Subject: Mathematics – II (303191151)

Semester: 2nd Sem. B.Tech Programme (All Branches)

Lecture Notes: Unit – 4, Fourier Integral

Introduction :

We have learnt Fourier series for periodic functions. There exist many practical problem in engineering which involve non-periodic functions. We can solve such problems on the basis of Fourier series technique by converting non-periodic functions in terms of sine and cosine functions. This conversion will lead to the extension of Fourier series to Fourier integral.

Fourier Integral :

Let $f(x)$ be a function which is piecewise continuous in every finite interval $(-\infty, \infty)$ and absolutely integrable in $(-\infty, \infty)$, then Fourier integral is given by following formula.

Formula :

- **Fourier Integral :**

$$f(x) = \int_0^\infty [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda$$

$$where\ A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\lambda v\,dv, \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin\lambda v\ dv$$

- **Fourier Cosine Integral :**

$$f(x) = \int_0^\infty A(\lambda)\cos\lambda x\,d\lambda, \qquad where\ A(\lambda) = \frac{2}{\pi}\int_0^\infty f(v)\cos\lambda v\,dv$$

- **Fourier Sine Integral:**

$$f(x) = \int_0^\infty B(\lambda)\sin\lambda x\,d\lambda, \qquad where\ B(\lambda) = \frac{2}{\pi}\int_0^\infty f(v)\sin\lambda v\ dv$$

- **Existence of Fourier Integral:**

The Fourier Integral of $f(x)$exists whenever x is a point of continuity of $f(x)$.

Otherwise when x is a point of discontinuity then $f(x)$ is replaced by $\dfrac{f(x+0)+f(x-0)}{2}$.

Examples

1. **Find the Fourier Integral representation of the function** $f(x) = \begin{cases} 1-x^2 & if\ \ |x| \le 1 \\ 0 & if\ \ |x| > 1 \end{cases}$

Solution:

$$f(x) = \int_0^\infty A(\lambda)\cos\lambda x\,d\lambda$$

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(v)\cos\lambda v\,dv$$

$$= \frac{2}{\pi}\int_0^\infty (1-v^2)\cos\lambda v \; dv$$

$$= \frac{2}{\pi}\left|(1-v^2)\frac{\sin\lambda v}{\lambda} - (-2t)\left(-\frac{\cos\lambda v}{\lambda^2}\right) + (-2)\left(-\frac{\sin\lambda v}{\lambda^3}\right)\right|_0^1$$

$$= \frac{2}{\pi}\left(-\frac{2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right) \quad [\because \sin 0 = 0]$$

$$= \frac{4}{\pi}\left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)$$

$$f(x) = \frac{4}{\pi}\int_0^\infty \left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right)\cos\lambda x \, d\lambda$$

2. **Find the Fourier cosine and sine Integral of $f(x) = e^{-kx}$, $(x > 0, k > 0)$**

Solution: (a)

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(v)\cos\lambda v \; dv$$

$$= \frac{2}{\pi}\int_0^\infty e^{-kv}\cos\lambda v \; dv$$

$$= \frac{2}{\pi}\left[\frac{e^{-kv}}{k^2+\lambda^2}\left(-\frac{\lambda}{k}\sin\lambda v + \cos\lambda v\right)\right]_0^\infty$$

$$= \frac{2}{\pi}\left[0 + \frac{k}{k^2+\lambda^2}\right]$$

$$= \frac{2k}{\pi(k^2+\lambda^2)}$$

We obtain the Fourier cosine integral representation

$$i.e.\, f(x) = \int_0^\infty A(\lambda)\cos\lambda x \; d\lambda$$

$$f(x) = e^{-kx} = \frac{2k}{\pi}\int_0^\infty \frac{\cos\lambda x}{(k^2+\lambda^2)}\, d\lambda, \quad (x > 0, k > 0)$$

From this, we see that

$$\int_0^\infty \frac{\cos\lambda x}{(k^2+\lambda^2)}\, d\lambda = \frac{\pi}{2k}e^{-kx}, \quad (x > 0, k > 0)$$

(b) Similarly we have,

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(v)\sin\lambda v \; dv$$

$$= \frac{2}{\pi} \int_0^\infty e^{-kv} \sin \lambda v \ dv$$

$$= \frac{2}{\pi} \left[\frac{-\lambda}{k^2 + \lambda^2} e^{-kv} \left(-\frac{k}{\lambda} \sin \lambda v + \cos \lambda v \right) \right]_0^\infty$$

$$= \frac{2\lambda}{\pi (k^2 + \lambda^2)}$$

We obtain the Fourier sine integral representation

$$i.e. f(x) = \int_0^\infty B(\lambda) \sin \lambda x \ d\lambda$$

$$f(x) = e^{-kx} = \frac{2k}{\pi} \int_0^\infty \frac{\lambda \sin \lambda x}{(k^2 + \lambda^2)} d\lambda, \quad (x > 0, k > 0)$$

$$\therefore \int_0^\infty \frac{\lambda \sin \lambda x}{(k^2 + \lambda^2)} d\lambda = \frac{\pi}{2} e^{-kx}, \quad (x > 0, k > 0)$$

3. **Find the Fourier Integral representation of the function** $\boldsymbol{f(x) = \begin{cases} 2 & if\ |x| < 2 \\ 0 & if\ |x| > 2 \end{cases}}$

Solution : Here

$$f(x) = \begin{cases} 2 & if\ |x| < 2 \\ 0 & if\ |x| > 2 \end{cases} \ and\ f(x) = \begin{cases} 2 & if\ |-x| < 2 \\ 0 & if\ |-x| > 2 \end{cases}$$

$\therefore f(-x) = f(x) \qquad \therefore f\ is\ an\ even\ function.$

$\therefore$ Fourier integral reduces to Fourier cosine integral of $f(x)$ $i.e. f(x) = \int_0^\infty A(\lambda) \cos \lambda x \ d\lambda$.

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos \lambda v \ dv = A(\lambda)$$

$$= \frac{2}{\pi} \int_0^\infty f(v) \cos \lambda v \ dv, \quad (\because f(x) is\ an\ even\ function)$$

$$= \frac{2}{\pi} \int_0^2 2 \ \cos \lambda v \ dv + \int_2^\infty 0. \cos \lambda v \ dv$$

$$= \frac{4}{\pi} \left[\frac{\sin 2\lambda}{\lambda} - 0 \right]$$

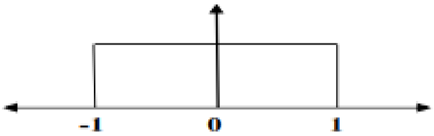
$$\therefore A(\lambda) = \frac{4 \sin 2\lambda}{\lambda \pi}$$

Now $f(x) = \int_0^\infty A(\lambda) \cos \lambda x \ d\lambda = \int_0^\infty \frac{4 \sin 2\lambda}{\lambda \pi} \cos \lambda x \ d\lambda$

4. Find the Fourier Integral representation of the function $f(x) = \begin{cases} 1 & if\ \ |x| < 1 \\ 0 & if\ \ |x| > 1 \end{cases}$

Hence evaluate $(1)\ \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\, d\lambda,$

$(2)\ \int_0^\infty \frac{\sin\lambda}{\lambda}\, d\lambda.$



Solution :

$$f(x) = \int_0^\infty A(\lambda)\cos\lambda x\, d\lambda$$

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(v)\cos\lambda v\, dv \ = \frac{2}{\pi}\int_0^1 \cos\lambda v\ \ dv \quad = \left[\frac{\sin\lambda v}{\lambda\pi}\right]_{-1}^{1} = \frac{2\sin\lambda}{\pi\lambda}$$

$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^\infty f(v)\sin\lambda v\, dv \ \ = \frac{1}{\pi}\int_{-1}^1 \sin\lambda v\ \ dv = \ 0$$

$$Hence,\quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda \ldots\ldots\ldots\ldots\ldots(1)$$

$$(i)\ \int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda}\ d\,\lambda = \frac{\pi}{2}f(x) \ = \begin{cases} \frac{\pi}{2} & |x| < 1 \\ 0 & |x| > 1 \end{cases}$$

At $|x| = 1, i.e. x = \pm 1, f(x)$ is discontinuous.

At $x = 1,\qquad f(x) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

At $x = -1,\qquad f(x) = \frac{1}{2}\left[\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right]$

$$= \frac{1}{2}(0+1)$$

$$= \frac{1}{2}$$

Hence from Eq. (1), $\displaystyle\int_0^\infty \frac{\sin\lambda\cos\lambda x}{\lambda} d\lambda = \begin{cases} \frac{\pi}{2} & if\ |x| < 1 \\ \frac{\pi}{4} & if\ x = 1 \\ 0 & if\ x > 1 \end{cases}$

(ii)putting $x = 0$ in Eq. (1) in $\displaystyle\int_0^\infty \frac{\sin\lambda}{\lambda} d\lambda = \frac{\pi}{2}f(0) = \frac{\pi}{2};\quad [\because f(0) = 1]$

5. Using the Fourier Integral representation prove that $\int_0^\infty \frac{\cos\lambda x+\lambda\sin\lambda}{1+\lambda^2}d\lambda = \begin{cases} 0 & if\ x<0 \\ \frac{\pi}{2} & if\ x=0 \\ \pi e^{-x} & if\ x>0 \end{cases}$

Solution:

Let $f(x) = \begin{cases} 0 & if\ x<0 \\ e^{-x} & if\ x>0 \end{cases}$ $at\ x=0, f(0) = \frac{f(0-0)+f(0+0)}{2} = \frac{0+1}{2} = \frac{1}{2}$

$f(x) = \int_0^{-\infty}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda.$

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos\lambda v\ dv$$

$$= \frac{1}{\pi}\left[\int_{-\infty}^{0} 0\ \cos\lambda v\ dv + \int_0^\infty e^{-v}\cos\lambda v\ dv\right]$$

$$= \frac{1}{\pi}\int_0^\infty e^{-v}\cos\lambda v\ dv$$

$$= \frac{1}{\pi}\left[\frac{e^{-v}}{1+\lambda^2}\left(-\cos\lambda v + \lambda\sin\lambda v\right)\right]_0^\infty$$

$$= \frac{1}{\pi(1+\lambda^2)}$$

$$B(\lambda) = \frac{1}{\pi}\int_0^\infty f(v)\sin\lambda v\ dv$$

$$= \frac{1}{\pi}\int_0^\infty e^{-v}\sin\lambda v\ dv$$

$$= \frac{1}{\pi}\left[\frac{e^{-v}}{1+\lambda^2}\left(-\sin\lambda v + \lambda\cos\lambda v\right)\right]_0^\infty$$

$$= \frac{\lambda}{\pi(1+\lambda^2)}$$

Hence

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x+\lambda\sin\lambda x}{\pi(1+\lambda^2)}d\lambda \ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore\ \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x+\lambda\sin\lambda x}{\pi(1+\lambda^2)}d\lambda = f(x) = \begin{cases} 0 & if\ x<0 \\ \frac{\pi}{2} & if\ x=0 \\ \pi e^{-x} & if\ x>0 \end{cases}$$

$$\therefore\ \frac{1}{\pi}\int_0^\infty \frac{\cos\lambda x+\lambda\sin\lambda x}{\pi(1+\lambda^2)}d\lambda = \begin{cases} 0 & if\ x<0 \\ \frac{\pi}{2} & if\ x=0 \\ \pi e^{-x} & if\ x>0 \end{cases}$$

To find $f(0)$ directly put $x = 0$ in (1)

$$f(0) = \frac{1}{\pi}\int_0^\infty \frac{1}{(1+\lambda^2)}d\lambda = \frac{1}{\pi}[\tan^{-1}\lambda]_0^\infty = \frac{1}{\pi}\left(\frac{\pi}{2}\right) = \frac{1}{2}.$$

6. **Using Fourier sin integral show that ,**

$$\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin x\lambda\, d\lambda = \begin{cases} \frac{\pi}{2} & if \quad 0 < x < \pi \\ 0 & if\ x > \pi \end{cases}$$

Solution:

We have to obtain sin integral

$$i.e. f(x) = \int_0^\infty B(\lambda)\sin\lambda x\, d\lambda$$

$$where\ B(\lambda) = \frac{2}{\pi}\int_0^\infty f(v)\sin\lambda v\, dv$$

$$= \frac{2}{\pi}\left[\int_0^\pi \sin\lambda v\, dv + \int_\pi^\infty 0.\sin\lambda v\, dv\right]$$

$$= \frac{2}{\pi}\left|-\frac{\cos\lambda v}{\lambda}\right|_0^\pi$$

$$= \frac{2}{\pi}\left(\frac{-\cos\lambda\pi + 1}{\lambda}\right) = \frac{2}{\pi}\left(\frac{1-\cos\lambda\pi}{\lambda}\right)$$

$$Hence\ f(x) = \frac{2}{\pi}\int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda v\, d\lambda$$

$$\therefore \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda v\, d\lambda = \frac{\pi}{2}f(x) = \begin{cases} \frac{\pi}{2} & if 0 < x < \pi \\ 0 & if\ x > \pi \end{cases}$$

At $x = \pi, f(x)$ is discontinuous.

$$f(x) = \frac{1}{2}\left[\lim_{x\to\pi^-} f(x) + \lim_{x\to\pi^+} f(x)\right]$$

$$= \frac{1}{2}(1+0)$$

$$= \frac{1}{2}$$

Hence from equation (1), $\therefore \int_0^\infty \frac{1-\cos\lambda\pi}{\lambda}\sin\lambda v\, d\lambda = \begin{cases} \frac{\pi}{2} & if 0 < x < \pi \\ \frac{\pi}{2} & if\ x = \pi \\ 0 & if\ x > \pi \end{cases}$

H.W. Ex.

7. **Find the Fourier Integral representation of the function** $f(x) = \begin{cases} x & if\ |x| < 1 \\ 0 & if\ |x| > 1 \end{cases}$

8. **Find the Fourier Integral representation of the function** $f(x) = \begin{cases} -e^{ax} & if\ x < 0 \\ e^{ax} & if\ x > 0 \end{cases}$