*Frobenious Method.*

Standard form of the Differential Equation

Consider a homogeneous linear second order differential equation with variable coefficients

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0$$

Here we develop the method of solving equations of the type

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0 \dots\dots\dots\dots (1)$$

Where $P_0(x), P_1(x) and\ P_2(x)$ are polynomials in x in terms of an infinite convergent series. $Assuming\ P_0(x)\neq 0, dividing\ (1)\ by\ P_0(x), we\ have$

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0 \dots\dots\dots\dots (2)$$

$$where\ P(x)=\frac{P_1(x)}{P_0(x)}\quad and\ Q(x)=\frac{P_2(x)}{P_0(x)}$$

The power series method is the standard basic method for solving linear differential equations with variable coefficients. It gives solution in the form of power series.

Definition:-Power Series

A power series in power of $(x-x_0)$ $is\ an\ infinite\ series\ of\ the\ form$

$$\sum_{k=0}^{\infty} a_k(x-x_0)^k=a_0+a_1(x-x_0)+a_2(x-x_0)^2+\cdots\dots.(3)$$

$where\ a_0, a_1, a_2 \dots..$ are constants, called the coefficients of the series x_0 is a constant called the centre of the series and x is a variable. If in particular $x_0 = 0$ we obtain a power series in power of x.

$$\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \dots \dots . (4)$$

Existence of Power Series solutions

Every differential equation of the form

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \dots \dots \dots \dots (1)$$

does not have series solution Assuming $P_0(x) \neq 0$,the above equation is written in the standard form as

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \dots \dots \dots \dots . (2)$$

$$where\ P(x) = \frac{P_1(x)}{P_0(x)} \quad and\ Q(x) = \frac{P_2(x)}{P_0(x)}$$

The behaviour of solutions of (2) near a point x_0 depends on the behavior of its coefficient functions $P(x)\ and\ Q(x)$ near this point x_0.

Classification of Singularities

Definition: Analytic

A function $f(x)$ is said to be analytic at $x_0\ if\ f(x)$ has Taylor' series expansion about x_0 given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x - x_0)^n$$

Exists and converges to $f(x)$ for all x in some open interval including x_0. If a function $f(x)$ is not analytic $at\ x_0$ then it is called Singular at x_0.

Definition: Ordinary Point

A point $x = x_0$ is said to be an ordinary point of differential equation (2) if both $P(x)$ and $Q(x)$ are analytic at x_0; that is, if both $P(x)\ and\ Q(x)$ have Taylor Series representations but $x = x_0$

Definition: Singular Point

A point x_0 is said to be a singular point of (2) if either $P(x)\ or\ Q(x)\ or\ both$ are not analytic at x_0

(OR) A point $x = x_0$ that is not an ordinary point of (1) is called a singular point.

Definition: Regular Singular Point (RSP)

A point $x = x_0$ of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \dots\dots\dots\dots (2)$$

Is said to be regular singular point if both the following terms are analytic at x_0.

$(\boldsymbol{i})(\ x - x_0)P(x)$ $(\boldsymbol{ii})\ (\ x - x_0)^2Q(x)$

NOTE: If either of the above terms or both are not analytic at x_0, then x_0 is called **An Irregular Singular Point.**

Question 1. Find ordinary point, singular point of given below two equations.

$(\boldsymbol{i})\ (\mathbf{1} - \mathbf{x^2})\boldsymbol{\frac{d^2y}{dx^2}} - \mathbf{6}\boldsymbol{x}\boldsymbol{\frac{dy}{dx}} - \mathbf{4}\boldsymbol{y} = \mathbf{0}$

Solution: $P(x) = \frac{P_1(x)}{P_0(x)} = \frac{-6x}{(1-x^2)}\ \ and\ Q(x) = \frac{-4}{(1-x^2)}$.

In this example the points $x = -1\ and\ x = 1$ are singular points of the equation. Except $x = -1\ and\ 1$ all other points are ordinary points.

$(\boldsymbol{ii})\ (\mathbf{x^2} + \mathbf{4})\boldsymbol{\frac{d^2y}{dx^2}} + \mathbf{2}\boldsymbol{x}\boldsymbol{\frac{dy}{dx}} - \mathbf{12}\boldsymbol{y} = \mathbf{0}$

Solution:$P(x) = \frac{P_1(x)}{P_0(x)} = \frac{2x}{(4+x^2)}\ and\ Q(x) = \frac{-12}{(4+x^2)}$

Singular points need not be real numbers. It has singular points $x = \pm 2i$.

Other than $x = \pm 2i$ all other points are ordinary points.

Question 2. Find singular points and classify them into regular singular point or irregular singular point.

$(i)\ x^2\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-3y=0$

Solution: The point x = 0 is a singular point of the equation. Since the factor $(x-0)$ occurs to only the first power in the denominator $of\ P(x)=-\frac{x}{x^2}=-\frac{1}{x}$ and to only the second power in $Q(x)=-\frac{3}{x^2}$, we conclude that $x=0$ is a regular singular point

$(ii)\ 2x(x-2)^2y''+3xy'+(x-2)y=0$

Solution: $P(x)=\frac{3}{2(x-2)^2}$, $Q(x)=\frac{1}{2x(x-2)}$

$P(x) or\ Q(x)\ or\ both$ are infinite at x = 0 and 2. They are not analytic at x = 0 and 2.

$Thus\ x=0\ and\ 2\ are\ singular\ point\ of\ the\ equation.$

Now, $xP(x)=\frac{3x}{2(x-2)^2}$ and $x^2Q(x)=\frac{x}{2(x-2)}$

Thus both $xP(x)$ and $x^2Q(x)$ are analytic at $x=0$.

The point $x=0$ is a regular singular point.

Now, $(x-2)P(x)=\frac{3}{2(x-2)}$ and $(x-2)^2Q(x)=\frac{x-2}{2x}$

Thus $(x-2)P(x)$ is an irregular singular point.

$(iii)\ x(x+1)^2y''+(2x-1)y'+x^2y=0$

Solution: $P(x)=\frac{2x-1}{x(x+1)^2}$, $Q(x)=\frac{x}{(x+1)^2}$

Since $P(x) or\ Q(x)\ or\ both$ are undefined at $x=0\ and\ x=-1$, they are singular point of the given equation.

Again $(x-0)P(x)=\frac{2x-1}{x(x+1)^2}$ and $(x-0)^2Q(x)=\frac{x^3}{(x+1)^2}$

$(x-0)P(x)$ and $(x-0)^2Q(x)$ are analytic at x=0.Thus x=0 is a regular singular point.

Again $(x+1)P(x)=\frac{2x-1}{x(x+1)}$, $(x+1)^2Q(x)=x^3$

But $(x+1)P(x)$ is not analytic at $x=-1$. Thus $x=-1$ is an irregular singular point.

POWER SERIES SOLUTION NEAR AN ORDINARY POINT

Let $x=0$ be an ordinary point of the equation

$$P_0(x)\frac{d^2y}{dx^2}+P_1(x)\frac{dy}{dx}+P_2(x)y=0\ \dots\dots\dots\dots(i)$$

$$\text{Or } \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

$$where\ P(x) = \frac{P_1(x)}{P_0(x)}\ and\ Q(x) = \frac{P_2(x)}{P_0(x)}\ , (P_0(x) \neq 0).$$

Let a solution of (i) be given as

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \qquad \dots\dots\dots\dots (ii)$$

$$\therefore \frac{dy}{dx} = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$and\ \frac{d^2y}{dx^2} = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = 2a_2 + 3 \cdot 2\, a_3 x + 4 \cdot 3 a_4 x^2 + \cdots$$

Substituting these values in (i), and equating coefficients of various powers of x to 0.

Equating the coefficient of x^n, we obtain the recurrence relation.

Assigning different values to n in this recurrence relation, we can determine the unknown coefficients, in (ii) successively and $a_i{}'s$ in terms of a_0 and a_1.

Using these values of $a_i{}'s$ in (i) we can obtain series solution of given differential equation.

EXAMPLE:-1: Solve the equation $y' - y = 0$ by the power series method.

Solution: given equation

$$y' - y = 0 \dots\dots\dots\dots (i)$$

Differentiating the power series

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \quad \dots\dots\dots\dots (ii)$$

term by term, we get

$$\therefore y'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \dots\dots\dots\dots (iii)$$

writing (i) as $y' = y\ and\ comparing\ (iii)$ with, we get

$$a_1 = a_0, \qquad 2a_2 = a_1, \qquad 3a_3 = a_2 \dots\dots\dots\ and\ so\ on.$$

In general we have

$$(k+1)a_{k+1} = a_k, \qquad k = 0,1,2,3 \dots$$

Therefore we can express a_{k+1} in terms of a_k as

$$a_{k+1} = \frac{1}{k+1} a_k\ , \qquad k = 0,1,2,3 \dots$$

Let us compute the first few coefficients explicitly

$$a_1 = \frac{1}{1}a_0, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2\cdot 1}a_0, \qquad a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2\cdot 1}a_0 \dots\dots$$

From here it is clear that

$$a_k = \frac{1}{k!}a_0$$

$$y(x) = \sum_{k=0}^{\infty}\frac{1}{k!}a_0x^k = a_0\sum_{k=0}^{\infty}\frac{x^k}{k!} = a_0\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) = a_0e^x$$

EXAMPLE:-2: Solve the equation $\frac{d^2y}{dx^2} + y = 0$ by the power series method.

Solution:

Let the series solution be

$$y(x) = \sum_{k=0}^{\infty} a_kx^k = a_0 + a_1x + a_2x^2 + \cdots \quad \dots\dots\dots\dots(i)$$

$$\therefore \frac{dy(x)}{dx} = \sum_{k=1}^{\infty} ka_kx^{k-1} = a_1 + 2a_2x + 3a_3x^2 + \cdots$$

$$and \ \frac{d^2y(x)}{dx^2} = \sum_{k=2}^{\infty} k(k-1)a_kx^{k-2} = 2a_2 + 3\cdot 2\,a_3x + 4\cdot 3a_4x^2 + \cdots$$

Substituting in given equation, we get

$$\sum_{k=2}^{\infty} k(k-1)a_kx^{k-2} + \sum_{k=0}^{\infty} a_kx^k = 0.$$

$$i.e.\,(2a_2 + 3\cdot 2\,a_3x + 4\cdot 3a_4x^2 + \cdots) + (a_0 + a_1x + a_2x^2 + \cdots) = 0$$

$$i.e.\,(2a_2 + a_0) + (3\cdot 2\,a_3 + a_1)x + (4\cdot 3a_4 + a_2)x^2 + \cdots$$
$$+\big((n+2)(n+1)a_{n+2} + a_n\big)x^n + \cdots = 0$$

Comparing the coefficients, we get

$$2a_2 + a_0 = 0,$$
$$3\cdot 2\,a_3 + a_1 = 0,$$
$$4\cdot 3a_4 + a_2 = 0,$$
$$\dots$$
$$(n+2)(n+1)a_{n+2} + a_n = 0$$

Solving these equations, we get

$$a_2 = -\frac{a_0}{2!}, a_3 = -\frac{a_1}{3!}, a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4!}, \dots,$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)} \dots\dots\dots\dots (ii)$$

Using (ii), for $n = 3 a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5!}$

Substituting these values in (i), we get

$$y = a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

Which is the required series solution of given differential equation.

Note: For above solution, considering the Maclaurin's series:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots; \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

We get

$$y = a_0 \cos x + a_1 \sin x$$

EXAMPLE:2. Solve the differential equation $(1 - x^2)y'' - 2xy' + 2y = 0$ using power series method.

Solution:

Here $x = 0$ is an ordinary point, and except $x = -1$ and $x = 1$ all other points are regular points.

Let the series solution be

$$y = \sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots \quad \dots\dots\dots\dots (i)$$

$$\therefore y' = \sum_{k=1}^{\infty} k a_k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$$

$$and\ \ y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} = 2a_2 + 3\cdot 2\, a_3 x + 4\cdot 3 a_4 x^2 + \cdots$$

Substituting in given equation, we get

$$(1 - x^2)\sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - 2x \sum_{k=1}^{\infty} k a_k x^{k-1} + 2\sum_{k=0}^{\infty} a_k x^k = 0.$$

$$i.e. \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2} - \sum_{k=0}^{\infty} [k(k-1) + 2k - 2] a_k x^k = 0$$

Collecting and comparing the coefficient of x^n, we get

$$(n+2)(n+1) a_{n+2} = (n+2)(n-1) a_n$$

$$(n+1) a_{n+2} = (n-1) a_n$$

$$a_{n+2} = \frac{n-1}{n+1} a_n$$

For $n = 0, a_2 = -a_0,$

$n = 1, a_3 = 0$, which gives $a_3 = a_5 = a_7 = a_9 = \cdots = 0$

Now, for even values of n, let $n = 2m$.

$$\therefore a_{2m+2} = \frac{2m-1}{2m+1} a_{2m} \quad for\ m = 0,1,2, \ldots$$

For $m = 0, a_2 = -a_0$

For $m = 1, a_4 = \frac{1}{3} a_2 = -\frac{1}{3} a_0,$

For $m = 2, a_6 = \frac{3}{5} a_4 = -\frac{1}{5} a_0$, etc.

Substitution in (i) gives

$$y = a_1 x + a_0 \left(1 - x^2 - \frac{1}{3} x^4 - \frac{1}{5} x^6 - \frac{1}{7} x^8 + \cdots\right)$$

Which is the required series solution of the given differential equation.

Example 3: Find the first four terms in each portion of the series solution around

$x_0 = 0$ for the following differential equation $(1 + \mathrm{x}^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = 0 \ldots (1)$

Solution: Here$p(x) = (1 + x^2),\ p(0) = 1 \neq 0.$

So $x_0 = 0$ is an ordinary point for this differential equation.

$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$y'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1} \qquad and \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} \ldots\ldots\ldots (2)$$

Putting above equation in (1)

$$(1 + \mathrm{x}^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 4x \sum_{n=1}^{\infty} n\, a_n x^{n-1} + 6 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=1}^{\infty} 4n\, a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} 4n\, a_n x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

At this point we could strip out some terms to get all the series starting at n=2, but that's actually more work than is needed. Let's instead note that we could start the third series at n=0 if we wanted to because that term is just zero. Likewise, the terms in the first series are zero for both n=1 and n=0 and so we could start that series at n=0. If we do this all the series will now start at n=0 and we can add them up without stripping terms out of any series.

$$\sum_{n=2}^{\infty}\{n(n-1)a_n + (n+2)(n+1)a_{n+2} - 4n\, a_n + 6a_n\}x^n = 0$$

$$\sum_{n=2}^{\infty}\{(n^2-5n+6)a_n + (n+2)(n+1)a_{n+2}\}x^n = 0$$

$$\sum_{n=2}^{\infty}\{(n-2)(n-3)a_n + (n+2)(n+1)a_{n+2}\}x^n = 0$$

$$(n-2)(n-3)a_n + (n+2)(n+1)a_{n+2} = 0, \qquad n = 0,1,2,\ldots.$$

$$a_{n+2} = -\frac{(n-2)(n-3)a_n}{(n+2)(n+1)} \quad n = 0,1,2,\ldots.$$

$n = 0, a_2 = -3a_0$

$n = 1, a_3 = -\frac{1}{3}a_0$

$n = 2, a_4 = 0$

$n = 3, a_5 = 0$

$y(x) = a_0(1-3x^2) + a_1(x - \frac{1}{3}x^3)$

H.W. Examples.

1. Solve the equation $y'' = y'$ by the power series method.

2. Find a power series solution in powers of x of $y' + 2xy = 0$.

Frobenius Method for Solution near a Regular Singular Point:

Just as the power series method, the Frobenius method is useful for solving second order differential equations with variable coefficients about a regular singular point of the equation.

$$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \ldots\ldots\ldots.(1) \quad OR \qquad \frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

$$where\ P(x) = \frac{P_1(x)}{P_0(x)}\ and\ Q(x) = \frac{P_2(x)}{P_0(x)}$$

Method of Solution

Let $x = 0$ be a regular singular point of equation (1), its solution can be represented in the form

$$y = x^m \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+m} = x^m(a_0 + a_1 x + a_2 x^2 + \cdots), \qquad a_0 \neq 0 \ldots \ldots \ldots (2)$$

$$Then, \frac{dy}{dx} = \sum_{k=0}^{\infty} (k+m) a_k x^{k+m-1} \quad and \quad \frac{d^2y}{dx^2} = \sum_{k=0}^{\infty} (k+m)(k+m-1) a_k x^{k+m-2},$$

Substitute the value of $y,\ \frac{dy}{dx}\ and\ \frac{d^2y}{dx^2}\ in\ (1)$

$$\therefore [m(m-1)\,a_0\,x^{m-2} + (m+1)m\,a_1\,x^{m-1} + (m+2)(m+1)\,a_2\,x^m + \cdots]$$

$$+ [p_0 x + p_1 x^2 + \cdots] \cdot [m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + \cdots]$$

$$+ [q_0 + q_1 x + \cdots] \cdot [a_0\,x^m + \ a_1\,x^{m+1} + a_2\,x^{m+2} + \cdots] = 0$$

Equate to zero the coefficient of lowest power of x. This gives quadratic equation in m, which is called the indicial equation of the differential equation (1).Equate to zero the coefficients of various powers of x and express $a_1, a_2,\ a_3$ … …in terms of a_0.

Substitute the values$of\ a_1, a_2, a_3\ in\ (2) to$ get solution of (1) having a_0 as arbitrary constant. One of the two solutions will always be the form (2), where m is a root of an indicial equation.

Let $m_1\ and\ \ m_2$ be the roots of an indicial equation, then we have the following

- **Distinct roots not differing by an integer.**

$$m_1 \neq m_2\ and\ m_1 - m_2\ is\ not\ an\ integer. The\ general\ solution\ is$$

$$y = c_1(y)_{m_1} + c_2(y)_{m_2}$$

EXAMPLE:-1: Solve in series the differential equation

$$\mathbf{4x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0}$$

Solution: Clearly $x = 0$ is a regular singular point.

$$y = x^m \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+m} = x^m(a_0 + a_1 x + a_2 x^2 + \cdots..) \ldots \ldots \ldots (1)$$

be the series solution of given equation.

$$Then\ \frac{dy}{dx}=\sum_{k=0}^{\infty}(k+m)a_kx^{k+m-1}, \qquad \frac{d^2y}{dx^2}=\sum_{k=0}^{\infty}(k+m)(k+m-1)a_kx^{k+m-2}$$

substituting the values of $y, \frac{dy}{dx}\ and\ \frac{d^2y}{dx^2}$in the given equation, we get

$$4x\sum_{k=0}^{\infty}(k+m)(k+m-1)a_kx^{k+m-2}+2\sum_{k=0}^{\infty}(k+m)a_kx^{k+m-1}+\sum_{k=0}^{\infty}a_kx^{m+k}=0$$

$$or\ 4x[m(m-1)a_0x^{m-2}+(m-1)ma_1x^{m-1}+(m+2)(m+1)a_2x^m+(m+3)(m+2)a_3x^{m+1}+\cdots]$$
$$+2[ma_0x^{m-1}+(m+1)a_1x^m+(m+2)a_2x^{m+1}+(m+3)a_3x^{m+2}+\cdots.]$$
$$+[a_0x^m+a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+\cdots\ldots]=0$$

The lowest power of $x\ is\ x^{m-1}$. Equating to zero the coefficient of x^{m-1}, we get

$$a_0[4m(m-1)+2m]=0,$$

Its roots are $m=0\ and\ m=\frac{1}{2}$, which are distinct and not differing by an integer.

Equating the coefficient of x^{k+m}, gives

$$a_{k+1}=-\left(\frac{a_k}{(2m+2k+2)(2m+2k+1)}\right), k=0,1,2,3\ldots.$$

First solution:-The solution corresponding to m= $\frac{1}{2}$is obtained from the recurrence relation.

$$a_{k+1}=-\left(\frac{a_k}{(2k+3)(2k+2)}\right)$$

Hence,

$$a_1=-\frac{a_0}{3.2}, a_2=-\frac{a_1}{5.4},\quad a_3=-\frac{a_2}{7.6}, etc,$$

$$\therefore a_1=-\frac{a_0}{3!},\quad a_2=\frac{a_0}{5!},\quad a_3=-\frac{a_0}{7!}\ldots\ldots$$

and in general $a_n=\frac{(-1)^n}{(2n+1)!}\quad as\ (n=0,1,2,3\ldots)$, The first solution is

$$y_1(x)=a_0x^{\frac{1}{2}}\left(1-\frac{1}{6}x+\frac{1}{120}x^2-\cdots\ldots.\right)$$

Second solution:-The solution corresponding to $m=0$ is obtained from the recurrence relation.

$$a_{k+1} = -\left(\frac{a_k}{(2k+3)(2k+2)}\right)$$

Hence,

$$a_1 = -\frac{a_0}{2.1}, a_2 = -\frac{a_1}{4.3}, \quad a_3 = -\frac{a_2}{6.5}, etc,$$

$$\therefore a_1 = -\frac{a_0}{2!}, \quad a_2 = \frac{a_0}{4!}, \quad a_3 = -\frac{a_0}{6!} \dots \dots$$

and in general $a_n = \frac{(-1)^n}{2n!} \quad as\ (n = 0,1,2,3 \dots),$

The second solution is

$$y_2 = a_0\left(1 - \frac{x}{2} + \frac{x^2}{24} - \cdots \dots . + \cdots .\right),$$

Hence general solution is

$y = c_1 y_1 + c_2 y_2$

$$= Ax^{\frac{1}{2}}\left(1 - \frac{1}{6}x + \frac{1}{120}x^2 - \cdots \dots .\right) + B\left(1 - \frac{x}{2} + \frac{x^2}{24} - \cdots \dots . + \cdots .\right),$$

$where\ A = c_1 a_0\ and\ B = c_2 a_0$

H.W.

EXAMPLE-3: Solve in series the differential equation $x^2y'' + xy' + (x^2 - 4)y = 0$

LEGENDRE POLYNOMIALS

$$P_n(x) = \sum_{r=0}^{N} (-1)^r \frac{(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!} x^{n-2r}$$

Where, $\quad N = \frac{n}{2}\ , when\ n\ is\ even, \quad N = \frac{n-1}{2}\ when\ n\ is\ odd$

From 2 we get the following set of polynomials:

$$P_0(x) = 1, \qquad P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)\,, P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), etc.$$

Each satisfies a Legendre differential equation in which n has the value indicated by the subscript.

Note: Rodrigue's Formula

$$P_n(x) = \frac{1}{2^n(n!)}\frac{d^n}{dx^n}(x^2-1)^n$$

Example 1: show that $x^3 = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)$.

Solution: we know that

$$P_n(x) = \frac{1}{2^n(n!)}\frac{d^n}{dx^n}(x^2-1)^n \dots\dots\dots\dots (1)$$

Putting, $n = 1 \; and \; n = 3 \; in \; 1 \; we \; get,$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = x$$

$$P_3(x) = \frac{1}{2^3(3!)}\frac{d^3}{dx^3}(x^2-1)^3$$

$$= \frac{1}{48}\frac{d^3}{dx^3}(x^6 - 3x^4 + 3x^4 - 1) \;= \left[\frac{1}{2}(5x^3 - 3x)\right]$$

$$\therefore \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) = \frac{2}{5}\left[\frac{1}{2}(5x^3 - 3x)\right] + \frac{3}{5}(x) \quad = \frac{1}{5}(5x^3 - 3x) + \frac{3}{5}x = x^3$$

Example 2: show that $x^4 = \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)]$.

Solution: we know that

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^3 + 3)$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad P_0(x) = 1.$$

$$\therefore \frac{1}{35}[8P_4(x) + 20P_2(x) + 7P_0(x)] = \frac{1}{35}[(35x^4 - 30x^3 + 3) + 10(3x^2 - 1) + 7] = x^4$$

Example 3: Express $f(x) = x^4 + 3x^3 - x^2 + 5x - 2$ is terms of Legendre's polynomials.

BESSEL FUNCTION:

The differential equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0------(1)$$

Is called **Bessel's equation** of order n and its particular solutions are called **Bessel functions** of order n.

Application: In vibration problems, electric fields, heat conduction, fluid flow.

A) Bessel Function of the first kind of order n .

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ \Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

where n is a positive real number or zero.

B) Bessel function of the first kind of order $-n$.

$$J_{-n}(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ \Gamma(n+k+1)}\left(\frac{x}{2}\right)^{-n+2k}$$

Hence the complete solution of ***Bessel's equation*** (1) may be expressed as

$$y=AJ_n(x)+BJ_{-n}(x)$$

When A and B are arbitrary constants.

C) Bessel equation the first kind of order zero.

$$J_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}$$

$$since\quad \Gamma(k+1)=k!$$

EXAMPLE 1 : Bessel functions $J_0(x) and\ J_1(x)$.

Solution:

Since , $$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ \Gamma(n+k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

So, $$J_0(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\ \Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}$$

$$=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k!)^2}\left(\frac{x}{2}\right)^{2k}\ \text{as ,}\qquad \Gamma(k+1)=k!$$

$$= 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots.$$

$$\therefore J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots.$$

$$J_1(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma(k+2)}\left(\frac{x}{2}\right)^{1+2k} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(k+1)!}\left(\frac{x}{2}\right)^{2k}$$

$$= \frac{x}{2} - \frac{1}{2!}\left(\frac{x}{2}\right)^3 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^5 - \cdots + \cdots.$$

$$\therefore J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} - \cdots + \cdots$$

In particular $J_0(0) = 1$ and $J_1(0) = 0$

EXAMPLE 2 : Prove that $\therefore J'_0(x) = -J_1(x)$

Solution: we have

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} + \cdots.$$

$$and\ J_1(x) = \frac{\mathrm{x}}{2} - \frac{\mathrm{x}^3}{2^2.4} + \frac{\mathrm{x}^5}{2^2.4^2.6} - \cdots + \cdots.$$

Now $\frac{d}{dx}J_0(x) = J_0'(x) = 0 - \frac{x}{2} + \frac{x^3}{2^2.4} - \frac{x^5}{2^2.4^2.6} + \cdots.$

$$= -\left(\frac{x}{2} - \frac{4x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} + \cdots.\right)$$

$$\therefore J'_0(x) = -J_1(x)$$

Prove that

$$(\boldsymbol{i})\ \boldsymbol{J_{\frac{1}{2}}(x)} = \sqrt{\frac{\boldsymbol{2}}{\boldsymbol{\pi x}}}\,\boldsymbol{sinx},\quad (\boldsymbol{ii})\ \boldsymbol{J_{\frac{1}{-2}}(x)} = \sqrt{\frac{\boldsymbol{2}}{\boldsymbol{\pi x}}}\,\boldsymbol{cosx}$$

Solution: (i)

$$J_{\frac{1}{2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma\left(\mathrm{k}+\frac{3}{2}\right)}\left(\frac{x}{2}\right)^{\left(\frac{1}{2}+2k\right)}$$

$$= \frac{x^{(-\frac{1}{2})}}{2^{(-\frac{1}{2})}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!\ \Gamma\left(\mathrm{k}+\frac{3}{2}\right)}$$

$$= \frac{x^{(-\frac{1}{2})}}{2^{(-\frac{1}{2})}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^{2k+1} k!\ \Gamma\left(\mathrm{k}+\frac{3}{2}\right)}$$

$$= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k k!\ \left(\mathrm{k}+\frac{1}{2}\right) \ldots..\Gamma(\frac{1}{2})}$$

$$= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2^k k!\ (2\mathrm{k}+1)(2\mathrm{k}-1) \ldots \ldots \ldots.\sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad = \sqrt{\frac{2}{\pi x}} \left(\frac{x}{1!} - \frac{x^3}{3!} + \cdots..\right) \Rightarrow J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} sinx$$

(ii) $J_{\frac{1}{-2}}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\ \Gamma\left(\mathrm{k}-\frac{1}{2}+1\right)} (\frac{x}{2})^{(\frac{-1}{2}+2k)}$

$$= \frac{x^{(-\frac{1}{2})}}{2^{(-\frac{1}{2})}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!\ \Gamma\left(\mathrm{k}+\frac{1}{2}\right)}$$

$$= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!\ \left(\mathrm{k}-\frac{1}{2}\right) \ldots..\Gamma(\frac{1}{2})}$$

$$= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!\ (2\mathrm{k}-1) \ldots \ldots.\sqrt{\pi}}$$

$$= \sqrt{\frac{2}{\pi x}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$= \sqrt{\frac{2}{\pi x}} (1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots) \Rightarrow J_{\frac{1}{-2}}(x) = \sqrt{\frac{2}{\pi x}} cosx$$

PROPRTIES OF BESSEL FUNCTION

Recurrence relations for J_n:

1. $(J_n'(x) + \left(\frac{n}{x}\right) J_n(x) = J_{n-1}(x)$
2. $J_n'(x) - \left(\frac{n}{x}\right) J_n(x) = -J_{n+1}(x)$
3. $2J_n\,'(x) = J_{n-1}(x) - J_{n+1}(x)$
4. $\left(\frac{2n}{x}\right) J_n(x) = J_{n-1}(x) + J_{n+1}(x)$