

1 Free Line Parametrization

A coordinate frame L_i is defined with origin ${}_W\mathbf{P}_L$ on the line and with minimum distance d_L to the frame origin. The x axis is aligned with the line and the z axis points away from the origin. The parametrization is as follows:

$$\mathbf{l} = [q_{WL}^T \quad d_L]. \quad (1.1)$$

The line anchor ${}_W\mathbf{P}_L$ can be calculated in the following way:

$${}_W\mathbf{P}_L = q_{WL} \times \begin{bmatrix} 0 \\ 0 \\ d_L \end{bmatrix} \quad (1.2)$$

And the line direction ${}_W\mathbf{x}_L$:

$${}_W\mathbf{x}_L = q_{WL} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad (1.3)$$

2 Measurement

Let \mathbf{f}_1 and \mathbf{f}_2 be the bearing vectors through the two endpoints of the detected line segment. $\tilde{\mathbf{n}} = \frac{\mathbf{f}_1 \times \mathbf{f}_2}{\|\mathbf{f}_1 \times \mathbf{f}_2\|}$ is the normalized normal vector of the plane spanned by the two bearing vectors and defines our measurement.

3 Error formulation

Given: Measurement $\tilde{\mathbf{n}}$ and line $\mathbf{l} = [q_{WL}^T \quad d_L]$.

3.1 Parallel constraint

The direction of the line is parallel to the plane defined with $\tilde{\mathbf{n}}$:

$$\varepsilon_1 = {}_C\tilde{\mathbf{n}}^T C_{CB} \hat{C}_{BW} {}_W\mathbf{x}_L. \quad (3.1)$$

ε_1 is related to the angle $\delta\phi$ between plane and the line \mathbf{l} in the following way:

$$\varepsilon_1 = \sin \delta\phi. \quad (3.2)$$

3.2 Distance constraint

The anchor point should be part of the measured plane. We define the second error term ε_2 as the sine of the angle between the bearing vectors through the

anchor point and the projection of that point on the measured plane.

$$\varepsilon_2 = \frac{{}_C\tilde{\mathbf{n}}^T C_{CB} \hat{C}_{BW} ({}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L)}{\|{}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L\|} \quad (3.3)$$

$${}_W\hat{\mathbf{p}}_C = -({}_W\hat{\mathbf{t}}_{CB} + {}_W\hat{\mathbf{t}}_{BW}) \quad (3.4)$$

$$= -\left(\hat{C}_{WB} C_{BC} {}_C\mathbf{t}_{CB} + \hat{C}_{WB} {}_B\hat{\mathbf{t}}_{BW}\right) \quad (3.5)$$

$$= -\hat{C}_{WB} (C_{BC} {}_C\mathbf{t}_{CB} + {}_B\hat{\mathbf{t}}_{BW}) \quad (3.6)$$

4 Jacobian w.r.t to T_{BW}

$$C_{BW} \leftarrow C_{BW} \text{Exp}(\boldsymbol{\delta}\phi) \quad (4.1)$$

$${}_B\mathbf{t}_{BW} \leftarrow {}_B\mathbf{t}_{BW} + C_{BW} \boldsymbol{\delta}\mathbf{t} \quad (4.2)$$

4.1 ε_1

$$\varepsilon_1 = {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \text{Exp}(\boldsymbol{\delta}\phi) {}_W\mathbf{x}_L \quad (4.3)$$

$$\approx {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} {}_W\mathbf{x}_L - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} {}_W\mathbf{x}_L^\wedge \boldsymbol{\delta}\phi \quad (4.4)$$

$$\frac{\partial \varepsilon_1}{\partial \boldsymbol{\delta}\phi} = -{}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} {}_W\mathbf{x}_L^\wedge \quad (4.5)$$

$$\frac{\partial \varepsilon_1}{\partial \boldsymbol{\delta}\mathbf{t}} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \quad (4.6)$$

4.2 ε_2

Let:

$$\mathbf{v} = \frac{{}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L}{\|{}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L\|} \quad (4.7)$$

Then:

$$\varepsilon_2 = {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \text{Exp}(\boldsymbol{\delta}\phi) \mathbf{v} \quad (4.8)$$

$$\approx {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v} - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge \boldsymbol{\delta}\phi \quad (4.9)$$

$$\frac{\partial \varepsilon_2}{\partial \boldsymbol{\delta}\phi} = {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta}\phi} - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge \quad (4.10)$$

$$\frac{\partial \varepsilon_2}{\partial \boldsymbol{\delta}\mathbf{t}} = {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta}\mathbf{t}} \quad (4.11)$$

$$\frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta}\phi} = \frac{\partial \mathbf{v}}{\partial {}_W\mathbf{p}_C} \frac{\partial {}_W\mathbf{p}_C}{\partial \boldsymbol{\delta}\phi} \quad (4.12)$$

$$\frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta}\mathbf{t}} = \frac{\partial \mathbf{v}}{\partial {}_W\mathbf{p}_C} \frac{\partial {}_W\mathbf{p}_C}{\partial \boldsymbol{\delta}\mathbf{t}} \quad (4.13)$$

Some derivatives:

$$\frac{d\|\mathbf{w}\|}{d\mathbf{w}} = \frac{1}{\|\mathbf{w}\|} \mathbf{w}^T \quad (4.14)$$

$$\frac{d\frac{1}{\|\mathbf{w}\|}}{d\mathbf{w}} = -\frac{1}{\|\mathbf{w}\|^3} \mathbf{w}^T \quad (4.15)$$

$$\begin{aligned} \frac{d\frac{\mathbf{w}}{\|\mathbf{w}\|}}{d\mathbf{w}} &= \frac{1}{\|\mathbf{w}\|} I - \frac{1}{\|\mathbf{w}\|^3} \mathbf{w} \mathbf{w}^T \\ &= \frac{1}{\|\mathbf{w}\|} \left(I - \frac{\mathbf{w}}{\|\mathbf{w}\|} \left(\frac{\mathbf{w}}{\|\mathbf{w}\|} \right)^T \right) \end{aligned} \quad (4.16)$$

It follows:

$$\frac{\partial \mathbf{v}}{\partial {}_w\mathbf{p}_C} = \frac{1}{\|{}_w\mathbf{p}_C - {}_w\mathbf{p}_L\|} (I - \mathbf{v} \mathbf{v}^T) \quad (4.17)$$

4.2.1 Derivative of camera position w.r.t to T_{BW}

$${}_w\mathbf{p}_C = -\text{Exp}(-\delta\phi) C_{WB} (C_{BC} \mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW} \delta\mathbf{t})) \quad (4.18)$$

$$\approx -[C_{WB} (C_{BC} \mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW} \delta\mathbf{t})) \quad (4.19)$$

$$-\delta\phi^\wedge C_{WB} (C_{BC} \mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW} \delta\mathbf{t}))] \quad (4.20)$$

$$\approx -C_{WB} (C_{BC} \mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW} \delta\mathbf{t})) \quad (4.21)$$

$$+ \delta\phi^\wedge C_{WB} (C_{BC} \mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW} \delta\mathbf{t})) \quad (4.22)$$

$$\approx -({}_w\mathbf{t}_{CB} + {}_w\mathbf{t}_{BW} + \delta\mathbf{t}) + \delta\phi^\wedge ({}_w\mathbf{t}_{CB} + {}_w\mathbf{t}_{BW} + \delta\mathbf{t}) \quad (4.23)$$

$$\approx -({}_w\mathbf{t}_{CB} + {}_w\mathbf{t}_{BW} + \delta\mathbf{t}) - ({}_w\mathbf{t}_{CB} + {}_w\mathbf{t}_{BW} + \delta\mathbf{t})^\wedge \delta\phi \quad (4.24)$$

$$(4.25)$$

$$\frac{\partial {}_w\mathbf{p}_C}{\partial \delta\phi} = -({}_w\mathbf{t}_{CB} + {}_w\mathbf{t}_{BW})^\wedge \quad (4.26)$$

$$\frac{\partial {}_w\mathbf{p}_C}{\partial \delta\mathbf{t}} = -I \quad (4.27)$$

4.2.2 Full jacobian

$$\frac{\partial \varepsilon_2}{\partial \delta \phi} = {}_C \tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \delta \phi} - {}_C \tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge \quad (4.28)$$

$$= {}_C \tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial {}_W \mathbf{p}_C} \frac{\partial {}_W \mathbf{p}_C}{\partial \delta \phi} - {}_C \tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge \quad (4.29)$$

$$= {}_W \tilde{\mathbf{n}}^T \frac{1}{\|{}_W \mathbf{p}_C - {}_W \mathbf{p}_L\|} (I - \mathbf{v} \mathbf{v}^T) \frac{\partial {}_W \mathbf{p}_C}{\partial \delta \phi} - {}_W \tilde{\mathbf{n}}^T \mathbf{v}^\wedge \quad (4.30)$$

$$= \frac{1}{\|{}_W \mathbf{p}_C - {}_W \mathbf{p}_L\|} {}_W \tilde{\mathbf{n}}^T (I - \mathbf{v} \mathbf{v}^T) (-({}_W \mathbf{t}_{CB} + {}_W \mathbf{t}_{BW})^\wedge) - {}_W \tilde{\mathbf{n}}^T \mathbf{v}^\wedge \quad (4.31)$$

$$= \frac{-1}{\|{}_W \mathbf{p}_C - {}_W \mathbf{p}_L\|} {}_W \tilde{\mathbf{n}}^T (I - \mathbf{v} \mathbf{v}^T) ({}_W \mathbf{t}_{CB} + {}_W \mathbf{t}_{BW})^\wedge - {}_W \tilde{\mathbf{n}}^T \mathbf{v}^\wedge \quad (4.32)$$

$$= -{}_W \tilde{\mathbf{n}}^T \left(\frac{1}{\|{}_W \mathbf{p}_C - {}_W \mathbf{p}_L\|} (I - \mathbf{v} \mathbf{v}^T) ({}_W \mathbf{t}_{CB} + {}_W \mathbf{t}_{BW})^\wedge + \mathbf{v}^\wedge \right) \quad (4.33)$$

$$\frac{\partial \varepsilon_2}{\partial \delta \mathbf{t}} = {}_C \tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \delta \mathbf{t}} \quad (4.34)$$

$$= {}_W \tilde{\mathbf{n}}^T \frac{\partial \mathbf{v}}{\partial {}_W \mathbf{p}_C} \frac{\partial {}_W \mathbf{p}_C}{\partial \delta \mathbf{t}} \quad (4.35)$$

$$= {}_W \tilde{\mathbf{n}}^T \frac{1}{\|{}_W \mathbf{p}_C - {}_W \mathbf{p}_L\|} (I - \mathbf{v} \mathbf{v}^T) \frac{\partial {}_W \mathbf{p}_C}{\partial \delta \mathbf{t}} \quad (4.36)$$

$$= \frac{1}{\|{}_W \mathbf{p}_C - {}_W \mathbf{p}_L\|} {}_W \tilde{\mathbf{n}}^T (I - \mathbf{v} \mathbf{v}^T) (-I) \quad (4.37)$$

$$= \frac{-1}{\|{}_W \mathbf{p}_C - {}_W \mathbf{p}_L\|} {}_W \tilde{\mathbf{n}}^T (I - \mathbf{v} \mathbf{v}^T) \quad (4.38)$$