The Fargues-Fontaine curve (relative to a perfectoid field.)

- $\mathbb{Q}_p$ will be replaced by X_C.

  where C is a perfectoid field.
- X_C is regular, Noetherian scheme of Krull dimension 1.

  $X_C \to \operatorname{Spec} \mathbb{Q}_p$.
- For any finite extension $\mathbb{Q}_p \subseteq E$.

  $X_{C,E} := X_C \times_{\operatorname{Spec}\mathbb{Q}_p} \operatorname{Spec} E$.

$$\left\{ \begin{array}{c} \text{finite étale} \\ \text{covers of } \operatorname{Spec}\mathbb{Q}_p \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{finite étale} \\ \text{covers of } X_C \end{array} \right\}.$$

Tilting.

- A : $\mathbb{Z}_p$-algebra $\longmapsto A^\flat := \varprojlim (A/p)$ perfect $\mathbb{F}_p$-algebra
- B : $\mathbb{F}_p$-algebra $\longmapsto W(B) := \varprojlim W_n(B)$ p^{th} root := shift to the right.

  p-adic. complete $\mathbb{Z}_p$-algebra.

Lem. They form an adjunction

$$W : \left\{ \begin{array}{c} \text{perfect} \\ \mathbb{F}_p\text{-algebras} \end{array} \right\} \rightleftarrows \left\{ \begin{array}{c} p\text{-adic complete} \\ \mathbb{Z}_p\text{-algebras} \end{array} \right\} : (\cdot)^\flat$$

Proof. A : perfect $\mathbb{F}_p$-alg.

B : p-adic complete $\mathbb{Z}_p$-algebra.

$$W(A) \to B \overset{(*)}{\Longrightarrow} A \to B/p \iff A \to \varprojlim_{x \mapsto x^p} (B/p)$$

Need to prove: any map like this has a unique lift.

$$\begin{tikzcd} W(A) \arrow[r, dashed] \arrow[d, "\mathrm{mod}\ p"'] & B \arrow[d] \\ A \arrow[r] & B/p \end{tikzcd}$$

Since B is p-adic complete, suffices to prove

$$\begin{tikzcd} W_n(A) \arrow[r, dashed] \arrow[d] & B/p^n \arrow[d] \\ A \arrow[r] & B/p \end{tikzcd}$$

$$\iff \begin{tikzcd} \mathrm{Spec}(B/p) \arrow[r] \arrow[d] & \mathrm{Spec}\, W_n(A) \arrow[d] \\ \mathrm{Spec}(B/p^n) \arrow[r] \arrow[ur, dashed] & \mathrm{Spec}(\mathbb{Z}/p^n) \end{tikzcd} \impliedby \mathbb{L}_{W_n(A)/(\mathbb{Z}/p^n)} \cong 0$$

- Reduce to $\mathbb{L}_{A/\mathbb{F}_p} = 0$ (i.e. $n=1$)

A perfect.

$$\begin{tikzcd} \operatorname{Spec} A \arrow[rr, "\varphi", "\sim"'] \arrow[dr] & & \operatorname{Spec} A \arrow[dl] \\ & \operatorname{Spec} \mathbb{F}_p & \end{tikzcd}$$

$$\varphi^* \Omega_{A/\mathbb{F}_p} \xrightarrow{\sim} \Omega_{A/\mathbb{F}_p}.$$

$$\varphi^* da = da^p = 0. \qquad \square$$

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unit : $A \xrightarrow{\sim} W(A)^\flat$

counit : B p-adic. complete $\mathbb{Z}_p$-algebra.

$$\begin{tikzcd} W(B^\flat) \arrow[r, "\theta"] \arrow[d, "\bmod p"'] & B \arrow[d] \\ B^\flat \arrow[r, "ev_0"'] & B/p \end{tikzcd}$$

x^{1/p^n}

$\mapsto$ x^t

$x^{1/p^n} \longmapsto$

~~But~~ also

$$\begin{tikzcd} B^\flat \arrow[r, equal] \arrow[d, "{[\cdot]}"'] & B^\flat \arrow[d, "(\cdot)^\#"] \\ W(B^\flat) \arrow[r] & B \end{tikzcd}$$

diagram of multiplicative monoids.

<u>Recall</u>. B p-adic complete.

$$\begin{tikzcd} \varprojlim\limits_{x\mapsto x^p} B \arrow[r, "\sim"] \arrow[d, "ev_0"'] & \varprojlim\limits_{x\mapsto x^p} B/p \arrow[dl, "(-)^\#"] \\ B & \end{tikzcd}
\qquad
\begin{tikzcd} \cdots \arrow[r] & B \arrow[r] \arrow[d] & B \arrow[d] \\ \cdots \arrow[r] & B/p \arrow[r] & B/p \\ & x_1 \arrow[u, phantom, "\in" description] & x_0 \arrow[u, phantom, "\in" description] \end{tikzcd}$$

$$(x)^\# = \lim_{n\to\infty} \begin{pmatrix} \text{arbitrary} \\ \text{lift of } x_n \\ \text{to } B \end{pmatrix}^{p^n}$$

$$= \lim_{n\to\infty} \begin{pmatrix} \text{arbitrary lift of the } 0^{th} \text{ component} \\ \text{of } x^{1/p^n} \text{ to } B \end{pmatrix}^{p^n}$$

$$= \theta\Bigg(\underbrace{\lim_{n\to\infty} \begin{pmatrix} \text{arbitrary lift of } x^{1/p^n} \\ \text{to } W(B^\flat) \end{pmatrix}^{p^n}}_{\begin{subarray}{c} \shortparallel \\ [x] \end{subarray}} \Bigg).$$

Any $f \in W(B^\flat)$ can be represented

by $f = \sum_{n\geq 0} [c_n] p^n \xmapsto[\theta]{} \sum_{n\geq 0} c_n^\# p^n$.

$c_n \in B^\flat$.

Perfectoid fields.

Def. A topological field K is <u>perfectoid</u> if its topology is induced from a complete non-arch. absolute value $|\cdot|_K : K \to \mathbb{R}^{\geq 0}$, satisfying

1) $|p|_K < |\varpi|_K < 1$ for some $\varpi \in K$.
2) Writing $\mathcal{O}_K := \{ x \in K \mid |x|_K \leq 1 \}$, then

$$\mathcal{O}_K/p \xrightarrow{(\cdot)^p} \mathcal{O}_K/p \text{ is surjective.}$$

<u>Notation</u>. - Let $K^\flat$ denote the fraction field of $\mathcal{O}_K^\flat$ (domain.)
call it the tilt of K.

- $K^\flat$ inherits an absolute value by

$$|x|_{K^\flat} = |x^\#|_K \text{ for } x \in \mathcal{O}_K^\flat.$$

(also, $K^\flat := \lim\limits_{x \mapsto x^p} K$ as a multiplicative monoid.)

Prop 1 K perfectoid $\Rightarrow K^\flat$ is perfectoid with respect to $|\cdot|_{K^\flat}$.

Lem. $|\cdot|_{K^\flat}$ and $|\cdot|_K$ have the same image in $\mathbb{R}^{\geq 0}$.

Example. For $|p|_K < |x|_K < 1$,

Let $y \in \mathcal{O}_K^\flat$ be any element with

$$y^\# \in x + p\mathcal{O}_K.$$

Then $$|y|_{K^\flat} = |x|_K.$$

- In particular, one can find $\varpi \in K^\flat$ with

$$0 < |\varpi| < 1. \text{ (pseudo-unif. of } K^\flat.)$$

Prop 2. K perfectoid. Then

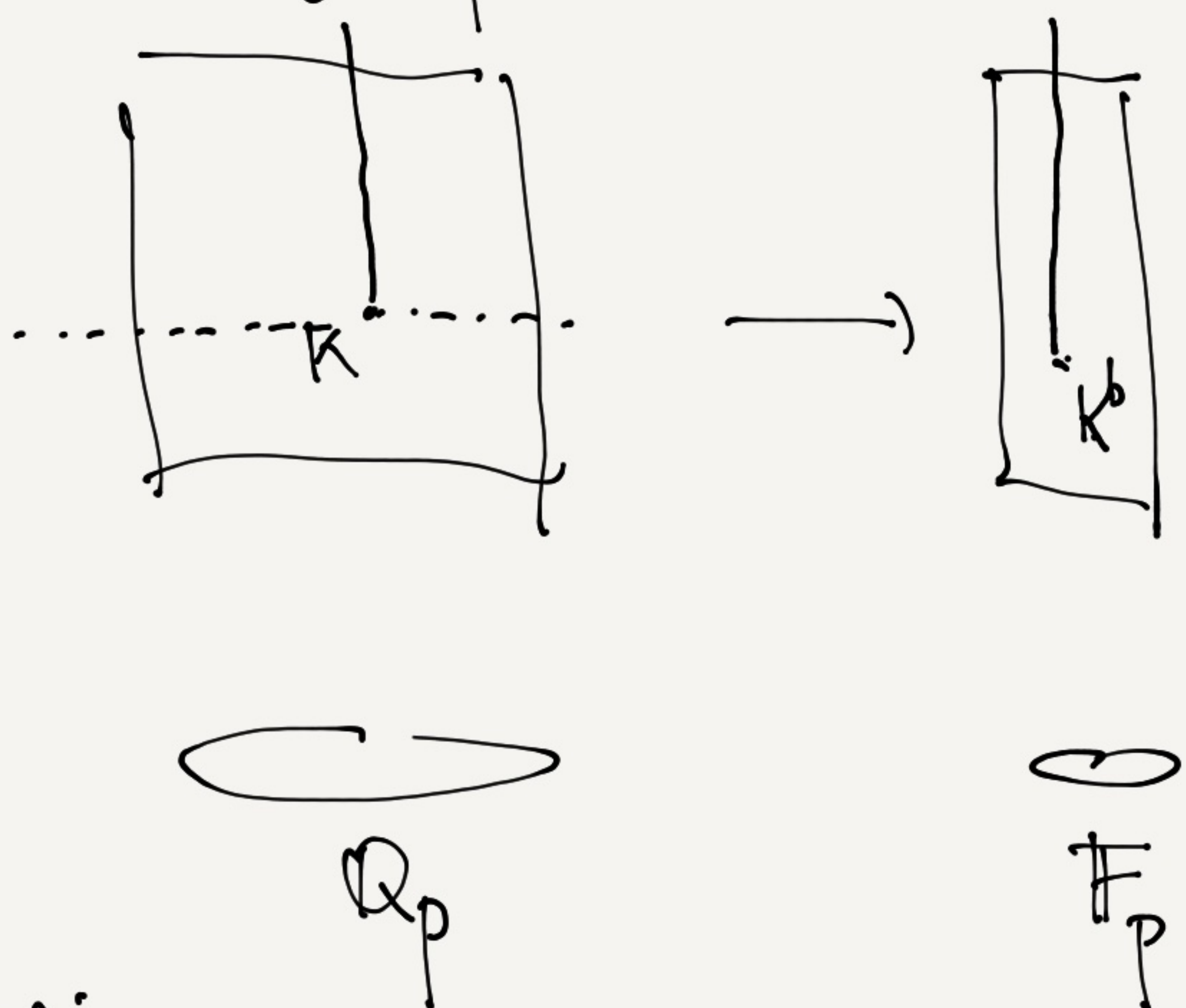
$$\left\{ \begin{array}{c} \text{perfectoid fields} \\ \text{containing } K \end{array} \right\} \xrightarrow[\sim]{(\cdot)^\flat} \left\{ \begin{array}{c} \text{perfectoid fields} \\ \text{containing } K^\flat \end{array} \right\}.$$

perfectoid spaces containing $K \xrightarrow{\sim}$ perfectoid spaces containing $K^\flat$.

Proof. Adjunction tells you that the inverse must send $C \supseteq K^\flat$ to the frac. field of

$$W(\mathcal{O}_C) \otimes_{W(\mathcal{O}_K^\flat)} \mathcal{O}_K.$$ Can check this is well-defined and unit and counit are isoms.

Rmk. $\left\{\begin{array}{c}\text{perfectoid fields}\\ \text{containing } \mathbb{Q}_p\end{array}\right\} \xrightarrow{(\cdot)^\flat} \left\{\begin{array}{c}\text{perfectoid fields}\\ \text{containing } \mathbb{F}_p\end{array}\right\}.$



Fargues-Fontaine curve

C perfectoid field of char. $= p$. $(|\cdot|_C, \varpi)$

Def. An untilt of C is a pair (K, ι) where K is a perfectoid field and $\iota : C \cong K^\flat$.

$$Y := \left\{ \text{char.} = 0 \text{ untilts of } C \right\} / \text{isomorphism}.$$

fiber of the functor above.

Candidate 1 for functions on Y:

Def. $\mathbb{A}_{inf} := W(\mathcal{O}_C)$. Then for any $y = (K, \iota) \in Y$,

have $\mathbb{A}_{inf} \underset{\iota}{\cong} W(\mathcal{O}_K^\flat) \xrightarrow{\theta} \mathcal{O}_K$.

$$\sum_{n \geq 0} [c_n] p^n \longmapsto \sum_{n \geq 0} c_n^\# p^n.$$

$$c_n \in \mathcal{O}_C$$

- Imagine $Y \approx \mathring{\mathbb{D}} = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$.

Radius function on $Y \to (0,1)$

- Fix $|\cdot|_K$ determined by $|x|_C = |x^\#|_K$ for all $x \in \mathcal{O}_C$.

Let radius of $y \in Y$ be $|p|_K$.

- Candidate 2

$$\mathbb{A}_{inf}\left[\frac{1}{p}, \frac{1}{[\varpi]}\right] = \left\{ \sum_{n \gg -\infty} [c_n] p^n ;\ c_n \in C \text{ with } \{|c_n|_C\} \text{ bounded} \right\}$$

- removes (C, id), but only gives meromorphic functions.

- Want to build $X_C := Y/\varphi^{\mathbb{Z}}$.

- φ acts Y by $(K, \iota) \mapsto (K, \iota \circ \varphi_C)$.

$$\varphi_C : C \xrightarrow{\sim} C$$
$$x \mapsto x^p.$$

- φ_C induces an endomorphism of $\mathbb{A}_{\inf} = W(\mathcal{O}_C)$ and $\mathbb{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$.

Try to form $\mathrm{Proj}\left(\bigoplus_{n \geq 0} \mathbb{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]^{\varphi = p^n} \right)$

This will not work.

$$\sum_{n \gg -\infty} [c_n] p^n = f \in \mathbb{A}_{\inf}[\tfrac{1}{p}, \tfrac{1}{[\varpi]}]^{\varphi = p}$$

$$\begin{tikzcd} \mathbb{A}_{\inf} \arrow[r] \arrow[d] & \mathcal{O}_K \arrow[d] \\ \mathbb{A}_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}] \arrow[r] & K \end{tikzcd}$$

$f \mapsto f(y).$

$$\sum_{n \gg -\infty} [c_n] p^n \mapsto \sum_{n \gg -\infty} c_n^{\#} p^n.$$

$$\varphi(f) = \sum_{n \gg -\infty} [c_n^p] p^n$$

$$p \cdot f = \sum_{n \gg -\infty} [c_n^p] p^{n+1}$$

$$\Rightarrow \boxed{c_n^p = c_{n-1} \text{ for all } n} \Rightarrow f = 0.$$

$\mathrm{Fun}(\mathbb{D}^\circ)^{\varphi = p}$ $\quad \varphi(f)(z) = f(z^p)$. $\quad \log(z)$.

Candidate 3 (True ring of functions on Y).

Def. (Gauss norm) $|\cdot|_\rho : \mathbb{A}_{inf}[\frac{1}{p}, \frac{1}{[\varpi]}] \to \mathbb{R}^{\geq 0}$

for any $\rho \in (0,1)$ real.

$$\left| \sum_{n \gg -\infty} [c_n] p^n \right|_\rho := \sup_n \left(|c_n|_C \cdot \rho^n \right).$$

Observations.

1) For any $y = (K, \iota)$ of radius ρ,

have $|f(y)|_K = \left| \sum_{n \gg -\infty} c_n^\# p^n \right|_K$

$$\leq \sup_n \{ |c_n^\#|_K |p^n|_K \}$$

$$= |f|_\rho .$$

2) If you have a sequence $f_n \in \mathbb{A}_{inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$. which is Cauchy w.r.t $|\cdot|_\rho$, then at any y of radius ρ, $f_n(y)$ is Cauchy w.r.t. $|\cdot|_K$.

$\Rightarrow f_n(y)$ has a limit in K.

Def. $B :=$ completion of $A_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ at the family of Gauss norms $|\cdot|_\rho$ $(0<\rho<1.)$

(i.e. topological $\mathbb{Q}_p$-vector space universal w.r.t. the property (receiving a map from $A_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$) that any sequence $f_n \in A_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$ which is Cauchy w.r.t. all $|\cdot|_\rho$ has a limit in B.)

Recall. $C^\infty([0,1])$ is complete w.r.t. $|f|_k := \sup_{[0,1]} (D^{(k)} f)$. $(k \geq 0.)$

Rmk. 1) B has the natural structure of a top. $\mathbb{Q}_p$-algebra.

2) any $\sum_{n\in\mathbb{Z}} [c_n] p^n$ can be viewed as a sequence in $A_{\inf}[\frac{1}{p}, \frac{1}{[\varpi]}]$, can ask: does it converge to an element in B?

Answer: $\sum_{n\in\mathbb{Z}} [c_n] p^n$ converges in B if and only if

1) $\limsup_{n\geq 0} |c_n|_c^{1/n} \leq 1$

2) $\lim_{n\to -\infty} |c_n|_c^{-1/n} = 0$.

- Actually, these characterize holomorphic functions
$$f(z) = \sum c_n z^n$$
on $\mathring{D} = \{ z \in \mathbb{C} \mid 0 < |z| < 1 \}$.

Rmk. Not all elements in B can be written uniquely as such.

Example. For any $x \in 1 + m_C$, the expression
$$\log([x]) := \sum_{n>0} \frac{(-1)^{n+1}}{n} ([x]-1)^n$$
converges in B.

Proof. Suffices to show that
$$\left| \frac{(-1)^{n+1}}{n} ([x]-1)^n \right|_\rho \to 0 \text{ for all } \rho \in (0,1).$$

observe:

$$\left| [x]-1 \right|_\rho = \alpha < 1.$$

$$\left| \frac{(-1)^{n+1}}{n} \right|_\rho = \frac{1}{|n|_\rho} = (\rho^{-1})^{v_p(n)}$$

$$\left| \frac{(-1)^{n+1}}{n} \right|_\rho \left| [x]-1 \right|_\rho^n = (\rho^{-1})^{v_p(n)} \alpha^n$$

$$\leq (\rho^{-1})^{\log_p(n)} \alpha^n \to 0 \qquad \square$$

Def. $X_C := \mathrm{Proj}\left(\bigoplus_{n \in \mathbb{Z}} B^{\varphi = p^n} \right).$

Thm. 1) For $n<0$, $B^{\varphi=p^n} = 0$

C algebraically closed.

2) For $n=0$, $B^{\varphi=id} = \mathbb{Q}_p$. $\subseteq \mathbb{A}_{inf}[\frac{1}{p}, \frac{1}{[\varpi]}] \subseteq B$

$$\mathbb{Q}_p = \left\{ \sum_{n \gg -\infty} [c_n] p^n, \ c_n \in \mathbb{F}_p \right\}$$

3) for $n>0$, any nonzero $f \in B^{\varphi=p^n}$ can be written uniquely (up to ...) as $\lambda \cdot \log([x_1]) \cdots \log([x_n])$, $\lambda \in \mathbb{Q}_p^\times$, $x_i \in 1+m_C$.

C alg. closed.

Thm. 1) X_C is regular, Noetherian, of Krull dim = 1.

2) $\left\{ \begin{matrix} \text{closed points} \\ \text{of } X_C \end{matrix} \right\} \cong \left\{ \begin{matrix} \text{ideals in } \bigoplus_{n\geq 0} B^{\varphi = p^n} \\ \text{of the form } (\log([x])) \\ \text{for } x \in 1 + m_C \end{matrix} \right\}$

$\cong \{ \text{char.} = 0 \text{ untilts of } C \} / \varphi^{\mathbb{Z}}$.

Important ingredient: $\mathrm{Div}(f)$ for $f \in B$.

- For any $y = (K, \iota) \in Y$, there is a "completed local ring" $B^+_{dR}(y) := \lim_n \left(A_{\inf}/(\xi^n)\left[\frac{1}{\varpi}\right] \right)$

(Here ξ generates the ideal $\underbrace{A_{\inf} \to \mathcal{O}_K}$.)

$\xi = \sum_{n\geq 0} [c_n] p^n$ then $|c_0| < 1$, $|c_1| = 1$.

- There is a commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & B^+_{dR}(y) \arrow[d] \\ & K \end{tikzcd}$$

for all y.

$f \mapsto f(y)$.

Thm. $B^+_{dR}(y)$ is a discrete valuation ring with residue field K (The image of ξ is a uniformizer.).

- For any $f \in B$, $\mathrm{Div}(f) := \sum_{y \in Y} \mathrm{ord}_y(f) \cdot y$.

- $\mathrm{Div}(\log([x])) = \sum_{y \in Y} y$.

the corresponding untilts

Given $x \in 1 + m_C$, can let $\xi \in A_{inf}$ be

$$\xi = 1 + [x^{1/p}] + \dots + [x^{p-1/p}].$$

is distinguished

$\rightsquigarrow (K, \iota)$