Verdier Duality.

X : variety. $\qquad \omega_X = a_X^! \Bbbk$.

$\downarrow a_X$

$*$

Recall: If X is smooth of dim n, then

$$\omega_X \cong \Bbbk_X[2n](n)$$

shift

Tate twist. $H_c^{2n}(\mathbb{A}^n, \Bbbk) = \Bbbk(-n)$.

Prop. $\omega_X \in D_c^b(X, \Bbbk)$. In particular, $\mathbb{D}: D_c^b(X;\Bbbk) \to D_c^b(X;\Bbbk)$.

pf. Prove by Noetherian induction.

$X = pt$ ✓.

For general X, pick a Zar. open subset $U \overset{j}{\subseteq} X$ s.t. U is sm.

$i: Z := (X/U) \hookrightarrow X$.

$$\rightsquigarrow \quad i_*i^!\omega_X \to \omega_X \to j_*j^*\omega_X \to \quad \text{DT}.$$

$i_*i^!\omega_X = i_*i^!a_X^!\Bbbk = i_*\omega_Z \in D_c^b$ by ind. hypo.

$j_*j^*\omega_X = j_*j^!a_X^!\Bbbk = j_*\omega_U = j_*\Bbbk_U[2n](n) \in D_c^b(X, \Bbbk)$

Coro. $\forall \mathcal{F} \in D^b_c(X, \Bbbk)$, $R\mathcal{H}om(\mathcal{F}, \omega_X) := \mathbb{D}(\mathcal{F}) \in D^b_c(X, \Bbbk)$.

$$\mathrm{Hom}_{D^b_c}(\mathbb{D}(\mathcal{F}) \overset{L}{\otimes} \mathcal{F}, \omega_X) = \mathrm{Hom}_{D^b_c}(\mathbb{D}(\mathcal{F}), \underbrace{R\mathcal{H}om(\mathcal{F}, \omega_X)}_{= \mathbb{D}(\mathcal{F})})$$

$$\begin{tikzcd}
\mathrm{Hom}_{D^b_c}(\mathbb{D}(\mathcal{F}) \overset{L}{\otimes} \mathcal{F}, \omega_X) \arrow[d, phantom, "\ni"] & \mathrm{Hom}_{D^b_c}(\mathbb{D}(\mathcal{F}), \mathbb{D}(\mathcal{F})) \arrow[d, phantom, "\ni"] \\
(\mathbb{D}(\mathcal{F}) \overset{L}{\otimes} \mathcal{F} \to \omega_X) \arrow[dr, mapsto] & \mathrm{id}_{\mathbb{D}(\mathcal{F})} \arrow[l, mapsto] \\
& (\mathcal{F} \xrightarrow{ev.} \mathbb{D}(\mathbb{D}(\mathcal{F}))) \arrow[d, phantom, "\in"] \\
\mathrm{Hom}_{D^b_c}(\mathbb{D}(\mathcal{F}) \overset{L}{\otimes} \mathcal{F}, \omega_X) \arrow[r, phantom, "="] & \mathrm{Hom}_{D^b_c}(\mathcal{F}, R\mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_X))
\end{tikzcd}$$

$$R\mathcal{H}om(\mathbb{D}(\mathcal{F}), \omega_X) = \mathbb{D}(\mathbb{D}(\mathcal{F}))$$

Main Goal: $\mathcal{F} \xrightarrow{ev} \mathbb{D}(\mathbb{D}(\mathcal{F}))$ is an isom.

Lem. Assume X is sm.

(1) $\forall \mathcal{F} \in D^b_{locf}(X, \Bbbk)$, we have $\underbrace{\mathbb{D}(\mathcal{F})_x}_{D^b_c(X,\Bbbk)} \xrightarrow{\sim} \underbrace{\mathbb{D}(\mathcal{F}_x)}_{D^b_c(pt,\Bbbk)}[2n](n)$.

(2) $\forall \mathcal{L} \in Loc^{ft}(X, \Bbbk)$ loc. free, $\mathbb{D}(\mathcal{L}) \cong \mathcal{L}^\vee[2n](n)$.

pf. (1) $X \underset{x.}{\overset{a_X}{\rightleftarrows}} pt.$ Taking stalk at x = pullback by x.

(cf. [Ac] 1.4)

$$\begin{array}{ccc}
x^* R\mathcal{H}om(\mathcal{F}, \omega_X) & \to & R\mathcal{H}om(x^*\mathcal{F}, x^*\omega_X). \\
\| & & \| \\
\mathbb{D}(\mathcal{F})_x & & R\mathcal{H}om(\mathcal{F}_x, \Bbbk[2n](n)) \\
& & \| \\
& & \mathbb{D}(\mathcal{F}_x)[2n](n).
\end{array}$$

$$\left(\text{Recall: } \quad \mathcal{F} \in D^b_c(X, \Bbbk). \quad H^k(\mathcal{F}_x) = \operatorname*{colim}_{U \ni x} H^k(R\Gamma(U, \mathcal{F})). \right)$$

$$H^k(\mathbb{D}(\mathcal{F})_x) = \operatorname*{colim}_{U \ni x} H^k\big(R\Gamma(U, \mathbb{D}(\mathcal{F}))\big)$$

$$= \operatorname*{colim}_{U \ni x} H^k\big(R\Gamma(U, R\mathcal{H}om(\mathcal{F}, \omega_X))\big)$$

$$= \operatorname*{colim}_{U \ni x} H^k\big(R\mathrm{Hom}(\mathcal{F}|_U, \Bbbk_U[2n](n))\big).$$

Assume U is contractible. Using homotopy invariance:

$$H^k(R\mathrm{Hom}(\mathcal{F}|_U, \Bbbk_U[2n](n)))$$
$$\downarrow \wr$$
$$H^k(R\mathrm{Hom}(\mathcal{F}_x, \Bbbk[2n](n))).$$

(2) $\mathbb{D}(\mathcal{L}) \simeq R\mathcal{H}om(\underset{\text{loc. free.}}{\mathcal{L}}, \Bbbk_X[2n](n)) = \mathcal{L}^\vee[2n](n)$. $\square$.

Coro. X sm. $\mathcal{F} \in D^b_{\mathrm{locf}}(X; \Bbbk) \Rightarrow \mathrm{ev}: \mathcal{F} \xrightarrow{\sim} \mathbb{D}(\mathbb{D}(\mathcal{F}))$.

pf. ① $R\mathcal{H}om(-,-)$ preserves D^b_{locf}. ② Verdier duality for a pt.

Recall: $f_* R\mathcal{H}om(\mathcal{F}, f^! \mathcal{G}) \simeq R\mathcal{H}om(f_! \mathcal{F}, \mathcal{G})$. (adjoint)

$f^! R\mathcal{H}om(\mathcal{F}, \mathcal{G}) \simeq R\mathcal{H}om(f^*\mathcal{F}, f^! \mathcal{G})$.

(equiv to the Projection Formula: $f_! \mathcal{F} \overset{L}{\otimes} \mathcal{G} = f_!(\mathcal{F} \overset{L}{\otimes} f^* \mathcal{G})$.)

$\Rightarrow \forall f: X \to Y,$ (take $\mathcal{G} = \omega_Y$).

$f_* \mathbb{D}_X(\mathcal{F}) \cong \mathbb{D}_Y(f_! \mathcal{F}).$ $\forall \mathcal{F} \in D^b_c(X; \mathbb{k})$

$f^! \mathbb{D}_X(\mathcal{F}) \cong \mathbb{D}_Y(f^* \mathcal{F}).$ $\forall \mathcal{F} \in D^b_c(Y; \mathbb{k}).$

Coro. $U \overset{j}{\hookrightarrow} X \xleftarrow[\text{proper}]{p} Z.$ $\mathcal{F} \in D^b_c(X; \mathbb{k}).$

$\mathbb{D}_X(\mathcal{F})|_U = \mathbb{D}_U(\mathcal{F}|_U).$

$p_* \mathbb{D}_Z(\mathcal{F}) = \mathbb{D}_X(p_* \mathcal{F}).$

Lem. X var. $j: U \hookrightarrow X$. s.t. U is sm. irr.

$\forall \mathcal{F} \in D^b_{locf}(U; k)$. We have a natural isom

$j_! \mathbb{D}_U(\mathcal{F}) \longrightarrow \mathbb{D}_X(j_* \mathcal{F}).$

pf. Step 0. Construct the nat. map.

$R\mathcal{H}om(\mathcal{F}, \omega_U) = R\mathcal{H}om(j^* j_* \mathcal{F}, j^! \omega_X) = j^! R\mathcal{H}om(j_* \mathcal{F}, \omega_X)$

adjoint $\Rightarrow$

$$\begin{array}{ccc} j_! R\mathcal{H}om(\mathcal{F}, \omega_U) & \to & R\mathcal{H}om(j_* \mathcal{F}, \omega_X) \\ \| & & \| \\ j_! \mathbb{D}_U(\mathcal{F}) & & \mathbb{D}_X(j_* \mathcal{F}). \end{array}$$

Step 1. Reduction to the case where U is dense in X.

$\bar{j}: U \hookrightarrow \bar{U} \overset{i}{\hookrightarrow} X,$

$\mathbb{D}_X(j_* \mathcal{F}) = \mathbb{D}_X(i_* \bar{j}_* \mathcal{F}) = i_* \mathbb{D}_{\bar{U}}(\bar{j}_* \mathcal{F}).$

$j_! \mathbb{D}_U(\mathcal{F}) = i_* \bar{j}_! \mathbb{D}_U(\mathcal{F}).$

So we may replace X by $\bar{U}$.

Step 2. Assume $\overline{U} = X$.

Resolution of singularity $\Rightarrow$ $p : \tilde{X} \to X$. s.t.

- p is proper, $\tilde{X}$ is sm.
- p induces an isom $f^{-1}(U) \to U$.
- $p^{-1}(X \setminus U) =: Z$ is a divisor of $\tilde{X}$ w/ simp. n.c.

$$\begin{tikzcd}
p^{-1}(U) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \tilde{X} \arrow[d, "p"] & Z \arrow[l, hook', "i"'] \\
U \arrow[ur, "\tilde{j}"] \arrow[r, hook, "j"'] & X &
\end{tikzcd}$$

$$D_X(j_* \mathcal{F}) = D_X(p_* \tilde{j}_* \mathcal{F}) = p_* D_{\tilde{X}}(\tilde{j}_* \mathcal{F}),$$

$$j_! D_U(\mathcal{F}) = p_* \tilde{j}_! D_U(\mathcal{F}),$$

So we may replace X by $\tilde{X}$.

Step 3. Assume X is sm, $\overline{U} = X$, $Z := X \setminus U$ is a div. w/ simp. n.c. $Z = Z_1 \cup \cdots \cup Z_m$.

$$\left(\hat{j}_! D_U(\mathcal{F})\right)_x = \begin{cases} D_U(\mathcal{F})_x & x \in U. \\ 0 & x \notin U. \end{cases}$$

$$\left(\mathbb{D}_X(j_*\mathcal{F})\right)\big|_U = \mathbb{D}_U(j^*j_*\mathcal{F}) = \mathbb{D}_U(\mathcal{F}).$$

STP: $x \in Z \Rightarrow \left(\mathbb{D}_X(j_*\mathcal{F})\right)_x = 0.$

$$J = \{ s \in \{1,\dots,k\} \mid x \in Z_s \} \neq \phi. \qquad |J| = j$$

$$H^k\left(\mathbb{D}_X(j_*\mathcal{F})\right)_x = \operatorname*{colim}_{V \ni x} H^k\left(R\Gamma(V, R\mathcal{H}om(j_*\mathcal{F}, \omega_X))\right)$$

$$= \operatorname*{colim}_{V \ni x} H^k(R\mathrm{Hom}(j_*\mathcal{F}|_V, \omega_V))$$

$$= \operatorname*{colim}_{V \ni x} \mathrm{Hom}\left((j_*\mathcal{F})|_V, a_V^! \Bbbk_{pt}[k]\right)$$

$$= \operatorname*{colim}_{V \ni x} \mathrm{Hom}\left(a_{V,*}\left((j_*\mathcal{F})|_V\right), \Bbbk_{pt}[k]\right)$$

$$\begin{tikzcd} U\cap V \arrow[r] \arrow[d, "j'"'] & U \arrow[d] \\ V \arrow[r] & X \end{tikzcd}$$

$$= \operatorname*{colim}_{V \ni x} \mathrm{Hom}\left(R\Gamma_c\left(\underbrace{(j_*\mathcal{F})|_V}_{= (j')_*\mathcal{F}|_{U\cap V}}\right), \Bbbk_{pt}[k]\right).$$

Let V be a normal crossing coordinate chart.

$$\rightsquigarrow \quad \begin{tikzcd} V\cap U \arrow[r, "\sim"] \arrow[d, phantom, "\cap" rotate=-90] & (\mathbb{C}^\times)^j \times \mathbb{C}^{n-j} \arrow[d, phantom, "\cap" rotate=-90] \arrow[r] & h \\ V \arrow[r, "\sim"] & \mathbb{C}^j \times \mathbb{C}^{n-j} & \end{tikzcd}$$

STP: $\forall \mathcal{G} \in D^b_{locf}((\mathbb{C}^\times)^j \times \mathbb{C}^{n-j})$, $R\Gamma_c(\mathcal{G}) = 0$.

c.f Lem B.2.6.

$\square$

Lem. $\mathcal{U} = (\mathbb{C}^\times)^k \times \mathbb{C}^{n-k} \xhookrightarrow{j} \mathbb{C}^n \xrightarrow{p} \{0\} \times \mathbb{C}^{n-k}$.

$\mathcal{F} \in D^b_{locf}(\mathcal{U}, \Bbbk)$. Then $p_! j_* \mathcal{F} = 0$. $(\Rightarrow R\Gamma_c(\mathcal{U}, j_*\mathcal{F}) = 0)$.

pf.

$$\begin{tikzcd}
\mathcal{U} \arrow[r, hook, "h"] \arrow[d, hook, "j"'] \arrow[dd, bend right=60, "\tilde{p}"'] & V = (\mathbb{C}^k \setminus \{0\}) \times \mathbb{C}^{n-k} \arrow[dl, hook', "v"] \\
\mathbb{C}^n \arrow[d, shift right, "p"'] \arrow[dr, "w"] & \\
\{0\} \times \mathbb{C}^{n-k} \arrow[u, shift right, "i"'] & \hat{\mathbb{C}}^k \times \mathbb{C}^{n-k} \arrow[l, "\hat{p}\ (\text{proper})"]
\end{tikzcd}$$

$$v_! v^* j_* \mathcal{F} \to j_* \mathcal{F} \to i_* i^* j_* \mathcal{F} \to ?$$
$$v^* j_* \mathcal{F} = h_* \mathcal{F}, \quad v_! h_* \mathcal{F}$$

$\boxed{\text{First try}}$ $p_! v_! h_* \mathcal{F} \to p_! j_* \mathcal{F} \to i^* j_* \mathcal{F}$. $\to$?

$$p_* v_! h_* \mathcal{F} \to p_* j_* \mathcal{F} \to i^* j_* \mathcal{F}.$$
$$p_* j_* \mathcal{F} = \tilde{p}_* \mathcal{F}$$

<u>Claim</u>. $\tilde{p}_* \mathcal{F} \to i^* j_* \mathcal{F}$ is an isom.

$\forall (0, z) \in \{0\} \times \mathbb{C}^{n-k}$, we have

$$H^k\big((\hat{p}_*\mathcal{F})_{(0,z)}\big) = \operatorname*{colim}_{\substack{W\ni z\\ W\subseteq \mathbb{C}^{n-k}}} H^k\big(R\Gamma(W, \hat{p}_*\mathcal{F})\big)$$

$$= \operatorname*{colim}_{W\ni z} H^k\Big(R\Gamma\big(\underbrace{\hat{p}^{-1}(W)}_{=(\mathbb{C}^*)^k\times W.}, \mathcal{F}|_{\hat{p}^{-1}(W)}\big)\Big).$$

$$R^k\Gamma(U,\mathcal{F}).$$

$$H^k\big((i^*j_*\mathcal{F})_{(0,z)}\big) = H^k\big((j_*\mathcal{F})_{(0,z)}\big)$$

$$= \operatorname*{colim}_{W'\times W\ni(0,z)} H^k\big(R\Gamma(W'\times W, j_*\mathcal{F})\big)$$

$$= \operatorname*{colim}_{W'\times W\ni(0,z)} R^k\Gamma\big((W'\cap U)\times W, \mathcal{F}\big)$$

$$\Rightarrow \quad p_* v_! h_* \mathcal{F} = 0.$$

$$s:\ (\mathbb{C}^k\setminus\{0\})\times\mathbb{C}^{n-k} \longrightarrow (\mathbb{C}^k\setminus\{0\})\times\mathbb{C}^{n-k}$$
$$(z,w) \longmapsto \left(\frac{z}{|z|^2}, w\right)$$

$s(U) = U.$

```tikzcd
V \arrow[r, "v"] \arrow[d, "vs"'] \arrow[dr, phantom, "\lrcorner"] & \mathbb{C}^n \arrow[d, "w"] & (0,z) \arrow[d, mapsto] \\
\mathbb{C}^n \arrow[r, "\bar{w}"] & \hat{\mathbb{C}}^k\times\mathbb{C}^{n-k}. & (0,z).
```

$$(0,z) \longmapsto (\infty, z).$$

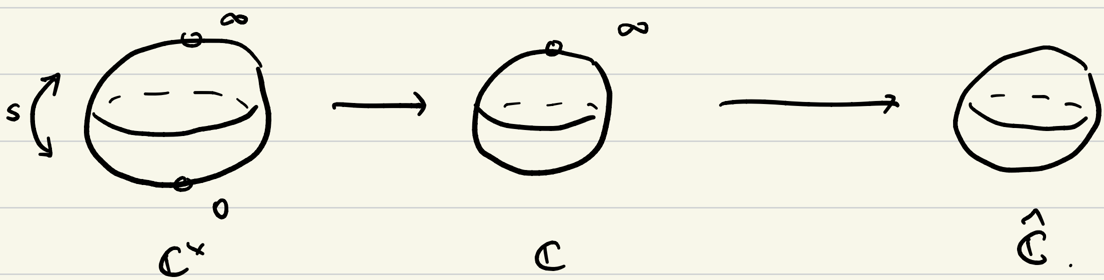
Claim: $\bar{w}_!\,(vs)_*\,(h_*\mathcal{F}) = w_*\,v_!\,(h_*\mathcal{F})$.

$$\bar{w}^*\,\bar{w}_!\,(vs)_* = (vs)_*,$$

$$\bar{w}^*\,w_*\,v_! \overset{\text{open BC}}{=\!=} (vs)_*\,v^*v_! = (vs)_*.$$

$$(w_*\,v_!\,h_*\mathcal{F})_{(0,z)} = (w_*\,v_!\,h_*\mathcal{F})_{w(0,z)}$$
$$= (v_!\,h_*\mathcal{F})_{(0,z)} = 0, \qquad \checkmark.$$

$$0 = p_*\,v_!\,h_*\mathcal{F} = \hat{p}_*\,w_*\,v_!\,h_*\mathcal{F}$$
$$= \hat{p}_*\,\bar{w}_!\,(vs)_*\,h_*\mathcal{F}.$$
$$= \hat{p}_!\,\bar{w}_!\,v_*\,s_*\,h_*\mathcal{F}.$$
$$= p_!\,v_*\,h_*\,(s(u))_*\mathcal{F}. \qquad \square.$$



Thm. $\mathcal{F}\in D^b_c(X;\mathbb{k})$. $ev: \mathcal{F}\to \mathbb{D}(\mathbb{D}(\mathcal{F}))$ is an isom.

pf. We have proved this w/ assumption that X is sm, $\mathcal{F}\in D^b_{loc}$.
For the general case, use the Noetherian induction.
$X = pt$. ✓.

Pick $U \subseteq X$ irr. sm. Zar. open, s.t. $\mathcal{F}|_U \in D^b_{loc.f}(U; \Bbbk)$.

$$U \xhookrightarrow[j]{\circ} X \xhookleftarrow[i]{} Z := X \setminus U.$$

$$\begin{tikzcd}
i_* i^! \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & j_* j^* \mathcal{F} \arrow[r] \arrow[d] & {} \\
\mathbb{D}^2(i_* i^! \mathcal{F}) \arrow[r] \arrow[d, dashed, "(*)"] & \mathbb{D}^2(\mathcal{F}) \arrow[r] \arrow[d, equal] & \mathbb{D}^2(j_* j^* \mathcal{F}) \arrow[r] \arrow[d, dashed, "(**)"] & {} \\
i_* i^! \mathbb{D}^2(\mathcal{F}) \arrow[r] & \mathbb{D}^2(\mathcal{F}) \arrow[r] & j_* j^* \mathbb{D}^2(\mathcal{F}) \arrow[r] & {}
\end{tikzcd} \qquad (\Delta)$$

Previous lemma: $\mathbb{D}^2(i_* i^! \mathcal{F}) \cong i_* \mathbb{D}^2(i^! \mathcal{F})$

$$\Rightarrow \operatorname{Hom}_{D^b_c}\left(\mathbb{D}^2(i_* i^! \mathcal{F}),\ j_* j^* \mathbb{D}^2(\mathcal{F})[n]\right) \overset{(i_*, i^!)}{=\!=} 0.$$

$\Rightarrow \exists!\ (*)$ making the diagram commute.

$$\mathbb{D}^2(j_* j^* \mathcal{F}) \overset{\substack{\text{assumption} \\ \text{on } U}}{\cong} j_* \mathbb{D}^2(j^* \mathcal{F}).$$

$$\Rightarrow \operatorname{Hom}_{D^b_c}\left(\mathbb{D}^2(j_* j^* \mathcal{F}),\ i_* i^! \mathbb{D}^2(\mathcal{F})[n]\right) = 0.$$

$\Rightarrow \exists!\ (**)$.

Characterizing property of (Δ) $\Rightarrow$ $(*)$, $(**)$ are isoms.