BAN250

HW1

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1>

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2.2. Given the matrices

$$\mathbf{A} = \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$$

perform the indicated multiplications.

(a) $5\mathbf{A}$

(b) $\mathbf{BA}$

(c) $\mathbf{A}'\mathbf{B}'$

(d) $\mathbf{C}'\mathbf{B}$

(e) Is $\mathbf{AB}$ defined?

a)

$$5A = 5 \cdot \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} -5 & 15 \\ 20 & 10 \end{bmatrix}$$

b)

$$BA = \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 3 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4(-1)+(-3)4 & 4\cdot 3+(-3)\cdot 2 \\ 1\cdot(-1)+(-2)4 & 1\cdot 3+(-2)\cdot 2 \\ (-2)(-1)+0\cdot 4 & (-2)\cdot 3+0\cdot 2 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & 6 \\ -9 & -1 \\ 2 & -6 \end{bmatrix}$$

c)

$$A'B' = \begin{bmatrix} -1 & 4 \\ 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & 1 & -2 \\ -3 & -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)4+4(-3) & (-1)\cdot 1+4(-2) & (-1)(-2)+4\cdot 0 \\ 3\cdot 4+2(-3) & 3\cdot 1+2(-2) & 3(-2)+2\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -4-12 & -1-8 & 2 \\ 12-6 & 3-4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} -16 & -9 & 2 \\ 6 & -1 & -6 \end{bmatrix}$$

d)

$$C'B = \begin{bmatrix} 5 & -4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -3 \\ 1 & -2 \\ -2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5\cdot 4+(-4)\cdot 1+2(-2) & 5(-3)+(-4)(-2)+2\cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 & -7 \end{bmatrix}$$

e) The matrix A is 2X2, while the matrix B is 3X2. Therefore, the multiplication is impossible because the inner dimensions are different.

2> **2.3.** Verify the following properties of the transpose when

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$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}, \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$$

(a) $(\mathbf{A}')' = \mathbf{A}$
(b) $(\mathbf{C}')^{-1} = (\mathbf{C}^{-1})'$
(c) $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
(d) For general $\underset{(m\times k)}{\mathbf{A}}$ and $\underset{(k\times \ell)}{\mathbf{B}}$, $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

a)

$$(A')' = \left(\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\right)' = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = A$$

So, we can proof that: $(\mathbf{A}')' = \mathbf{A}$

b)

Set: $(C')^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad C^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$\because (C')^{-1} \cdot (C') = I$

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{cases} a+4b=1 \\ 3a+2b=0 \\ c+4d=0 \\ 3c+2d=1 \end{cases} \Rightarrow \begin{cases} a=-0.2 \\ b=0.3 \\ c=0.4 \\ d=-0.1 \end{cases}$$

Therefore $(C')^{-1} = \begin{bmatrix} -0.2 & 0.3 \\ 0.4 & -0.1 \end{bmatrix}$

$\because$ Same Way $C^{-1} \cdot C = I$

$\therefore \begin{bmatrix} w & x \\ y & z \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{cases} w+3x=1 \\ 4w+2x=0 \\ y+3z=0 \\ 4y+2z=1 \end{cases} \Rightarrow \begin{cases} w=-0.2 \\ x=0.4 \\ y=0.3 \\ z=-0.1 \end{cases}$$

Therefore $C^{-1} = \begin{bmatrix} -0.2 & 0.4 \\ 0.3 & -0.1 \end{bmatrix}$

$(C^{-1})' = \begin{bmatrix} -0.2 & 0.3 \\ 0.4 & -0.1 \end{bmatrix}$

So, we can proof that: $(\mathbf{C}')^{-1} = (\mathbf{C}^{-1})'$

c)

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 2 \\ 5 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+5 & 8 & 4+3 \\ 1+15 & 4 & 2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 8 & 7 \\ 16 & 4 & 11 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 1 & 5 \\ 4 & 0 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2+5 & 1+15 \\ 8 & 4 \\ 4+3 & 2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 16 \\ 8 & 4 \\ 7 & 11 \end{bmatrix}$$

Therefore: $(AB)' = B'A'$

d): For general $\underset{(m\times k)}{\mathbf{A}}$ and $\underset{(k\times \ell)}{\mathbf{B}}$, $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$.

For general $\underset{(m\times k)}{A}$ and $\underset{(k\times l)}{B}$, Let's Set:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{m1} & \cdots & a_{mk} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1l} \\ \vdots & & \vdots \\ b_{k1} & \cdots & b_{kl} \end{bmatrix}$$

To Proof: $\underset{l\times m}{(AB)'} = \underset{l\times k}{B'}\ \underset{k\times m}{A'}$

$l \times m$

$$AB = (AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$$(AB)' = (AB)_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki}$$

$$B'A' = (B'A')_{ij}$$

$= [i^{th}$ row of $B']$ by $[j^{th}$ column of $A']$

$= [i^{th}$ column of $B]$ by $[j^{th}$ row of $A]$

$$= \sum_{k=1}^{n} b_{ki} a_{jk}$$

$$= \sum_{k=1}^{n} a_{jk} b_{ki} = (AB)'$$

3>

NOT (b)

2.7. Let **A** be as given in Exercise 2.6.

(a) Determine the eigenvalues and eigenvectors of **A**.

(b) Write the spectral decomposition of **A**.

(c) Find $\mathbf{A}^{-1}$.

(d) Find the eigenvalues and eigenvectors of $\mathbf{A}^{-1}$.

$$\mathbf{A} = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

a)

We can find two eigenvalues, which are 10 and 5.

The eigenvectors they are corresponding to are:

(-0.8944272, 0.4472136) and

(-0.4472136, -0.8944272)

```
> eigen(A)
eigen() decomposition
$values
[1] 10  5

$vectors
           [,1]       [,2]
[1,] -0.8944272 -0.4472136
[2,]  0.4472136 -0.8944272
```

c)

We can find the inverse of matrix A by using solve() function:

```
> solve(A)
     [,1] [,2]
[1,] 0.12 0.04
[2,] 0.04 0.18
```

d)

There are two eigenvalues: 0.2 and 0.1, which correspond to the eigenvectors (0.4472136, 0.8944272) and (-0.8944272, 0.4472136) respectively.

```
> inverse_A = solve(A)
> eigen(inverse_A)$values
[1] 0.2 0.1
> eigen(inverse_A)$vectors
          [,1]       [,2]
[1,] 0.4472136 -0.8944272
[2,] 0.8944272  0.4472136
```

4>

2.20. Determine the square-root matrix $\mathbf{A}^{1/2}$, using the matrix $\mathbf{A}$ in Exercise 2.3. Also, determine $\mathbf{A}^{-1/2}$, and show that $\mathbf{A}^{1/2}\mathbf{A}^{-1/2} = \mathbf{A}^{-1/2}\mathbf{A}^{1/2} = \mathbf{I}$.

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix},$$

```
> A = matrix( c(2,1,1,3), nrow = 2, byrow = T)
> A
     [,1] [,2]
[1,]    2    1
[2,]    1    3
> A_sqrt = sqrt(A)  # Or: A ^ (1/2)
> A_sqrt
         [,1]     [,2]
[1,] 1.414214 1.000000
[2,] 1.000000 1.732051
> A_sqrt_inv = solve(sqrt(A))
> A_sqrt_inv
           [,1]       [,2]
[1,]  1.1949383 -0.6898979
[2,] -0.6898979  0.9756630
> A_sqrt %*% A_sqrt_inv
             [,1] [,2]
[1,] 1.000000e+00    0
[2,] 2.220446e-16    1
> A_sqrt_inv %*% A_sqrt
     [,1]         [,2]
[1,]    1 2.220446e-16
[2,]    0 1.000000e+00
```

5> **2.21.** (See Result 2A.15) Using the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

(a) Calculate $\mathbf{A}'\mathbf{A}$ and obtain its eigenvalues and eigenvectors.

(b) Calculate $\mathbf{A}\mathbf{A}'$ and obtain its eigenvalues and eigenvectors. Check that the nonzero eigenvalues are the same as those in part a.

a)

A'A has two eigenvalues: 10 and 8, which correspond to the eigenvectors:

(0.7071068, 0.7071068) and

(-0.7071068, 0.7071068) respectively.

```
> A_trans = t(A)
> B = A_trans %*% A
> eigen(B)
eigen() decomposition
$values
[1] 10  8

$vectors
          [,1]       [,2]
[1,] 0.7071068 -0.7071068
[2,] 0.7071068  0.7071068
```

b)

AA' has three eigenvalues: 10, 8 and 0, which respectively correspond to the eigenvectors:

(-0.4472136, 0, -0.8944272),

(0, -1, 0) and

(0.8944272, 0, -0.4472136).

The non-zero eigenvalues are the same as those for the matrix A'A.

```
> C = A %*% A_trans
> eigen(C)
eigen() decomposition
$values
[1] 1.000000e+01 8.000000e+00 3.552714e-15

$vectors
           [,1] [,2]       [,3]
[1,] -0.4472136    0  0.8944272
[2,]  0.0000000   -1  0.0000000
[3,] -0.8944272    0 -0.4472136

> format(eigen(C)$values, scientific = F,digits = 0)
[1] "10" "8"  "0"
```

6> **2.24.** Let $\mathbf{X}$ have covariance matrix

$$\boldsymbol{\Sigma} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find

(a) $\boldsymbol{\Sigma}^{-1}$

(b) The eigenvalues and eigenvectors of $\boldsymbol{\Sigma}$.

(c) The eigenvalues and eigenvectors of $\boldsymbol{\Sigma}^{-1}$.

a)

We can get the inverse of the covariance matrix Sigma

by using solve() function, which assigned to Sigma_inv.

```
> Sigma_inv = solve(Sigma)
> Sigma_inv
     [,1]      [,2] [,3]
[1,] 0.25 0.0000000    0
[2,] 0.00 0.1111111    0
[3,] 0.00 0.0000000    1
> Sigma %*% Sigma_inv
     [,1] [,2] [,3]
[1,]    1    0    0
[2,]    0    1    0
[3,]    0    0    1
```

b)

The eigenvalues for Sigma are 9, 4, and 1, which respectively response to the eigenvectors:

(0, 1, 0)

(1, 0, 0) and

(0, 0, 1)

```
> eigen(Sigma)
eigen() decomposition
$values
[1] 9 4 1

$vectors
     [,1] [,2] [,3]
[1,]    0    1    0
[2,]    1    0    0
[3,]    0    0    1
```

c)

The eigenvalues for Sigma_inv are 1, 0.25, and 0.11, which respectively response to the eigenvectors:

(0, 0, 1)

(1, 0, 0) and

(0, 1, 0)

```
> eigen(Sigma_inv)
eigen() decomposition
$values
[1] 1.0000000 0.2500000 0.1111111

$vectors
     [,1] [,2] [,3]
[1,]    0    1    0
[2,]    0    0    1
[3,]    1    0    0
```

7>

2.25. Let $\mathbf{X}$ have covariance matrix

$$\mathbf{\Sigma} = \begin{bmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

(a) Determine $\boldsymbol{\rho}$ and $\mathbf{V}^{1/2}$.

(b) Multiply your matrices to check the relation $\mathbf{V}^{1/2}\boldsymbol{\rho}\mathbf{V}^{1/2} = \mathbf{\Sigma}$.

a)

```
> Sigma = matrix( c(25,-2,4,-2,4,1,4,1,9), nrow = 3, byrow = T)
> Sigma
     [,1] [,2] [,3]
[1,]   25   -2    4
[2,]   -2    4    1
[3,]    4    1    9
> sqrt(diag(Sigma))
[1] 5 2 3
> V_sqrt = matrix ( c(5,0,0,0,2,0,0,0,3), nrow = 3, byrow = T)
> V_sqrt
     [,1] [,2] [,3]
[1,]    5    0    0
[2,]    0    2    0
[3,]    0    0    3
> Row = solve(V_sqrt) %*% Sigma %*% solve(V_sqrt)
> Row
           [,1]       [,2]      [,3]
[1,]  1.0000000 -0.2000000 0.2666667
[2,] -0.2000000  1.0000000 0.1666667
[3,]  0.2666667  0.1666667 1.0000000
```

b)

```
> V_sqrt %*% Row %*% V_sqrt
     [,1] [,2] [,3]
[1,]   25   -2    4
[2,]   -2    4    1
[3,]    4    1    9
```

8>

2.41. You are given the random vector $\mathbf{X}' = [X_1, X_2, X_3, X_4]$ with mean vector $\boldsymbol{\mu}'_{\mathbf{X}} = [3, 2, -2, 0]$ and variance–covariance matrix

$$\boldsymbol{\Sigma}_{\mathbf{X}} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

(a) Find $E(\mathbf{AX})$, the mean of $\mathbf{AX}$.

(b) Find $\mathrm{Cov}(\mathbf{AX})$, the variances and covariances of $\mathbf{AX}$.

(c) Which pairs of linear combinations have zero covariances?

a)

```
> Mean_X = matrix( c(3,2,-2,0), nrow = 4, byrow = F)
> A = matrix( c(1,-1,0,0,1,1,-2,0,1,1,1,-3), nrow = 3, byrow = T)
> A %*% Mean_X
     [,1]
[1,]    1
[2,]    9
[3,]    3
```

b)

```
> ## for covariance matrix for (AX)
> Sigma_x = matrix( c(3,0,0,0,0,3,0,0,0,0,3,0,0,0,0,3), nrow = 4, byrow = T)
> Cov_AX = A %*% Sigma_x %*% t(A)
> ## for variances of (AX)
> diag(Cov_AX)
[1]  6 18 36
```

```
> Cov_AX
     [,1] [,2] [,3]
[1,]    6    0    0
[2,]    0   18    0
[3,]    0    0   36
```

c) According to the covariance matrix Cov_AX, all numbers are zero except for the diagonal, which means these three variables (AX1, AX2, AX3) are independent with each other, they all have zero covariances with others.