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2025.10.19 Sunday 21:09:42

Lem. 2.3.19. $f : X \to Y$ is finite, then f_* preserves constructibility. And if $\mathcal{F} \in Sh_c(X)$, then $\dim\operatorname{supp} f_*\mathcal{F} = \dim\operatorname{supp}\mathcal{F}$ wrt. mdsupp.

Pf. $X \to f(X) \hookrightarrow Y$ WLOG assume f surj.

$\exists$ smooth open connected $V \subset Y$ s.t. $f : f^{-1}(V) \to V$ is finite and etale. Then $\exists$ open stratum $U \subset f^{-1}(V)$ and $\dim U = \dim f^{-1}(V)$.

$X_s \leq X_t$ iff $X_S \subset \bar{X}_t$. V is irred. $\dim V = \dim f^{-1}(V) = \dim U > \dim(\bar{U}\backslash U)$. so $f(\bar{U}\backslash U) \cap V \subsetneq V$. $V' := V\backslash f(\bar{U}\backslash U)$, then V' is conn. $U' := U \cap f^{-1}(V') = \bar{U} \cap f^{-1}(V')$.

So U' is both open and closed in $f^{-1}(V)$. $f|_{U'} : U' \to V'$, hence surj. fininte etale, hence is covering map.

$\mathcal{L} := \mathcal{F}|_{U'}$ local system of finite type. $(f|_{U'})_*\mathcal{L} \in Loc^{ft}(V')$ $\operatorname{grade}((f|_{U'})_*\mathcal{L}) = \operatorname{grade}(\mathcal{L})$.

$j : U' \hookrightarrow X, h : V' \hookrightarrow Y, i : Z := X\backslash U \hookrightarrow X$.

$0 \to j_! j^*\mathcal{F} \to \mathcal{F} \to i_* i^*\mathcal{F} \to 0$,

$0 \to f_* j_! j^*\mathcal{F} \to f_*\mathcal{F} \to (f \circ i)_* i^*\mathcal{F} \to 0$.

$$\begin{tikzcd} U' \arrow[r, "j"] \arrow[d, "f|_{U'}"'] & X \arrow[d, "f"] \\ V' \arrow[r, "h"] & Y \end{tikzcd}$$

$f_* j_!\mathcal{L} = h_!(f|_{U'})_*\mathcal{L}$.

$\dim\operatorname{supp}(f \circ i)_* i^*\mathcal{F} = \dim\operatorname{supp} i^*\mathcal{F} = \dim\operatorname{supp} i_* i^*\mathcal{F}$.

$\dim\operatorname{supp} h_!(f|_{U'})_*\mathcal{L} = \dim\operatorname{supp}(f|_{U'})_*\mathcal{L} = \dim\operatorname{supp}\mathcal{L}$.

Good stratification defn. omitted. ref. Achar.

Lemma 2.3.22: $(X_s)_{s\in\mathscr{S}}$ is good strat, $Y \subset X$ locally closed, Y is union of strata $h : Y \hookrightarrow X$, then $h^*, h_*, h_!, h^!$ preserves constructibility.

Pf. $h^*, h_!$ triv. for embedding.

For h_*, induction on number of strata on Y. If $Y = X_s$, then for $\mathcal{F} \in D^b_{loc.f.}(X_s)$. $\tau^{\leq n}$ truncation functor, $\tau^{\leq n}\mathcal{F} \to \mathcal{F} \to \tau^{\geq n+1}\mathcal{F} \xrightarrow{+1}$.

Only need to prove that Y have one strata. So we can assume $\mathcal{F}$ is a sheaf.

Generally, $\exists$ closed stratum $X_s \subset Y$, let $Y' = Y\backslash X_s$, $j : Y' \hookrightarrow Y$, $i : X_s \hookrightarrow Y$, $h' := h \circ j : Y' \hookrightarrow X$

$i_* i^!\mathcal{F} \to \mathcal{F} \to j_* j^*\mathcal{F} \xrightarrow{+1} \cdot$.

And $j_* j^*\mathcal{F} = (h'_* j^*\mathcal{F})|_Y$, lies in $D^b_{\mathscr{S}}(Y)$ by induction hyp.

Hence $i_* i^!\mathcal{F} \in D^b_{\mathscr{S}}(Y)$. But $i^!\mathcal{F} \simeq i^*(i_* i^!\mathcal{F}) \in D^b_{\mathscr{S}}(Y) = D^b_{locf}(X_s)$.

$h_* i_* i^!\mathcal{F} \to h_*\mathcal{F} \to h'_* j^*\mathcal{F} \xrightarrow{+1}$.

$h_* i_* = (j_s)_*$ hence $h_* i_* i^!\mathcal{F} \in D^b_{\mathscr{S}}(Y)$, and by induction hyp. $h'_* j^*\mathcal{F} \in D^b_{\mathscr{S}}(Y)$, hence for h_*.

For $h^!$, WLOG assume Y is a closed embedding in X, $j : X\backslash Y \hookrightarrow X$, $h_* h^!\mathcal{F} \to \mathcal{F} \to j_* j^*\mathcal{F} \xrightarrow{+1}$. ($\mathcal{F} \in D^b_{\mathscr{S}}(X)$), by proven, $j_* j^*\mathcal{F} \in D^b_{\mathscr{S}}(X)$. Hence $h_* h^!\mathcal{F} \in D^b_{\mathscr{S}}(X)$, so $h^!\mathcal{F} \simeq h^* h_* h^!\mathcal{F} \in D^b_{\mathscr{S}}(X)$.

Starts from 1.3

$h : Y \hookrightarrow X$ locally closed.

h^* always exact. $(h^*\mathcal{F})_x = \mathcal{F}_x$.

$({}^0 h_*\mathcal{F})(U) = \mathcal{F}(h^{-1}(U))$ is left exact.

$({}^0 h_!\mathcal{F})(U) = \{s \in \mathcal{F}(h^{-1}(U)) | h|_{\operatorname{supp} s} \text{ is proper}\} = \{s \in \mathcal{F}(h^{-1}(U)) | h(\operatorname{supp} s) \text{ is closed in } U\}$

$({}^0h_!\mathcal{F})_x = \begin{cases} \mathcal{F}_x & x \in Y \\ 0 & x \notin U \end{cases}$ extend by zero.
${}^0h_!$ exact functor $\implies h_! = {}^0 h_! : D(Y) \to D(X)$.
Eg. skyscraper sheaf: $\mathcal{F} = h_!(\underline{\Bbbk}_Y)$.
$Y \subset X$ closed $\implies h_! = h_*$, ${}^0h^!\mathcal{F}(U) = \lim_{V \cap \bar{Y} = U}\{s \in \mathcal{F}(V) | \operatorname{supp} s \subset U\}$ restriction with support.
We have adj. pair $(h_!, h^!)$.
$Y \subset X$ open, $h^! = h^*$.
$h^!\mathcal{F} \hookrightarrow h^*\mathcal{F}$
Prop. 1.3.9. $\mathcal{F} \xrightarrow{\sim} h^!h_!\mathcal{F} \xrightarrow{\sim} h^*h_!\mathcal{F}$, $h^!h_*\mathcal{F} \xrightarrow{\sim} h^*h_*\mathcal{F} \to \mathcal{F}$.
Thm. 1.3.10. $Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U$, Z closed, U open.

$$\begin{tikzcd} D^+(Z) \arrow[r, "i_!=i_*"] & D^+(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, leftrightarrow, "j^*=j^!"] & D^+(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

(1) $j^*i_! = j^!i_! = j^*j_* = j^!i_* = 0$,
(2) $i^!j_! = i^*j_! = i^!j_* = 0$,
(3) dist. tri.

$$j_!j^!\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \xrightarrow{+1},$$

(4) dist. tri.

$$i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F} \xrightarrow{+1},$$

Pf. (1) and (2) check by stalk, for (3), with $\mathcal{F}$ sheaf,

$$0 \to j_!j^*\mathcal{F} \to \mathcal{F} \to i_!i^*\mathcal{F} \to 0.$$

$x \in U$,

$$0 \to \mathcal{F}_x \to \mathcal{F}_x \to 0 \to 0,$$

$x \in Z$,

$$0 \to 0 \to \mathcal{F}_x \to \mathcal{F}_x \to 0.$$

For (4), $\mathcal{F}$ sheaf,

$$0 \to i_!i^!\mathcal{F} \to \mathcal{F} \to j_*j^*\mathcal{F}$$

Eg. $X = \mathbb{R}, U = (-\infty, 0), Z = [0, \infty)$.
(3): $0 \to \underline{\Bbbk}_U \to \underline{\Bbbk}_X \to \underline{\Bbbk}_Z \to 0$,
(4): $0 \to \underline{\Bbbk}_{\mathring{Z}} \to \underline{\Bbbk}_X \to \underline{\Bbbk}_{\bar{U}} \to 0$.
Eg. $X = [0,1], U = (0,1), Z = \{0,1\}$,
${}^0j_*\underline{\Bbbk}_U = \underline{\Bbbk}_X$, by (4), $i^!\underline{\Bbbk}_X = 0$.
Eg. $X = \mathbb{R}, U = \mathbb{R}\backslash\{0\}, Z = \{0\}$,
$({}^0i^!\underline{\Bbbk}_X)_0 = \Bbbk^2$, by (4),

$$0 \to 0 \to \Bbbk \to \Bbbk^2$$

$$\Bbbk \to 0 \to \cdots$$

Hence $i^!\underline{\Bbbk}_X = \underline{\Bbbk}_Z[-1]$.
$\mathcal{F} \in D^b(X)$ complex, $\mathcal{H}^n(X, \mathcal{F}) \in \mathrm{Sh}(X)$ cohomology sheaf.
$D^b(X) \xrightarrow{R\Gamma} D^b(\mathrm{pt}) = D^b(\Bbbk\text{-mod})$.
$\mathbb{H}^n(X, \mathcal{F}) = H^n(a_*\mathcal{F}) = R^n\Gamma(X, \mathcal{F})$.
Lemma. $(\mathcal{H}^n\mathcal{F})_x = \lim_{U \ni x} \mathbb{H}^n(U, \mathcal{F}|_U)$.
Pf. $(\mathcal{H}^n\mathcal{F})_x = H^n(\mathcal{F}_x) = H^n(\lim_{U\ni x} \mathcal{F}(U)) = \lim_{U \ni x} H^n(\mathcal{F}(U)) = \lim_{U\ni x} \mathbb{H}^n(U, \mathcal{F}|_U)$.

$X \in \mathbb{C}, U = \mathbb{C}^\times, Z = \mathrm{pt}$,

$j_* \underline{\Bbbk}_U$:

$(\mathcal{H}^n(j_* \underline{\Bbbk}_U))_0 = \lim_{V \ni 0} \mathbb{H}^n(V, j_* \underline{\Bbbk}_U|_V) = \lim_{V \ni 0} \mathbb{H}^n(j^{-1}(V), \underline{\Bbbk}) = \begin{cases} \Bbbk^2 & n = 0 \\ 0 & n = 0 \end{cases}.$