Henselian Ring

(One) Motivation: given a scheme X, and a point $x \in X$, the local ring at x.

$$\mathcal{O}_{X,x} = \varinjlim_{\substack{x \in U \to X \\ \text{zariski open}}} \mathcal{O}_X(U)$$

Question: when we are considering étale topology, what property should it have?

And we also expect it truly like some "topologically trivial disk".

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1. Definition of Henselian rings Idea: lift the property on fiber to the whole neighborhood

Def / Thm: Let (A, m) be a local ring. Denote $X = \operatorname{Spec} A$, $k = A/m$, $x = [m] \in X$. Also view it as a closed subscheme of X, $x: \operatorname{Spec} k(x) \hookrightarrow \operatorname{Spec} A$.

TFAE:

① Any finite A-algebra B is a direct product of local rings $\prod_{i=1}^{r} B_i$

①' For any finite morphism $\underset{\operatorname{Spec} B}{Y} \to X$, each connected component of Y has a unique closed point which maps to x

①'' For any finite morphism $\underset{\operatorname{Spec} B}{Y} \to X$, $\{$connected components of $Y\} \xleftrightarrow{1:1} \{$connected components of $Y_x\}$

② For any étale morphism $f: Y \to X$, (write $Y_x := \operatorname{Spec} k(x) \times_X Y$)
then any section of $f_x: Y_x \to \operatorname{Spec} k(x)$ is induced by a section of f.

this data consists of a point $y \in Y$ s.t. $k(y) \xleftarrow{\sim} k(x)$

Rmk: in fact this ② can be extended to all smooth $f: Y \to X$

important

(Recall: for finite étale & separated $f: Y \to X$,

$\{$sections of $f\} \xleftrightarrow{1:1} \{$connected component of Y mapping isomorphically to $X\}$, so sections are determined by a value at one point. So the term ② says

$$\{\text{sections of } f: Y \to X\} \xleftrightarrow{1:1} \{\text{sections of } f_x: Y_x \to \operatorname{Spec} k(x)\}$$)

③ For any $f(t) \in A[t]$, denote $\bar{f}(t) \in k[t]$ for its image in this quotient.

If $\bar{f}(t) = \bar{g}(t)\bar{h}(t)$ for some relatively prime monic polynomials $\bar{g}(t), \bar{h}(t)$

Then $\exists$ uniquely determined lifting $g(t), h(t) \in A[t]$ s.t. $f(t) = g(t)h(t)$, and $(g(t), h(t)) = A[t]$

③' $f_1, \dots, f_n \in A[T_1, \dots, T_n]$; if $\exists$ $a = (a_1, \dots, a_n) \in k^n$, s.t. $\bar{f_i}(a) = 0$ $i = 1, \dots, n$, and $\det(\frac{\partial f_i}{\partial T_j})(a) \neq 0$, then $\exists$ $b \in A^n$, s.t. $\bar{b} = a$ and $f_i(b) = 0$, $i = 1, \dots, n$

Some part of proof:

- ① $\Leftrightarrow$ ①' $\Leftrightarrow$ ①'' is direct.

(① $\Rightarrow$ ①' : $Y = \operatorname{Spec} B = \coprod_{i=1}^{r} \operatorname{Spec} B_i$. each $\operatorname{Spec} B_i$ is connected component and $\operatorname{Spec} B_i \to \operatorname{Spec} A$ finite $\Rightarrow$ closed map so the unique closed pt of $\operatorname{Spec} B_i$ will map to x

①' $\Rightarrow$ ①'' : For each connected component Y_i, ①' says $Y_i = \operatorname{Spec} B_i$, and $A \to B_i$ is a local ring homo, so $A/m \to B_i/mB_i$ is also local ring homo

①'' $\Rightarrow$ ① : For each connected component Y_i, ①'' says $Y_i = \operatorname{Spec} B_i$, and the finite k-algebra B_i/mB_i is connected, which can only be local ring (using the structure theorem of Artinian ring, [Atiyah-MacDonald, Thm 8.7])

So B_i is local ring.

- ① $\Rightarrow$ ②.

Unramified requires locally of finite type. After specialize Y at some nbhd of y, we assume Y affine & $f: Y \to X$ is finite type, then unramified implies quasi-finite since $Y \to X$ is separated, X quasicpt, Zariski main thm implies $Y \to X$ factors as $Y \xrightarrow[j]{\text{open immersion}} Y' \xrightarrow[\bar{f}]{\text{finite}} X$. ① implies $Y' \cong \coprod_i \operatorname{Spec} B_i$, where B_i are local, finite over A. y belongs to some $\operatorname{Spec} B_i$, then y must be the closed point in B_i (a version of going up? [Atiyah-MacDonald, Cor 5.9])

Then $\operatorname{Spec} B_i \subseteq Y$ since Y is open so contains all points specialize to y

We know $A \to B_i$ is finite étale, in particular finite flat $\Rightarrow$ free of finite rank

And $\underset{\substack{\| \\ k(x)}}{A/m} \to \underset{\substack{\| \\ k(y)}}{B_i/mB_i}$ has a section so $A/m \to B_i/mB_i$ is isomorphism

$\Rightarrow$ B_i is a free A-mod of rank 1, i.e. $A \xrightarrow{\sim} B_i$

Then $\operatorname{Spec} A \xrightarrow{\sim} \operatorname{Spec} B_i \hookrightarrow Y$ gives a section

- ② ⇒ ③' Follows from the structure thm of étale morphism, i.e. étale morphism locally look like $\operatorname{Spec}(A/f(t))_b$

- ③' ⇒ ③ a technical trick, see [Milne Thm 4.2 $(d')\Rightarrow(e)$]
  (I even didn't read ...)

- ③ ⇒ ①/①'/①'' idea: ③ assures lifting the good property of fiber to the whole neighborhood, and at fiber they are Artinian.

First prove B of the form $A[T]/f(T)$, then

$$\left(A[T]/f(T)\right)\otimes_A A/m = k[T]/\bar{f}(T) = \prod_i k[T]/\bar{f}_i(T) \quad \text{where } \bar{f}(T)=\prod \bar{f}_i(T)$$

$\bar{f}_i(T)$ power of irred poly
$\bar{f}_i, \bar{f}_j$ coprime

$\bar{f}(T)=\prod \bar{f}_i(T)$ lifts to $f(T)=\prod f_i(T)$,

$(f_i(T), f_j(T)) = A[T] \quad (i\neq j)$

By Chinese remainder ⇒ $A[T]/f(T) = \prod_i A[T]/f_i(T)$

Each $A[T]/f_i(T)$ is local because $\left(A[T]/f_i(T)\right)\otimes A/m$ is local

(Here we use going up, so any maximal ideal in $A[T]/f_i(T)$ lies over m)

Second, let B be an arbitrary finite A-algebra. Then B/mB is a finite $k=A/m$-alg, so is Artinian. By the structure thm of Artinian ring,

$B/mB \cong \prod_{i=1}^{r} \bar{B}_i$ of local ring. Let $\bar{e}_i = (0,\cdots,1,\cdots,0)$ be idempotents.

Take $b_i \in B$ st. $\bar{b}_i = \bar{e}_i$, and since B is finite over A ⇒ integral ⇒ ∃ monic polynomial $f_i(T) \in A[T]$ st. $f_i(b_i)=0$.

Consider $C_i = A[T]/f_i(T) \to B$, $\quad C_i/mC_i \to B/mB$
$T \longmapsto b_i$

idempotents of B/mB in the image of $C_i/mC_i \to B/mB$ easily lifts to C_i/mC_i
(by the structure of Artinian rings)

Then lift b_i to an idempotent in C_i/mC_i

Then according to first step, lift this idempotent to an idempotent in C_i

<u>Cor 1</u>: If A is Henselian, then so is any finite local A-algs B and any quotient ring A/I.

<u>Prop 1</u>: If A is Henselian, then we have equivalence

$$\{\text{cat of finite étale } A\text{-alg}\} \xrightarrow{\sim} \{\text{cat of finite étale } k\text{-alg}\}$$
$$B \longmapsto B \otimes_A k$$

Or more geometrically

$$\{\text{cat of finite étale } Y \to \underset{\substack{\| \\ \operatorname{Spec} X}}{\operatorname{Spec} A}\} \xrightarrow{\sim} \{\text{cat of finite étale } \overline{Y} \to \operatorname{Spec} k\}$$
$$Y \longmapsto Y_x \quad , x = m \in \operatorname{Spec} A$$

<u>Proof</u>: ① Apply Ref/Thm ①, only need to consider finite étale local A-algs / $\operatorname{Spec}(\cdots)$

② .. Need to show $\operatorname{Hom}_X(Y', Y) \xrightarrow{\sim} \operatorname{Hom}_x(Y'_x, Y_x)$

In fact, we can show a more general lemma. (useful; and can also generalize: B can be a colim of étale A-algs)

<u>Lem 1</u>: Let $A \to R$ be a ring map, <u>with (R, m) Henselian local ring</u>. Let B be an étale A algebra. Write $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $Z = \operatorname{Spec} R$, Let $x \in X$ st. $z = [m]$ lies over x; $Y_x = \operatorname{Spec} B \otimes_A k(x)$

Then $\operatorname{Hom}_X(Z, Y) \simeq \operatorname{Hom}_x(z, Y_x)$

In other word, give a prime P_y of B st. m lies over P_y, and a $k(x)$-alg map $k(y) \to k(z)$ we can uniquely determine a X-morphism $Z \to Y$, or A-algebra map $B \to R$

<u>Proof of Lem</u>: Notice that $\operatorname{Hom}_X(Z, Y) = \{\text{sections of projection } Z \times_X Y \xrightarrow{\text{étale}} Z\}$

$$\overset{\text{specialize at } z \in Z}{=} \{\text{sections of projection } z \times_X Y \to z\}$$
$$\overset{(z \times_X Y = z \times_x Y_x)}{=} \{\text{sections of projection } z \times_x Y_x \to z\}$$
$$\overset{(\text{Def/Thm ②})}{=} \operatorname{Hom}_x(z, Y_x)$$

Come back, in our case Y' is Henselian, and $Y'_x = y' Y_x = y$ since they are étale local ring over X

③. essentially surjective: any local finite étale k-alg K has the form

$K = k[T]/\bar{f}(T)$ $\quad$ $\bar{f}(T)$ monic irreducible separable

so $\bar{f}(T)' \neq 0$

Lift $\bar{f}(T)$ to a monic $f(T) \in A[T]$. Then $A[T]/f(T)$ is a finite local ring over A, and $f'(T)$ is not in the maximal ideal of $A[T]$

so is a unit in this local ring

$\Rightarrow$ $A[T]/(f(T))$ is finite étale over A. $\square$

In particular, if k is already separably closed, any finite étale A-algebra is copies of A

Def 2: (A, m) is strictly Henselian, if (A, m) is Henselian, and $k = A/m$ is separably closed.

Important example:

Prop 2: Any complete local ring is Henselian ($\Rightarrow$ Hensel's lemma in number theory)

Proof using ①: ([Fu, Prop 2.8.4, First method])

First Method: Let us verify that the condition 2.8.3 (i) holds. Let A be a finite R-algebra. Then A is complete with respect to the $\mathfrak{m}$-adic topology. Any maximal ideal of A lies over the maximal ideal $\mathfrak{m}$ of R. Since $A/\mathfrak{m}A$ is finite over $R/\mathfrak{m}$, it is artinian. It follows that A has finitely many maximal ideals. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ be all the maximal ideals of A. Then the nilpotent radical of $A/\mathfrak{m}A$ is $(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n)/\mathfrak{m}A$. So there exists a positive integer k such that

$$(\mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n)^k \subset \mathfrak{m}A \subset \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n.$$

Combined with the Chinese remainder theorem, we get

$$A \cong \varprojlim_i A/\mathfrak{m}^i A \cong \varprojlim_i A/\mathfrak{m}_1^i \cdots \mathfrak{m}_n^i \cong \varprojlim_i (A/\mathfrak{m}_1^i \times \cdots \times A/\mathfrak{m}_n^i) \cong \widehat{A}_{\mathfrak{m}_1} \times \cdots \times \widehat{A}_{\mathfrak{m}_n}$$

So A is a direct product of local rings.

(The notation is a little bit different)

Proof using ②: ([Milne, Prop 4.5]) $\quad \underset{\operatorname{Spec} B}{\overset{\parallel}{Y}} \to X = \operatorname{Spec} A$, $Y_x \to \operatorname{Spec} k(x)$ has a section

$\rightsquigarrow \operatorname{Spec} A/m \to Y_x \hookrightarrow Y$

Since $\operatorname{Hom}_A(B, A/m) = \operatorname{Hom}_A(B, A/m^2) = \operatorname{Hom}_A(B, A/m^3) \cdots\cdots$

$\Rightarrow \operatorname{Hom}_A(B, A/m) = \varprojlim_n \operatorname{Hom}_A(B, A/m^n) = \operatorname{Hom}_A(B, \varprojlim A) = \operatorname{Hom}_A(B, A)$

So $\operatorname{Hom}_A(B, A) \neq \phi$

Proof using ③': 牛顿迭代法.

Another fun fact: If X is analytic mfld over $\mathbb{C}$, then the local ring at a point x of X is Henselian.

Next goal is to answer the question at the beginning.

Let (A, m) be a local ring, $k = A/m$. $X = \operatorname{Spec} A$, $x = [m] \in X$

An étale neighborhood of (A, m) (or (X, x)) consists of the following data (B, q): B is an étale A-algebra, q is a prime ideal lying over m. We say this étale nbhd elementary (follow notation in [Stacks])

if $A \to B$ induces an isomophism of residue field at q, $k(x) \xrightarrow{\sim} k(q)$

Caution: (B, q) may not be

(1). Lem2: The set of elementary étale nbhds form a filtered system.

Proof: (i) Exists étale nbhd

(ii) $(\operatorname{Spec} B, q)$, $(\operatorname{Spec} B', q')$ are two elementary étale nbhds

Then consider $\operatorname{Spec} B \times_{\operatorname{Spec} A} \operatorname{Spec} B' = \operatorname{Spec} B \otimes_A B' =: B''$

and a prime $q'' := q \otimes B' + B \otimes q' \subseteq B \otimes_A B'$ gives an elementary étale nbhd (B'', q'') of A, and $(A, m) \to (B'', q'') \to (B, q)$, $(B'', q'') \to (B', q')$

(iii) Two morphisms between étale nbhds $(B, q) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (B', q')$

Let $\operatorname{Spec} B''$ be the connected component of $\operatorname{Spec} B'$ containing $[q']$

Then (B'', q') is an étale nbhd of (A, m), and the composition

$(B, q) \overset{\varphi}{\underset{\psi}{\rightrightarrows}} (B', q') \to (B'', q')$ must be equal according to [Milne, Cor 3.13]

(consider sections of projection $\operatorname{Spec} B'' \times_{\operatorname{Spec} A} \operatorname{Spec} B \to \operatorname{Spec} B''$) $\square$

Def3. Define $A^h := \operatorname{colim}_{(B, q)} B$ to be the filtered colimit along elementary étale nbhds

(It consists of elements (B,q,f), where $f\in B$, modulo equivalence that $(B,q,f)=(B',q',f')$ if $\exists$ $(B,q)\xrightarrow{\varphi}(B'',q'')$, $(B',q')\xrightarrow{\psi}(B'',q'')$ s.t. $\varphi(f)=\psi(f')$)

Prop3: (1) A^h is a local ring with maximal ideal mA^h, and $A/m \xrightarrow{\sim} A^h/m^h$

(2) A^h is Henselian, and for any other Henselian ring $(\tilde{A},\tilde{m})$, and

$$\mathrm{Hom}_{loc}(A^h,\tilde{A}) = \mathrm{Hom}_{loc}(A,\tilde{A})$$, which is the universal property of A^h

means local ring map

Proof: (1). Since B_q can also be written as filtered colimit $\operatorname{colim}_{b\in B\setminus q} B_b$

and (B_b, qB_q) are all elementary étale nbhds, we know

$A^h \cong \operatorname{colim}_{(B,q)} B_q$. Every maps $(B_q,q)\to(B'_{q'},q')$ are local.

so this is a filtered colimit of local rings, thus A^h is local ring

with maximal ideal $m^h := \operatorname{colim}_{(B,q)} qB_q$ (Reason: $A^h - m^h = \operatorname{colim}(B_q - qB_q)$ consists of units)

Since $A\to B$ is étale $\Rightarrow$ $qB_q = mB_q$ $\Rightarrow$ $m^h = \operatorname{colim}_{(B,q)} mB_q = mA^h$

And $A^h/m^h = \operatorname{colim}_{(B,q)} B_q/qB_q = k = A/m$

(2).

- We first show (A^h, m^h) is Henselian. We want to verify Def/Prop ②.

Let $Y\to \operatorname{Spec} A^h$ be an étale morphism with a section at the fibre of $[m^h]$

Let $y\in Y$ be the image of this section; we also know $k(y)=k$. Shrink Y we assume Y is of standard form, which means $Y=\operatorname{Spec} R_b$, where $R=A^h[T]/(f(T))$, and $b=\overline{b(T)}\in R$

s.t. $f'(T)$ is unit in R_b. We can find some elementary étale nbhd (B,q)

s.t. the coefficients of $f(T)$ and $b(T)$ and all comes from B.

Then we can define $R' := B[T]/(f(T))$, $b=\overline{b(T)}\in R'$, and $Y' := \operatorname{Spec} R'_b$

$\varphi_{y'} := m R'_b$, $y' = [\varphi_{y'}]$

So $\operatorname{Spec} A^h \times_{\operatorname{Spec} B} Y' = Y$, $k(y') = R'_b / mR'_b = R_b / mR_b = k$

Since Y' is étale over $\operatorname{Spec} B$, so is étale over $\operatorname{Spec} A$.

So (Y', y') is also an elementary étale nbhd of $(\operatorname{Spec} A, [m])$

Then we have a map $(\operatorname{Spec} A^h, [m^h]) \to (Y', y')$. The graph of this map is is a section of $Y \to \operatorname{Spec} A^h$

- Second, $\operatorname{Hom}_{loc}(A^h, \tilde{A}) = \operatorname{Hom}_{loc}(\operatorname{colim}_{(B,q)} B_q, \tilde{A}) = \lim_{(B,q)} \operatorname{Hom}_{loc}(B_q, \tilde{A})$ (*)

  $\operatorname{Hom}_{loc}(B_q, \tilde{A})$ consists of ring map $B \xrightarrow{\varphi} \tilde{A}$ st. $\varphi^{-1}(\tilde{m}) = q$

  If we give a local homo $A \to A^h$, according to Lem 1, it uniquely determine a map $B \to \tilde{A}$ st.
$$\begin{tikzcd} B \arrow[r] & \tilde{A} \\ & A \arrow[ul] \arrow[u] \end{tikzcd}$$ ✓

  Combine the equation (*), we get a unique element in $\operatorname{Hom}_{loc}(A^h, \tilde{A})$ $\square$

We call (A^h, m^h) the henselization of (A, m).

For any scheme X, $x \in X$ Similar as before, define the cat of (elementary) étale nbhd (U, u) at x whose opposite cat also forms a filtered system, We can also form $\operatorname{colim}_{(U,u)} \mathcal{O}_X(U)$, which can be viewed as stalk of structure sheaf at x under étale topology.

<u>Claim</u>: $\operatorname{colim}_{(U,u)} \mathcal{O}(U) = \mathcal{O}^h_{X,x}$.

The proof is not very hard, since the localization for a ring can also be constructed by colimit. For more details, see [Stacks, Tag 05KS]

There is also a similar notion called strict Henselization.

In this situation, we are considering geometric point: For a scheme X, a geometric point is a morphism $x: \operatorname{Spec} K \to X$ for some separably closed field K.

Now let (A, m) be a local ring with notations as before, i.e. $x = [m] \in X = \operatorname{Spec} A$, $k = A/m$

Fix a separably closed field $k^{sep} \supseteq k$. This sentense means we fix a geometric point $\operatorname{Spec} k^{sep} \to X$ with image x. We also denote this geometric point $\bar{x}$.

We can define the notion of étale nbhd for a geometric point. This notion consists of data (B, q, α), st B is étale A-algebra, q lies over m, α is a field map $\alpha: k(q) \to k^{sep}$ st.

$$\begin{tikzcd} k(q) \arrow[r, "\alpha"] & k^{sep} \\ & k \arrow[ul] \arrow[u] \end{tikzcd}$$

commutes

Similar as before, one can show the category of étale nbhd at a geometric point form a filtered system, and we can also define

$$A^{sh} := \operatorname*{colim}_{(B, q, \alpha)} B$$

We can similarly prove

(1) A^{sh} is a local ring with maximal ideal $m^{sh} := mA^{sh}$, and a natural geometric point $\bar{x}^{sh}: \operatorname{Spec} k^{sep} \to \operatorname{Spec} A^{sh}$ (in fact, $k(m^{sh}) \simeq k^{sep}$)

We have Cartesian diagrams

$$\begin{tikzcd} k \arrow[r] & k \arrow[r] & k^{sep} \\ A \arrow[r] \arrow[u] & A^h \arrow[r] \arrow[u] & A^{sh} \arrow[u] \end{tikzcd}$$

(2) A^{sh} is strict Henselian and for any other strict Henselian ring $(\tilde{A}, \tilde{m})$

$$\operatorname{Hom}_{\text{loc w/ geom pt}}(A^{sh}, \tilde{A}) = \operatorname{Hom}_{\text{loc w/ geom pt}}(A, \tilde{A})$$, which is the universal property of A^{sh}

→ this means local ring homo, st. compatible with their natural geometric points.

We will call (A^{sh}, m^{sh}) the strict Henselization of (A, m).

Another construction of A^{sh} using A^h: According to Prop 1, to any finite separable extension $k \subseteq k' (\subseteq k^{sep})$, we can construct a finite étale A^h-algebra $A^h(k')$ s.t. $A^h(k')/\mathfrak{m}A^h(k') = k'$. We can construct A^{sh} using $\bigcup_{k \subseteq k' \subseteq k^{sep}} A^h(k')$

For any scheme X, $x : \operatorname{Spec} K \to X$ is a geometric point. Similar as before, define the cat of étale nbhd (U, u, α), where $U \to X$ is étale, $u \in U$,
$$\begin{tikzcd} & \operatorname{Spec} K \arrow[dl, "\alpha"'] \arrow[d, "x"] \\ U \arrow[r] & X \end{tikzcd}$$
, image of α is u.

The opposite cat also forms a filtered system, We can also form $\operatorname{colim}_{(U,u,\alpha)} \mathcal{O}_X(U)$, which can be viewed as stalk of structure sheaf at the geom point x under étale topology.

Use similar argument as before, we have $\operatorname{colim}_{(U,u,\alpha)} \mathcal{O}(U) = \mathcal{O}^{sh}_{X,x}$

Some properties of (strict) Henselization.

- Functorality ([Stacks, tag 0BSK])

[Stacks, tag 07QL]

- $A \to A^h \to A^{sh}$ are faithfully flat ring maps ;
- $A \to A^h$, $A^h \to A^{sh}$, $A \to A^{sh}$ are formally étale

(since they are colim of étale A-algs)

- A is Noetherian $\Leftrightarrow$ A^h is Noetherian $\Leftrightarrow$ A^{sh} is Noetherian

If in this case we have : $\hat{A} \simeq \widehat{(A^h)}$, $\hat{A} \to \widehat{(A^{sh})}$ is formally smooth

$(\hat{A})^{sh} = \hat{A} \otimes_{A^h} A^{sh}$, $\widehat{((\hat{A})^{sh})} = \widehat{(A^{sh})}$ ....

- A is reduced $\Leftrightarrow$ A^h is reduced $\Leftrightarrow$ A^{sh} is reduced.

(normal)	(normal)	(normal	
(Cohen-Macaulay)	(Cohen-Macaulay)	(Cohen-Macaulay)	for Noetherian local A
(regular)	(regular)	(regular)	for Noetherian local A
(DVR)	(DVR)	(DVR)	for Noetherian local A

- $\dim(A) = \dim(A^h) = \dim(A^{sh})$

$\operatorname{depth}(A) = \operatorname{depth}(A^h) = \operatorname{depth}(A^{sh})$ for Noetherian local A

Example:

Examples 4.10. (a) Let A be normal; let K be the field of fractions of A, and let K_s be a separable closure of K. The Galois group G of K_s over K acts on the integral closure B of A in K_s. Let $\mathfrak{n}$ be a maximal ideal of B lying over $\mathfrak{m}$, and let $D \subset G$ be the decomposition group of $\mathfrak{n}$, that is, $D = \{\sigma \in G \mid \sigma(\mathfrak{n}) = \mathfrak{n}\}$. Let A^{h} be the localization at $\mathfrak{n}^D$ of the integral closure B^D of A in K_s^D. (Here

$$B^D = \{b \in B \mid \sigma(b) = b \text{ all } \sigma \in D\}$$

etc.) I claim that A^{h} is the Henselization of A.

Indeed, if A^{h} were not Henselian, there would exist a monic polynomial $f(T)$ that is irreducible over A^{h} but whose reduction $\bar{f}(T)$ factors into relatively prime factors. But from such an f one can construct a finite Galois extension L of K_s^D such that the integral closure A' of A^{h} in L is not local. This is a contradiction since the Galois group of L over K_s^D permutes the prime ideals of A' lying over $\mathfrak{n}^D$ and hence cannot be a quotient of D. To see that A^{h} is the Henselization, one only has to show that it is a union of étale neighborhoods of A, but this is easy using (3.21).