E non-Arch local field w/ residue field $\mathbb{F}_q$.

$\underset{\pi}{\text{ш}}$ uniformiser.

S perfectoid space over $\mathbb{F}_q$.

$\overset{\text{ф}}{\mathrm{Perf}}_{\mathbb{F}_q} = \mathrm{Perf}$.

$S = \mathrm{Spa}(R, R^+)$ $\varpi \in R^+$ uniformizer.

$$\mathcal{Y}_S := \mathrm{Spa}\, \underline{W_{\mathcal{O}_E}(R^+)} \setminus \{[\varpi] = 0\}$$

$$\bigcup$$

$$Y_S := \text{"} \qquad \text{"} \setminus \{\pi[\varpi] = 0\}$$

$$\downarrow$$

$$X_S := Y_S / \mathrm{Frob}$$

If C alg. closed. perf. field

X_C "$\sim$" genus 0 complete curve.

$\cup$

U. affine & Connected.

$\Gamma(U, \mathcal{O})$ is a PID.

<u>Goal</u>: Study vector bundles on X_S.

classification. global sections ...

geometrize local class field theory.

Vector bundles on X_S

$$X_S = Y_S / \mathrm{Frob}.$$

$$\begin{array}{ccc}
\{\text{v.b. on } X_S\} & & \circlearrowleft \\
\wr\!\!\shortparallel & & \big\uparrow \text{descent} \\
\{\text{Frob-equ. v.b. on } Y_S\} & & V \underset{E}{\otimes} \mathcal{O}_{Y_S},\ \varphi \otimes \mathrm{Frob} \\
\big\uparrow & & \big\updownarrow \\
\boxed{\left\{ \begin{array}{l} E\text{-module } V \text{ w/} \\ \varphi : V \xrightarrow{\sim} V\ \sigma\text{-linear} \end{array} \right\}} & & V
\end{array}$$

$$\begin{array}{ccc}
E & \longrightarrow & \mathcal{O}_{Y_S} \\
\shortparallel & & \uparrow \\
\mathcal{O}_E[\pi^{-1}] & \longrightarrow & W_{\mathcal{O}_E}(R^+)
\end{array}$$

Def: For any perfect field $k/\mathbb{F}_q$.

$\mathrm{Isoc}_k = \{$ isocrystals over $k \}$

$$:= \left\{ \begin{array}{l} \underline{W_{\mathcal{O}_E}(k)[\frac{1}{\pi}]} \text{-module } V \\ \varphi : V \xrightarrow{\sim} V \quad \sigma\text{-linear} \\ V \text{ is f.d free module} \end{array} \right\}$$

Ex: 1) $\operatorname{rank} V = 1 \qquad \varphi = \pi^n \sigma$

$n \in \mathbb{Z}$.

2) $\lambda = s/r \in \mathbb{Q} \qquad r > 0 \ / \ \underline{(s,r) = 1}$

$\operatorname{rank} V_\lambda = r$

$$\varphi = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ \pi^s & & & & 0 \end{pmatrix}$$

defines $V_\lambda \in \mathrm{Isoc}_k$.

$\mathrm{Slope}(V_\lambda) = \lambda$.

<u>Def</u>: $\lambda \in \mathbb{Q}$

$V_\lambda \in \mathrm{Isoc}_{\overline{\mathbb{F}}_q} \rightsquigarrow \underline{\mathcal{O}(-\lambda)} \in \mathrm{Bun}_n(X_S)$

<u>Ex</u>: $\underline{\Gamma(X_S, \mathcal{O}(1)) = \Gamma(Y_S, \mathcal{O})^{\varphi = \pi}}$

For schematic version

$\Gamma(Y_S, \mathcal{O}) = B$

$A_{\inf, S} = W_{\mathcal{O}_E}(R^+)$

$B = A_{\inf, S}[\pi^{-1}, [\varpi]^{-1}]^\wedge$

"complete for all Gaussian norms".

$[a, b] \subset (0, \infty)$

$$B_{[a,b]} = (\qquad)^{\wedge}$$

Gamma? nums. between $[a,b]$

$$B = \bigcup B_{[a,b]}$$

$$X_S^{alg} = \mathrm{Proj} \bigoplus_{d \geq 0} B^{\varphi = \pi^d}$$

<u>Thm</u> (Dieudonne-Manin) $\underline{k = \bar{k}}$

$Isoc_k$ is semi-simple w/ simple objects $\{ V_\lambda, \lambda \in \mathbb{Q} \}$

$\mathrm{End}(V_\lambda) = D_\lambda$ the central simple $\underline{E}$-algebra $\mathrm{Br}(D_\lambda) = \lambda$
$\in \mathbb{Q}/\mathbb{Z}$.

$k = \bar{\mathbb{F}}_q$

$$\begin{array}{ccc} \mathcal{O}_E[\pi^{-1}] & \hookrightarrow & W_{\mathcal{O}_E}(k)[\pi^{-1}] \\ \wr & & \wr \\ E & \hookrightarrow & \breve{E} \end{array}$$

$\breve{E}$ is the maximal unramified extension of E if $k = \overline{\mathbb{F}}_q$

If $S \in \mathrm{Perf}_{/k}$

$$\begin{array}{ccc} \breve{E} & \longrightarrow & \mathcal{O}_{Y_S} \\ & \searrow & \uparrow \\ & & W_{\mathcal{O}_E}(R^+)[\pi^{-1}] \end{array}$$

$$\mathrm{Isoc}_k \longrightarrow \mathrm{Bun}(X_S)$$

(If S is defined over k)

<u>Exercise</u>, If S/k

$\lambda \in \mathbb{Q}$

$V = V_{\lambda, \mathbb{F}_q} \in \mathrm{Isoc}_{\mathbb{F}_q} \rightsquigarrow \mathcal{O}(-\lambda)$ on X_S.

$V_{\lambda, \underline{k}} \in \mathrm{Isoc}_{\underline{k}} \rightsquigarrow \mathcal{O}(-\lambda)$ on X_S

They are the same..

$$V \underset{E}{\otimes} \mathcal{O}_{Y_S} \simeq \breve{V} \underset{\breve{E}}{\otimes} \mathcal{O}_{Y_S}$$

Thm 1: (k residue field of C)

1) If C alg. clo. perf. field

$$\mathrm{Isoc}_k \longrightarrow \mathrm{Bun}(X_C)$$

is bijective on isom. classes.

In particular, $\mathrm{Pic}(X_C) = \mathbb{Z}$

we can define $\deg(\mathcal{E})$, $\mathrm{slope}(\mathcal{E})$

semi-stable, stable, ...

2) $\mathcal{O}(\lambda)$ is the unique stable vector bundle with slope λ.

Recall: If $\mathcal{E}$ is v.b.

$\mathcal{E}$ is semi-stable if $\forall\, \mathcal{F} \subseteq \mathcal{E}$

$\mathrm{slope}(\mathcal{F}) \leq \mathrm{Slope}(\mathcal{E})$

$\mathcal{E}$ is <u>stable</u> if

$\mathrm{slope}(\mathcal{F}) < \mathrm{slope}(\mathcal{E})$

3) $\forall\, \mathcal{E}$, $\mathcal{E} \simeq \bigoplus \mathcal{O}(\lambda)$

(Harder-Narasimhan filtration splits)

<u>Thm 2</u>:

S perfectoid $/\mathbb{F}_q$.

$\mathcal{E}$ is a v.b. on X_S

1)

$$\begin{tikzcd}
\{\text{geometric points of } S\} \arrow[d] & \mathrm{Spa}\, C \xrightarrow{s} S \arrow[d, mapsto] \\
\boxed{\{\text{H-N polygons}\}} & \mathrm{H\text{-}N}(\mathcal{E}_s)
\end{tikzcd}$$

$\big(\ X_s := X_S \times_S \mathrm{Spa}\, C$

$\mathcal{E}_s := \mathcal{E}|_{X_s}$

$\underline{\text{H-N}(\mathcal{E}_s)}$ is the H-N polygon for $\mathcal{E}_s$

is upper semi-continuous.

2) If $s \longmapsto \text{H-N}(\mathcal{E}_s)$ is constant,

$\exists$ H-N filtration on $\mathcal{E}$, which splits as $\bigoplus \mathcal{O}(\lambda)$ on pro-étale covers of S.

Thm 3: (GAGA)

$$X_S^{alg} := \mathrm{Proj} \bigoplus_{n \geq 0} H^0(X_S, \mathcal{O}(n))$$

$\exists\ X_S \to X_S^{alg}$ between locally ringed spaces.

$$\mathrm{Bun}(X_S^{alg}) \xrightarrow{\sim} \mathrm{Bun}(X_S)$$

Morrow

$$\mathcal{E}^{alg} \longleftrightarrow \mathcal{E}$$

$$RP(\mathcal{E}^{alg}) \simeq RP(\mathcal{E})$$

The main technique:

Banach–Colmez space

$\mathcal{E}_S$ on X_S

$\underline{\mathcal{BC}(\mathcal{E}_S)}$ "prestack of sections of $\mathcal{E}_S$"

Prop:

$$\begin{array}{ccc} T & \longmapsto & R\Gamma(X_T, \mathcal{E}_T) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathrm{Perf}_{/S} & \longrightarrow & \mathrm{Vect} \end{array}$$

is a v-sheaf on $\mathrm{Perf}_{/S}$

Def: $\mathcal{BC}(\mathcal{E}_S) := \{T \mapsto H^0(X_T, \mathcal{E}_T)\}$

Def: If $H^0(X_T, \mathcal{E}_T) = 0 \quad \forall T,$

$$\mathrm{BC}(\mathcal{E}_S[1]) := \{T \mapsto H^1(X_T, \mathcal{E}_T)\}.$$

<u>Def</u>: For $\lambda \in \mathbb{Q}$.

$$\mathrm{BC}(\mathcal{O}(\lambda)) := \{\underset{\underset{\text{Perf}}{\cap}}{S} \mapsto H^0(X_S, \mathcal{O}(\lambda))\}$$

$(\lambda \geq 0)$

$$\mathrm{BC}(\mathcal{O}(\lambda)[1]) = \{\underset{\underset{\text{Perf}}{\cap}}{S} \mapsto H^1(X_S, \mathcal{O}(\lambda))\}.$$

$\lambda < 0$

<u>Thm 4</u>:

1) If $\lambda < 0$, $H^0(X_S, \mathcal{O}(\lambda)) = 0$

<u>$\mathrm{BC}(\mathcal{O}(\lambda)[1])$</u> is well-defined.

It's a nice diamond.

locally spatial

$\boxed{\text{partially proper}}$ [proper without quasi-cpt]

cohomologically smooth

($\underline{\text{Spd } E \text{ is partially proper}}$).

2) $\lambda = 0$

$$\underline{E} = BC(\mathcal{O})$$

$\Big(S \longmapsto H^1(X_S, \mathcal{O}_{X_S})$ vanishes after pro-étale sheafification$\Big)$

3) $\lambda > 0$ $\quad H^1(X_S, \mathcal{O}_{X_S}) = 0$

$BC(\mathcal{O}(\lambda))$ is a nice diamond.

4) $\mathrm{BC}(\mathcal{O}(1)) = \mathrm{Spd}\ k[\![x_1^{1/p^\infty}]\!]$

$\lambda \leq [E:\mathbb{Q}_p]$

$\mathrm{BC}(\mathcal{O}(\lambda)) = \mathrm{Spd}\ k[\![x_1^{1/p^\infty}, \dots, x_r^{1/p^\infty}]\!]$

$r = \mathrm{rank}\ \mathcal{O}(\lambda)$

$$\mathrm{Div}^1 = \underline{\mathrm{Spd}\, E / \varphi^{\mathbb{Z}}}$$

$$\{ S \to \mathrm{Div}^1 \}$$

$$\|$$

$$\{ S^\sharp,\ \underline{S^\sharp \to \mathrm{Spa}}\, E \} / \varphi^{\mathbb{Z}}$$

$$\{ S^\sharp,\ \underset{\text{Cartier divisor.}}{S^\sharp \to Y_S} \}$$

$$\{ S^\sharp,\ \underset{\text{Cartier divisor.}}{S^\sharp \to Y_S \to X_S} \}$$

$$\mathrm{Div}^d = (\mathrm{Div}^1)^d / S_d$$

<u>Fact</u>: Div^d is a diamond.

<u>Classical story</u>:

rank 1 local system E_θ on X.

$E_\theta^{(d)}$ rank 1 local system on Div^d.

$$X^{(d)} \xrightarrow{AJ} \mathrm{Pic}^d_X$$

if $d > 2g-2$, then $E_\theta^{(d)}$ can descend to an ℓ-adic sheaf $\mathcal{F}_\theta^d$ on Pic^d_X.

$\underbrace{\text{(AJ's fibers is simple connected)}}$

Want to do this for X_S.

<u>Thm</u>:

$$\underline{\mathrm{Div}}^d = \{ \mathcal{BC}(\mathcal{O}(d)) \setminus \{0\} \} / \underline{E}^\times$$

$\left(\underline{E} \simeq \underline{BC}(0) \curvearrowright BC(\mathcal{O}(d)) \right)$

$\underline{\text{Cor}}$: $\boxed{BC(\mathcal{O}(d)) \setminus \{0\}}$

$\downarrow$ $\quad$ $\underline{E}^{*}$-torsor

Div^{d}

$AJ^{d}: \mathrm{Div}^{d} \longrightarrow [*/\underline{E}^{*}]$

$\underline{\text{Thm}}$: If $d \geq 2$, then $BC(\mathcal{O}(d)) \setminus \{0\}$ is simply-connected.

$\boxed{\underline{\text{Thm}}: \pi_1(\mathrm{Div}^1_k) \simeq W_E}$

$$
\begin{tikzcd}
1 \arrow[r] & I_E \arrow[r] \arrow[d] & W_E \arrow[r] \arrow[d, hook] & \mathbb{Z}\,\mathrm{Frob} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & I_E \arrow[r] & \mathrm{Gal}(\bar{E}/E) \arrow[r] & \hat{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}
$$

$$
\begin{tikzcd}
W_E \arrow[d] & = & \pi_1(\mathrm{Div}^1_{\mathbb{F}_q}) \arrow[d, "\pi_1(AJ^1)"] \\
E^\times & = & \pi_1(\mathrm{Div}^1)
\end{tikzcd}
$$

<u>Thm</u>: This map is the Artin reciprocity map in number theory.