$\mathscr{A}$ abelian cat., $K(\mathscr{A})$: Ob: complex, by "localization", $D(\mathscr{A})$:

$$\cdots \to X^{n-1} \overset{d^{n-1}}{\to} X^n \overset{d^n}{\to} X^{n+1} \to \cdots$$

the truncation functor

$$(\tau^{\leq n} X)^i := \begin{cases} X^i & i < n \\ \ker(d^n) & i = n \\ 0 & i > n \end{cases}$$

And then

$$(\tau^{\geq n} X)^i := \begin{cases} 0 & i < n \\ X^n / \operatorname{im} d^{n-1} & i = n \\ X^i & i > n \end{cases}$$

$$\tau^{\geq n}\tau^{\leq n} X \leftarrow \tau^{\leq n} X \hookrightarrow X \to \tau^{\geq n} X.$$

Defn. of Distinguished triangle.

$$
\begin{array}{ccccccc}
X & -f\to Y & \overset{g}{\longrightarrow} & Z & \overset{h}{\longrightarrow} & X[1] \\
\| & \| & & \downarrow u & & \| \\
X & -f\to Y & \longrightarrow & cone(f) & \longrightarrow & X[1]
\end{array}
$$

Upper triangle is a distinguished triangle if there is u s.t. the diag. comm.

For $F : \mathscr{A} \to \mathscr{B}$ functor, can define $RF : D^+(\mathscr{A}) \to D^+(\mathscr{B})$, $X \mapsto F(Q_X)$, Q_X resolution, preserves distinguished triangle.

$$0 \to X \to Y \to Z \to 0,$$

Start 2.3, Constructible sheaf.

Def. X is a var., a stratification of X is a collection of strata $(X_s)_{s \in \Phi}, |\Phi| < \infty$ s.t. $X = \sqcup X_S$, each X_s is smooth conn, locally closed subvar. s.t. $\bar{X}_s = \sqcup X_t$.

Eg. G reductive gp, Bruhat decomp. $G/B = \sqcup_{w \in W} BwB/B$.

Def. A filtration of X by smooth var. is a collection $(X_s)_{s \in \Phi}$ s.t. $X = \sqcup X_S$, each X_s is smooth conn, locally closed subvar s.t. $\bar{X}_s = \sqcup X_t$, and there is a rearrangement $(X_{s_i})_{1 \leq i \leq n}$ s.t. $\sqcup_{i=1}^{j} X_{s_i}$ are all closed.

Fact: Any var. X has filtration by smooth var.

Find nontrivial open U, consider $X \backslash U$ and use Noetherian induction.

Lemma: (1) Φ is a filtration, then there is refinement of Φ that it's also a stratification

$(X_s)_{s \in \Phi}$, choose $U \subset X_s$ be open smooth conn. $(X_s \cap (\bar{U} \backslash U), X_s \backslash \bar{U})$ can be refined into a filtrtion of $X \backslash U$. By noetherian induction it can be refined into a stratification $(X_t)_{t \in S}$ of $X \backslash U$. $\bar{U} = U \sqcup (\sqcup X_s \cap (\bar{U} \backslash U))$ finishes the proof.

(2) Any two stratification has a common stratification.

$(X_s), (X_t)$. $X_1 \subset X$ be irred. comp., $\eta \in X_1$ be generic pt, $\eta \in X_t \cap X_t$, then Noetherian induction.

(Another way, take $(X_s \cap X_t)$, and consider its filtration.)

(3) Y locally closed subvar. of X, $\exists$ stratification of X s.t. Y is union of strata.

Take "$Y = Y_2 - Y_1$".

Def. $(X_s)_{s\in\Phi}$ is a stratification, $\mathcal{F} \in Sh(X,\Bbbk)(\overset{\Delta}{=} Sh(X))$ is constructible w.r.t. Φ, if $\mathcal{F}|_{X_s}$ are local system of finite type. $\mathcal{F}|_{X_s} = j_s^*\mathcal{F}, j_S : X_s \hookrightarrow X$.

For $\mathcal{F} \in D^b(X)$ is constructible w.r.t. Φ, if $H^k(\mathcal{F})$ is constructible w.r.t. Φ.

Def. $(X_s)_{s\in P}$ modified dimension of support of $\mathcal{F} \in Sh_P(X)$, $\mathrm{mdsupp}\,\mathcal{F} = \sup_{s\in P}(\dim X_s - \mathrm{grade}(\mathcal{F}|_{X_s}))$.

Lemma 2.3.11: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is S.E.S in $Sh_C(X)$, then $\mathrm{mdsupp}\,\mathcal{F} = \max\{\mathrm{mdsupp}\,\mathcal{F}', \mathrm{mdsupp}\,\mathcal{F}''\}$. Homology algebra.

Cor. 2.3.12: $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to$ be distinguished triangle in $D_c^b(X)$, if $\exists n$ s.t. $\forall k$, $\mathrm{mdsupp}\,H^k(\mathcal{F}'), \mathrm{mdsupp}\,H^k(\mathcal{F}'') < n - k$, then $\mathrm{mdsupp}\,H^k(\mathcal{F}) < n - k$.

Lemma 2.3.13: $D_P^b(X), D_c^b(X)$ are full triangulate subset of $D^b(X)$.

Prop. 2.3.14: $f : X \to Y, \forall \mathcal{F} \in D_c^b(Y), f^*\mathcal{F} \in D_c^b(X)$.

Pf. $(Y_s), (f^{-1}(Y_s))$ is "filtration" of X, hence can be redined by stratification (X_t), $(f^*\mathcal{F})|_{X_t} \in D_{locf}^b(X_t)$.

Prop. 2.3.15: $h : Y \hookrightarrow X$ be locally closed embedding. Then $h_! : D_c^b(Y) \subset D_c^b(X)$.

Pf: $\mathcal{F} \in D_{P|_Y}^b(Y), P = (X_s)_{s\in P}, Y \in \sqcup X_s$.

$$(h_!\mathcal{F})|_{X_t} = \begin{cases} \mathcal{F}|_{X_t} & X_t \subset Y \\ 0 & X_t \cap Y = \emptyset \end{cases}.$$

Lemma 2.3.16: $U \subset X$ open, $\mathcal{F} \in D^b(X), \mathcal{F}|_U, \mathcal{F}|_{U^c}$ constructible, then $\mathcal{F} \in D_c^b(X)$.

Pf: $j_!j^*\mathcal{F} \to \mathcal{F} \to i_*i^*\mathcal{F} \to \cdots, j : U \hookrightarrow X, i : U^c \hookrightarrow X$.

Lemma 2.3.17: X is smooth conn. var., then $D_{locf}^b(X)$ is closed under $\cdot \otimes^L \cdot$ and $RHom(\cdot, \cdot)$.

Pf: $\mathcal{F} \otimes^L \mathcal{G} = Q_\mathcal{F} \otimes^L Q_\mathcal{G}, (\mathcal{F} \otimes^L \mathcal{G})|_U = \underline{M}_U \otimes^L \mathcal{G}|_U$.

Prop. 2.3.18: $D_c^b(X), D_P^b(X)$ is stable under $\cdot \otimes^L \cdot$.

Lemma 2.3.19: $f : X \to Y$ finite, if $\mathcal{F} \in D_c^b(X)$, then $f_*\mathcal{F} \in D_c^b(Y)$. And $\dim \mathrm{supp}\, f_*\mathcal{F} = \dim \mathrm{supp}\,\mathcal{F}$, $\mathrm{mdsupp}\, f_*\mathcal{F} = \mathrm{mdsupp}\,\mathcal{F}$..

Pf: $^\circ f_* : Sh(X) \to Sh(Y)$ is exact. WLOG, $\mathcal{F} \in Sh_C(X)$. $\exists V \subset Y$ open s.t. $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is finite and etale. $\emptyset \neq U$ is a strata and $U \subset f^{-1}(V)$. Shrink U to U' s.t. $f|_{U'} : U' \to V'$ is homeo. $\mathcal{L} := \mathcal{F}|_{U'} \in Loc^{ft}(U'), (f|_{U'})_*\mathcal{L} \in Loc^{ft}(V')$. $f : U' \hookrightarrow X, h : V' \hookrightarrow Y, i : Z = X\backslash U' \hookrightarrow X$.

$$f_*j_!j^*\mathcal{F} \to f_*\mathcal{F} \to f_*i_*i^*\mathcal{F} \to \cdots.$$

Consider

$$
\begin{array}{ccc}
U' & \overset{j}{\longrightarrow} & X \\
\downarrow{\scriptstyle f|_{U'}} & & \downarrow{\scriptstyle f} \\
V' & \overset{h}{\longrightarrow} & Y
\end{array}
$$

$f_*j_!\mathcal{L} \simeq h_!(f|_{U'})_*\mathcal{L}$, $h_!$ fix, $f|_{U'}$ homeo.

$\dim \mathrm{supp}\, h_!(f|_{U'})_*\mathcal{L} = \dim \mathrm{supp}\, j_!\mathcal{L}$.

$\dim \mathrm{supp}(f \circ i)_*i^*\mathcal{F} = \dim \mathrm{supp}\, i^*\mathcal{F} = \dim \mathrm{supp}\, i_*i^*\mathcal{F}$.

$\mathrm{mdsupp}(f \circ i)_*i^*\mathcal{F} = \mathrm{mdsupp}\, i^*\mathcal{F} = \mathrm{mdsupp}\, i_*i^*\mathcal{F}$. Noetherian induction.

Def. (X_s) is good stratification if $\forall \mathcal{L} \in Loc^{ft}(X_s), (j_s)_*\mathcal{L} \in D_P^b(X_s)$.

Lemma 2.3.22: (X_s) is a good stratification , $Y \subset X$ locally closed union of strata. $h : Y \hookrightarrow X$.