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Tate twist

$\Bbbk$, $\Bbbk(1) = \mathrm{Hom}(\mathbb{H}^2_c(\mathbb{A}^1_{\mathbb{C}}; \Bbbk), k)(\simeq \Bbbk)$, not canonical!

Canonically, there is $\mathrm{ev} : \mathbb{H}^0(\mathbb{C}, \Bbbk) \to \Bbbk$.

$\sigma \mapsto \sigma([M])$ induces $\mathrm{ev} : \mathbb{H}^2_c(\mathbb{C}, \Bbbk) \to \Bbbk$, depend on the choice of $[M]$.

Kunneth formula: $R\Gamma_c(X \times Y, \underline{\Bbbk}_{X\times Y}) \simeq R\Gamma_c(X, \underline{\Bbbk}_X) \otimes R\Gamma_c(Y, \underline{\Bbbk}_Y)$.

$\mathbb{H}^{2n}_c(\mathbb{C}^n; \underline{\Bbbk}) \simeq \mathbb{H}^2_c(\mathbb{C}^1; \underline{\Bbbk})^{\otimes n} \simeq \underline{\Bbbk(-n)}$.

$f : X \to Y$ smooth of rel. dim. d. $f^* \leftrightarrow f^!$.

Prop 2.2.8. $or_f \simeq \underline{\Bbbk}_X$.

Thm 2.2.9. $f^! \simeq f^*[2d](d)$.

Defn. $or_{f,pre}(U) := \mathrm{Hom}(H^{2d}(f_!\underline{\Bbbk}_U))$

Remark. hyper-cohom. $\mathbb{H}^{\cdot} : D(X, \Bbbk) \to \Bbbk$-mod.

cohomology sheaf. $H^{\cdot} : D(X, \Bbbk) \to \mathrm{Sh}(X, \Bbbk)$.

Proof of Prop:

Step 1.

$$H^{2d}(f_!\underline{\Bbbk}_U) \simeq \underline{\mathbb{H}^{2d}((\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U)\times M})}_{f(U)}$$

$$\simeq \underline{\mathbb{H}^{2d}_c(M, \Bbbk)}_{f(U)} \simeq \underline{\mathbb{H}^{2d}_c(\mathbb{C}^d, \Bbbk)}_{f(U)} \simeq \underline{\Bbbk}_{f(U)}(-d).$$

Step 2. restriction indep. with M

Step 3. Gluing.

Step 1:

$$\begin{tikzcd}
U_\alpha \arrow[r, "\phi_\alpha"'] \arrow[d, "f"] & f(U_\alpha) \times M_\alpha \arrow[d, "\mathrm{pr}_1"'] \\
f(U_\alpha) \arrow[r, equal] & f(U_\alpha)
\end{tikzcd}$$

$$f = \mathrm{pr}_1 \circ \phi \implies f_! \simeq (\mathrm{pr}_1)_! \circ (\phi_\alpha)_! \implies f_!\underline{\Bbbk}_{U_\alpha} \simeq (\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\alpha)\times M_\alpha}.$$

$$\begin{tikzcd}
f(U_\alpha) \times M_\alpha \arrow[r, "\mathrm{pr}_2"'] \arrow[d, "\mathrm{pr}_1"] \arrow[dr, phantom, "\lrcorner", very near start] & M_\alpha \arrow[d, "a_{M_\alpha}"'] \\
f(U_\alpha) \arrow[r, "a_{f(U_\alpha)}"] & \mathrm{pt}
\end{tikzcd}$$

$$(\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\alpha)\times M_\alpha} \simeq (\mathrm{pr}_1)_!(\mathrm{pr}_2)^*\underline{\Bbbk}_{M_\alpha}$$

$$\simeq (a_{f(U_\alpha)})^*(a_{M_\alpha})_!\underline{\Bbbk}_{M_\alpha} \simeq \underline{R\Gamma_c(M, \Bbbk)}_{f(U_\alpha)}.$$

"concrete iteration of four functors": $h : Y \hookrightarrow X$ locally closed

h^*: good,

h_*: extra stalks,

$h_!$: extension by 0, good,
$h^!$: lose stalks.

$$\begin{tikzcd}
U \arrow[r, hook, "j"'] \arrow[d, "f'"] & X \arrow[d, "f"'] \\
V \arrow[r, hook, "h"] & Y
\end{tikzcd}$$

We have "identity":

$$\begin{aligned}
(f')_!\underline{\Bbbk}_U &\simeq h^! h_! (f')_! \underline{\Bbbk}_U, \\
&\simeq h^! f_! j_! \underline{\Bbbk}_U \to h^! f_! \underline{\Bbbk}_X.
\end{aligned}$$

Remark. For $U \overset{j}{\hookrightarrow} X$, can induces $H_c^*(U;\Bbbk) \xrightarrow{j_\sharp} H_c^*(X;\Bbbk)$.
$(\phi_\alpha)_\sharp : f_!\underline{\Bbbk}_{U_\alpha} \xrightarrow{\sim} (\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\alpha)\times M_\alpha} \simeq \underline{R\Gamma_c(M_\alpha, \underline{\Bbbk}_{M_\alpha})}_{f(U_\alpha)}$.

$$\begin{tikzcd}
f(U_\alpha) \times M_\alpha \arrow[r, "\phi_*"'] \arrow[d, "f"] & f(U_\alpha) \times \mathbb{C}^d \arrow[d, "\mathrm{pr}_1"'] \\
f(U_\alpha) \arrow[r, equal] & f(U_\alpha)
\end{tikzcd}$$

By this diagram, $(j_\alpha)_\sharp : (\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\alpha)\times M_\alpha} \to (\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\alpha)\times \mathbb{C}^d}$. By Appendix B, this is an isom.
Step 2.

$$\begin{tikzcd}
U_\beta \arrow[rrr, hook, "h"] \arrow[dr, "\phi_\beta"] \arrow[ddd, "f"] & & & U_\alpha \arrow[dr, "\phi_\alpha"] \arrow[ddd, "f"] & & \\
& f(U_\beta) \times M_\beta \arrow[rrr, hook, "q"] \arrow[dr, hook] \arrow[ddd, "\mathrm{pr}_1"] & & & f(U_\alpha) \times M_\alpha \arrow[dr, hook] \arrow[ddd, "\mathrm{pr}_1"] & \\
& & f(U_\beta) \times \mathbb{C}^d \arrow[ddd, "\mathrm{pr}_1"] & & & f(U_\alpha) \times \mathbb{C}^d \arrow[ddd, "\mathrm{pr}_1"] \\
f(U_\beta) \arrow[rrr, hook, "k"] \arrow[dr, equal] & & & f(U_\alpha) \arrow[dr, equal] & & \\
& f(U_\beta) \arrow[rrr, hook, "k"] \arrow[dr, equal] & & & f(U_\alpha) \arrow[dr, equal] & \\
& & f(U_\beta) \arrow[rrr, hook, "k"] & & & f(U_\alpha)
\end{tikzcd}$$

it induces

$$\begin{tikzcd}
U_\beta \arrow[r, "h"] \arrow[d, "\phi_\beta"] & U_\alpha \arrow[d, "\phi_\alpha"] \\
f(U_\beta) \times M_\beta \arrow[r, "q"] \arrow[d, "j_\beta"] & f(U_\alpha) \times \mathbb{C}^d \arrow[d, "j_\alpha"] \\
f(U_\beta) \times \mathbb{C}^d & f(U_\alpha) \times \mathbb{C}^d
\end{tikzcd}$$

with

$$
\begin{tikzcd}
f_!\underline{\Bbbk}_{U_\beta} \arrow[r, "h_\sharp"] \arrow[d, "(\phi_\beta)_\sharp"] \arrow[dr, phantom, "\circlearrowright"] & (f_!\underline{\Bbbk}_{U_\alpha})|_{f(U_\beta)} \arrow[d, "(\phi_\alpha)_\sharp"] \\
(\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\beta)\times M_\beta} \arrow[r, "q_\sharp"] \arrow[d, "(j_\beta)_\sharp"] \arrow[dr, "i_\sharp"] & (\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\alpha)\times M_\alpha}|_{f(U_\beta)} \arrow[d, "(j_\alpha)_\sharp"] \\
(\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\beta)\times \mathbb{C}^d} \arrow[r, equal] & (\mathrm{pr}_1)_!\underline{\Bbbk}_{f(U_\beta)\times \mathbb{C}^d}|
\end{tikzcd}
$$

Step 3. Sheaf theory, by $or_f|_{U_\alpha} \simeq \underline{\Bbbk}_{U_\alpha}(d)$. $\square$

Thm 2.2.9. $f^! \simeq f^*[2d](d)$.

Consider $\mathbb{C}^d \to \mathrm{pt}$, $(a_{\mathbb{C}^d})^!\underline{\Bbbk}_{\mathrm{pt}} = \underline{\Bbbk}_{\mathbb{C}^d}[2d](d)$,

this is for $(a_{\mathbb{C}^d})_!\underline{\Bbbk}_{\mathbb{C}^d} = \underline{\Bbbk}_{\mathrm{pt}}[2d](d)$.

And $k[-2n](-n) = H^k((a_{\mathbb{C}^d})_!\underline{\Bbbk}_{\mathbb{C}^d}) \simeq \mathbb{H}^k_c(\mathbb{C}^d, \Bbbk)$.

Smooth pair.

$f : X \to S$ smooth of rel. dim. d, (Z, X) is a smooth pair of codim r, iff $f|_Z : Z \to S$ smooth of rel. dim. $d - r$.

(Z, X) smooth pair, $\mathcal{F} \in \mathrm{Sh}(X)$, $\exists i^!\mathcal{F} \to i^*\mathcal{F}[-2r](-r)$.

Moreover, it is an isom. if one of those holds

(1) $\mathcal{F}$ local system (local const.)

(2) $\mathcal{F} = f^*\mathcal{G}, \mathcal{G} \in \mathrm{Sh}(S)$.