The Tilting Equivalence.

(References: Scholze's paper
Bhatt's notes
Matthew Morrow's notes.)

Def. R topological ring.
R is integral perfectoid if
$\exists$ non-zero-divisor π s.t.

- R has the π-adic topology and is complete and if.
- $p \in \pi^p R$
- Frob: $R/\pi R \to R/\pi^p R$

given by $a \mapsto a^p$ is iso.

e.g. $\widehat{\mathbb{Z}_p[p^{1/p^\infty}]}$.

Def. a complete Tate ring R is a perfectoid ring if R uniform, R° (power-bounded elements) is an integral perfectoid ring.

Remark In this case (page 15 of Morrow's notes).
have $R \cong R^\circ[\frac{1}{\pi}]$.

Remark

Perfectoid Field (as defined last time) is (by a result of

Kedlaya) equivalently a perfectoid ring that is a field.

Tilt of a perfectoid ring A.

$$:= A^\flat := \lim_{x \mapsto x^p} A.$$

<u>Claim:</u> there exists a pseudo-uniformizer $\pi^\flat$ for $A^\flat$ s.t. it exhibits $A^\flat$ as a perfectoid ring.

subclaim: up to multiplying by a unit, π (pu for A) admits p^n-th roots. (Lemma 1.3 of Morrow).

then take

$$\pi^\flat := (\pi, \pi^{1/p}, \pi^{1/p^2}, \dots).$$

addative structure of $A^\flat$:

$$x+y := \lim_{n\to\infty}\left(x^{1/p^n} + y^{1/p^n}\right)^{p^n}.$$

$$\lim_{x\mapsto x^p} R^0/\pi \xleftarrow{\sim} \lim_{x\mapsto x^p} R^0. \quad \text{(claims)}.$$

$$\left(\lim_{x\mapsto x^p} R^0\right)\left[\frac{1}{\pi^\flat}\right] \simeq \lim_{x\mapsto x^p} R.$$

Main Claim 1:

Fix A a perfectoid ring.

then $\bullet^\flat$ gives an equivalence of categories

$$\left(\begin{array}{c}\text{perfectoid rings}\\ /A\end{array}\right) \xrightarrow{\sim} \left(\begin{array}{c}\text{perfectoid}\\ \text{rings } /A^\flat\end{array}\right).$$

To retrieve Yifei,

need the fact that

A perfectoid field

$\widehat{\Downarrow}$

$A^\flat$ - - - - - - - .

How to do this? Two ways.

- writing down the inverse (Gabber-Romero)

• Almost mathematics
(Scholze).

Almost Math.

From now on, $A = K$ is a perfectoid field.

Idea: $K = \mathcal{O}_K[\frac{1}{\pi}]$.

so K-mod

$$\simeq \mathcal{O}_K\text{-mod}/S.$$

S is the Serre subcategory of modules where every element is annihilated by some power of π.

But one can take other choices of S e.g.

$$S' := \left\{ M \in \mathcal{O}_K\text{-mod (discrete)} \;\middle|\; \mathfrak{m} M = 0 \right\}$$

$\mathfrak{m}$:= maximal ideal

$\simeq$ { topologically nilpotent elements }

Claim: S' is a Serre subcategory.

(key fact : $\mathfrak{m} \simeq \mathfrak{m}^2$).

Def. $\mathcal{O}_K^a\text{-mod} := \mathcal{O}_K\text{-mod}/S'$.

$$\begin{tikzcd} \mathcal{O}_K\text{-mod} \arrow[r] & \mathcal{O}_K^a\text{-mod} \arrow[l, bend right, "(\ )_!"'] \arrow[l, bend left, "(\ )_*"] \arrow[d, dashed, shift left] \\ & K\text{-mod} \arrow[u, dashed, shift left] \end{tikzcd}$$

(Todo: ex where $(\ )_! = (\ )_*$).

Think of $\mathcal{O}_K^a$-mod as sitting between $\mathcal{O}_K$-mod and K-mod.

$\mathcal{O}_K^a$-mod is again a sym. mon. abelian category.

$$\mathcal{O}_K^a\text{-alg} := \mathrm{CAlg}(\mathcal{O}_K^a\text{-mod}).$$

and other concepts similarly.

General Paradigm:

1)

easy for char p

(almost) Deformation theory

- usual version over K
- almost version over $\mathcal{O}_K$
- almost version over $\mathcal{O}_K/\pi$

(Question: is there a function field analogue?)

2) $\mathcal{O}_K/\pi \simeq \mathcal{O}_{K^b}/\pi^b$.

(Hw: is $D(\mathcal{O}_K^a\text{-mod}) \simeq \mathcal{O}_K\text{-mod}/S$?)

no

For claim 1:

① Perfectoid algebra $/K$.

$\wr\!|$

② $\mathcal{O}_K^a\text{-PerfAlg} :=$

$\{$ $\mathcal{O}_K^a$-algebras A which

are π-adically complete,

flat,

i.e. Frob : $A/\pi^{1/p} \longrightarrow A/\pi$

is an iso $\}$

(flat: A $\mathcal{O}_K^a$-alg

M is an A-mod

($\mathcal{O}_K^a$-alg).

M is flat if

$\cdot \otimes_A M$ is exact.)

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③ $\mathcal{O}_K^a/\pi$-Perfalg =

$\{ \mathcal{O}_K^a/\pi$-alg $\bar{A}$,

flat, s.t.

Frob: $\bar{A}/\pi^{1/p} \simeq \bar{A} \}$.

② $\Leftrightarrow$ ③:

② $\to$ ③ is the obvious one.

③ $\to$ ②

$\mathcal{O}_K\text{-mod} \xrightarrow{j^*} \mathcal{O}_K^a\text{-mod}$ is

sym. mon.

$$\mathrm{CAlg}(\mathcal{O}_K\text{-mod}) \underset{j^*}{\overset{(j^*)^L =: ()_{!!}}{\longleftrightarrow}} \mathrm{CAlg}(\mathcal{O}_K^a\text{-mod})$$

Define $C_n =$

$\{ \mathcal{O}_K/\pi^n$-alg B,

flat, s.t.

relative Frob: $\mathcal{O}_K/\pi \to B/\pi$

is an iso $\}$.

Claim: by Deformation.

$$C_1 \simeq C_2 \simeq \cdots$$

$$C := \lim_n C_n$$

$\simeq \{ \mathcal{O}_K$-alg B s.t.

π-adically complete, flat

and s.t. rel. Frob $\mathcal{O}_K/\pi \to B$

is iso }

given $A \in \mathcal{O}_K^a$-PerfAlg.

(wlog) can assume

$$(A)_{!!} \in C_1 \simeq C.$$

so can take

$$j^*[(A)_{!!}]$$

Claim: this falls within $\mathcal{O}_K^a$-PerfAlg.

and provides the inverse.

Q: why is "almost math" useful?

$$K = \mathbb{Q}_p(p^{1/p^\infty}).$$

$$L = K(p^{1/2}).$$

L/K is not unramified

but is "almost" unramified.

$$K_n := \mathbb{Q}_p(p^{1/p^n})$$

$$L_n := K_n(p^{1/2})$$

$$\mathcal{O}_{L_n} \simeq \mathcal{O}_{K_n}(p^{1/2p^n}).$$

then relative different

$$\mathcal{D}_{L_n/K_n} \simeq \left(p^{1/2p^n}\right).$$

so becomes "less ramified" as $n \to \infty$.

A perfectoid ring

$$\{\text{perf Alg}/A\} \simeq \{\text{perfectoid Alg}/A^\flat\}.$$

Q: How does this interact with finite etale morphisms?

Main Claim: (Almost Purity)

R is perfectoid ring.

S finite étale R-alg.

then S is perfectoid.

Moreover, $(\ )^\flat$ induces an equivalence

$$\begin{pmatrix} \text{finite étale} \\ R\text{-alg} \end{pmatrix} \simeq \begin{pmatrix} \text{finite étale} \\ R^\flat\text{-alg} \end{pmatrix}.$$

Strategy:

$$\begin{pmatrix} \text{finite étale} \\ R \text{ alg} \end{pmatrix} \overset{\boxed{\star}}{\simeq} \begin{pmatrix} \text{almost finite} \\ \text{étale } R^\circ\text{-alg} \end{pmatrix}$$

$$\simeq \left(\text{a.f.é. } R^\circ/\pi\text{-alg}\right).$$

(*) let's check if

T is finite étale $R^\flat$-alg.

then $R^{\flat\circ} \longrightarrow T^\circ$ is

<u>almost unramified.</u>

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{ $A \to B$ of R°-alg is

almost unramified if

$\exists$ an element $e \in (B \otimes_A B)_*$.

s.t. $e^2 = e$,

$\mathrm{mult}(e) = 1$ and $\ker(\mathrm{mult}) \cdot e = 0$)

where $\mathrm{mult} : B \otimes_A B \longrightarrow B$ }.

$R^b \to T$ unramified

then $\exists$ $e \in T \otimes_{R^b} T$ idempotent

s.t. it generates $\ker(\text{mult})$

but $R^b \simeq R^{b\circ}[\frac{1}{\pi^b}]$.

$T \simeq T^\circ[\frac{1}{\pi^b}]$.

so $\pi^{bp^N} e \in T^\circ \otimes_{R^{b\circ}} T^\circ$.

$\exists N \gg 0$

but Frob is an iso on

T°, $R^{b\circ}$, and fixes e.

so

$$(\pi^b)^{1/p^m} e \simeq \mathrm{Frob}^{-N-m}\left(\pi^{b p^N} e\right)$$

$$\in T^\circ \otimes_{R^{b\circ}} T^\circ.$$

true for all $m \geq 0$.

$$\Rightarrow e \in \left(T^\circ \otimes_{R^{b\circ}} T^\circ\right)_*$$

is the desired element.

For general case, need filtering intervals nicely with the analytic topology

X is an perfectoid space

(i.e. an adic space locally
given by $\mathrm{Spa}(R, R^+)$
where R is a perfectoid ring).

then can define
$X^\flat$ locally as $\mathrm{Spa}(R^\flat, R^{+\flat})$.

(glue using tilting equivalence).

(Implicitly using: $\mathrm{Spa}(R, R^+)$
for R perfectoid is
sheafy).

$\exists$ tilting map

$b : X \longrightarrow X^b$

$x : R \longrightarrow \Gamma \cup \{0\}$

then $(x)^b(f) := x(f^\#)$

$f^\#$ is the untilt of f.

Claim: $(x)^b$ is a continuous valuation

and this sends rational subsets

to - - - - - .

<u>Prop</u>.

1) $b : X \longrightarrow X^b$ is a homeomorphism.

2) if $V \subseteq X$ is a rational

subset, $U = (V)^\flat$, then

- $\mathcal{O}_X(V)$ is perfectoid algebra.
- same for $\mathcal{O}_{X^\flat}(U)$
- $\exists$ an iso $\mathcal{O}_X(V)^\flat \simeq \mathcal{O}_{X^\flat}(U)$.