Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme (All Branches)

Mathematics – II (203191152)

Unit – 5(a) Vector Calculus (Lecture Note)

Scalar point function:

- If to each point (x, y, z) of a region R in space there corresponds a number or a scalar $f = f\ (x, y, z)$ then, f is called a scalar point function and R is called a scalar field.
- For example
  (i) the temperature field in a body.
  (ii) The pressure field of the air in the earth's atmosphere.
  (iii)The density of a body.
  These quantities take different values at different points.

Note: A scalar field which is independent of time is called a stationary or steady-state scalar field.

Vector point function:

If to each point (x, y, z) of a region R in space there corresponds a vector $v\ (x, y, z) = v_1 i + v_2\, j + v_3\, k$ then, v is called a vector point function and R is called a vector field.

For example

(i) the velocity of a moving fluid at any instant.

(ii) The gravitational force.

(iii)The electric and magnetic field intensity.

Note: A vector field which is independent of time is called a stationary or steady-state vector field.

<u>Vector differential operator</u> -

The vector differential operator is denoted by ∇ (del or nabla) and is defined as

$$\nabla = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

Gradient of a scalar field: - For a given scalar function $\emptyset\ (x, y, z\ )$ the gradient of $\emptyset$ is denoted by $grad\ \emptyset$ or $\nabla\emptyset$ is defined as

$$\nabla\emptyset = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

Example: **Find the gradient of** $\emptyset = \boldsymbol{3x^2\, y - y^3 z^2}$ **at the point** $(\mathbf{1}, -\mathbf{2}, \mathbf{1})$.

Sol:

$$\nabla\emptyset = \hat{\imath}\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\frac{\partial\emptyset}{\partial y} + \hat{k}\frac{\partial\emptyset}{\partial z}$$
$$= \hat{\imath}(6xy) + \hat{\jmath}\,(3x^2 - 3y^2z^2) + \hat{k}\,(-2y^3z)$$

At the point $(1, -2,1)$

$\nabla\emptyset = -12\,\hat{\imath} - 9\,\hat{\jmath} - 16\,\hat{k}$.

Example: Evaluate $\boldsymbol{\nabla e^{r^2}}$, $\boldsymbol{where\ r^2 = x^2 + y^2 + z^2}$

Solution:

$$r^2 = x^2 + y^2 + z^2$$

Differentiating r partially with respect to x, y, z

$$2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y \quad \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

$$2r\frac{\partial r}{\partial z} = 2z \quad \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla e^{r^2} = \; i\frac{\partial e^{r^2}}{\partial x} + j\,\frac{\partial e^{r^2}}{\partial y} + \; k\,\frac{\partial e^{r^2}}{\partial z}$$

$$= \; i\frac{\partial e^{r^2}}{\partial r}\frac{\partial r}{\partial x} + j\,\frac{\partial e^{r^2}}{\partial r}\frac{\partial r}{\partial y} + \; k\,\frac{\partial e^{r^2}}{\partial r}\frac{\partial r}{\partial z}$$

$$= \; i(2r\,e^{r^2})\frac{x}{r} + j\,(2r\,e^{r^2})\frac{y}{r} + \; k\,2r\,e^{r^2}\frac{z}{r}$$

$$= 2e^{r^2}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})$$

Example: Find a unit normal vector to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$

Sol.

$$\emptyset(x, y, z) = \; x^3 \; + \; y^3 \; + \; 3xyz - \; 3 = 0$$

$$\nabla\emptyset \; = \; \hat{\imath}\,\frac{\partial\emptyset}{\partial x} + \; \hat{\jmath}\,\frac{\partial\emptyset}{\partial y} + \; \hat{k}\,\frac{\partial\emptyset}{\partial z}$$

$$= \hat{\imath}\,(3x^2 \; + \; 3yz\,) + \hat{\jmath}\,(3y^2 + \; 3xz\,) + \hat{k}\,(3xy)$$

At the point $(1,2,-1)$

$$\nabla\emptyset \; = \; -3\hat{\imath} + \; 9\hat{\jmath} \; + \; 6\hat{k}$$

$$\hat{n} \; = \frac{\nabla\emptyset}{|\nabla\emptyset|} = \frac{-3\hat{\imath} \; + \; 9\hat{\jmath} \; + 6\hat{k}}{\sqrt{126}}$$

Examples for Practice:

1. Find a unit normal vector to the surface $\boldsymbol{x^2y + 3xz^2 = 8}$ at the point $(\mathbf{1}, \mathbf{0}, \mathbf{2})$
2. Find the unit normal to the surface $\boldsymbol{x^2 + xy + y^2 + xyz}$ at the point $(\mathbf{1}, -\mathbf{2}, \mathbf{1})$.

Directional Derivative: -

The directional derivative of scalar point function $\emptyset(x, y, z)$ in the direction of vector $\hat{a}$, is the component of $\nabla\emptyset$ in the direction of $\hat{a}$.

If $\hat{a}$ is the unit vector in the direction of a, *then* the directional derivative of $\emptyset$ in the direction of a is $D\emptyset = \nabla\emptyset * \hat{a}$

Examples: Find the directional derivative of $\boldsymbol{\emptyset(x, y, z) = x^3 - xy^2 - z}$ at point (1,1,0) in the direction of $\boldsymbol{v = 2\hat{\imath} - 3\hat{\jmath} + 6\hat{k}}$

Sol. Here, $\emptyset\,(x, y, z) = x^3 - xy^2 - z$

$$\nabla\emptyset = \hat{\imath}\,\frac{\partial\emptyset}{\partial x} + \hat{\jmath}\,\frac{\partial\emptyset}{\partial y} + \hat{k}\,\frac{\partial\emptyset}{\partial z}$$

$$= \hat{\imath}(3x^2 - y^2) + \hat{\jmath}\,(-2xy) + \hat{k}\,(-1)$$

$$At\ the\ point\ (1,1,0)$$

$$\nabla\emptyset = 2\hat{\imath} - 2\hat{\jmath} - \hat{k}$$

The direction derivative of $\emptyset$ at point P (1,1,0) in the direction of is v is

$$D\phi = \nabla\emptyset * \hat{v} = \nabla\emptyset * \frac{v}{|v|}$$

$$(2\hat{\imath} - 2\hat{\jmath} - \hat{k}) * (2\hat{\imath} - 3\hat{\jmath} + 6\hat{k})/\sqrt{49}$$

$$= \frac{4}{7}.$$

Example: Find the directional derivative of $\boldsymbol{\emptyset = 6x^2y + 24y^2z - 8z^2x}$ at (1,1,1) in the direction of $\boldsymbol{v = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}}$. Hence, find the maximum value.

Solution: $grad\ \phi = i\frac{\partial}{\partial x}(6x^2y + 24y^2z - 8z^2x) + j\frac{\partial}{\partial y}(6x^2y + 24y^2z - 8z^2x) + k\frac{\partial}{\partial z}(6x^2y + 24y^2z - 8z^2x)$

$$\nabla\phi = (12xy - 8z^2)i + (6x^2 + 48yz)j + (24y^2 - 16zx)k$$

$$\nabla\phi_{1,1,1} = 4i + 54j + 8k$$

Directional derivative in the direction of $v = (2i - 2j + k)$ at the point (1,1,1)

$$= \nabla\phi \frac{v}{|v|}$$

$$= (4i + 54j + 8k)\frac{2i - 2j + k}{|2i - 2j + k|}$$

$$= (4i + 54j + 8k)\frac{(2i - 2j + k)}{\sqrt{4 + 4 + 1}}$$

$$= \frac{8 - 108 + 8}{3} = -\frac{92}{3}$$

Maximum value of directional derivative = $|\nabla\phi|$

Example for Practice

1. Find the directional derivative of $\boldsymbol{\varphi(x, y, z)} = \boldsymbol{xy^2} + \boldsymbol{yz^3}$ at the point P($\mathbf{2}, -\mathbf{1}, \mathbf{1}$) in the direction of PQ where Q is the point ($\mathbf{3}, \mathbf{1}, \mathbf{3}$)
2. In what direction from ($-\mathbf{1}, \mathbf{1}, \mathbf{2}$) is the directional derivative of $\boldsymbol{\varphi} = \boldsymbol{xy^2}\ \boldsymbol{z^3}$ a maximum? Find also the magnitude of this maximum.
3. Find the directional derivative of the scalar function $\boldsymbol{\varphi} = \boldsymbol{xyz}$ in the direction of the outer normal to the surface $\boldsymbol{z} = \boldsymbol{xy}$ at the point ($\mathbf{3}, \mathbf{1}, \mathbf{3}$).
4. Find the directional derivative of $\boldsymbol{\varphi} = \boldsymbol{xy} + \boldsymbol{yz} + \boldsymbol{zx}$ at ($\mathbf{1}, \mathbf{2}, \mathbf{0}$) in the direction of $\boldsymbol{\hat{\imath} + 2\hat{\jmath} + 2\hat{k}}$.

Divergence of a vector function:

Let $F = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a vector function then, divergence of F is

div F OR $\nabla F = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}\right) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$

Note:

1. If $\nabla . F = 0$ then, the vector function F is called solenoidal or incompressible
2. In hydrodynamics (the study of fluid motion), a velocity field that is divergence free is called **incompressible**.
3. In the study of electricity and magnetism, a vector field that is divergence free is called **solenoidal**.

Example: If $\boldsymbol{F = x^2\, z\, \hat{\imath} - 2y^3\, z^3\, \hat{\jmath} + xy^2 z\, \hat{k}}$ then, find divergence of $\boldsymbol{F}$ at $(\mathbf{1}, -\mathbf{1}, \mathbf{1})$

Sol. Here, $F = x^2\, z\, \hat{\imath} - 2y^3\, z^3\, \hat{\jmath} + xy^2 z\, \hat{k}$

$$\nabla F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial (x^2\, z)}{\partial x} + \frac{\partial (-2y^3\, z^3)}{\partial y} + \frac{\partial (xy^2 z)}{\partial z}$$

$$= 2xz - 6y^2\, z^3 + xy^2$$

At (1,–1,1)

$$\nabla F = -3$$

Example: Show that A = $\boldsymbol{3y^4z^2 i + 4x^3z^2 j - 3x^2y^2 k}$ is a solenoidal.

Solution:

$$\nabla \bar{\mathrm{A}} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}$$

$$= \frac{\partial (3y^4 z^2)}{\partial x} + \frac{\partial (4x^3 z^2)}{\partial y} + \frac{\partial (-3x^2 y^2)}{\partial z} = 0$$

Hence Given function is solenoidal.

Example for Practice:

1. Determine the constant a such that $\boldsymbol{A} = (\boldsymbol{ax^2y + yz})\hat{\boldsymbol{\imath}} + (\boldsymbol{xy^2 + xz^2})\hat{\boldsymbol{\jmath}} + (\boldsymbol{2xyz - 2x^2y^2})\widehat{\boldsymbol{k}}$ is solenoidal.
2. Find div $\vec{\mathbf{F}}$, where $\vec{\mathbf{F}}$ = grad$(\boldsymbol{x}^3+\boldsymbol{y}^3+\boldsymbol{z}^3-3\boldsymbol{xyz})$
3. If $\vec{\boldsymbol{F}} = \boldsymbol{xy^2}\hat{\boldsymbol{\imath}} + \boldsymbol{2x^2yz}\hat{\boldsymbol{\jmath}} - \boldsymbol{3yz^2}\widehat{\boldsymbol{k}}$ find $\boldsymbol{\nabla}.\boldsymbol{F}$ at a point **(1,-1,1)**
4. If $\vec{\boldsymbol{F}} = (\boldsymbol{x^2 - y^2 + 2xz})\hat{\boldsymbol{\imath}} + (\boldsymbol{xz - xy + yz})\hat{\boldsymbol{\jmath}} - (\boldsymbol{z^2 + x^2})\widehat{\boldsymbol{k}}$ then find $\boldsymbol{\nabla}.\boldsymbol{F}$.

Curl

Let $F= F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k}$ be a vector function then, curl of F is

$$\text{curl F or } \nabla \times F = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Note: - If $\nabla \times F = 0$ then, the vector function F is called Irrotational or conservative.

Example: If $\boldsymbol{F} = \boldsymbol{xz^3}\hat{\boldsymbol{\imath}} - \boldsymbol{2x^2yz}\hat{\boldsymbol{\jmath}} + \boldsymbol{2yz^4}\widehat{\boldsymbol{k}}$ then, find curl of F at (1,-1,1)

Sol. Here, $F = xz^3\hat{\imath} - 2x^2yz\hat{\jmath} + 2yz^4\hat{k}$

$$\nabla \times F = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{bmatrix}$$

$$= \hat{\imath}(2z^4 + 2x^2y) + \hat{\jmath}\,(3xz^2) + \hat{k}\,(-4xyz)$$

At point $(1, -1, 1)$

$\nabla \times F = 3\hat{\jmath} + 4\hat{k}$.

Example: Show that $\boldsymbol{r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}}$ is Irrotational.

Solution:

$$\nabla \times F = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & \mathrm{z} \end{bmatrix}$$

$$= 0\,\hat{\imath} + 0\hat{\jmath} + 0\hat{k}$$

Therefore, $r = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$ is Irrotational

Example: Find curl of $\boldsymbol{A = e^{xyz}(\hat{\imath} + \hat{\jmath} + \hat{k})}$ at a point(1,2,3)
Solution:

$$\nabla \times A = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{bmatrix}$$

$$= \hat{\imath}\left(e^{xyz}(xz) - e^{xyz}(xy)\right) - \hat{\jmath}\left(e^{xyz}(yz) - e^{xyz}(xy)\right) + \hat{k}\left(e^{xyz}(yz) - e^{xyz}(xz)\right)$$

$$= e^{xyz}(\hat{\imath}(xz - xy) - \hat{\jmath}(yz - xy) + \hat{k}(yz - xz))$$

At
(1,2,3)

$$= e^{6}(\hat{\imath} - 4\hat{\jmath} + 3\hat{k})$$

Example for Practice:

1. If $\boldsymbol{\vec{F} = xy^2\hat{\imath} + 2x^2yz\hat{\jmath} - 3yz^2\hat{k}}$ find $\boldsymbol{\nabla \times F}$ at a point **(1,-1,1)**
2. If $\boldsymbol{\vec{F} = (x^2 - y^2 + 2xz)\hat{\imath} + (xz - xy + yz)\hat{\jmath} - (z^2 + x^2)\hat{k}}$ then find $\boldsymbol{\nabla \times F}$.
3. Find **div(grad $\boldsymbol{\varphi}$)** and **curl(grad $\boldsymbol{\varphi}$)** at (1,1,1) for $\boldsymbol{\varphi = x^2y^3z^4}$

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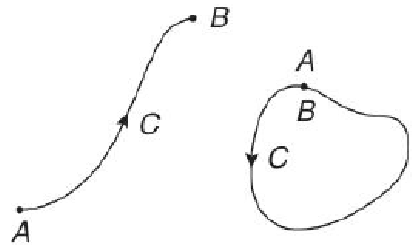
Mathematics – 2 (303191151)
Unit–5(b) Vector Calculus (Lecture Notes)

LINE-INTEGRAL:

The line integral is a simple generalization of a definite integral $\int_a^b f(x)\ dx$ which is integrated from $x = a$ ($point\ A$) $to\ x = b$($point\ B$) along the $x - axis$.



In a line integral the integration is done along a curve C in space.

Let **$\bar{F}(\bar{r})$ be a vector function** defined at every point of a curve C. If $(\bar{r})$is the position vector of the point $P(x, y, z)$ on the curve C then the line integral of $\bar{F}(\bar{r})$ over a curve C is defined by

$$\int_C \bar{F}(\bar{r})d\bar{r} = \int_C (F_1 dx + F_2 dy + F_3 dz), \qquad where\ \bar{F} = F_1\hat{\imath} + F_2\hat{\jmath} + F_3\hat{k} \,\&\, \bar{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

If the curve **C** is represented by parametric representation, $\bar{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

Then the line integral along the curve **C** from $t = a$ to $t = b$ is

$$\int_C \bar{F}(\bar{r})d\bar{r} = \int_a^b \bar{F}\frac{d\bar{r}}{dt}dt = \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt$$

If **C** is closed curve, then the symbol of the line integral $\int_C$ is replaced by $\oint_c$

Examples:

1) **If $\overline{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$, evaluate $\int_C \overline{F} \cdot d\overline{r}$ where C is the curve $y = 2x^2$ from (0,0) to (1,2).**

Solution: Given $\overline{F} = 3xy\hat{\imath} - y^2\hat{\jmath}$, $d\overline{r} = dx\hat{\imath} + dy\hat{\jmath} \Longrightarrow \overline{F} \cdot d\overline{r} = 3xy - y^2$

Given **C** is $y = 2x^2 \Longrightarrow dy = 4x\,dx$

Along C, x varies from 0 to 1

$$\int_C \overline{F} \cdot d\overline{r} = \int_0^1 3x(2x^2)\,dx - 4x^4(4x\,dx)$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx = \left[6\frac{x^4}{4} - 16\frac{x^6}{6}\right] = \frac{6}{4} - \frac{16}{6} = -\frac{\mathbf{7}}{\mathbf{6}}\ \textit{units}.$$

2) **Find the work done when a force $\overline{F} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$, moves a particle from the origin to the point (1, 1) along $y^2 = x$.**

Solution: Given $\overline{F} = (x^2 - y^2 + x)\hat{\imath} - (2xy + y)\hat{\jmath}$, $d\overline{r} = dx\hat{\imath} + dy\hat{\jmath}$

$\Longrightarrow \overline{F} \cdot d\overline{r} = (x^2 - y^2 + x)dx - (2xy + y)dy$

Given **C** is: $y^2 = x \Longrightarrow 2ydy = dx$

Along **C**, x varies from 0 to 1,

$$\int_C \overline{F} \cdot d\overline{r} = \int_0^1 ((y^2)^2 - y^2 + y^2)2ydy - (2y^3 + y)dy$$

$$= \int_0^1 (2y^5 - 2y^3 + 2y^3 - 2y^3 - y)\,dy$$

$$= \int_0^1 (2y^5 - 2y^3 - y)\,dy = \left[\frac{2y^6}{6} - \frac{2y^4}{4} - \frac{y^2}{2}\right]_0^1 = \frac{2}{6} - \frac{2}{4} - \frac{1}{2} = -\frac{\mathbf{2}}{\mathbf{3}}$$

3) **Find the work done in moving a particle in the force field $\overline{F} = 3x^2\hat{\imath} + (2xz - y)\hat{\jmath} - z\hat{k}$ from t=0 to 1 along the curve $x = 2t^2, y = t, z = 4t^3$.**

4) **Find $\int_C \overline{F} \cdot d\overline{r}$ where C is the circle $x^2 + y^2 = 4$ in the xy-plane where**

$$\overline{F} = (2xy + z^3)\hat{\imath} + x^2\hat{\jmath} - 3xz^2\hat{k}.$$

SURFACE INTEGRAL:

An integral which is evaluated over a surface is called a surface integral. Consider a surface S. Let $\overline{F}$ be a vector valued function which is defined at each point on the surface and let P be any point on the surface and $\overline{n}$ be the unit outward normal to the surface at P. The normal component of $\overline{F}$ at P is $\overline{F} \cdot \overline{n}$.

The integral of the normal component of $\overline{\boldsymbol{F}}$is denoted by $\iint_S \overline{\boldsymbol{F}}\cdot\overline{\boldsymbol{n}}\,\boldsymbol{ds}$

EVALUATION OF SURFACE INTEGRAL

If $\boldsymbol{R_1}$be the projection of S on the xy-plane, $\widehat{\boldsymbol{k}}$ is the unit vector normal the xy-planethen $\boldsymbol{ds} = \frac{\boldsymbol{dx\,dy}}{|\overline{\boldsymbol{n}}\widehat{\boldsymbol{k}}|}$

$$\therefore \iint_S \overline{\boldsymbol{F}}\cdot\overline{\boldsymbol{n}}\,\boldsymbol{ds} = \iint_{R_1} \overline{\boldsymbol{F}}\cdot\overline{\boldsymbol{n}}\frac{\boldsymbol{dx\,dy}}{|\overline{\boldsymbol{n}}\widehat{\boldsymbol{k}}|}$$

If $\boldsymbol{R_2}$be the projection of S on the yz-plane, $\hat{\boldsymbol{\imath}}$ is the unit vector normal the yz-plane then $\boldsymbol{ds} = \frac{\boldsymbol{dy\,dz}}{|\overline{\boldsymbol{n}}\hat{\boldsymbol{\imath}}|}$

$$\therefore \iint_S \overline{\boldsymbol{F}}\cdot\overline{\boldsymbol{n}}\,\boldsymbol{ds} = \iint_{R_1} \overline{\boldsymbol{F}}\cdot\overline{\boldsymbol{n}}\frac{\boldsymbol{dy\,dz}}{|\overline{\boldsymbol{n}}\hat{\boldsymbol{\imath}}|}$$

If $\boldsymbol{R_3}$be the projection of S on the xz-plane, $\hat{\boldsymbol{\jmath}}$ is the unit vector normal the xz-plane then $\boldsymbol{ds} = \frac{\boldsymbol{dx\,dz}}{|\overline{\boldsymbol{n}}\hat{\boldsymbol{\jmath}}|}$

$$\therefore \iint_S \overline{\boldsymbol{F}}\cdot\overline{\boldsymbol{n}}\,\boldsymbol{ds} = \iint_{R_1} \overline{\boldsymbol{F}}\cdot\overline{\boldsymbol{n}}\frac{\boldsymbol{dx\,dz}}{|\overline{\boldsymbol{n}}\hat{\boldsymbol{\jmath}}|}$$

Problems based on Surface Integral

1) **Evaluate $\iint_S \overline{F}\cdot\hat{n}\;ds$ if $\overline{F} = (x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\widehat{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant.**

Solution:

Given $\overline{\boldsymbol{F}} = (\boldsymbol{x}+\boldsymbol{y}^2)\hat{\boldsymbol{\imath}} - 2\boldsymbol{x}\hat{\boldsymbol{\jmath}} + 2\boldsymbol{yz}\widehat{\boldsymbol{k}}$

Let $\boldsymbol{\varphi} = 2x + y + 2z + 6\;\; then$

$$\nabla\varphi = \hat{\imath}\frac{\partial\varphi}{\partial x} + \hat{\jmath}\frac{\partial\varphi}{\partial y} + \widehat{k}\frac{\partial\varphi}{\partial z} = 2\hat{\imath} + 1\hat{\jmath} + 2\widehat{k} \text{ and } |\nabla\varphi| = \sqrt{4+1+4} = \sqrt{9} = 3$$

$$\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\hat{\imath} + 1\hat{\jmath} + 2\widehat{k}}{3}$$

$$\overline{F}\cdot\hat{n} = \left[(x+y^2)\hat{\imath} - 2x\hat{\jmath} + 2yz\widehat{k}\right]\cdot\left(\frac{2\hat{\imath} + 1\hat{\jmath} + 2\widehat{k}}{3}\right)$$

$$= \frac{1}{3}[2(x+y^2) - 2x + 4yz]$$

$$= \frac{2}{3}[y^2 + 2yz]$$

$$= \frac{2}{3} y[y + 2z]$$

$$= \frac{2}{3} [y + 6 - 2x - y] \qquad [\because 2z = 6 - 2x - y]$$

$$= \frac{2}{3} [6 - 2x]$$

$$= \frac{4}{3} y[3 - x]$$

Let R be the projection of S on the xy – plane

$$\therefore ds = \frac{dx\,dy}{|\bar{n}\hat{k}|}$$

$$\bar{n} \cdot \hat{k} = \left(\frac{2\hat{\imath} + 1\hat{\jmath} + 2\hat{k}}{3}\right) \cdot \hat{k} = \frac{2}{3}$$

$$\therefore \iint_S \bar{F} \cdot \bar{n}\, ds = \iint_R \bar{F} \cdot \bar{n} \frac{dx\,dy}{|\bar{n}\hat{k}|}$$

$$= \iint_R \frac{4}{3} y[3 - x] \cdot \frac{dx\,dy}{\left(\frac{2}{3}\right)}$$

$$= 2 \iint [3 - x]\, y\, dx\, dy$$

In $R_1 (2x + y = 6)$, x varies from 0 to $\frac{6-y}{2}$

y varies from 0 to 6

$$= 2 \int_0^6 \int_0^{\frac{6-y}{2}} y(3 - x)\, dx\, dy$$

$$= 2 \int_0^6 \left[3x - \frac{x^2}{2}\right]_0^{\frac{6-y}{2}} dy$$

$$= 2 \int_0^6 \frac{1}{2}(18y - 3y^2) - \frac{1}{8}(6 - y^2) dy$$

$$= \frac{2}{2}\left[\frac{18y^2}{2} - \frac{3y^3}{3} - \frac{(6 - y)^3}{8(3)(-1)}\right]$$

$$= \left[9(6)^2 - (6)^3 + \frac{1}{12}(0)\right] - \left[0 - 0 + \frac{1}{12}(6^2)\right] \quad = 81\, units.$$

2) **Evaluate $\iint_S 6xy\, ds$ where S is the portion of the plane $x+y+z=1$ that lies in front of yz plane.**

Solution: We are looking for portion of the plane ABC that lies in front of the yz–plane, Therefore ,we write equation of the surface in the form $x = f(y,z)$

For the points on the surface we have $x = 1 - y - z$

$$\iint_S 6xy\, ds = \iint_S 6(1-y-z)y\sqrt{3}\, dA$$

$$= 6\sqrt{3}\int_0^1\int_0^{1-y} 6(1-y-z)y dz\, dy$$

$$= 6\sqrt{3}\int_0^1 \left[yz - y^2z - \frac{1}{2}yz^2\right]_0^{1-y} dy$$

$$= 6\sqrt{3}\left[\frac{1}{4}y^2 - \frac{1}{3}y^3 + \frac{1}{8}y^4\right]_0^1 = \frac{\sqrt{3}}{4}$$

3) **Evaluate $\iint_S 6xy\, ds$ where S is the portion of the plane $x+y+z=1$ that lies in front of yz plane.**

GREEN'S THEOREM IN PLANE:

Statement:-

If $M(x,y), N(x,y), \frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ be continuous every where in a region R of xy plane bounded by a closed curve c then $\oint_c M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$

1) **Verify Green's Theorem for $\oint_c [(x^2-2xy)dx + (x^2y+3)dy]$ where C is the boundary of the region bounded by the parabola $y = x^2$ and the line $y = x$.**

Solution: The points of intersection of the parabola $y=x^2$ and the line $y=x$ are obtained as $x=x^2$, $x=0,1$ and $y=0,1$.

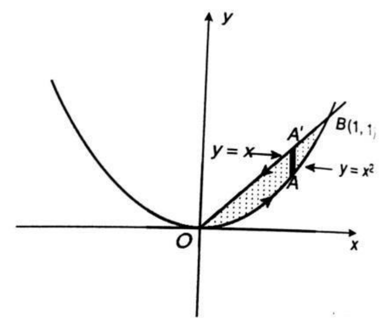
Hence, O (0,0) and B (1,1) are the points of intersection.

Here $M = x^2 - 2xy, \quad N = x^2y + 3$

$$\frac{\partial M}{\partial y} = -2x, \qquad \frac{\partial N}{\partial x} = 2xy$$

$$\oint_c (M\,dx + N\,dy) = \int_{OAB} (M\,dx + N\,dy) + \int_{BO} (M\,dx + N\,dy) \ldots\ldots\ldots\ldots (1)$$

$\Longrightarrow$ Along $OAB : y = x^2, dy = 2x\,dx, x$ varies from 0 to 1



$$\int_{OAB} (M\,dx + N\,dy) = \int_{OAB} [(x^2 - 2xy)dx + (x^2y + 3)dy]$$

$$= \int_0^1 [(x^2 - 2x.x^2)dx + (x^2x^2 + 3)\,2x\,dx]$$

$$= \int_0^1 (x^2 - 2x^3 + 2x^5 + 6x)\,dx = \frac{19}{6}$$

$\Longrightarrow$ Along $BO : y = x,\ dy = dx, x$ varies from 1 to 0

$$\int_{BO} (M\,dx + N\,dy) = \int_{BO} [(x^2 - 2xy)dx + (x^2y + 3)dy]$$

$$= \int_0^1 [(x^2 - 2x^2)dx + (x^3 + 3)\,2x\,dx] = -\frac{35}{12}$$

Substituting in (1) $\Longrightarrow \oint_c (M\,dx + N\,dy) = \frac{19}{6} - \frac{35}{12} = \boldsymbol{\frac{1}{4}} \ldots\ldots\ldots\ldots (2)$

Let R be the region bounded by the line $y = x$ and the parabola $y = x^2$ along the vertical strip AA'.

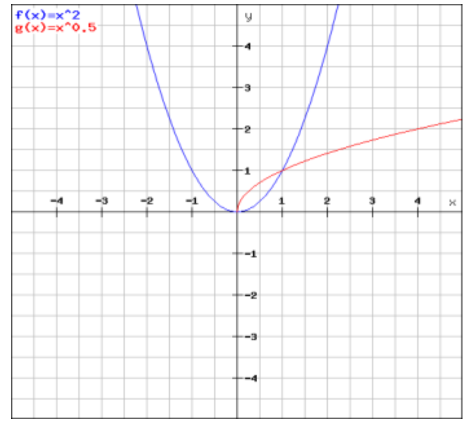
$\Longrightarrow y$ varies from x^2 to x and in the region R, x varies from 0 to 1

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_{x^2}^{x} (2xy + 2x)dy\,dx$$

$$= \int_0^1 [xy^2 + 2xy]_{x^2}^{x} = \int_0^1 (x^3 + 2x^2 - x^5 - 2x^3)\,dx = \frac{1}{4} \ldots\ldots\ldots\ldots (3)$$

From equation (2) and (3)$\Longrightarrow \oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \boldsymbol{\frac{1}{4}}$

2) **By using Green's Theorem evaluate $\oint_c [(3x^2 - 8y^2)dx + (4y - 6y)dy]$,where C is the boundary of the region bounded by$y^2 = x$and the line $y = x^2$.**



Solution: $y^2=x$ and $y=x^2$ are two parabolas intersecting at

(0,0) and (1,1).Here, $M=3x^2- 8y^2$, $N=4y-6xy$

$$\Longrightarrow \frac{\partial M}{\partial y} = -16y, \quad \frac{\partial N}{\partial x} = -6y$$

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} 10y\,dx\,dy = \int_0^1 [5y^2]_{x^2}^{\sqrt{x}} = 5\int_0^1 (x - x^4)\,dx = \frac{3}{2}$$

3) **Prove that the area bounded by a simple closed curve C is given by $\frac{1}{2}\int_C (x\,dy - y\,dx)$.Hence find the area of the ellipse$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ by using green's theorem.**

Solution:

By Green's Theorem , $\oint_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

Let $M = -y \quad and \quad N = x \Longrightarrow \frac{\partial M}{\partial y} = -1 \;\; and \;\; \frac{\partial N}{\partial y} = 1$

$$\int_C (x\,dy - y\,dx) = \iint_R (1+1)dx\,dy = 2\iint_R dx\,dy = 2 \qquad (Area\ enclosed\ by\ C)$$

$$\therefore\ Area\ enclosed\ by\ C = \frac{1}{2}\int_C (x\,dy - y\,dx)$$

Parametric equation of the ellipse

$x = a\cos\theta\,, y = b\sin\theta,\ dx = -a\sin\theta\,d\theta,\ \ dy = b\cos\theta\,d\theta\,, where\ 0 \le \theta \le 2\pi$

Area of the ellipse $= \frac{1}{2}\int_0^{2\pi} ((a\cos\theta)(b\cos\theta) - (b\sin\theta)(-a\sin\theta))\,d\theta$

$$= \frac{1}{2}ab\int_0^{2\pi} ((\cos\theta)^2 + (\sin\theta)^2)\,d\theta = \frac{1}{2}ab\int_0^{2\pi} d\theta = \frac{1}{2}ab[\theta]_0^{2\pi} = \boldsymbol{\pi ab}$$

4) **Evaluate $\frac{1}{2}\int_C (x^2 - 2y)dx + (4x + y^2)dy$ by Green's theorem where C is the boundary of the region bounded by $y = 0, y = 2x\ and\ x + y = 3$.** $(ans. = 6)$

5) Verify Green's theorem in plane for $\frac{1}{2}\int_C (x^2 - 2xy)dx + (x^2y + 3)dy$, where C is the boundary of the region bounded by the parabola $y^2 = 8x$ and the line $x = 2$. $\left(ans. = \frac{128}{5}\right)$

GAUSS-DIVERGENCE THEOREM: (Convert surface integral to volume integral)

Statement: If $\underline{F}$ be a vector point function having continuous partial derivatives in the region bounded by a closed surface S, then

$$\iint_s F.nds = \iiint_v divFdv$$

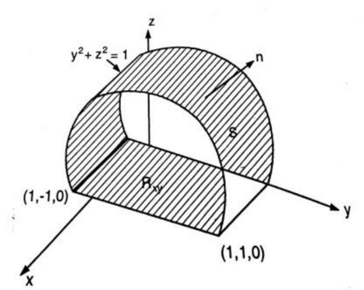
where n is the unit outward normal at any point of the surface S.

1) Find the flux of $F = yzj + z^2k$ outward through the surface s cut from the cylinder $y^2 + z^2 = 1, z \geq 0$ by the plane $x = 0$ & $x = 1$.

Solution: The outward normal field on **S** calculated from the gradient of $g(x, y, z) = y^2 + z^2$

to be $n = \frac{\nabla g}{|\nabla g|} = \frac{2y\, j + 2z\, k}{\sqrt{4y^2+4z^2}} = \frac{2y\, j + 2z\, k}{2\sqrt{1}} = yj + 2k$

$$dS = \frac{|\nabla g|}{|\nabla g.k|} dA = \frac{2}{|2z|} dA = \frac{1}{z} dA$$



Since $z \geq 0$ on S, $F.n = (yzj + z^2k).(yj + zk)$

$$= y^2z + z^3$$

$$= z(y^2 + z^2) = z$$

Therefore, the flux F outward through S is

$$\iint_S F.n\, dS = \iint_S z\left(\frac{1}{z} dA\right) = \iint_{R_{xy}} dA = area\ R_{xy} = 2$$

2) Find the flux of $F = 4xzi - y^2 j + yzk$ **outward through the surface of the cube cut from the first octant by the planes** $x = 1, y = 1, z = 1$

Solution: Here $F = 4xzi - y^2 j + yzk$

$$\nabla \cdot F = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz) = 4z - 2y + y$$

$$\therefore \nabla \cdot F = 4z - y$$

Over the interior of cube:

$$Flux = \underset{\substack{\text{Cube}\\\text{surface}}}{\iint} F.n\, ds = \underset{\substack{\text{Cube}\\\text{interior}}}{\iint} \nabla.F\, dV = \int_0^1 \int_0^1 \int_0^1 (4z - y)\, dxdydz$$

VOLUME INTEGRAL:

$$\iiint_V \phi dv = \iiint_v \phi(x, y, z) dxdydz = \iiint_v F dv$$

1.) If $\phi = 45x^2 y$ **then evaluate** $\iiint_v \phi dv$ **where v denote the closed region bounded by the planes** $4x + 2y + z = 8, x = 0, y = 0, z = 0$

Solution: $\iint \int \emptyset\, dV = \int_{x=0}^{2} \int_{y=0}^{4-2x} \int_{z=0}^{8-4x-2y} 45\, x^2 y\, dx\, dy\, dz$

$$= 45 \int_{x=0}^{2} \int_{y=0}^{4-2x} x^2 y\, (8 - 4x - 2y) dy\, dx$$

$$= 45 \int_{x=0}^{2} \frac{1}{3} x^2 (4 - 2x)^3\, dx \;\; = 128$$

STOKE'S THEOREM:

If s is an open two sided surface bounded by a closed non intersecting curve and if a vector function $F(x, y, z)$ has continuous first partial derivatives in a domain in a space containing s. Then

$$\oint_c F.dr = \iint_s (curlF).nds = \iint_s (\nabla \times F) ds$$

Where c is described in positive (anti clock wise) direction and $\hat{n}$ is a unit positive(outward drawn) normal to **S.**

1.) Verify Stoke's theorem for $A = (2x - y)i - yz^2j - y^2zk,$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution: The boundary C of S is a circle in the xy-plane of radius unity and centre at the origin. Let $x = cos\ cos\ t,\ \ y = sin\ sin\ t,\ \ z = 0, 0 \leq t \leq 2\pi$ be the parametric equations of C

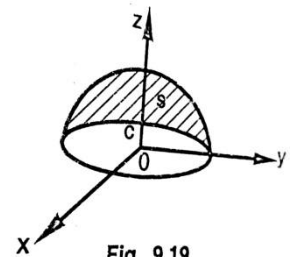


Fig 0.10

Then, $\oint_C \quad A.dr = \oint_C \quad [(2x - y)dx - yz^2dy - y^2zdz]$

$$= \int_0^{2\pi} \quad (2\ cos\ cos\ t - sin\ sin\ t)(-\ sin\ sin\ t)dt$$

$$= \int_0^{2\pi} \quad (-2\ sin\ sin\ t\ cos\ cos\ t + t)dt \quad = \pi$$

Also, $\nabla\ X\ A = \left| i\ j\ k\ \frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\ 2x - y\ \ - yz^2\ \ - y^2z \right| = k$

$$curl\ A\ .n = k.k = 1$$

$\iint_S (curl\ A).n\ dS = \iint_R dx\ dy,$ where R is the projection of S on the xy-plane

$$= \int_{-1}^{1} \quad \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \quad dy\ dx$$

$$= \int_{-1}^{1} \quad 2\sqrt{1 - x^2}dx$$

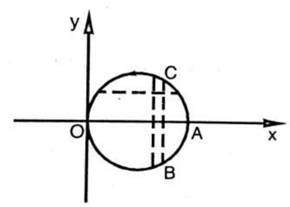
$$= 4\int_0^1 \quad \sqrt{1 - x^2}dx = \pi$$

Hence, Stoke's Theorem is verified.

2.) Evaluate $\iint_S (\nabla\, X\, F).dS$ taken over the portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve in the plane z=0 and $F = (y^2 + z^2 - x^2)i + (z^2 + x^2 - y^2)j + (x^2 + y^2 - z2k$

Solution: The given surface meets the plane $z = 0$ in the circle

$$x^2 + y^2 - 2ax = 0, z = 0$$



$$F = (y^2 + z^2 - x^2)i + (z^2 + x^2 - y^2)j + (x^2 + y^2 - z^2)k$$

$$\nabla X\, F = \left| i\; j\; k\; \frac{\partial}{\partial x}\, \frac{\partial}{\partial y}\, \frac{\partial}{\partial z}\; y^2 + z^2 - \; x^2\; z^2 + x^2 - y^2\; x^2 + y^2 - z^2 \right|$$

$$= i(2y - 2z) + j(2z - 2x) + k(2x - 2y)$$

The surface integral of $\nabla\, X\, F$ over the given surface is the same as the surface integral of $\nabla\, X\, F$ over the area of the circle $x^2 + y^2 - 2ax = 0, z = 0$

$$dS = n\, dS = k\, dx\, dy$$

$$\iint_S (\nabla\, X\, F).dS = \int_0^{2a} \int_{-\sqrt{2ax - x^2}}^{\sqrt{2ax - x^2}} (2x - 2y)dy\, dx$$

$$= \int_0^{2a} \int_{-\sqrt{2ax - x^2}}^{\sqrt{2ax - x^2}} 2xdy\, dx - \int_0^{2a} \int_{-\sqrt{2ax - x^2}}^{\sqrt{2ax - x^2}} 2y\, dy\, dx$$

$$= 2\int_0^{2a} \int_0^{\sqrt{2ax - x^2}} 2x\, dy\, dx$$

$$= 4\int_0^{2a} x[y]_0^{\sqrt{2ax - x^2}} = 2\pi a^2$$

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