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Non-Dynamical Stochastic Resonance: Theory and Experiments with White and Arbitrarily Coloured Noise.

Z. GINGL (*), L. B. KISS (*)(**) and F. MOSS (***)

(*) *Institute for Experimental Physics, JATE University*
Dóm tér 9, H-6720 Szeged, Hungary
(**) *Solid State Physics Department, Institute of Technology*
Uppsala University - Box 534, S-75121 Uppsala, Sweden
(***) *Department of Physics, University of Missouri at St. Louis*
St. Louis, MO 63121, USA

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Abstract. – We describe the simplest system which shows stochastic resonance. Theoretical results for white and (almost) arbitrarily coloured noise are presented. The new system has new, unique properties which originate from its *non-dynamical* character; for example, the strength and phase shift of periodic response of the system is independent of the frequency. Experiments have been carried out with the following noise processes: (physical) white noise, (physical) Lorentzian noise and (physical) $1/f$ noise. With a small extension of the system, its linear-response regime can be significantly increased. As the system is similar to some simple models of neurons, the new results might have not only physical but also biological importance.

Introduction. – In the last decade's physics literature, stochastic-resonance (SR) effect has been one of the most interesting phenomena taking place in noisy non-linear dynamical systems (see, *e.g.*, [1-14]). The input of *stochastic resonators* [12] (non-linear systems showing SR) is fed by a Gaussian noise and a sinusoidal signal with frequency f_0, that is, a random excitation and a periodic one are acting on the system. There is an optimal strength of the input noise, such that the system's output power spectral density, at the signal frequency f_0, has a maximal value. This effect is called SR. It can be viewed as: the transfer of the input sinusoidal signal through the system shows a «resonance» *vs.* the strength of the input noise. It is a very interesting, and somewhat paradoxial effect, because it indicates that in these systems the existence of a certain amount of «indeterministic» excitation is necessary to obtain the optimal «deterministic» response. There are certain indications [2, 13, 14] that the principle of SR may be applied by nature in biological systems in order to optimise the transfer of neural signals.

Until last year, it was a common belief that SR phenomena occur only in (bistable, sometimes monostable [10] or multistable) *dynamical* systems [1-14]. Very recently, Wiesenfeld *et al.* [15] have proposed that certain systems with threshold-like properties should also show SR effects.

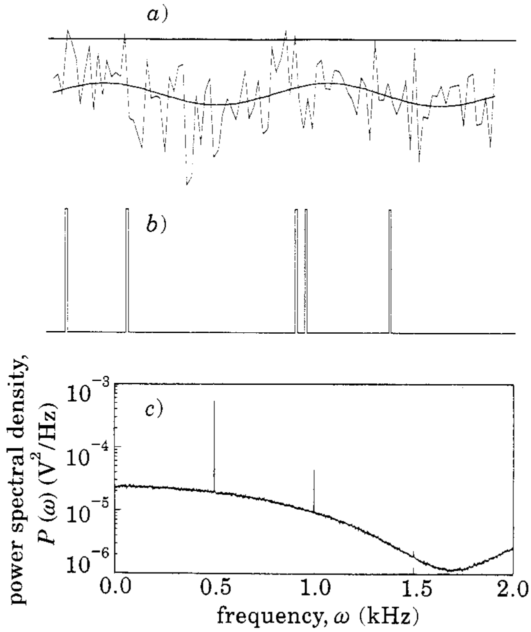

Fig. 1. – A numerical representation of non-dynamical stochastic resonance. *a*) The coherent signal, a sine wave, plus band-limited Gaussian noise, underlying a threshold shown by the straight line. The distance between the threshold line and the average of the sine wave is equal to the height of the threshold. *b*) A pulse train marking the undirectional (in this case, positive going) threshold-crossing events. *c*) The averaged power spectrum of the pulse train.

We present here an extremely simple system, invented by Moss, which displays SR. It consists only of a threshold and a subthreshold coherent signal plus noise as shown in fig. 1*a*). It is not a dynamical system, instead there is a single rule: whenever the signal plus the noise crosses the threshold unidirectionally, a narrow pulse of standard shape is written to a time series, as shown in fig. 1*b*). The power spectrum of this series of pulses is shown in fig. 1*c*). It shows all the familiar features of SR systems previously studied [1, 2, 7, 16], in particular, the narrow, delta-like signal features riding on a broad-band noise background from which the signal-to-noise ratio (SNR) can be extracted. This system can be easily realized electronically as a level-crossing detector (LCD). There is a simple and very physically motivated theory of this phenomenon (due to Kiss), see below. Other, more detailed studies of various aspects of threshold-crossing dynamics have been made by Fox *et al.* [17], Jung [18] and Bulsara *et al.* [19].

We have experimentally realised and developed this simple SR system and carried out extensive analog and computer simulations on it. The theory of Kiss has been verified for the case of white and several sorts of coloured noises. Until now, the description of this new SR system, its physical realisation and the original theory have not appeared in the open literature, so in this letter we shall describe the new system and its developments made by us, present the outline and the main results of the theory and finally show some interesting experimental results.

Description of the realised systems showing non-dynamical SR.

Asymmetric system. The asymmetric system consists of an LCD of the following kind: whenever the instantaneous strength of the input excitation (noise and small sinusoidal signal) crosses the positive threshold level U_t *in increasing direction*, the LCD produces a positive, short pulse with amplitude A and duration τ_0 at its output. The resulting output response of the system is a random time-sequence $u(t)$ of uniform, positive pulses (fig. 1).

Symmetric system. The symmetric system consists of an LCD of the following kind: whenever the instantaneous voltage (noise and small sinusoidal signal) crosses the positive threshold level U_t *in increasing direction*, the LCD produces a positive, short pulse with amplitude A and duration τ_0 at its output; on the other hand, whenever the instantaneous strength of the input excitation (noise and sinusoidal signal) crosses the negative threshold level $-U_t$ *in decreasing direction*, the LCD produces a negative, short pulse with amplitude $-A$ and duration τ_0 at its output. The resulting output response of the system is a random time sequence $u(t)$ of uniform, positive and negative pulses with zero time average.

It follows from the above definitions that the role of the sinusoidal input signal can be viewed as a periodic modulation of the threshold level(s) which causes a periodic modulation of the mean frequency ν of the threshold crossings of the noise, that is the mean repetition frequency ν of pulses at the output of the system.

Theoretical results. – The outline of the theory for asymmetric LCD follows (details described elsewhere [20]). The first theoretical problem is to determine the noise spectrum of a random sequence of pulses, when the frequency ν is weakly and slowly modulated. The time average U_{av} of randomly repeated, uniform, sufficiently short pulse sequence $u(t)$ is proportional to ν, $U_{av} = \langle u(t) \rangle_t = \nu A \tau_0$. The slow and small modulation of ν yields the linear modulation of the time average, $U_{av}(t) = \nu(t) A \tau_0$ which results in the transfer of the modulating signal through the system. In the case of sinusoidal modulation of ν with frequency f_0, the requirement of good transfer of the modulating signal requires: $f_0 \ll \nu(t)$. This condition also implies that $f_0 \ll \tau_{corr}^{-1}$, where τ_{corr} is the correlation time of the input noise. Moreover, when $\nu \ll \tau_{corr}^{-1}$ the time sequence of pulses can be considered purely random. At the above conditions, we applied Campbell's pulse-noise theorem to calculate the spectrum of the output response provided the time function $\nu(t)$ is known.

The second theoretical problem is to determine the unknown $\nu(t)$. In order to do that, we generalised the Rice theory of zero-crossings [21] for the crossings of arbitrary levels.

Finally, by combining the above-described methods, in the linear limit (much smaller signal than the r.m.s. and the U_t), we obtained the following results for the square P_s of the output sinusoidal signal (after subtracting the background noise which is the output power density spectrum without input sinusoidal signal):

$$P_s = (B \nu_0 A \tau_0 U_t)^2 \sigma^{-4} \exp[-(U_t/\sigma)^2], \tag{1}$$

and the phase shift of the first harmonic is zero compared to the input sinusoidal signal. The «modified» «signal-to-noise ratio» SNR $= P_s / S(f_0)$, where $S(f)$ is the output background noise

$$\mathrm{SNR} = \nu_0 (B U_t)^2 \sigma^{-4} \exp[-(U_t/\sigma)^2/2]. \tag{2}$$

In the above equations, B is the amplitude of the input periodic signal, ν_0 is the mean frequency of unidirectional zero-crossings of the input noise, and σ is the root-mean-square (r.m.s.) amplitude of input noise. The value of ν_0 is obtained from the original Rice formula [21]:

$$\nu_0 = \sigma^{-1} \sqrt{\int_0^\infty f^2 S(f)\, \mathrm{d}f}. \tag{3}$$

Note that the last square-root term is the r.m.s. velocity (time derivative) of the noise amplitude. Furthermore, analysing eq. (1), it is obvious that in the limit of our approximation, the *only important properties* of the input noise are its r.m.s. amplitude σ and its r.m.s. velocity, the particular structure of its spectrum $S(f)$ does not have influence on the strength of the output signal. Moreover, the frequency of the input signal does *not* play any role in eqs. (1) and (2). It is important to note that these properties are present only in this threshold system, even the very recent threshold SR systems proposed by Jung [18] and Bulsara *et al.* [19] do show frequency-dependent behaviour. These properties are completely new in the field of SR, and they originate from the fundamental and non-dynamical character of this system, which is successfully represented by the theory outlined above and verified by our experiments, see below.

Experiments. – Both the symmetric and the asymmetric LCD systems have been realised electronically, using comparators and monostable multivibrators and practical Gaussian noise sources (details described elsewhere [20]). The values of pulse parameters were $A = 5$ V and $\tau_0 = 1\ \mu$s. In fig. 2, the experimental results for P_s and the SNR with physical (band-limited from above) white noise are presented. It is clear that there is no observable difference between the data obtained at 38 Hz and 305 Hz signal frequencies; moreover, the fit of the data by eqs. (1) and (2) is excellent. In fig. 3, experimental results for P_s and the SNR with a *strongly coloured* noise, a physical (band-limited from below and above) $1/f$ noise, are presented. The fit of the signal data by eq. (1) is excellent and the fit of the SNR by eq. (2) is fairly good. We have observed the same behaviour for different kinds of Lorentzian noises (to be shown elsewhere [20]).

Note that, when the approximations of the theory, as outlined above, were *not* fulfilled, we have naturally found deviations from eqs. (1) and (2): for example, at very small input noise, the linear approach breaks down; and at very large input noise, correlations between level-crossing times can cause deviations on the SNR curve. These deviations will be shown and analysed elsewhere [20].

Summary. – We have described the simplest stochastic resonator. The new system produces a non-dynamical response, that is, no frequency dependence can be observed under the assumed conditions. The results of the outlined adiabatic and linear theory describe well the behaviour around the SR peak for both white and coloured noise.

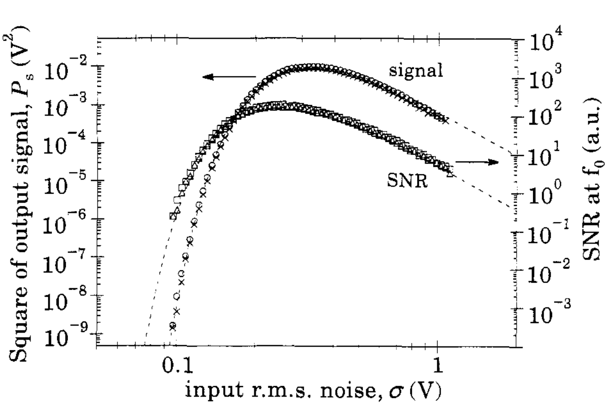

Fig. 2. – Representative plot of white-noise experiments on an asymmetric LCD system. Input signal: $B = 0.1$ V, $f_0 = 38$ Hz and 305 Hz; input noise: white, upper cut-off 12 kHz; $U_t = 0.45$ V, $\times$ and $\triangle$ 305 Hz; $\circ$ and $\square$ 38 Hz; curve fits: dashed lines by eqs. (1) and (2).

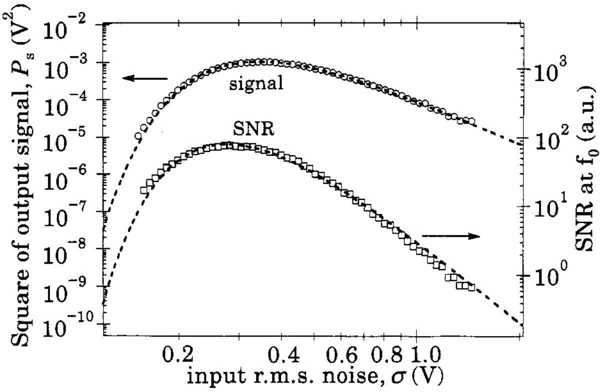

Fig. 3. – Representative plot of *strongly coloured noise* experiments on symmetric LCD system. Input signal: $B = 0.1$ V, $f_0 = 38$ Hz; input noise: Gaussian $1/f$ (lower and upper cut-offs: 300 Hz and 10 kHz); $U_t = 0.5$ V; curve fits: dashed lines by eqs. (1) and (2).

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