1 Free Line Parametrization

A coordinate frame L_i is defined with origin ${}_W\mathbf{p}_L$ on the line and with minimum distance d_L to the frame origin. The x axis is aligned with the line and the z axis points away from the origin. The parametrization is as follows:

$$\mathbf{l} = \begin{bmatrix} q_{WL}^T & d_L \end{bmatrix}. \tag{1.1}$$

The line anchor ${}_W\mathbf{p}_L$ can be calculated in the following way:

$${}_W\mathbf{p}_L = q_{WL} \times \begin{bmatrix} 0 \\ 0 \\ d_L \end{bmatrix} \tag{1.2}$$

And the line direction ${}_W\mathbf{x}_L$:

$${}_W\mathbf{x}_L = q_{WL} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{1.3}$$

2 Measurement

Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the bearing vectors through the two endpoints of the detected line segment. $\tilde{\mathbf{n}} = \frac{\mathbf{f}_1 \times \mathbf{f}_2}{||\mathbf{f}_1 \times \mathbf{f}_2||}$ is the normalized normal vector of the plane spanned by the two bearing vectors and defines our measurement.

3 Error formulation

Given: Measurement $\tilde{\mathbf{n}}$ and line $\mathbf{l} = [q_{WL}^T \quad d_L]$.

3.1 Parallel constraint

The direction of the line is parallel to the plane defined with $\tilde{\mathbf{n}}$:

$$\varepsilon_1 = {}_C\tilde{\mathbf{n}}^T C_{CB} \hat{C}_{BW\,W}\mathbf{x}_L. \tag{3.1}$$

ε_1 is related to the angle $\delta\phi$ between plane and the line $\mathbf{l}$ in the following way:

$$\varepsilon_1 = \sin \delta\phi. \tag{3.2}$$

3.2 Distance constraint

The anchor point should be part of the measured plane. We define the second error term ε_2 as the sine of the angle between the bearing vectors through the anchor point and the projection of that point on the measured plane.

$$
\begin{aligned}
\varepsilon_2 &= \frac{{}_C\tilde{\mathbf{n}}^T C_{CB} \hat{C}_{BW} \left({}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L \right)}{||{}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L||} && (3.3) \\
{}_W\hat{\mathbf{p}}_C &= -\left({}_W\hat{\mathbf{t}}_{CB} + {}_W\hat{\mathbf{t}}_{BW} \right) && (3.4) \\
&= -\left(\hat{C}_{WB} C_{BC\,C}\mathbf{t}_{CB} + \hat{C}_{WB\,B}\hat{\mathbf{t}}_{BW} \right) && (3.5) \\
&= -\hat{C}_{WB} \left(C_{BC\,C}\mathbf{t}_{CB} + {}_B\hat{\mathbf{t}}_{BW} \right) && (3.6)
\end{aligned}
$$

4 Jacobian w.r.t to T_{BW}

$$
\begin{aligned}
C_{BW} &\leftarrow C_{BW}\mathrm{Exp}(\boldsymbol{\delta\phi}) && (4.1) \\
{}_B\mathbf{t}_{BW} &\leftarrow {}_B\mathbf{t}_{BW} + C_{BW}\boldsymbol{\delta t} && (4.2)
\end{aligned}
$$

4.1 $\boldsymbol{\varepsilon_1}$

$$
\begin{aligned}
\varepsilon_1 &= {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathrm{Exp}(\boldsymbol{\delta\phi})\, {}_W\mathbf{x}_L && (4.3) \\
&\approx {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW\,W}\mathbf{x}_L - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW\,W}\mathbf{x}_L{}^\wedge \boldsymbol{\delta\phi} && (4.4)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varepsilon_1}{\partial \boldsymbol{\delta\phi}} &= -{}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW\,W}\mathbf{x}_L{}^\wedge && (4.5) \\
\frac{\partial \varepsilon_1}{\partial \boldsymbol{\delta t}} &= \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} && (4.6)
\end{aligned}
$$

4.2 $\boldsymbol{\varepsilon_2}$

Let:

$$
\mathbf{v} = \frac{{}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L}{||{}_W\hat{\mathbf{p}}_C - {}_W\mathbf{p}_L||} \qquad (4.7)
$$

Then:

$$
\begin{aligned}
\varepsilon_2 &= {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathrm{Exp}(\boldsymbol{\delta\phi}) \mathbf{v} && (4.8) \\
&\approx {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v} - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge \boldsymbol{\delta\phi} && (4.9)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \varepsilon_2}{\partial \boldsymbol{\delta\phi}} &= {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta\phi}} - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge && (4.10) \\
\frac{\partial \varepsilon_2}{\partial \boldsymbol{\delta t}} &= {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta t}} && (4.11)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta\phi}} &= \frac{\partial \mathbf{v}}{\partial_W\mathbf{p}_C} \frac{\partial_W\mathbf{p}_C}{\partial \boldsymbol{\delta\phi}} && (4.12) \\
\frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta t}} &= \frac{\partial \mathbf{v}}{\partial_W\mathbf{p}_C} \frac{\partial_W\mathbf{p}_C}{\partial \boldsymbol{\delta t}} && (4.13)
\end{aligned}
$$

Some derivatives:

$$\frac{d\,||\mathbf{w}||}{d\mathbf{w}} = \frac{1}{||\mathbf{w}||}\mathbf{w}^T \tag{4.14}$$

$$\frac{d\frac{1}{||\mathbf{w}||}}{d\mathbf{w}} = -\frac{1}{||\mathbf{w}||^3}\mathbf{w}^T \tag{4.15}$$

$$\begin{aligned}
\frac{d\frac{\mathbf{w}}{||\mathbf{w}||}}{d\mathbf{w}} &= \frac{1}{||\mathbf{w}||}I - \frac{1}{||\mathbf{w}||^3}\mathbf{w}\mathbf{w}^T \\
&= \frac{1}{||\mathbf{w}||}\left(I - \frac{\mathbf{w}}{||\mathbf{w}||}\left(\frac{\mathbf{w}}{||\mathbf{w}||}\right)^T\right) \qquad (4.16)
\end{aligned}$$

It follows:

$$\frac{\partial\mathbf{v}}{\partial_W\mathbf{p}_C} = \frac{1}{||_W\mathbf{p}_C - {}_W\mathbf{p}_L||}\left(I - \mathbf{v}\mathbf{v}^T\right) \tag{4.17}$$

4.2.1 Derivative of camera position w.r.t to T_{BW}

$$\begin{aligned}
{}_W\mathbf{p}_C &= -\mathrm{Exp}(-\boldsymbol{\delta\phi})C_{WB}\left(C_{BC\,C}\mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW}\boldsymbol{\delta t})\right) && (4.18)\\
&\approx -\big[C_{WB}\left(C_{BC\,C}\mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW}\boldsymbol{\delta t})\right) && (4.19)\\
&\quad -\boldsymbol{\delta\phi}^\wedge C_{WB}\left(C_{BC\,C}\mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW}\boldsymbol{\delta t})\right)\big] && (4.20)\\
&\approx -C_{WB}\left(C_{BC\,C}\mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW}\boldsymbol{\delta t})\right) && (4.21)\\
&\quad + \boldsymbol{\delta\phi}^\wedge C_{WB}\left(C_{BC\,C}\mathbf{t}_{CB} + ({}_B\mathbf{t}_{BW} + C_{BW}\boldsymbol{\delta t})\right) && (4.22)\\
&\approx -({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW} + \boldsymbol{\delta t}) + \boldsymbol{\delta\phi}^\wedge\left({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW} + \boldsymbol{\delta t}\right) && (4.23)\\
&\approx -({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW} + \boldsymbol{\delta t}) - ({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW} + \boldsymbol{\delta t})^\wedge\,\boldsymbol{\delta\phi} && (4.24)\\
& && (4.25)
\end{aligned}$$

$$\frac{\partial_W\mathbf{p}_C}{\partial\boldsymbol{\delta\phi}} = -({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW})^\wedge \tag{4.26}$$

$$\frac{\partial_W\mathbf{p}_C}{\partial\boldsymbol{\delta t}} = -I \tag{4.27}$$

4.2.2 Full jacobian

$$\frac{\partial \varepsilon_2}{\partial \boldsymbol{\delta\phi}} = {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta\phi}} - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge \tag{4.28}$$

$$= {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial {}_W\mathbf{p}_C} \frac{\partial {}_W\mathbf{p}_C}{\partial \boldsymbol{\delta\phi}} - {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \mathbf{v}^\wedge \tag{4.29}$$

$$= {}_W\tilde{\mathbf{n}}^T \frac{1}{||{}_W\mathbf{p}_C - {}_W\mathbf{p}_L||} \left(I - \mathbf{v}\mathbf{v}^T\right) \frac{\partial {}_W\mathbf{p}_C}{\partial \boldsymbol{\delta\phi}} - {}_W\tilde{\mathbf{n}}^T \mathbf{v}^\wedge \tag{4.30}$$

$$= \frac{1}{||{}_W\mathbf{p}_C - {}_W\mathbf{p}_L||} {}_W\tilde{\mathbf{n}}^T \left(I - \mathbf{v}\mathbf{v}^T\right) \left(-({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW})^\wedge\right) - {}_W\tilde{\mathbf{n}}^T \mathbf{v}^\wedge \tag{4.31}$$

$$= \frac{-1}{||{}_W\mathbf{p}_C - {}_W\mathbf{p}_L||} {}_W\tilde{\mathbf{n}}^T \left(I - \mathbf{v}\mathbf{v}^T\right) \left({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW}\right)^\wedge - {}_W\tilde{\mathbf{n}}^T \mathbf{v}^\wedge \tag{4.32}$$

$$= -{}_W\tilde{\mathbf{n}}^T \left(\frac{1}{||{}_W\mathbf{p}_C - {}_W\mathbf{p}_L||} \left(I - \mathbf{v}\mathbf{v}^T\right) \left({}_W\mathbf{t}_{CB} + {}_W\mathbf{t}_{BW}\right)^\wedge + \mathbf{v}^\wedge \right) \tag{4.33}$$

$$\frac{\partial \varepsilon_2}{\partial \boldsymbol{\delta t}} = {}_C\tilde{\mathbf{n}}^T C_{CB} C_{BW} \frac{\partial \mathbf{v}}{\partial \boldsymbol{\delta t}} \tag{4.34}$$

$$= {}_W\tilde{\mathbf{n}}^T \frac{\partial \mathbf{v}}{\partial {}_W\mathbf{p}_C} \frac{\partial {}_W\mathbf{p}_C}{\partial \boldsymbol{\delta t}} \tag{4.35}$$

$$= {}_W\tilde{\mathbf{n}}^T \frac{1}{||{}_W\mathbf{p}_C - {}_W\mathbf{p}_L||} \left(I - \mathbf{v}\mathbf{v}^T\right) \frac{\partial {}_W\mathbf{p}_C}{\partial \boldsymbol{\delta t}} \tag{4.36}$$

$$= \frac{1}{||{}_W\mathbf{p}_C - {}_W\mathbf{p}_L||} {}_W\tilde{\mathbf{n}}^T \left(I - \mathbf{v}\mathbf{v}^T\right) (-I) \tag{4.37}$$

$$= \frac{-1}{||{}_W\mathbf{p}_C - {}_W\mathbf{p}_L||} {}_W\tilde{\mathbf{n}}^T \left(I - \mathbf{v}\mathbf{v}^T\right) \tag{4.38}$$