POSET DIMENSION AND SINGULAR CARDINALS

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1. On a question of Kierstead and Milner

Definition 1.1. For a poset $(P, \leq)$, the *dimension* of P is the least cardinal λ such that there exists a collection $\mathcal{H}$ of linear extensions of P with

- $|\mathcal{H}| = \lambda$,
- for any $a, b \in P$, $a \leq b$ iff for all $\preceq \in \mathcal{H}$, $a \preceq b$.

Definition 1.2. Fix cardinals α, κ.

(1) Let $d(\alpha, \kappa)$ denote the dimension of the partially ordered set $([\kappa]^{<\alpha}, \subseteq)$.
(2) Let $d_{<\alpha}(\kappa)$ denote the density of 2^κ with α-box product topology. In other words, $d_{<\alpha}(\kappa)$ is the least cardinal λ such that there exists a collection $\{f_i \in {}^\kappa 2 : i < \lambda\}$ with the property that for any $g : a \to 2$ where $a \in [\kappa]^{<\alpha}$, there is $i < \lambda$ such that $g \subset f_i$.

For any cardinal τ, define $\log(\tau)$ to be the least γ such that $2^\gamma \geq \tau$. In [2], Kierstead and Milner showed that for any infinite κ, $d(\omega, \kappa) = \log\log(\kappa)$. More generally, they showed if $\alpha \leq \mu =_{def} \log\log(\kappa)$, then $\mu \leq d(\alpha, \kappa) \leq \mu^{<\alpha}$. Komm [3] showed that for any cardinal κ, $d(\kappa^+, \kappa) = \kappa$. Therefore, we know how to compute $d(\alpha, \kappa)$ under GCH when κ is singular or $\log\log(\kappa)$ is regular. Kierstead and Milner asked what happens in the remaining case, namely, κ is regular and $\log\log(\kappa)$ is singular. In other words, $\kappa = \mu^+$ or $\kappa = \mu^{++}$ for some singular μ. In particular, they asked under GCH, is $d(\aleph_1, \aleph_{\omega+1})$ equal to $\aleph_\omega$ or $\aleph_{\omega+1}$. We address this question in this section.

The the rest, fix cardinals α, κ.

Lemma 1.3. *$d(\alpha, \kappa)$ equals the least cardinal λ such that there exist linear orders on κ, $\{\leq_i : i < \lambda\}$, such that for any $B \in [\kappa]^{<\alpha}$ and $y \notin B$, there is some $i < \lambda$ such that for any $x \in B$, $x <_i y$.*

Proof. Suppose $\{\leq_i : i < \theta\}$ is a collection of linear extensions witnessing $d(\alpha, \kappa) = \theta$. Define $<_i'$ on κ such that $x <_i' y$ iff $\{x\} <_i \{y\}$. For any $B \in [\kappa]^{<\alpha}$ and $y \notin B$, there exists $i < \theta$ such that $B <_i \{y\}$. Fix $x \in B$. We know that $\{x\} \leq_i B <_i \{y\}$, which implies $x <_i' y$.

Suppose $\{\leq_i' : i < \theta\}$ is a collection of linear orders on κ satisfying the property as in the lemma. For each i, we find a linear extension $\leq_i$ of $([\kappa]^{<\alpha}, \subset)$ such that $B <_i \{y\}$ whenever $B \in [\kappa - \{y\}]^{<\alpha}$ and for any $x \in B$, $x <_i' y$. This is possible by Lemma 3.1 of [2]. To see that $\{\leq_i : i < \theta\}$ realizes $([\kappa]^{<\alpha}, \subset)$, notice that for any $A \not\subset B$, we can find $y \in A - B$. Find $i < \theta$ such that $x <_i' y$ for all $x \in B$. Then $B <_i \{y\} \leq_i A$. $\square$

Theorem 1.1. *Fix regular uncountable κ. Let $\mu =_{def} \log\log\kappa < \kappa$. If $\kappa \leq \mu^{++}$ and $\mathrm{cf}(\mu) < \alpha$, then $d(\alpha, \kappa) > \mu$.*

Proof. If $\alpha \geq \kappa$, then $\kappa \geq d(\alpha,\kappa) \geq \alpha \geq \kappa > \mu$. We may assume $\alpha < \kappa$. In particular, $\alpha \leq \mu^+$.

We first assume that $\mathrm{cf}(\mu) < \mathrm{cf}(\alpha)$. Suppose for the sake of contradiction that $d(\alpha,\kappa) = \mu$. By Lemma 1.3, we know there are linear orders on κ, $\{\leq_i : i < \mu\}$, such that for any $B \in [\kappa]^{<\alpha}$ and $y \notin B$, there exists $i < \mu$ such that for any $x \in B$, $x <_i y$. Let $\mathrm{cf}(\mu) = \theta$ and fix an increasing sequence $\langle \mu_i : i < \theta\rangle$ converging to μ.

For each $i < \theta$, consider the set $A_i = \{y \in \kappa : \exists B \in [\kappa - \{y\}]^{<\alpha} \forall j < \mu_i \; \exists x \in B \; y <_j x\}$. We claim that A_i^c must have size $< \kappa$. Suppose for the sake of contradiction $X =_{def} A_i^c$ has size κ. By the definition of A_i, we know for any $B \in [X]^{<\alpha}$ and $y \in X - B$, there is $j < \mu_i$ such that for any $x \in B$, $x <_j y$. By Lemma 1.3, we know the dimension of $([X]^{<\alpha}, \subseteq)$ is less than or equal to μ_i. This contradicts with our assumption that $d(\alpha,\kappa) \geq \mu$.

Since κ is a regular cardinal and $\theta < \kappa$, we know that $|\bigcup_{i<\theta} A_i^c| < \kappa$. Pick $y \in \bigcap_{i<\theta} A_i$. For each $i < \theta$, let $B_i \in [\kappa - \{y\}]^{<\alpha}$ be such that for any $j < \mu_i$, there is some $x \in B$ with $y <_j x$. Consider $B = \bigcup_{i<\theta} B_i$. Note that $|B| < \alpha$ since $\theta < \mathrm{cf}(\alpha)$. Recall there is some $j < \mu$ such that any $x \in B <_j y$. Fix such j and some i such that $j < \mu_i$. By the definition of B_i, there is some $x \in B_i$, $y <_j x$. This is a contradiction.

Suppose now only that $\mathrm{cf}(\mu) < \alpha$. We may assume that $\mathrm{cf}(\mu) \geq \mathrm{cf}(\alpha)$ as otherwise by above we are done. In particular α is singular. Let $\alpha' < \alpha$ and $\alpha' > \mathrm{cf}(\mu)$ be a regular cardinal. By the argument above, we know $d(\alpha',\kappa) > \mu$. Notice that since $\alpha' \leq \alpha$, we have that $\mu < d(\alpha',\kappa) \leq d(\alpha,\kappa)$. □

A special case of the theorem above is that if $\aleph_\omega$ is a strong limit cardinal and $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$, then $d(\aleph_1, \aleph_{\omega+1}) = \aleph_{\omega+1}$.

Under GCH, we have the following complete answer ($\lambda =_{def} \log(\kappa), \mu =_{def} \log\log(\kappa)$):

$$d(\alpha,\kappa) = \begin{cases} \kappa & \text{if } \alpha \geq \kappa \text{ or if } \mathrm{cf}(\kappa) < \kappa \\ \mu & \text{else if } \mathrm{cf}(\kappa) = \kappa, \alpha \leq \mu \text{ and } \mathrm{cf}(\mu) = \mu \\ \lambda & \text{else if } \mathrm{cf}(\kappa) = \kappa, \alpha = \lambda \text{ and } \mathrm{cf}(\mu) = \mu \\ \mu^+ & \text{else if } \mathrm{cf}(\kappa) = \kappa, \mathrm{cf}(\mu) < \alpha \leq \mu^+ \text{ and } \mathrm{cf}(\mu) < \mu \\ \mu & \text{else if } \mathrm{cf}(\kappa) = \kappa, \mathrm{cf}(\mu) \geq \alpha \text{ and } \mathrm{cf}(\mu) < \mu \end{cases}$$

2. Consistency Results

Lemma 2.1. *Suppose in V, $d(\alpha,\lambda) \leq \theta$, then $d(\alpha,\lambda) \leq |\theta|$ is true in any forcing extension satisfying the α-covering property.*

Proof. Let $\{\leq_i : i < \theta\}$ be a collection of linear orders on λ witnessing $d(\alpha,\lambda) \leq \theta$. Work in $V[G]$. For any $B \in [\lambda]^{<\alpha}$ and $y \in \lambda - B$, by the α-covering property, there is $B' \in V \cap [\lambda]^{<\alpha}$ such that $B \subset B'$. We may assume $y \notin B'$. In V, there exists $i < \theta$ such that for all $x \in B'$, $x <_i y$. This clearly implies $d(\alpha,\lambda) \leq \theta$ in $V[G]$ by Lemma 1.3. □

Kierstead and Milner [2] actually showed $d(\alpha, 2^\kappa) \leq d_{<\alpha}(\kappa)$. It is then a natural question to ask if the inequality can be reversed. We note that it is possible to separate $d(\aleph_1,\kappa)$, $d_{<\aleph_1}(\kappa)$ and 2^κ.

Corollary 2.2. *The following are consistent:*

1. *there is some regular uncountable κ such that $d(\aleph_1, 2^\kappa) < d_{<\aleph_1}(\kappa)$,*

(2) *there is some singular strong limit κ of countable cofinality such that* $d(\aleph_1, 2^\kappa) < d_{<\aleph_1}(\kappa) < 2^\kappa$,
(3) *there is some singular strong limit κ of countable cofinality such that* $d(\aleph_1, 2^\kappa) < d_{<\aleph_1}(\kappa) = 2^\kappa$

Proof. Work in a ground model satisfying GCH. To see (1), fix a regular uncountable cardinal κ. Force with $\mathrm{Add}(\omega, \kappa^{++})$. Then in the forcing extension, $d_{<\aleph_1}(\kappa) \geq 2^\omega = \kappa^{++}$ and $d(\aleph_1, 2^\kappa) = d(\aleph_1, \kappa^{++}) \leq \kappa$ by Lemma 2.1. To see (2), with further large cardinal assumptions on κ, Gitik-Shelah [1] showed it is possible via some κ^{++}-c.c cardinal preserving extension to get a model where κ is a singular strong limit of countable cofinality, $2^\kappa = \kappa^{+4}$ and $d_{<\aleph_1}(\kappa) = \kappa^{+3}$. Since in the ground model, we have $d(\kappa^{++}, \kappa^{+4}) = \kappa^{++}$, by Lemma 2.1, in the Gitik-Shelah model, we have $d(\aleph_1, 2^\kappa) \leq d(\kappa^{++}, 2^\kappa) = d(\kappa^{++}, \kappa^{+4}) = \kappa^{++} < d_{<\aleph_1}(\kappa) = \kappa^{+3} < \kappa^{+4} = 2^\kappa$. To see (3), further force with $\mathrm{Coll}(\kappa^{+3}, \kappa^{+4})$ over the model in (2), since $d(\kappa^{++}, \kappa^{+3}) \leq \kappa^{++}$ in the ground model and the inequality continues to hold in the final model by Lemma 2.1. □

Remark 2.3. Start with a ground model with $\kappa < \lambda$ where κ is Laver indestrucibly supercompact and λ is a measurable cardinal. It is possible to find a cardinal preserving extension in which κ is a singular strong limit cardinal of countable cofinality, $2^\kappa = \lambda$ and $d(\kappa^+, 2^\kappa) = 2^\kappa$.

It seems that $d_{<\aleph_1}(\kappa)$ is largely connected with the pcf structure of the model, as evidenced by the following:

Lemma 2.4 (Shelah, [4])**.** *Let κ be a strong limit cardinal with $\mathrm{cf}(\kappa) = \omega$. Suppose the following are given: an increasing sequence of cardinals $\langle \kappa_n : n \in \omega \rangle$ converging to κ and $\langle \lambda_n : n \in \omega \rangle$ such that for all $n \in \omega$, $\lambda_n \leq d_{<\aleph_1}(\kappa_n)$. Then for any χ, if $\Pi_{n\in\omega}([2^{\kappa_n}]^{<\lambda_n}, \subseteq)$ is χ-directed under $\leq^*$, we have $d_{<\aleph_1}(\kappa) \geq \chi$.*

The hypotheses of Lemma 2.4 hold in many models. For example, if V satisfies $2^\kappa = \lambda$ and has a normal measure U such that the ultrapower $j =_{def} j_U : V \to M$ satisfies that ${}^{<\lambda} j(\kappa) \subset M$, then in the generic extension by Prikry forcing with U, for any $\chi < \lambda$, we can find $\langle \kappa_n : n \in \omega \rangle$ and $\langle \lambda_n : n \in \omega \rangle$ with the χ^+-directedness property as in Lemma 2.4. In particular in this model, $d_{<\aleph_1}(\kappa) = \lambda = 2^\kappa$. In contrast, the pcf structure in the model does not seem to affect $d(\aleph_1, 2^\kappa)$ in the similar way.

References

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