Geometrization of the Local Langlands

§1 : What's local Langlands.

- E is a local field $\left(\text{e.g. } \mathbb{R}, \mathbb{C}, \mathbb{Q}_p, \mathbb{F}_p((t))\right)$

- $G_{/E}$ is a split reductive group over E.

- $\widehat{G}_{/\mathbb{Z}}$ is Langlands dual of G

- W_E : Weil group of E

<u>Conjecture</u> : There's a map between sets:

$$\left\{ \begin{array}{l} \text{irreducible objects} \\ \text{in } \mathrm{Rep}_{\mathbb{C}}(G(E)) \end{array} \right\} \longrightarrow \left\{ W_E \to \widehat{G}(\mathbb{C}) \right\}$$

$$\pi \longmapsto \varphi_\pi \quad \text{"L-parameter of } \pi\text{"}.$$

subjects to some compatibilities.

Rmk. (0) This is not a bijection

(1). From easy to hard for E:

archimedian, char p non-arch, char 0 non-arch.
($\mathbb{R}$, $\mathbb{C}$) (e.g. $\mathbb{F}_q((t))$) ($\mathbb{Q}_p$)

(2). $G(E)$ is a topological group
we require π to be "smooth"
(any vector is fixed by some quasi-cpt open of $G(E)$)

(3). W_E is a modification of $\mathrm{Gal}(\bar{E}/E)$

- $W_{\mathbb{C}} = \mathbb{C}^\times$
- $1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1$
- For non-arch E, residue field $k = \mathbb{F}_q$.

$$\begin{tikzcd}
1 \arrow[r] & I_E \arrow[r] \arrow[d, equal] & W_E \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "1 \mapsto \mathrm{Frob}"] & 1 \\
1 \arrow[r] & I_E \arrow[r] & \mathrm{Gal}(\bar{E}/E) \arrow[r] & \mathrm{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \arrow[r] & 1
\end{tikzcd}$$

inertia group

Warning: The topology on W_E is not the subspace topology induced from $\mathrm{Gal}(\bar{E}|E)$.

The correct topology: equip I_E with the subspace topology, and force it to be open inside W_E.

Explain W_E: It can already be seen in local class field theory.

["class formation"]

(4). We require φ_π to be continuous.

(5). Part of the compatibilities for $\pi \mapsto \varphi_\pi$ is about L-factors and ε-factors.

§2: What's geometrization?

<u>Answer</u>: Do global geometric Langlands on the <u>Fargues-Fontaine curve</u>, which behaves like a genus 0 curve in nonarchimedean geometry.

$E = \mathbb{Q}_p$. (For $\mathbb{F}_q((t))$, the story is easier)

<u>Analogy</u>

	alg. geo.	analytic geometre
affine	$\mathrm{Spec}\, R$, $R \in \mathrm{CAlg}$	$\mathrm{Spa}(R, R^+)$ (R, R^+) is a <u>Huber</u> pair $R^+ \subset R \in \mathrm{CAlg}(\mathrm{Top})$ satisfies .....
globalisation	Schemes as locally ringed spaces	(pre-)adic-space as locally <u>topologically</u> ringed space

point	Spec k, k field	$\mathrm{Spa}(K, K^+)$. (K, K^+): <u>affinoid field</u> analytic = non-discrete | non-analytic = discrete

<u>Def</u>: <u>Analytic affinoid field</u>:

K has a complete topology given by a non-arch valuation $K \to \mathbb{R}^+ \cup \{0\}$, ...

<u>Not</u>: In most cases people consider

$$\mathrm{Spa}\, R := \mathrm{Spa}(R, R^\circ)$$

$R^\circ \subset R$ subring of power-bounded objects.

(<u>Ex</u>: $R = \mathbb{Q}_p$, $R^\circ = \mathbb{Z}_p$).

<u>Rmk</u>: Among all the pre-adic spaces,

$$\{\text{perfectoid spaces}\} \subset \{\text{pre-adic spaces}\}$$

behave very well when you want to connect char 0 and char p geometry.

$\{$affine perfectoid spaces$\}^{op}$

$\simeq$ $\{$ perfectoid rings$\}$

<u>Ex</u>: 1) $\mathbb{F}_p((t^{1/p^\infty}))$

$:= \widehat{\bigcup_n \mathbb{F}_p((t^{1/p^n}))}$

2) $\mathbb{Q}_p^{cycl}$

$:= \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$

perfectoid rings are "super" ramified.

<u>Thm</u>: 1) There is no final object in $Perfd = \{$perfectoid spaces$\}$.

2) Inside $Perfd_p = \{$char p perfectoid spaces$\}$, we have products.

(but any perfectoid space is over $\mathbb{Q}_p$)

Analogy:

	Global GL for function fields	geometrization
geometry	alg. geo. over $\mathbb{F}_p$	"perfectoid geometry"
test objects.	schemes / $\mathbb{F}_p$	char p perfectoid spaces $S \in \mathrm{Perf}_{\mathbb{F}_p}$.
spaces	prestacks $\supset$ fppf-stacks $\supset$ alg. spaces.	prestacks $\supset$ v-stack $\supset$ diamonds
absolute curve	X curve over $\mathbb{F}_p$.	Not exist/sci-fi
relative curve	$\forall S$ scheme $X_S = X \times_{\mathbb{F}_p} S$	$X_S = Y_S/\mathrm{Frob}^{\mathbb{Z}}$ $Y_S = S \dot{\times} \operatorname{Spa} \mathbb{Q}_p$

Question: S is of char p

$\mathbb{Q}_p$ is of char 0.

What's $S \dot{\times} \operatorname{Spa} \mathbb{Q}_p$

Answer: It's not a product. $\dot{\times}$

Ex: If $S = \mathrm{Spa}(R, R^+)$

$$S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p \underset{\text{open}}{\subset} \boxed{\mathrm{Spa}\, W(R^+)}$$

$W(R^+)$: ring of p-Witt-vectors in R^+

is the open locus where p and $[\varpi]$ are invertible
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \ \ \ \cap$
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad W(R^+)$

In particular, $S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$ is of char 0.

Question: If $X_S := (S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p)/\mathrm{Frob}_S$

is of char 0, then how to study it using char p testing objects?

Ex: How to define the notion of Cartier divisors on X_S?

What's the base?

It can't be $\underbrace{S \text{ or } S/\mathrm{Frob}\,S}_{\text{char } p}$.

Answer: In fact, we do not use X_S.

Instead $(X_S)^\diamond$: the associated diamond of it.

(of char p).

Construction (Tilting):

$\forall\, R \in \mathrm{CAlg}$. (p)-tilt:

$$R^\flat := \lim\left(\cdots R \xrightarrow{\mathrm{Frob}} R \xrightarrow{\mathrm{Frob}} R\right).$$

A priori, $R^\flat$ is only a multiplicative monoid.

However: if R is nice enough

(e.g. R is perfectoid),

you can define $(R^\flat, +)$

$(x^{(0)}, x^{(1)}, \ldots) \in R^\flat$

$\mathrm{Frob}(x^{(n+1)}) = x^{(n)}$.

$(x^{(0)}, x^{(1)}, \ldots) + (y^{(0)}, y^{(1)}, \ldots)$

$= (z^{(0)}, \ldots\ldots)$

$$z^{(i)} := \lim_{n\to\infty} (x^{(i+n)} + y^{(i+n)})^{p^n}$$

$\Rightarrow$ For R perfectoid.

$R^\flat$ is a perfect ring of char p.

$\mathrm{Spd}(R, R^+) := \mathrm{Spa}(\underset{\sim}{R^\flat}, \underset{\sim}{(R^+)^\flat})$

(this is a char p perfectoid space).

<u>Ex</u>: If R is of char p, and perfectoid. $R^\flat = R$

Ex: $\mathbb{Q}_p^{cycl} = \widehat{\mathbb{Q}_p(\mu_{p^\infty})}$

$(\mathbb{Q}_p^{cycl})^\flat = \mathbb{F}_p((t^{1/p^\infty}))$

$= \widehat{\cup \mathbb{F}_p((t^{1/p^n}))}$

Thm: (Tilting equivalence)

$$Y \longmapsto Y^\flat$$

$$\mathrm{Perfd} \longmapsto \mathrm{Perfd}_p$$

It induce isomorphisms for over categorys

$$\mathrm{Perfd}_{/X} \xrightarrow[\simeq]{Y \mapsto Y^\flat} (\mathrm{Perfd}_p)_{/X^\flat}.$$

Moreover: preserves (finite) étale maps.

Def: For any prestack $X \in \mathrm{Fun}(\mathrm{Perfd}^{op}, \mathrm{Spc})$

we can define

$$X^\diamond \in \mathrm{PerfStk}_p := \mathrm{Fun}(\mathrm{Perfd}_p^{op}, \mathrm{Sets})$$

$$X^\diamond(S) := \coprod_{S^\# \in \mathrm{Untilt}(S)} \mathrm{Map}(S^\#, X)$$

$$(\ (S^\#)^\flat \simeq S \ ).$$

Ex: If $X \in \mathrm{char}\ p$, $X^\diamond = X$

$$\begin{tikzcd} \mathrm{PreStk} \arrow[r, "\diamond"] & \mathrm{PerfStk}_p \\ \mathrm{Perfd} \arrow[u, hook] \arrow[r, "\flat"'] & \mathrm{Perfd}_p \arrow[u, hook] \end{tikzcd}$$

$\diamond$ is the unique colimit preserving functor making it commute.

<u>Rmk</u>: (Scholze)

"$\diamond$ only remembers topological information"

(<u>Thm</u>: If X is ana-adic space

$$|X^\diamond| = |X|$$)

Ex: $(\mathrm{Spa}\,\mathbb{Z}_p)^\diamond = \mathrm{Spd}\,\mathbb{Z}_p$

classifies all the untilts.

$(\mathrm{Spa}\,\mathbb{Q}_p)^\diamond = \underline{\mathrm{Spd}\,\mathbb{Q}_p}$

classifies all the char 0 untilts.

Recall: $X_S := Y_S / \mathrm{Frob}_S$

$Y_S := S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p$.

Thm: $(Y_S)^\diamond \overset{!}{=} (S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p)^\diamond$

(It's not formal)

$= S^\diamond \times (\mathrm{Spa}\,\mathbb{Q}_p)^\diamond$

$= S \times \mathrm{Spd}\,\mathbb{Q}_p$.

(products taken in PreStk$_p$).

Rmk: $\mathrm{Spd}\,\mathbb{Q}_p$ is not a perfectoid space. It's a diamond.

Ex: $\mathbb{Q}_p \longrightarrow \mathbb{Q}_p^{cycl}$

$\mathrm{Spa}\, \mathbb{Q}_p^{cycl} \longrightarrow \mathrm{Spa}\, \mathbb{Q}_p$

is a pro-etale cover

char p

$$\mathrm{Spd}\, \mathbb{Q}_p = \mathrm{Spd}\, \mathbb{Q}_p^{cycl} / \underline{\mathbb{Z}_p^\times}$$

(here $\mathbb{Z}_p^\times$ is a discrete group.)

$$= \mathrm{Spa}\, \mathbb{F}_p((t^{1/p^\infty})) / \underline{\mathbb{Z}_p^\times}$$

$\left(\begin{array}{l} \text{For any topological space } T, \\ \underline{T} \in \mathrm{PreStk}_p. \\ \underline{T}(S) := \mathrm{Map}_{cont}(|S|, T) \end{array} \right)$

We can finally define Cartier divisors.

$$X_S := Y_S / \mathrm{Frob}_S = (S \dot{\times} \mathrm{Spa}\, \mathbb{Q}_p) / \mathrm{Frob}_{S_1}$$

$$X_S^\diamond = (Y_S/\mathrm{Frob}_S)^\diamond$$

$$= (S \times \mathrm{Spd}\,\mathbb{Q}_p)/\mathrm{Frob}_S$$

For $S \in \mathrm{Perf}_{\mathbb{F}_p}$, $\{S \to \mathrm{Spd}\,\mathbb{Q}_p\}$

$$\| $$

$\{$char 0 untilts of $S\}$

$$\downarrow$$

$\{S^\# \to \mathrm{Spa}\,\mathbb{Q}_p\}$

$\downarrow$ "taking graphs"

$\{S^\# \to Y_S : S \dot{\times} \mathrm{Spa}\,\mathbb{Q}_p\}$

This is a codim 1 closed subspace.

$$\downarrow$$

$\{S^\# \to Y_S \to X_S := Y_S/\mathrm{Frob}\}$

<u>Fact</u>: $S^\# \to X_S$ only depends

$$S \to \mathrm{Spd}\,\mathbb{Q}_p \to \mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$$

$\rightsquigarrow$) $\mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$ should be viewed as the moduli presheaf of deg 1 Cartier divis. on FF curve.

<u>Recall</u>: In usual GL, we use not $\mathbb{F}_p$-pts to do Hecke-modifications we use $\overline{\mathbb{F}}_p$-pts.

We want to use $\mathrm{Perfd}_{/\overline{\mathbb{F}}_p}$

$\rightsquigarrow$ replace $\mathrm{Spd}\,\mathbb{Q}_p/\mathrm{Frob}$

by $\mathrm{Spd}\,\breve{\mathbb{Q}}_p^{un}/\mathrm{Frob}$

$$\left(\qquad k(\mathbb{Q}_p^{un}) = \overline{\mathbb{F}}_p \quad \right)$$

<u>Def</u>: $\mathrm{Div}^1 := \mathrm{Spd}\,\mathbb{Q}_p^{un}/\mathrm{Frob}.$

"diamond of degree 1 closed Cartier divisors on FF curve"

$\left\{ \begin{array}{l} \text{relative FF curve} \quad X_S := (S \times \operatorname{Spec} \mathbb{Q}_p)/\mathrm{Frob} \\ \text{absolute FF curve} \quad \text{not exist} \end{array} \right.$

Div^1 is not the absolute FF curve.!

char p

<u>Rmk:</u> $\underline{\mathrm{Div}^1 \times \mathrm{Div}^1}$ makes sense in PerfStk$_p$ and is non trivial

However. $\operatorname{Spa} \mathbb{Q}_p \times \operatorname{Spa} \mathbb{Q}_p$ is boring.

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§3: Why FF curve ?

read I.11

<u>Thm:</u> $\pi_1(\mathrm{Div}^1) = W_E$

<u>Rmk</u>: $\mathrm{Div}^1 = \underline{\mathrm{Spd}\, \mathcal{O}_p^{un}} / \mathrm{Frob}.$

$$\pi_1(\mathrm{Spd}\, \mathcal{O}_p^{un}) = \mathrm{Gal}(\overline{\mathcal{O}_p} / \mathcal{O}_p^{un})$$
$$= I_{\mathcal{O}_p}$$

$$1 \to I_{\mathcal{O}_p} \to \pi_1(\mathrm{Div}^1) \to \mathbb{Z}\langle \mathrm{Frob} \rangle \to 1$$
$$\shortparallel$$
$$W_E.$$

<u>Question</u>: Did you say that FF curve behaves like genus 0.

X_S is not over Spa S

If C is a geometric point,

the

$$\Gamma(X_C, \mathcal{O}) = \mathbb{Q}_p.$$

$\mathrm{Bun}_G(S) :=$ " G-tors on X_S ".

<u>Thm</u>: Bun_G has a stratification by a poset $B(G)$ such that each stratum is of the form $*/H$, H is a grp diamond.

$$1 \to \underbrace{\text{unipotent}} \to H \to \underline{M(\mathbb{Q}_p)} \to 1$$

M is an $\overset{\text{inner form of}}{\text{Levi}}$ of G (depends on the stratum).

$\Rightarrow$

<u>Cor</u>: $\underline{\mathrm{Shv}(\mathrm{Bun}_G, \bar{\mathbb{Q}}_\ell)}$ can be

glued from $\mathrm{Shv}(*/M)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

$\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(M(\mathbb{Q}_p))$.

<u>Rmk</u>: What else?

p-adic Hodge theory