- Prop. M flat as A-module $\Leftrightarrow$ $\forall$ ideal $\mathfrak{a} \subseteq A$, $\mathfrak{a} \otimes_A M \to M$ is injective.
$$a \otimes m \mapsto m$$

- (Local criteria of flatness) TFAE.
  (1) M flat over A.
  (2) $M_{\mathfrak{p}}$ flat over $A_{\mathfrak{p}}$, $\forall \mathfrak{p} \in \operatorname{Spec} A$.
  (3) $M_{\mathfrak{m}}$ flat over $A_{\mathfrak{m}}$, $\forall \mathfrak{m} \in \operatorname{Specm} A$.

- Prop. (1) $f: A \to B$ flat. $S \subseteq A$, $T \subseteq B$ multiplicative sets $f(S) \subseteq T$.
  $\Rightarrow \tilde{f}: S^{-1}A \to T^{-1}B$ flat.
  (2) $A_{f^{-1}(\mathfrak{n})} \to B_{\mathfrak{n}}$ flat, $\forall \mathfrak{n} \in \operatorname{Specm} B \Rightarrow A \to B$ flat.

Pf. (1) $A \to B$ flat $\Rightarrow S^{-1}A \to S^{-1}B$ flat.
$S^{-1}B \simeq [f(S)]^{-1}B$. $T^{-1}B \simeq T^{-1}([f(S)]^{-1}B)$
$\Rightarrow S^{-1}B \to T^{-1}B$ flat $\Rightarrow \tilde{f}: S^{-1}A \to T^{-1}B$ flat.

(2) $N' \hookrightarrow N$ A-modules. Goal: $N' \otimes_A B \to N \otimes_A B$ inj.

$$\begin{array}{ccc} & \Updownarrow & \\ N' \otimes_A B \otimes_B B_{\mathfrak{n}} & \hookrightarrow & N \otimes_A B \otimes_B B_{\mathfrak{n}} \\ \wr\wr & & \wr\wr \\ N' \otimes_A B_{\mathfrak{n}} & & N \otimes_A B_{\mathfrak{n}} \\ \wr\wr & & \wr\wr \\ N' \otimes_A (A_{f^{-1}(\mathfrak{n})} \otimes_{A_{f^{-1}(\mathfrak{n})}} B_{\mathfrak{n}}) & & N \otimes_A (\cdots) \\ \wr\wr & & \wr\wr \\ N'_{f^{-1}(\mathfrak{n})} \otimes_{A_{f^{-1}(\mathfrak{n})}} B_{\mathfrak{n}} & & N_{f^{-1}(\mathfrak{n})} \otimes_{A_{f^{-1}(\mathfrak{n})}} B_{\mathfrak{n}} \end{array}$$

- Thm. (Equation Principle of flatness) TFAE.
  (1) M is flat over A.
  (2) $\forall a_1, \dots, a_n \in A$ and $x_1, \dots, x_n \in M$ with $\sum_{i=1}^n a_i x_i = 0$,
  $\exists y_1, \dots, y_m \in M$, $a_{ij} \in A$, $i = 1, \dots, n$, $j = 1, \dots, m$ s.t.
  ① $\sum_{i=1}^n a_i a_{ij} = 0$, $\forall j = 1, \dots, m$;

② $x_i=\sum_{j=1}^{m} a_{ij} x_j$, $\forall i=1,\dots,n$.

Formally:

$$(a_1 \cdots a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=0 \Rightarrow \exists \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \text{ s.t.}$$

$$(a_1 \cdots a_n)\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}=0, \quad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}=\begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}\begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

- Prop. M f.g. A-mod. (A noeth.) TFAE:
  - (a) M is flat.
  - (b) $M_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module, $\forall \mathfrak{m} \in \operatorname{Specm} A$.
  - (c) $\widetilde{M}$ is a locally free sheaf on $\operatorname{Spec} A$.
  - (d) M is projective as an A-module.

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  - (e) (if A is a domain) $\dim_{k(\mathfrak{p})} \underbrace{M \otimes_A k(\mathfrak{p})}_{= M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}} \equiv \text{const}$.

Pf. (a)⇔(b) local criteria of flatness.

- Lemma. $(A,\mathfrak{m})$ local ring. M f.g. & flat over $A \Rightarrow M$ free.

  Pf. $0 \to K \to A^n \twoheadrightarrow M \to 0$ exact $\overset{\otimes A/\mathfrak{m}}{\rightsquigarrow}$

  $$\underset{=0}{\operatorname{Tor}^1(M, A_{\mathfrak{m}})} \to K/\mathfrak{m}K \to (A/\mathfrak{m})^n \to M/\mathfrak{m}M=0.$$

  $\bar{x}_1,\dots,\bar{x}_n$ a $A/\mathfrak{m}$-basis of $M/\mathfrak{m}M \xRightarrow{\text{Nakayama}}$

  $x_1,\dots,x_n$ generate M over A. We can define $A^n \twoheadrightarrow M$, $e_i \mapsto x_i$

  $\Rightarrow (A/\mathfrak{m})^n \xrightarrow{\sim} M/\mathfrak{m}M$

  $\Rightarrow K/\mathfrak{m}K=0 \xRightarrow{\text{Nakayama}} K=0 \Rightarrow M \simeq A^n$.

(b)⇒(c) Fix $\mathfrak{m} \in \operatorname{Spec} A$. Take an $A_{\mathfrak{m}}$-basis of $M_{\mathfrak{m}}$:

$x_1, \dots, x_n$ (WLOG, WMA $x_1, \dots, x_n \in M$)

$\varphi : A^n \longrightarrow M \Rightarrow (\ker\varphi)_{\mathfrak{m}} = (\operatorname{im}\varphi)_{\mathfrak{m}} = 0$

$e_i \longmapsto x_i$ (exactness of localization)

$\ker\varphi$, $\operatorname{im}\varphi$ f.g. over $A \Rightarrow$ supp is closed

- $M = A\langle x_1, \dots, x_n\rangle \Rightarrow$

$$\begin{aligned}\operatorname{supp} M &= \{\mathfrak{p} \in \operatorname{Spec} A \mid M_{\mathfrak{p}} \neq 0\} \\ &= \{\mathfrak{p} \in \operatorname{Spec} A \mid \exists i = 1, \dots, n \text{ s.t. } \operatorname{Ann}(x_i) \subseteq \mathfrak{p}\} \\ &= \bigcup_{i=1}^{n} V(\operatorname{Ann}(x_i)) \text{ closed}\end{aligned}$$

$\Rightarrow \exists f \in A - \mathfrak{m}$ s.t. $\varphi_f : A_f^n \xrightarrow{\sim} M_f$.

(c) $\Rightarrow$ (d) Suppose $\exists f_1, \dots, f_m \in A$ s.t. $(f_1, \dots, f_m) = (1)$ & M_{f_i} free over A_{f_i}.

$\pi : A^n \twoheadrightarrow M$, $e_i \longmapsto x_i$ $\rightsquigarrow$ $\pi_i : A_{f_i}^n \longrightarrow M_{f_i}$

$\exists \varphi_i : M_{f_i} \longrightarrow A_{f_i}^n$ s.t. $\pi_i \circ \varphi_i = \mathrm{id}_{M_{f_i}}$

Claim: $\exists N$ s.t. $f_i^N \varphi_i$ can be lifted to $\tilde{\varphi}_i : M \longrightarrow A^n$,

i.e. the diagram

$$\begin{array}{ccc} M & \xrightarrow{\tilde{\varphi}_i} & A^n \\ \downarrow & & \downarrow \\ M_{f_i} & \xrightarrow{f_i^N \varphi_i} & A_{f_i}^n \end{array}$$

commutes.

Pf of claim. Write $M = A\langle x_1, \dots, x_n\rangle$ $\varphi(x_j) = \dfrac{y_j}{f_i^M}$, $y_j \in A^n$

Hope to find some M' s.t. the A-mod homomorphism

$\tilde{\varphi}_i : M \longrightarrow A^n$ is well-defined $\Rightarrow N = M + M'$ is OK.

$x_j \longmapsto f_i^{M'} y_j$

$(\Leftrightarrow \exists (a_1, \dots, a_n) \in \ker\pi,\ f_i^{M'} \sum_j a_j y_j = 0)$

Write $\ker\pi = A\langle v_1, \dots, v_s\rangle$ $v_k = (a_{k1}, \dots, a_{kn})$

$\Rightarrow \varphi_i\left(\sum_j a_{kj} x_j\right) = \sum_j \dfrac{a_{kj} y_j}{f_i^M} = 0$ in $A_{f_i}^n$, $\forall k = 1, \dots, s$

$\varphi_i(0)$

$\Rightarrow \exists M'$, s.t. $f_i^{M'}\sum_j a_{kj} y_j = 0,\ \forall k=1,\dots,s.$ $\Rightarrow \square$.

Take $\varphi: M \longrightarrow A^n$, where $g_i \in A$, $\sum_{i=1}^m g_i f_i^N = 1$

$x \longmapsto \sum_{i=1}^m g_i \tilde{\varphi}_i(x)$ ⌈partition of unity in AG⌋

$\Rightarrow \pi(\varphi(x)) = \sum_{i=1}^m g_i \pi(\tilde{\varphi}_i(x))$.

Localize at $f_i \Rightarrow \pi(\varphi(x)) = \sum_{i=1}^m g_i \pi_i(f_i^N \varphi_i(x)) = \sum_{i=1}^m g_i f_i^N x = x$ in M_{f_i}

$\Rightarrow \pi \circ \varphi = \mathrm{id}_M$. M projective.

(d)$\Rightarrow$(a) M projective $\Rightarrow$ M is direct summand of $A^n \Rightarrow M$ flat.

(c)$\Rightarrow$(e) $\dim_{k(\mathfrak{p})} M \otimes_A k(\mathfrak{p})$ locally constant $\xRightarrow{\text{connectedness}}$ globally const.

(e)$\Rightarrow$(c) (Hartshorne II Ex 5.8)

$\bar{x}_1, \dots, \bar{x}_n$ a $k(\mathfrak{p})$-basis of $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \xRightarrow{\text{Nakayama}}$

$x_1, \dots, x_n$ a minimal set of $A_\mathfrak{p}$-generators of $M_\mathfrak{p}$. (WLOG $x_1, \dots, x_n \in M$)

Consider $\varphi: A^n \longrightarrow M$ $\quad \varphi_\mathfrak{p}$ is surj $\Rightarrow (\operatorname{coker}\varphi)_\mathfrak{p} = 0$.

$e_i \longmapsto x_i$

φ_f surj.

(closed support) $\Rightarrow \exists f \in A - \mathfrak{p}$ s.t. $x_1, \dots, x_n$ generate M_f over A_f

If $\exists a_1, \dots, a_n \in A_f$, $(a_1, \dots, a_n) \neq (0, \dots, 0)$ s.t. $\sum_{i=1}^n a_i x_i = 0$

(WLOG, WMA $a_1 \neq 0$) (A integral) $\Rightarrow a_1$ not nilpotent

$\Rightarrow \exists$ prime ideal $\mathfrak{q}_f \subseteq A_f$ s.t. $a_1 \notin \mathfrak{q}_f$

$\Rightarrow a_1$ is invertible in $A_\mathfrak{q}$ $\Rightarrow x_1 = -\sum_{i=2}^n \frac{a_i}{a_1} x_i$ in $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q} = M \otimes_A k(\mathfrak{q})$

$\Rightarrow \dim_{k(\mathfrak{q})} M \otimes_A k(\mathfrak{q}) < \dim_{k(\mathfrak{p})} M \otimes_A k(\mathfrak{p})$. A contradiction.

Rmk 1. When A is not noeth. or M is not f.g.,

then M is flat $\not\Rightarrow$ M is projective.

Ex. ① (A is not noeth) (2023 Yau-Contest Q4(3))

$$A=\prod_{\mathbb{N}}\mathbb{Z},\ I=\bigoplus_{\mathbb{N}}\mathbb{Z}\subseteq A,\quad M=A/I.$$

② (M is not f.g.)

$$A=\mathbb{Z},\ M=\mathbb{Q}.$$

Rmk 2. (Lazard) A ring (not necessarily noeth.)

M A-module.

M is flat $\Leftrightarrow$ $M=\varinjlim\{\text{finite free module}\}$.

• Def. $f: Y\to X$ is a flat morphism if $\forall y\in Y$, $\mathcal{O}_{Y,y}$ is a flat $\mathcal{O}_{X,x}$-algebra. $x=f(y)$.

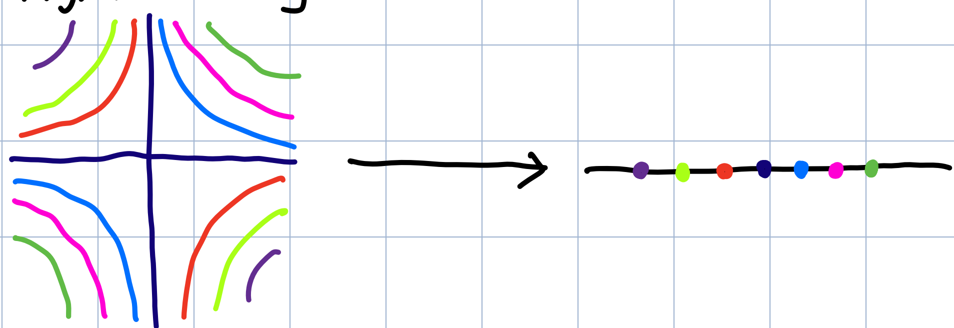
Rmk. (Equiv.) (1) $\forall$ open affine $U\subseteq X$, $V\subseteq Y$, $f(V)\subseteq U$:

$\Gamma(V,\mathcal{O}_Y)$ is flat over $\Gamma(U,\mathcal{O}_X)$.

(2) $\forall$ <u>closed</u> point $y\in Y$, $\mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,x}$.

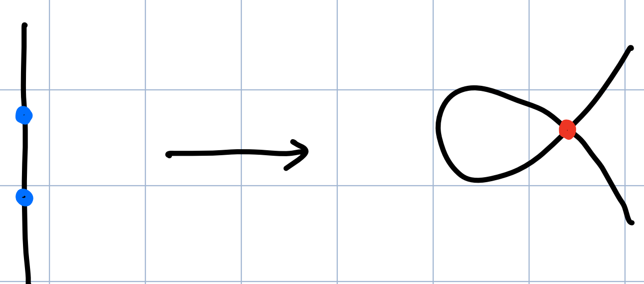
• Ex. (1) $f: \mathbb{A}^2 \longrightarrow \mathbb{A}$ is flat.

$(x,y)\longmapsto xy$



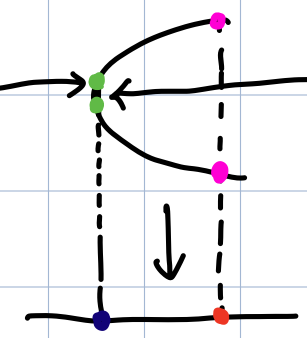
(2) $f: \operatorname{Spec} k[t] \longrightarrow \operatorname{Spec} k[x,y]/(x^2+x^3-y^2)$ is not flat.

$t \longmapsto (t^2-1,\ t(t^2-1))$



(3) $f : \operatorname{Spec} k[x,y]/(x-y^2) \longrightarrow \operatorname{Spec} k[x]$ is flat.

$(x,y) \longmapsto x$



but not unramified.

$k[x,y]/(x-y^2) \otimes_{k[x]} k[x]/(x)$

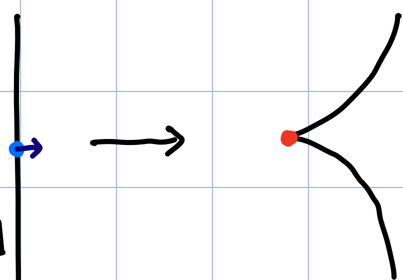
$\simeq k[\varepsilon]/\varepsilon^2.$

[double point]

(4) $f : \operatorname{Spec} k[t] \longrightarrow \operatorname{Spec} k[x,y]/(x^3-y^2)$ is not flat.

$t \longmapsto (t^2, t^3)$

[a point with a tangent orientation]



- $f : Y \longrightarrow X$ flat $\Rightarrow \forall y \in Y,\ x = f(y)$:

$\dim Y_x = \dim \mathcal{O}_{Y,y} - \dim \mathcal{O}_{X,x}$ (Hartshorne Chapter III Prop. 9.5)

- Rmk. In general, X integral. $f : Y \to X$ projective morphism. Then f flat $\Leftrightarrow$ All fibers $Y_x \longrightarrow \operatorname{Spec} k(x)$ have the same Hilbert polynomial (Hartshorne Chapter III Thm 9.9)

See EGA IV Chap. 12 for more properties of flat families.

- Prop. (1) Open immersions are flat.

(2) Composition of two flat morphisms is flat.

(3) Flatness is stable under base extension.

Pf. (3) $Y \times_X X' \longrightarrow X'$ $\quad A \otimes_S B \longleftarrow B$

$y' \longmapsto x'$

$Y \longrightarrow X \qquad A \longleftarrow S$

$S \to A$ flat $\Rightarrow B \to A \otimes_S B$ flat.

- Prop. $f: A \to B$ flat algebra.

  $b \in B$ s.t. $\bar{b}$ is not a zero-divisor in $B/\mathfrak{m}B, \forall \mathfrak{m} \in \operatorname{Specm} A$

  $\Rightarrow A \to B/(b)$ flat.

Pf. ① Localization. Goal: $\forall \mathfrak{n} \in \operatorname{Specm} B$ with $b \in \mathfrak{n}$, $\mathfrak{m} = f^{-1}(\mathfrak{n})$,

$A_{\mathfrak{m}} \to (B/(b))_{\mathfrak{n}} \simeq B_{\mathfrak{n}}/(b)$ is flat.

$A \to B$ flat $\Rightarrow A_{\mathfrak{m}} \to B_{\mathfrak{n}}$ flat.

If $\mathfrak{m} \subsetneq \mathfrak{m}' \subsetneq A \Rightarrow \mathfrak{m}B = B \Rightarrow \bar{b} = 0$ in $B/\mathfrak{m}B$ (contradiction)

$\Rightarrow \mathfrak{m} \in \operatorname{Specm} A$.

b in $B_{\mathfrak{n}}/\mathfrak{m}B_{\mathfrak{n}} \simeq (B/\mathfrak{m}B)_{\mathfrak{n}}$ is not a zero-divisor.

$\Rightarrow$ WMA $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ local homomorphism.

② We will show b is not a zero-divisor in B.

Suppose $bc = 0$ in $B \Rightarrow c \in \mathfrak{m}B$.

Goal: Prove $c \in \mathfrak{m}^r B, \forall r \geq 1$ by induction

$\Rightarrow c \in \bigcap_{r \geq 1} \mathfrak{m}^r B = 0$ (Krull's intersection thm)

If we have proved $c \in \mathfrak{m}^r B$.

Choose a minimal set of generators of $\mathfrak{m}^r = (a_1, \ldots, a_k)$

Write $c = \sum a_i b_i$, $b_i \in B \Rightarrow \sum a_i b_i b = 0$

(Equation principle of flatness) $\Rightarrow \exists a_{ij} \in A, b_j' \in B$ s.t.

$$\begin{pmatrix} b_1 b \\ \vdots \\ b_k b \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kl} \end{pmatrix} \begin{pmatrix} b_1' \\ \vdots \\ b_l' \end{pmatrix}$$

and $$(a_1 \ \cdots \ a_k)\begin{pmatrix} a_{11} & \cdots & a_{1l} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kl} \end{pmatrix} = 0.$$

If $a_{ij} \in A - \mathfrak{m} = A^\times$: $0 = \sum_h a_h a_{hj} \Rightarrow a_i = -a_{ij}^{-1} \sum_{h \neq i} a_h a_{hj}$

$\Rightarrow \{a_1, \ldots, \hat{a_i}, \ldots, a_k\}$ is a smaller set of generators (X)

$\Rightarrow a_{ij} \in \mathfrak{m}, \forall i,j \Rightarrow b_i b \in \mathfrak{m}B, \forall i \Rightarrow b_i \in \mathfrak{m}B, \forall i$

(b is not a zero-divisor in $B/\mathfrak{m}B$)

$\Rightarrow c = \sum a_i b_i \in \mathfrak{m}^{r+1}B$. □.

③ Take any ideal $\mathfrak{a} \subseteq A$.

A replaced by $A/\mathfrak{a}$, B replaced by $B/\mathfrak{a}B$.

The same proof as in ② $\Rightarrow$ b is not a zero-divisor in $B/\mathfrak{a}B$.

$$\mathfrak{a} \otimes_A B/(b) \simeq \mathfrak{a} \otimes_A (B \otimes_B B/(b)) \simeq (\mathfrak{a} \otimes_A B) \otimes_B B/(b)$$

$\|$ (flatness of B over A)

$\mathfrak{a}B \otimes_B B/(b) \simeq \mathfrak{a}B/b\mathfrak{a}B$

$\downarrow$

$B/(b)$

need to prove $\mathfrak{a}B \cap (b) = b\mathfrak{a}B$

(hence $\mathfrak{a}B/b\mathfrak{a}B \hookrightarrow B/(b)$ injective).

Take any $x \in \mathfrak{a}B \cap (b)$. Write $x = by, y \in B$.

b is not a zero-divisor in $B/\mathfrak{a}B \Rightarrow (x \in \mathfrak{a}B \Rightarrow y \in \mathfrak{a}B)$

$\Rightarrow x = by \in b\mathfrak{a}B$.

- Geometric interpretation

$f: Y \to X$ flat. $Z \subseteq Y$ a 「hypersurface」

If $\forall$ closed point $x \in X$, Z doesn't contain any generic point of irreducible component of fiber Y_x 「$\dim(Z \cap Y_x) < \dim Y_x$」, then the composition $Z \to X$ is flat.

- Thm. $f: Y \to X$ flat, locally of finite type $\Rightarrow$ f open.

Sketch. ① Reduce to X, Y affine. Need to prove $f(Y)$ open in X.

② $f(Y)$ is constructible; ③ flatness $\Rightarrow$ going-down.

- Thm. (generic flatness) $f: Y \longrightarrow X$ locally of finite type

$\Rightarrow \{y \in Y \mid \mathcal{O}_{Y,y}$ is flat over $\mathcal{O}_{X,f(y)}\}$ is open in Y.

(if X is integral) $\Rightarrow \exists$ non-empty open $U \subseteq X$ s.t. f flat on $f^{-1}(U)$.

$f^{-1}(U)$ may be empty.

Sketch. • Lemma: A noeth. $A \longrightarrow B$ finite type.

M f.g. B-mod $\Rightarrow \exists f \in A$ s.t. M_f is free over A_f.

Take $M = B$.

- Def. (faithfully flat) $f: A \longrightarrow B$ is faithfully flat if:

① $f: A \to B$ flat; ② $\forall$ A-mod M with $B \otimes_A M = 0 \Rightarrow M = 0$.

- Prop. $f: A \longrightarrow B$ flat, $A \neq 0$. TFAE:

(a) f faithfully flat;

(b) $B \otimes_A M' \longrightarrow B \otimes_A M \longrightarrow B \otimes_A M''$ exact $\Rightarrow M' \to M \to M''$ exact

(c) $f^{\#}$: Spec $B \longrightarrow$ Spec A surj.

(d) $\forall \mathfrak{m} \in \operatorname{Specm} A$, $\mathfrak{m}B \neq B$.

(Cor. $f: A \to B$ flat local homomorphism $\Rightarrow f$ faithfully flat).

Pf. (a) $\Rightarrow$ (b): $M' \xrightarrow{g_1} M \xrightarrow{g_2} M''$ s.t.

$B \otimes_A M' \longrightarrow B \otimes_A M \longrightarrow B \otimes_A M''$ exact

$\Rightarrow B \otimes_A \operatorname{im}(g_2 g_1) = \operatorname{im}((1 \otimes g_2)(1 \otimes g_1)) = 0$

$\Rightarrow \operatorname{im}(g_2 g_1) = 0.$ flatness

$\Rightarrow \operatorname{im} g_1 \subseteq \ker g_2$.

$B \otimes_A (\ker g_2 / \operatorname{im} g_1) = \ker(1 \otimes g_2) / \operatorname{im}(1 \otimes g_1) = 0$

$\Rightarrow \ker g_2 / \operatorname{im} g_1 = 0 \Rightarrow \operatorname{im} g_1 = \ker g_2$.

(b)⇒(a): $B \otimes_A M = 0 \Rightarrow B\otimes_A M \xrightarrow{1\otimes 0} B\otimes_A M \longrightarrow 0$ exact

$\Rightarrow M \xrightarrow{0} M \longrightarrow 0$ exact $\Rightarrow M=0$.

(Rmk. $\mathcal{F}: \mathcal{C} \longrightarrow \mathcal{D}$ exact functor. TFAE:

① $\mathcal{F}$ is faithfully exact, i.e. $\mathcal{F}(A)=0 \Rightarrow A=0$.

② $\mathcal{F}$ reflects exact sequences, i.e.

$\mathcal{F}(A') \to \mathcal{F}(A) \to \mathcal{F}(A'')$ exact $\Rightarrow A' \to A \to A''$ exact.)

(a)⇒(c). $\forall \mathfrak{p} \in \operatorname{Spec} A$, the fiber $B \otimes_A A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} \neq 0$ as A-module

$\Rightarrow f^\#: \operatorname{Spec} B \to \operatorname{Spec} A$ surj. 「non-empty」

(c)⇒(d). $\forall \mathfrak{m} \in \operatorname{Spec} A$, $\exists \mathfrak{n} \in \operatorname{Spec} B$ s.t. $f^{-1}(\mathfrak{n}) = \mathfrak{m}$

$\Rightarrow f(\mathfrak{m}) \subseteq \mathfrak{n} \Rightarrow \mathfrak{m}B \subseteq \mathfrak{n}B = \mathfrak{n} \subsetneq B$.

(d)⇒(a). $B \otimes_A M = 0$. Take any $x \in M$.

(flatness of B) $Ax \hookrightarrow M \rightsquigarrow B \otimes_A Ax \hookrightarrow B \otimes_A M = 0$

$\Rightarrow B \otimes_A Ax = 0$. Write $Ax \simeq A/\mathfrak{a}$.

Assume $\mathfrak{a} \notin A \Rightarrow$ Take any maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$,

$B \otimes_A A/\mathfrak{m} \simeq B/\mathfrak{m}B \neq 0$

But $A/\mathfrak{a} \twoheadrightarrow A/\mathfrak{m} \Rightarrow \underset{\substack{\| \\ 0}}{B \otimes_A Ax} \twoheadrightarrow \underset{\substack{\nparallel \\ 0}}{B \otimes_A A/\mathfrak{m}}$ contradiction

- Geometric interpretation. a faithfully flat morphism $f: Y \to X$ is an analogy of 「covering map」.

- Def. $f: Y \longrightarrow X$ is faithfully flat iff f is flat & surjective.

- Lemma. $f: A \to B$ faithfully flat ⇒ long exact seq

$$0 \to A \xrightarrow{f} B \xrightarrow{d^0} B^{\otimes 2} \xrightarrow{d^1} B^{\otimes 3} \longrightarrow \cdots B^{\otimes r} \xrightarrow{d^{r-1}} B^{\otimes r+1} \longrightarrow \cdots$$

where $d^{r-1}(b_0\otimes\cdots\otimes b_{r-1}) = \sum_{i=0}^{r}(-1)^i\, b_0\otimes\cdots\otimes b_{i-1}\otimes 1\otimes b_i\otimes\cdots\otimes b_{r-1}$.

Pf. $d^r d^{r-1}=0$ (✓)

We firstly assume that $\exists g: B\to A$ s.t. $gf=1$

$$\Rightarrow \begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "d^0"] \arrow[dl, dashed, "k_{-1}"'] & B^{\otimes 2} \arrow[r, "d^1"] \arrow[dl, dashed, "k_0"'] & B^{\otimes 3} \arrow[r] \arrow[dl, dashed, "k_1"'] & B^{\otimes 4} \arrow[r] \arrow[dl, dashed, "k_2"'] & \cdots \\ 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "d^0"] & B^{\otimes 2} \arrow[r, "d^1"] & B^{\otimes 3} \arrow[r] & B^{\otimes 4} \arrow[r] & \cdots \end{tikzcd}$$

chain homotopy.

$k_r(b_0\otimes\cdots\otimes b_{r+1}) = g(b_0)b_1\otimes b_2\otimes\cdots\otimes b_{r+1}$

$\Rightarrow k_{r+1}d^{r+1}+d^r k_r = \mathrm{id}_{B^{\otimes r+2}}$, $r\geq -1$

$\Rightarrow \mathrm{id} \underset{\text{chain homotopy}}{\sim} 0 \Rightarrow$ exact. [faithful flatness is not required in this step]

For general case: Define $f': B\longrightarrow B\otimes_A B$

$b\longmapsto b\otimes 1$

Inverse: $g': B\otimes_A B\longrightarrow B$

$b_1\otimes b_2\longmapsto b_1b_2$

$\Rightarrow 0\to B\to B\otimes_A B\longrightarrow (B\otimes_A B)^{\otimes_B 2}\longrightarrow (B\otimes_A B)^{\otimes_B 3}\longrightarrow\cdots$ exact

$(B\otimes_A B)^{\otimes_B 2}\cong B^{\otimes 2}\otimes_A B$, $(B\otimes_A B)^{\otimes_B 3}\cong B^{\otimes 3}\otimes_A B$

$\xRightarrow{\text{faith. flat}} 0\to A\to B\to B^{\otimes 2}\to B^{\otimes 3}\to\cdots$ exact.

• Rmk. Similar argument. $f: A\to B$ faith. flat M A-mod

$\Rightarrow 0\to M\longrightarrow M\otimes_A B\longrightarrow M\otimes_A B^{\otimes 2}\longrightarrow M\otimes_A B^{\otimes 3}\longrightarrow\cdots$

is exact.

• Descent theory

(Part I, Comm. Alg. Version)

$f: A \to B$ ring homomorphism

M A-mod $\rightsquigarrow M' = M \otimes_A B$ B-mod.

Can we recover M from M'?

ANS: Under assumption that $f: A \to B$ faithfully flat, we can recover M from M' together with some "extra data".

Why extra data?

$$0 \to M \to \boxed{M \otimes_A B} \to \boxed{M \otimes_A B^{\otimes 2}} \to \boxed{M \otimes_A B^{\otimes 3}} \to \cdots$$

$$M \otimes_A B = M', \quad M \otimes_A B^{\otimes 2} = M' \otimes_A B, \quad M \otimes_A B^{\otimes 3} = M' \otimes_A B^{\otimes 2}$$

$d^0 = m \otimes b \mapsto m \otimes 1 \otimes b - m \otimes b \otimes 1$ $\Rightarrow M \cong \ker d^0$.

If we only know M' is a B-mod, but don't know $M' = M \otimes_A B$, then d^0 can't be defined intrinsically!

That is, given $m \otimes b$, "$m \otimes 1$" = ?

Geometrically: $f: Y \to X$ faithfully flat. E.g. $G \curvearrowright Y$ free $X = Y/G$.

$\mathcal{M}$ a qcoh. $\mathcal{O}_X$-module on X. $\mathcal{M}' = f^* \mathcal{M}$ pull-back.

Try to recover $\mathcal{M}$ from $\mathcal{M}'$.

if $f(y_1) = f(y_2)$, we need to give an isom $\mathcal{M}'_{y_1} \xrightarrow{\sim} \mathcal{M}'_{y_2}$

Formally, $Y \times_X Y$ "=" $\{(y_1, y_2) \in Y \mid f(y_1) = f(y_2)\}$

$$\begin{array}{c} Y \times_X Y \\ p_1 \downarrow \downarrow p_2 \\ Y \end{array}$$

need to give an isom of $\mathcal{O}_{Y \times_X Y}$-modules

$$\varphi: p_1^* \mathcal{M}' \xrightarrow{\sim} p_2^* \mathcal{M}'$$

(stalks at (p_1,p_2): $\mathcal{M}'_{y_1} \xrightarrow{\sim} \mathcal{M}'_{y_2}$)

$$B \underset{p_2}{\overset{p_1}{\rightrightarrows}} B \otimes_A B$$

need to give a $B\otimes_A B$-mod isomorphism

$$\varphi: M' \otimes_{B,p_1} (B \otimes_A B) \xrightarrow{\sim} M' \otimes_{B,p_2} (B\otimes_A B)$$

$$\begin{array}{ccc} M' \otimes_{B,p_1} (B \otimes_A B) & \xrightarrow{\sim} & M' \otimes_{B,p_2} (B\otimes_A B) \\ \text{IS} & & \text{IS} \\ M' \otimes_A B & \xrightarrow{\sim} & M' \otimes_A B \end{array}$$

(different $B\otimes_A B$-module structures!)

$(b'\otimes b)(m'\otimes b_0) = b'm \otimes bb_0$ $\qquad$ $(b'\otimes b)(m'\otimes b_0) = bm'\otimes b'b_0$

If M' is obtained by $M\otimes_A B$, φ acts like

$$(m\otimes b)\otimes b' \longmapsto (m\otimes b')\otimes b$$

$\Rightarrow d^0(m') = \varphi(m'\otimes 1) - m'\otimes 1$, $m'\in M'$ is determined.

$M = \ker d^0$ can be recovered.

However, the isomorphism φ can't be given arbitrarily!

Note that

$$\begin{aligned} d^1: M\otimes_A B^{\otimes 2} &\longrightarrow M\otimes_A B^{\otimes 3} \\ m\otimes b'\otimes b &\longmapsto m\otimes 1\otimes b'\otimes b - m\otimes b'\otimes 1\otimes b \\ &\qquad + m\otimes b'\otimes b\otimes 1 \end{aligned}$$

Write $m' = m\otimes b' \rightsquigarrow d^1: M'\otimes_A B \longrightarrow M'\otimes_A B^{\otimes 2}$ $\quad$ d^1 can be determined by φ.

$$d^1(m'\otimes b) = \varphi(m'\otimes 1)\otimes b - m'\otimes 1\otimes b + m'\otimes b\otimes 1, \quad m'\in M'.$$

At least we require $d^1 d^0 = 0$!

Geometric interpretation. $f: Y\to X$ faithfully flat

$\mathcal{M}$ qcoh. $\mathcal{O}_X$-module. $\qquad \mathcal{M}' = f^*\mathcal{M}$.

For $y_1, y_2, y_3 \in Y$ with $f(y_1) = f(y_2) = f(y_3)$, the diagram of "glueing isomorphisms" $\mathcal{M}'_{y_2}$ commutes

$$\mathcal{M}'_{y_1} \xrightarrow{\sim} \mathcal{M}'_{y_3} \quad \rightsquigarrow$$

That is, in $Y \times_X Y \times_X Y$ "=" $\{(y_1,y_2,y_3) \in Y^3 \mid f(y_1)=f(y_2)=f(y_3)\}$:

$$\boxed{p_{13}^*(\varphi) = p_{23}^*(\varphi)\, p_{12}^*(\varphi)}$$

$$Y \times_X Y \times_X Y \overset{p_{13}}{\underset{p_{12}}{\rightrightarrows}} Y \times_X Y$$

$p_{13}: (y_1,y_2,y_3) \mapsto (y_1,y_3)$

$p_{23}: (y_1,y_2,y_3) \mapsto (y_2,y_3)$

$p_{12}: (y_1,y_2,y_3) \mapsto (y_1,y_2)$

$$\boxed{\varphi: p_1^*\mathcal{M}' \xrightarrow{\sim} p_2^*\mathcal{M}'}$$

Fact. if $\varphi: M' \underset{B,p_1}{\otimes} (B \underset{A}{\otimes} B) \xrightarrow{\sim} M' \underset{B,p_2}{\otimes} (B \underset{A}{\otimes} B)$
$B \underset{A}{\otimes} B$-module isomorphism

satisfies $\varphi \underset{p_{13}}{\otimes} B^{\otimes 3} = (\varphi \underset{p_{23}}{\otimes} B^{\otimes 3}) \circ (\varphi \underset{p_{12}}{\otimes} B^{\otimes 3})$,

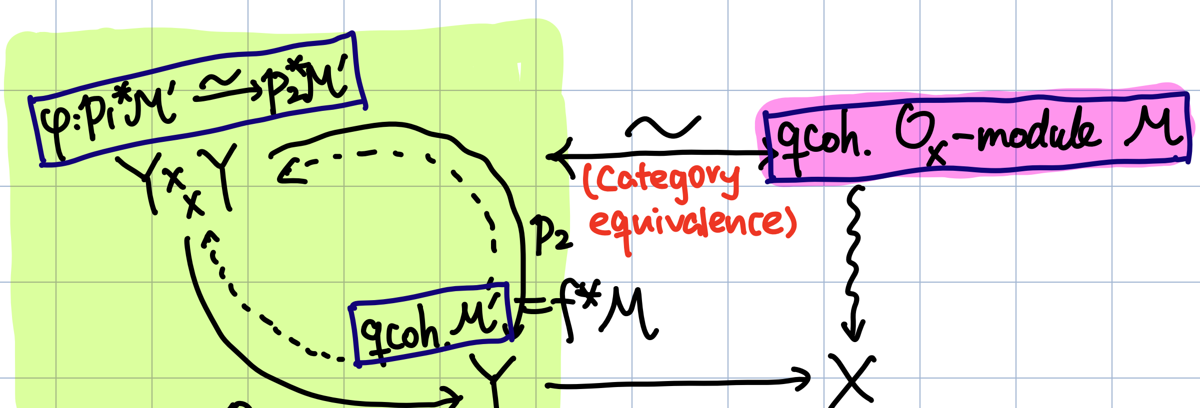
$M = \ker d^0 = \{m' \in M' \mid \varphi(m' \otimes 1) - m' \otimes 1 = 0\}$

then $M \underset{A}{\otimes} B \longrightarrow M'$ is an isomorphism.

$m \otimes b \longmapsto bm$ M can be recovered from (M', φ).

(Part I, Geometric Version I)

- Prop. $f: Y \to X$ faithfully flat & quasi-compact.



p_1

↳ s.t. $Y \times_X Y \times_X Y$ "=" $\{(y_1,y_2,y_3)\in Y^3 \mid f(y_1)=f(y_2)=f(y_3)\}$.

$$
\begin{tikzcd}
Y \times_X Y \times_X Y \arrow[d, bend right=60, "{p_{13}:(y_1,y_2,y_3)\mapsto(y_1,y_3)}"'] \arrow[d, "{p_{23}\ (y_1,y_2,y_3)\mapsto(y_2,y_3)}"] \arrow[d, bend left=60, "{p_{12}\ (y_1,y_2,y_3)\mapsto(y_1,y_2)}"] \\
Y \times_X Y
\end{tikzcd}
$$

$$p_{13}^*(\varphi) = p_{23}^*(\varphi)\, p_{12}^*(\varphi)$$

"$M'_{y_1} \xrightarrow{\sim} M'_{y_3}$" "$M'_{y_2} \xrightarrow{\sim} M'_{y_3}$" "$M'_{y_1} \xrightarrow{\sim} M'_{y_2}$"

(Part I, Geometric Version II)

- Thm. $f: Y \to X$ faithfully flat & quasi-compact.

$$
\begin{tikzcd}
p_2^* Z' & & & \\
p_1^* Z' \arrow[u, bend right, "\boxed{\simeq\varphi}"'] \arrow[r] \arrow[d] & Z' = Z\times_X Y \arrow[d, dotted] \arrow[rr, leftrightarrow, "\sim", "\text{(Category equivalence)}"'] & & Z \arrow[d, dotted, "\text{(affine morphism)}"] \\
Y \times_X Y \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & Y \arrow[rr] & & X
\end{tikzcd}
$$

$$p_{13}^*(\varphi) = p_{23}^*(\varphi)\, p_{12}^*(\varphi).$$

Fact. There's a category equivalence

$$\{Z \longrightarrow X \text{ affine}\} \xrightarrow{\sim} \{\text{qcoh. } \mathcal{O}_X\text{-algebra}\}$$

(Part II) $Y' = Y \times_X X' \longrightarrow Y$

$$\begin{tikzcd} {} \arrow[d, "f'"'] & {} \arrow[d, "f"] \\ X' \arrow[r] & X \end{tikzcd}$$

base change.

In what cases, ⌈$f': Y' \longrightarrow X'$ satisfies some properties (finite, finite-type, quasi-finite, flat, étale, etc.)⌋ $\Rightarrow$ ⌈$f: Y \to X$ satisfies the same properties⌋ ?

ANS. If $X' \longrightarrow X$ is faith.flat & quasi-compact $\Rightarrow$ ✓.