

STATIONARY AND CLOSED RAINBOW SUBSETS

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ABSTRACT. We study the structured rainbow Ramsey theory at uncountable cardinals. When compared to the usual rainbow Ramsey theory, the variation focuses on finding a rainbow subset that not only is of a certain cardinality but also satisfies certain structural constraints, such as being stationary or closed in its supremum. In the process of dealing with cardinals greater than ω_1 , we uncover some connections between versions of Chang's Conjectures and instances of rainbow Ramsey partition relations, addressing a question raised in [16].

0. INTRODUCTION

The study of rainbow Ramsey theory is dual to that of the usual Ramsey theory. A typical problem is that: given $f : [\kappa]^n \rightarrow \theta$ satisfying certain constraints, we are asked to find $y \subseteq \kappa$ satisfying certain constraints, such that y is *f-rainbow*, namely, $f \upharpoonright [y]^n$ is one-to-one.

Definition 0.1. We use $\lambda \rightarrow^{\text{poly}} (\kappa)_{<i-bdd}^n$ to abbreviate: for any $f : [\lambda]^n \rightarrow \lambda$ that is *< i-bounded*, namely for any $\alpha \in \lambda$, $|f^{-1}\{\alpha\}| < i$, there exists an *f-rainbow* $A \subset \lambda$ of order type κ .

We will let $\lambda \rightarrow^{\text{poly}} (\kappa)_{i-bdd}^n$ abbreviate $\lambda \rightarrow^{\text{poly}} (\kappa)_{<i^+-bdd}^n$.

Initial results in the area of infinite rainbow Ramsey theory were obtained by Galvin, followed by a series of results that appeared in [15], [2], [1], [16]. For a short history and an account of basic facts, we direct the reader to [2].

In this paper, we are concerned with structural strengthenings of the usual rainbow Ramsey partition relations, loosely motivated by the study of *topological partition relations*, see [13] for example.

Boundedness conditions based on the cardinality as in Definition 0.1 are the most studied constraints for the given coloring. We will encounter other types of boundedness conditions, as an immediate step to solve problems involving the cardinality-based boundedness conditions. The following general definition captures the main problems we would like to work on. Recall

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that a set $A \subset \text{Ord}$ is *closed*, if whenever $\sup A \cap \alpha = \alpha < \sup A$, then $\alpha \in A$.

Definition 0.2. Let \mathcal{P} be a property for colorings on $[\lambda]^n$.

- (1) We use $\lambda \rightarrow^{\text{poly}} (\alpha - cl)_{\mathcal{P}}^n$ to abbreviate: for any $f : [\lambda]^n \rightarrow \lambda$ satisfying \mathcal{P} , there exists a *closed* f -rainbow subset of order type α .
- (2) Similarly, $\lambda \rightarrow^{\text{poly}} (\alpha - st)_{\mathcal{P}}^n$ abbreviates: for any $f : [\lambda]^n \rightarrow \lambda$ satisfying \mathcal{P} , there exists a f -rainbow subset A of order type α that is *stationary* in $\sup A$.

Clearly, $\lambda \rightarrow^{\text{poly}} (\alpha - st)_{\mathcal{P}}^n$ only makes sense if $\text{cf}(\alpha) > \omega$. Todorćević in [15] showed that it is consistent relative to the consistency of ZFC that $\omega_1 \rightarrow^{\text{poly}} (\omega_1 - st)_{<\omega-bdd}^2$ and Abraham, Cummings and Smyth in [2] showed that under strong forcing axioms, namely Martin's Maximum, $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{<\omega-bdd}^2$. We will consider the improvements where the boundedness conditions are relaxed. Clearly, $\omega_1 \not\rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$ outright in ZFC. We show the next natural instance, after enlarging the source cardinal, is consistent.

Our main result reads:

Theorem 0.3. *Relative to the existence of a huge cardinal, it is consistent that $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$.*

The consistency of $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{\omega-bdd}^2$ relative to any reasonable assumption is still open. However, we are able to demonstrate an improvement to the Abraham-Cummings-Smyth result by relaxing the boundedness condition.

An interesting role that versions of Chang's Conjectures plays in certain "stepping-up" arguments has also been uncovered. Let us recall some definitions. Suppose that $\theta, \kappa, \lambda, \mu$ are infinite cardinals. *Chang's Conjecture* $(\mu, \lambda) \twoheadrightarrow (\kappa, \theta)$ is the following assertion: for every countable first-order language \mathcal{L} and for every \mathcal{L} -structure $A = (\mu, \lambda, \dots)$ there exists an $X \prec A$ such that $|X| = \kappa$ and $|X \cap \lambda| = \theta$. The special case $(\omega_2, \omega_1) \twoheadrightarrow (\omega_1, \omega)$ is called simply the *Chang's Conjecture* (CC).

In [2] and [1], it was shown that if κ is a regular cardinal satisfying $\kappa^{<\kappa} = \kappa$, then $\kappa^+ \rightarrow^{\text{poly}} (\eta)_{<\kappa-bdd}^2$ for all $\eta < \kappa^+$. Furthermore, these positive relations persist in certain forcing extensions. A question regarding the consistency of the partition relations when κ is singular was asked. In [16], the following were shown: for any singular κ with $\lambda = \text{cf}(\kappa) < \kappa$,

- (1) $\kappa^+ \not\rightarrow^{\text{poly}} (\eta)_{<\kappa-bdd}^2$, whenever $\eta > \lambda^+$,
- (2) $\kappa^+ \rightarrow^{\text{poly}} (\eta)_{<\kappa-bdd}^2$ for all $\eta < \lambda^+$ if in addition $\kappa^{<\lambda} = \kappa$ and
- (3) $S_{\lambda^+}^{\kappa^+} \in I[\kappa^+]$ implies $\kappa^+ \not\rightarrow^{\text{poly}} (\lambda^+)_{<\kappa-bdd}^2$.

Here $I[\kappa^+]$ is Shelah's *approachability ideal*. See [5] for more information on these ideals. In particular, (3) above shows that $\kappa^+ \rightarrow^{\text{poly}} (\lambda^+)_{<\kappa-bdd}^2$ has high large cardinal strength, if consistent at all. A question regarding the

consistency of $\kappa^+ \rightarrow^{\text{poly}} (\omega_1)_{<\kappa-bdd}^2$ for some singular κ of countable cofinality was asked in [16, Question 3.16]. We will answer this question positively utilizing a version of the singular Chang's Conjectures whose consistency was first established in [10].

The structure of the paper is:

- (1) Section 1 contains some general discussion on obtaining closed rainbow subsets of various order types,
- (2) Section 2 contains the proof for Theorem 0.3,
- (3) Section 3 demonstrates that PFA does not imply $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$,
- (4) Section 4 illustrates a strengthening of $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{<\omega-bdd}^2$ under MM,
- (5) Section 5 concludes with some open questions.

Unless otherwise stated, we will assume all the colorings are *normal*, in the sense that a coloring f whose domain is $[\kappa]^n$ is normal, if whenever $a, b \in [\kappa]^n$ satisfy that $f(a) = f(b)$, we have $\max a = \max b$.

1. CLOSED RAINBOW SETS

We begin with discussing some ZFC constraints.

Proposition 1.1. *For any cardinal κ with $\text{cf}(\kappa) > \omega$, $\kappa \not\rightarrow^{\text{poly}} (\text{cf}(\kappa) - cl)_{2-bdd}^2$.*

Recall that it is a theorem of Shelah [14, Claim 2.3] that if $\kappa = \text{cf}(\kappa) \geq \omega_2$ then there is a club-guessing sequence supported on S_ω^κ . The consequence that we will use is that there exists a sequence $\langle C_\alpha : \alpha \in S_\omega^\kappa \rangle$ where each C_α of order type ω is cofinal in α , so that any club $D \subset \kappa$, there exists some $\alpha \in S_\omega^\kappa$ such that $C_\alpha \subset D$.

Lemma 1.2. *Suppose that $\kappa = \text{cf}(\kappa) \geq \omega_2$. Then $\kappa \not\rightarrow^{\text{poly}} (\kappa - cl)_{2-bdd}^2$.*

Proof.

Let $S = S_\omega^\kappa$ and let $\bar{A} = (A_\delta : \delta \in S)$ be a club guessing sequence. We define a normal 2-bounded coloring f by induction on $\delta \in \kappa$, making sure that for every $\delta \in S$, there are $\alpha_\delta \neq \beta_\delta \in A_\delta$ satisfying $f(\alpha_\delta, \delta) = f(\beta_\delta, \delta)$.

Let $E \subseteq \kappa$ be a club and let $S_E = \{\delta \in S : A_\delta \subseteq E\}$. Since \bar{A} guesses clubs, S_E is stationary. Choose $\delta \in \lim(E \cap S_E) \cap (E \cap S_E)$. By the construction we have $\alpha_\delta \neq \beta_\delta \in A_\delta \subseteq E$ so that $f(\alpha_\delta, \delta) = f(\beta_\delta, \delta)$. Since $\alpha_\delta, \beta_\delta, \delta \in E$, we conclude that E is not f -rainbow.

□_{1.2}

If there is a club guessing sequence $(A_\delta : \delta \in S_\omega^{\omega_1})$ then $\omega_1 \not\rightarrow^{\text{poly}} (\omega_1 - cl)_{2-bdd}^2$ by the same reasoning. However, ZFC does not prove the existence of such club guessing sequences. Hence, we employ a different proof in the case of a successor cardinal.

Lemma 1.3. *If κ is an infinite cardinal then $\kappa^+ \not\rightarrow^{\text{poly}} (\kappa^+ - cl)_{2-bdd}^2$.*

Proof.

Suppose for the sake of contradiction that $\kappa^+ \rightarrow^{\text{poly}} (\kappa^+ - cl)_{2-bdd}^2$. For each ordinal $\beta \in \kappa^+ - \kappa$, fix a bijection $f_\beta : \beta \rightarrow \kappa$. For every pair $(i, j) \in \kappa \times \kappa$, we define a 2-bounded normal coloring $g_{ij} : [\kappa^+ - \kappa]^2 \rightarrow \kappa^+$ such that if $\alpha < \gamma < \beta$, $f_\beta(\alpha) = i$ and $f_\beta(\gamma) = j$ then $g_{ij}(\alpha, \beta) = g_{ij}(\gamma, \beta)$. Such g_{ij} can be easily constructed by recursion.

By our assumption, for every $(i, j) \in \kappa \times \kappa$ there is a club C_{ij} of κ^+ that is g_{ij} -rainbow. Let $C = \bigcap \{C_{ij} : (i, j) \in \kappa \times \kappa\}$, so C is a club subset of κ^+ as well. Pick up any triple $\alpha, \beta, \gamma \in C$ such that $\alpha < \gamma < \beta$. Set $i = f_\beta(\alpha)$ and $j = f_\beta(\gamma)$. It follows that $g_{ij}(\alpha, \beta) = g_{ij}(\gamma, \beta)$, by the definition of g_{ij} . However, g_{ij} is supposed to be one-to-one on C since $C \subseteq C_{ij}$. We have arrived at a contradiction.

□_{1.3}

Remark 1.4. The key point in the proof of Lemma 1.3 is that the club filter over κ^+ is κ^+ -complete. Any such filter can replace the club filter in the proof.

Proposition 1.1 now follows from Lemma 1.3 and Lemma 1.2.

Proposition 1.5. *If $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{<\omega-bdd}^2$, then for every $\sigma \in \omega_1$, $\omega_2 \rightarrow^{\text{poly}} ((\omega_1 + \sigma) - cl)_{<\omega-bdd}^2$.*

Proof.

Fix a countable limit ordinal $\sigma \in \omega_1$ and a $< \omega$ -bounded coloring $f : [\omega_2]^2 \rightarrow \omega_2$. Apply the hypothesis, we can find two f -rainbow closed sets $C_0 < C_1$, both of order type ω_1 . For each $\beta \geq \sup C_0$, there exists a club $D_\beta \subset C_0$ such that $f \upharpoonright [D_\beta]^2 \times \{\beta\}$ is injective. The reason is that for each $\alpha \in C_0$, since the color class $t_{\alpha\beta} =_{\text{def}} \{\eta : f(\eta, \beta) = f(\alpha, \beta)\}$ is finite, there exists $\gamma_\alpha \in C_0$ such that $C_0 - \gamma_\alpha$ is disjoint from $t_{\alpha\beta}$. Then the elements from C_0 that are closure points for $\alpha \mapsto \gamma_\alpha$ will form D_β . Similarly, for any $\beta_0, \beta_1 \geq \sup C_0$, there is a club $E_{\beta_0, \beta_1} \subset C_0$ disjoint from $t_{\beta_0\beta_1}$.

Let $\delta = \sup C_0$. For each $\alpha \in \lim C_1$, there exists $\beta_\alpha \in C_1 \cap \alpha$ such that $t_{\delta\alpha}$ is disjoint from $\alpha - \beta_\alpha$ since $t_{\delta\alpha}$ is finite. Apply Fodor's lemma within C_1 , we get a stationary $S \subset C_1$ in $\sup C_1$ and some $\xi \in C_1$ such that for all $\alpha \in S$, $\alpha - \xi$ is disjoint from $t_{\delta\alpha}$. As S is essentially a stationary subset of ω_1 , it is *fat*. This in particular means we can find a closed subset F of S of order type σ . Notice that $\{\delta\} \cup F$ is f -rainbow. Let

$$C^* = \bigcap_{\beta \in F \cup \{\delta\}} D_\beta \cap \bigcap_{\{\beta_0, \beta_1\} \in [F \cup \{\delta\}]^2} E_{\beta_0, \beta_1},$$

which is a club in $\sup C_0$. It is easy to see that $C^* \cup \{\delta\} \cup F$ is the f -rainbow subset as desired.

□_{1.5}

In the above theorems, we have seen that prediction principles are very useful when trying to prove negative rainbow statements about *closed* sets.

However, parallel negative statements concerning stationary sets cannot be proved in full generality.

2. VERSIONS OF CHANG'S CONJECTURES AND RAINBOW SETS

The following boundedness condition naturally arises when versions of Chang's Conjectures are applied to study rainbow Ramsey theory.

Definition 2.1. Let κ and α be two ordinals. A function $f : [\kappa]^2 \rightarrow \kappa$ is $< \alpha$ -type bounded iff there is an ordinal $\gamma < \alpha$ so that $\text{otp}(t_{\alpha\beta}) \leq \gamma$ whenever $\alpha < \beta < \kappa$, where $t_{\alpha\beta} = \{\xi \in \beta : f(\xi, \beta) = f(\alpha, \beta)\}$.

From now on, the domains of the colorings considered are of the form $[C]^2$, where C is a subset of the ordinals.

Type boundedness is not as robust as the cardinality-based boundedness conditions in the sense that if f on $[A]^2$ is $< \alpha$ -type bounded, and $h : A \rightarrow B$ is a bijection, then f' defined on $[B]^2$ such that $f'(\alpha, \beta) = f(h^{-1}(\alpha), h^{-1}(\beta))$ does not need to be $< \alpha$ -type bounded. However, if in addition, we know that h is an order isomorphism, then f' is indeed $< \alpha$ -type bounded.

Recall Definition 0.2, $\kappa \rightarrow^{\text{poly}} (\theta)_{<\alpha-t-bdd}^2$ iff for every $f : [\kappa]^2 \rightarrow \kappa$ which is $< \alpha$ -type bounded one can find an f -rainbow set of order type θ . Similar notation like $\kappa \rightarrow^{\text{poly}} (\theta - st)_{<\alpha-t-bdd}^2$, $\kappa \rightarrow^{\text{poly}} (\theta - cl)_{<\alpha-t-bdd}^2$ should be interpreted accordingly.

Definition 2.2. Given a boundedness condition \mathcal{P} , let

$$\kappa \rightarrow^{\text{poly}} (\text{par}(< \alpha))_{\mathcal{P}}^2$$

abbreviate that for any $f : [\kappa]^2 \rightarrow \kappa$ satisfying \mathcal{P} , there exists a partition $g : \kappa \rightarrow \gamma$ for some $\gamma < \alpha$ such that for any $i \in \gamma$, $g^{-1}\{i\}$ is f -rainbow.

Theorem 2.3 (Todorćević [15], see also [2]). (1) *If ZFC is consistent, so is $\omega_1 \rightarrow^{\text{poly}} (\text{par}(< \omega_1))_{<\omega_1-t-bdd}^2$.*
 (2) *PFA implies $\omega_1 \rightarrow^{\text{poly}} (\text{par}(< \omega_1))_{<\omega_1-t-bdd}^2$.*

□_{2.3}

Remark 2.4. The theorem proved in [15] was not exactly phrased as the above: it only considers $< \omega$ -bounded colorings. However, the proof is essentially the same. The reader is invited to reconstruct the proof either from [15],[2] or from the proof of Claim 4.2 in the later section.

The following proposition connects Chang's Conjectures with rainbow Ramsey partition relations.

Proposition 2.5. *Let $\kappa, \lambda, \mu \in \text{cof}(> \omega), \rho, \theta$ be cardinals. Consider the following statements:*

- (a) $\kappa \rightarrow (\mu)_{\lambda, < \rho}^2$, namely for any $f : [\kappa]^2 \rightarrow \lambda$, there exists $A \subset \kappa$ of order type μ such that $|f''[A]^2| < \rho$.
- (b) $\kappa \rightarrow (\mu - st)_{\lambda, < \rho}^2$, the same as (a) except that we also require A to be stationary in its supremum.

- (c) $\mu \rightarrow^{\text{poly}} (\theta)_{<\rho-t-bdd}^2$.
- (d) $\mu \rightarrow^{\text{poly}} (\text{par}(<\text{cf}(\mu)))_{<\rho-t-bdd}^2$

Then

- (1) (a) + (c) implies $\kappa \rightarrow^{\text{poly}} (\theta)_{<\lambda^+-t-bdd}^2$ and
- (2) (b) + (d) implies $\kappa \rightarrow^{\text{poly}} (\mu - st)_{<\lambda^+-t-bdd}^2$.

Proof.

First we show (1). Assume that $f : [\kappa]^2 \rightarrow \kappa$ is $<\lambda^+$ -type bounded, as witnessed by $\gamma < \lambda^+$. Recall that we are assuming that f is normal. Let $h : \gamma \rightarrow \lambda$ be a bijection.

Define $g : [\kappa]^2 \rightarrow \lambda$ by letting $g(\alpha, \beta) = \xi$ iff α is the $h^{-1}(\xi)$ -th element of $t_{\alpha\beta} = \{\eta \in \beta : f(\alpha, \beta) = f(\eta, \beta)\}$. Applying (a), one can fix $A \subseteq \kappa$ such that A has order type μ and $g''[A]^2$ is a subset of λ of size $< \rho$, call it C'_A . Let $C_A = h^{-1}(C'_A)$.

Let $\delta = \text{otp}(C_A)$, so $\delta < \rho$. We claim that the restriction $f \upharpoonright [A]^2$ is $<\rho$ -type bounded as witnessed by δ . Indeed, given $\alpha < \beta \in A$, let $s_{\alpha\beta} = t_{\alpha\beta} \cap A$. Define $h_{\alpha\beta} : s_{\alpha\beta} \rightarrow C_A$ by $h_{\alpha\beta}(\gamma) = h^{-1}(g(\gamma, \beta))$. It follows from the definition of g that $h_{\alpha\beta}$ is one-to-one and order preserving, hence $\text{otp}(s_{\alpha\beta}) \leq \text{otp}(C_A) = \delta$ for every $\alpha, \beta \in A$. Therefore, $f \upharpoonright [A]^2$ is $<\rho$ -type bounded. By (c) and the fact that $<\rho$ -type-boundedness is preserved under order isomorphisms, we obtain a f -rainbow subset A' of A of order type θ .

The proof for (2) is basically the same. By applying (b), one gets A as above satisfying additionally that A is stationary in $\sup A$. Apply (d) instead, we can find a partition of A into $<\text{cf}(\mu)$ many f -rainbow subsets. One of them must have order type μ and is stationary in its supremum.

□_{2.5}

Corollary 2.6. *If the Chang's Conjecture and $\omega_1 \rightarrow^{\text{poly}} (\omega_1)_{<\omega_1-t-bdd}^2$ hold, then $\omega_2 \rightarrow^{\text{poly}} (\omega_1)_{\omega-bdd}^2$.*

□_{2.6}

Remark 2.7. The consistency of $\omega_2 \rightarrow^{\text{poly}} (\omega_1)_{\omega-bdd}^2$ does not require large cardinals. In fact, in [2], it was shown that it holds in any model obtained by forcing over a model of CH with any poset satisfying the ω_1 -covering property. Corollary 2.6 gives a different scenario not covered previously that $\omega_2 \rightarrow^{\text{poly}} (\omega_1)_{\omega-bdd}^2$ can hold, for example, under Martin's Maximum.

The following addresses the question raised in [16] regarding the rainbow partition relations at the successor of a singular cardinal, as promised in the introduction.

Corollary 2.8. *Suppose that:*

- (N) $\aleph_{\omega+1} \rightarrow (\omega_1 - st)_{\aleph_{\omega}, <\omega_1}^2$ and
- (Q) $\omega_1 \rightarrow^{\text{poly}} (\text{par}(<\omega_1))_{<\omega_1-t-bdd}^2$.

Then $\aleph_{\omega+1} \rightarrow^{\text{poly}} (\omega_1 - st)_{<\aleph_{\omega+1}-t-bdd}^2$.

□_{2.8}

Remark 2.9. Note that (\aleph) is a consequence of a stationary variation of $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$. Here is one way to demonstrate the hypothesis of Theorem 2.8 is consistent relative to the existence of large cardinals. The argument is essentially due to Levinski, Magidor and Shelah [10], while the large cardinal hypothesis needed is improved by Hayut [7]. By [10] and [7], with the appropriate large cardinal hypothesis (for example, the existence of an $(\omega + 1)$ -subcompact cardinal), we may suppose in the ground model, GCH holds and there are two strongly inaccessible cardinals $\kappa < \lambda$ satisfying $(\lambda^{+\omega+1}, \lambda^{+\omega}) \rightarrow (\text{stat}(\kappa^{+\omega+1} \cap \text{cof}(\omega)), \kappa^{+\omega})$. More precisely, the symbol means for any regular $\mu \geq \lambda^{+\omega+1}$ and any countable language L , there exists $M \prec (H(\mu), L)$ such that $|M \cap \lambda^{+\omega+1}| = \kappa^{+\omega+1}$ and $|M \cap \lambda^{+\omega}| = \kappa^{+\omega}$ and in addition, $M \cap \lambda^{+\omega+1} \cap \text{cof}(\omega)$ contains a set A of order type $\kappa^{+\omega+1}$ stationary in its supremum. We argue that the hypothesis of Theorem 2.8 holds in the forcing extension by $R =_{\text{def}} \text{Coll}(\omega, \kappa^{+\omega}) * P$, where P is a λ -c.c proper forcing in $V^{\text{Coll}(\omega, \kappa^{+\omega})}$ of size λ turning λ to ω_2 while forcing $\omega_1 \rightarrow^{\text{poly}} (\text{par}(< \omega_1))^2_{< \omega_1 - t - bdd}$ (see for example the proof of Theorem 2.3). Clearly we can assume $R \subset V_\lambda$. Notice that in the forcing extension by R , $(\kappa^{+\omega+1})^V = \omega_1$ and $\lambda = \omega_2$.

Let $\dot{c} : [\lambda^{+\omega+1}]^2 \rightarrow \lambda^{+\omega}$ be an R -name. Let μ be a sufficiently large regular cardinal and consider the structure $K = (H(\mu), \lambda^{+\omega+1}, \lambda^{+\omega}, R, \dot{c})$. Apply the hypothesis in the ground model, we get $M \prec K$ such that $|M \cap \lambda^{+\omega+1}| = \kappa^{+\omega+1}$ and $|M \cap \lambda^{+\omega}| = \kappa^{+\omega}$ and some $A \subset M \cap \lambda^{+\omega+1} \cap \text{cof}(\omega)$ of order type $\kappa^{+\omega+1}$ stationary in its supremum. Let $G \subset R$ be generic over V .

First notice that in $V[G]$, A is a subset of order type ω_1 stationary in its supremum in $V[G]$. The reason is that $\text{Coll}(\omega, \kappa^{+\omega})$ satisfies $\kappa^{+\omega+1}$ -c.c, which in turn implies that A remains stationary in its supremum whose cofinality becomes ω_1 in $V^{\text{Coll}(\omega, \kappa^{+\omega})}$. As P is proper in $V^{\text{Coll}(\omega, \kappa^{+\omega})}$, A remains stationary in $\sup A$ in $V[G]$.

Finally, we note that $|M[G] \cap \lambda^{+\omega}| \leq |\kappa^{+\omega}| = \aleph_0$, which in turn implies $c''[A]^2$ is countable since $c''[A]^2 \subset c''[M[G] \cap \lambda^{+\omega+1}]^2 \subset M[G] \cap \lambda^{+\omega}$. Since $\lambda^{+\omega}$ is singular of countable cofinality, it suffices to show for each $n \in \omega$, $M[G] \cap \lambda^{+n}$ is countable. Since R is of size λ satisfying λ -c.c, a simple calculation reveals that the collection $T(\lambda^{+n})$ of nice R -names for an ordinal in λ^{+n} has size $\lambda^{\lambda^{+n}} < \lambda^{+\omega}$, by the GCH assumption. Therefore, in V , $M \models |T(\lambda^{+n})| < \lambda^{+\omega}$, in particular, $|T(\lambda^{+n}) \cap M| \leq \kappa^{+\omega}$. Hence in $V[G]$, we have that $M[G] \cap \lambda^{+n}$ is countable.

Proposition 2.5 shows that $\omega_2 \rightarrow^{\text{poly}} (\omega_1)_{\omega - bdd}^2$ is consistent. A natural question is whether we can enlarge the rainbow set found. The rest of this section is dedicated to the proof of Theorem 0.3, which basically says it is consistent to ask for a rainbow subset of order type ω_1 stationary in its supremum.

2.1. The role and property of the stationary Chang's Conjecture.

Definition 2.10. We shall say that the *stationary Chang's Conjecture* holds iff for every $e : [\omega_2]^{<\omega} \rightarrow \omega_1$ one can find $A \subseteq \omega_2$ such that $\text{otp}(A) = \omega_1$, A is stationary in $\sup A$ and $e''[A]^{<\omega}$ is countable.

Our definition is phrased with respect to colorings of all finite subsets, but for all the results in this section we need just the parallel statements applied to colorings of pairs. It is known that Chang's Conjecture is indestructible under *ccc* forcing notions. Let us show that stationary Chang's Conjecture is preserved by *ccc* forcing notions as well.

Claim 2.11. *Suppose that Stationary Chang's conjecture holds in V and \mathbb{P} is a *ccc* forcing notion. If $G \subseteq \mathbb{P}$ is V -generic then $V[G]$ satisfies stationary Chang's conjecture.*

Proof.

Let $e : [\omega_2]^{<\omega} \rightarrow \omega_1$ be a \mathbb{P} -name. For every $t \in [\omega_2]^{<\omega}$ let \mathcal{A}_t be a maximal antichain which determines the possible values of $e(t)$. By our assumptions on \mathbb{P} , each \mathcal{A}_t is countable. Define $d : [\omega_2]^{<\omega} \rightarrow \omega_1$ in V by letting $d(t) = \sup\{\alpha \in \omega_1 : \exists p \in \mathcal{A}_t, p \Vdash e(t) = \check{\alpha}\}$. Since each \mathcal{A}_t is countable we see that $d(t) \in \omega_1$ for every t , so d is well-defined.

Apply stationary Chang's Conjecture to d in the ground model, and choose $A \subseteq \omega_2$ of order type ω_1 so that A is stationary in $\sup(A)$ and $d''[A]^{<\omega}$ is countable. Fix $\delta \in \omega_1$ which bounds $d''[A]^{<\omega}$ and notice that $\Vdash_{\mathbb{P}} \sup e''[A]^{<\omega} \leq \delta$ as well. Since \mathbb{P} is *ccc* we see that A remains stationary in its supremum in $V[G]$, hence the proof is accomplished.

□_{2.11}

Remark 2.12. Proposition 2.5 with $\kappa = \omega_2$ and $\lambda = \mu = \rho = \omega_1$ implies that if the stationary Chang's Conjecture and $\omega_1 \rightarrow^{\text{poly}} (\text{par}(< \omega_1))_{<\omega_1-t-bdd}^2$ both hold, then $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{<\omega_2-t-bdd}^2$ holds, in particular, $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$ holds.

For the discussion to follow, it will be convenient to use

$$\omega_1 \rightarrow_{\aleph_1-ccc}^{\text{poly}} (\text{par}(< \omega_1))_{<\omega_1-t-bdd}^2$$

to abbreviate: for any \aleph_1 -sized *ccc* poset Q and any Q -name \check{f} for a $< \omega_1$ -type bounded coloring on $[\omega_1]^2$, there exists a Q -name \check{R} for a *ccc* forcing that adds a partition of ω_1 into countably many \check{f} -rainbow sets.

2.2. Forcing $\omega_1 \rightarrow_{\aleph_1-ccc}^{\text{poly}} (\text{par}(< \omega_1))_{<\omega_1-t-bdd}^2$ with finite/Easton mixed-support iterations. We first review the properties of the forcing used for Theorem 2.3 from [15] and then we modify it for our purpose.

Definition 2.13 (Jensen's fast club forcing). Let \mathbb{J}_{ω_1} be the poset consisting of (σ, C) where σ is a countable closed subset of ω_1 and C is a club subset of ω_1 . The order is: $(\tau, D) \leq (\sigma, C)$ iff $D \subset C$, τ end-extends σ and $\tau - \sigma \subset C$.

Let us summarize some properties of this forcing:

- (1) $|\mathbb{J}_{\omega_1}| = 2^{\omega_1}$,
- (2) \mathbb{J}_{ω_1} is countably closed,
- (3) \mathbb{J}_{ω_1} collapses the continuum of the ground model to ω_1 and
- (4) \mathbb{J}_{ω_1} adds a club in ω_1 that is almost contained (except for an initial segment) in any club from the ground model.

Theorem 2.14 (Magidor [11]). *Let κ be regular and $\lambda \geq \kappa$. If \mathbb{P} is a separative forcing poset such that \mathbb{P} is κ -closed, $|\mathbb{P}| = \lambda$, every condition in \mathbb{P} has λ incompatible extensions and \mathbb{P} adds a surjection from κ to λ . Then \mathbb{P} is equivalent to $\text{Coll}(\kappa, \lambda)$.*

As a result, if V satisfies $2^\omega = 2^{\omega_1} = \omega_2$, then by Theorem 2.14, \mathbb{J}_{ω_1} is forcing equivalent to $\text{Coll}(\omega_1, \omega_2)$.

We will make use of the following theorem, essentially due to Todorćević [15, Lemma 1].

Lemma 2.15. *Assume that:*

- (a) \mathbb{Q} is a ccc forcing notion in V .
- (b) $f : [\omega_1]^2 \rightarrow \omega_1$ is a \mathbb{Q} -name for a $< \omega_1$ -type bounded coloring.
- (c) The set $(G_0 * G_1) \times G_{\mathbb{Q}}$ is V -generic with respect to the forcing notion $(\text{Add}(\omega, \omega_1) * \mathbb{J}_{\omega_1}) \times \mathbb{Q}$.

*Then in $V[(G_0 * G_1) \times G_{\mathbb{Q}}]$, there is a ccc forcing notion which adds a partition of ω_1 into countably many f -rainbow subsets.*

$\square_{2.15}$

Let E be the class of even ordinals and O be the class of odd ordinals. Given $p \in \text{Coll}(\omega_1, \omega_2)$, let the *height* of p be denoted as $ht(p) = \sup \text{dom}(p)$.

The main idea behind the proof of the following theorem is to use a mixed support iteration of Cohen forcing and Lévy Collapse. This was the strategy of Mitchell in his classical proof of the consistency of the tree property at \aleph_2 , see [12].

The support for the forcing iterations used in [15] and [12] is finite on the Cohen part and countable on the countably closed part. We will use *Easton support* on the countably closed part and in addition, we require the conditions to be of certain shape. This will be useful for the lifting argument in Subsection 2.3.

Here is a short history of how these ideas originated: Silver, in his proof of the consistency of the Chang's Conjecture from a ω_1 -Erdős cardinal, invented the *Silver Collapse*, which is a modification of the usual Lévy Collapse requiring each condition to be of a specific shape. The method was later adapted by Kunen [8] to establish the consistency of the existence of a saturated ideal on ω_1 from the existence of a huge cardinal. Then Laver [9] modified Kunen's method further, using the Easton support, to establish the consistency of an $(\aleph_2, \aleph_2, \aleph_0)$ -saturated ideal on ω_1 .

A set $A \subseteq \lambda$ is an *Easton set* iff $|A \cap \rho| < \rho$ for every limit regular cardinal $\rho < \lambda$. The following theorem was basically proved in [15]. We include the proof adapted for the modified context for completeness.

Lemma 2.16. *If λ is 2-Mahlo then there is a forcing notion \mathbb{S} with the following properties:*

- (a) \mathbb{S} is proper and λ -cc.
- (b) \mathbb{S} forces $\lambda = \omega_2$.
- (c) \mathbb{S} forces $\omega_1 \rightarrow_{\aleph_1\text{-ccc}}^{\text{poly}} (\text{par}(<\omega_1))^2_{<\omega_1-t\text{-bdd}}$.

Proof.

Let E be the class of even ordinals and let O be the class of odd ordinals. We define a mixed support iteration $(\mathbb{S}_\alpha, \mathbb{T}_\beta : \alpha \leq \lambda, \beta < \lambda)$ as follows:

- (α) If $\alpha \in E$ then $\Vdash_{\mathbb{S}_\alpha} \mathbb{T}_\alpha = \text{Add}(\omega, \omega_1)$.
- (β) If $\alpha \in O$ then $\Vdash_{\mathbb{S}_\alpha} \mathbb{T}_\alpha = \text{Coll}(\omega_1, \omega_2)$.
- (γ) If α is a limit regular cardinal, then \mathbb{S}_α is the direct limit of $(\mathbb{S}_\beta : \beta \in \alpha)$.
- (δ) If α is not a limit regular cardinal, then $t \in \mathbb{S}_\alpha$ iff $t \restriction \beta \in \mathbb{S}_\beta$ for every $\beta \in \alpha$, $|\{\gamma \in E \cap \alpha : t(\gamma) \neq \emptyset\}| < \aleph_0$ and there exists $\nu \in \omega_1$ such that for every $\gamma \in O \cap \alpha$, $\Vdash_{\mathbb{S}_\gamma} \text{ht}(t(\gamma)) < \nu$.

If $t \in \mathbb{S}_\alpha$, in the case of (δ), then the least $\nu \in \omega_1$ which bounds $\text{ht}(t(\gamma))$ for every $\gamma \in O \cap \alpha$ will be denoted by $\text{ht}(t)$. We shall say that t has *bounded height*. Property (δ) implies that if $\alpha \leq \lambda$, t belongs to the inverse limit of $(\mathbb{S}_\beta : \beta \in \alpha)$ and t is of bounded height then $t \in \mathbb{S}_\alpha$ iff $O \cap \text{supp}(t)$ is an Easton set and $E \cap \text{supp}(t)$ is finite.

We claim that $\mathbb{S}_\lambda = \mathbb{S}$ satisfies all the requirements of the theorem. By design, it is easy to see \mathbb{S} satisfies λ -cc and forces $\lambda = \omega_2$. Let us show that \mathbb{S}_α is proper for every $\alpha \leq \lambda$.

Firstly, for every $\alpha \leq \lambda$ let $D_\alpha = \{p \in \mathbb{S}_\alpha : \forall \gamma \in E \cap \alpha, \exists s_\gamma \in \text{Add}(\omega, \omega_1), p \restriction \gamma \Vdash_{\mathbb{S}_\alpha} \check{s}_\gamma = p(\gamma)\}$. By induction on α it follows that D_α is dense (in \mathbb{S}_α) for every $\alpha \leq \lambda$. Hence we may assume, without loss of generality, that all our conditions belong to the pertinent D_α . For every $\alpha \leq \lambda$, one can see that \mathbb{S}_α projects onto $\text{Add}(\omega, \omega_1 \times (E \cap \alpha))$. Each element q of $\text{Add}(\omega, \omega_1 \times (E \cap \alpha))$ is a function from a finite subset of $E \cap \alpha$ to $\text{Add}(\omega, \omega_1)$, ordered by reverse inclusion. Hence for \mathbb{S} can be identified with the two-step iteration $\text{Add}(\omega, \omega_1 \times (\lambda \cap E)) * \mathbb{S} / \text{Add}(\omega, \omega_1 \times (\lambda \cap E))$.

Fix $\alpha \leq \lambda$. Given $p \in \mathbb{S}_\alpha$ we denote the Cohen part of p by $\sigma(p)$. Explicitly, the domain of $\sigma(p)$ is $\text{supp}(p) \cap E$ and if $\gamma \in \text{supp}(p) \cap E$ then $\sigma(p)(\gamma) \in \text{Add}(\omega, \omega_1)$ and $p \restriction \gamma \Vdash_{\mathbb{S}_\gamma} \sigma(p)(\gamma) = p(\gamma)$. The main disadvantage of the Cohen part is the lack of completeness, but we can define the direct extension \leq_α^* on \mathbb{S}_α by $q \leq_\alpha^* p$ iff $q \leq_{\mathbb{S}_\alpha} p$ and $\sigma(q) = \sigma(p)$. Observe that \leq_α^* is ω_1 -closed. In fact, any countable decreasing \leq_α^* -sequence has a greatest lower bound.

If $p \in \mathbb{S}_\alpha$ and $t \leq_{\text{Add}(\omega, \omega_1 \times (E \cap \alpha))} \sigma(p)$ then we define an amalgamation $p \wedge t$ of p and t as follows. Let $q = p \wedge t$ where $\text{dom}(q)$ is $\text{dom}(p) \cup \text{dom}(t)$ and if $\gamma \in \text{dom}(p) - \text{dom}(t)$, then $q(\gamma) = p(\gamma)$ and if $\gamma \in \text{dom}(t)$, $q(\gamma) = t(\gamma)$. Notice that $q \leq_{\mathbb{S}_\alpha} p, t$. We proceed now to properness. The following comes from [15], but we unfold the proof for completeness.

Claim 2.17. *Suppose that $\alpha \leq \lambda, p \in \mathbb{S}_\alpha$ and β is an \mathbb{S}_α -name of an ordinal. Then there exists $q \in \mathbb{S}_\alpha$ such that $q \leq_\alpha^* p$ and there are a maximal antichain $\{t_i : i \in \omega\} \subseteq \text{Add}(\omega, \omega_1 \times (\alpha \cap E))$ below $\sigma(p)$ and $B = \{\beta_i : i \in \omega\} \subseteq \text{Ord}$ such that $q \wedge t_i \Vdash_{\mathbb{S}_\alpha} \beta = \beta_i$ for every $i \in \omega$.*

Proof.

By induction on $i \in \omega_1$, we define $p_i \in \mathbb{S}_\alpha$, $t_i \in \text{Add}(\omega, \omega_1 \times \alpha)$ and find some $\delta < \omega_1$ such that $\{t_i : i \in \delta\}$ is a maximal antichain below $\sigma(p)$ and if $i < j < \delta$ then $p_j \leq_\alpha^* p_i$.

Arriving at $\gamma < \omega_1$, if $\{t_i : i \in \gamma\}$ is a maximal antichain below $\sigma(p)$ then we let $\delta = \gamma$ and we are done. If not, let q be the greatest lower bound of $(p_i : i \in \gamma)$ with respect to \leq_α^* , and let $t_\gamma \leq \sigma(p)$ be such that $t_i \perp t_\gamma$ for every $i \in \gamma$. Choose $r \in \mathbb{S}_\alpha$ such that $r \leq_{\mathbb{S}_\alpha} q \wedge t_\gamma$ and r forces a value to β , say β_γ . From r we can extract p_γ, s_γ such that $p_\gamma \leq_\alpha^* q$ and $s_\gamma \leq_{\text{Add}(\omega, \omega_1 \times (E \cap \alpha))} t_\gamma$. Notice that $p_\gamma \wedge s_\gamma \Vdash \beta = \beta_\gamma$. Since $\text{Add}(\omega, \omega_1 \times (E \cap \alpha))$ is *ccc*, the process must be terminated at some $\delta \in \omega_1$, so letting $B = \{\beta_i : i \in \delta\}$ we are done. $\square_{2.17}$

We conclude from the above claim that each \mathbb{S}_α is proper. In particular, \mathbb{S} is proper, thus (a) and (b) of our theorem are established and we can proceed to (c).

Claim 2.18 (Page 713-714, [15]). *If $\mathbb{Q} \in V$ is *ccc* then $\Vdash_{\mathbb{S}} \mathbb{Q}$ is *ccc*.*

$\square_{2.18}$

Observe that if $\alpha < \lambda$ is Mahlo then \mathbb{S}_α forces that $\mathbb{S}_{[\alpha, \lambda]}$ is forcing equivalent to \mathbb{S} as defined in $V^{\mathbb{S}_\alpha}$. An easy modification of Theorem 5.4 in Baumgartner's survey on iterated forcing [3] will give a proof.

To establish (c), suppose that \mathbb{Q} is a *ccc* forcing notion of size \aleph_1 and $f : [\omega_1]^2 \rightarrow \omega_1$ is a \mathbb{Q} -name of a $< \omega_1$ -type bounded function. Let $H \subseteq \mathbb{S}$ be V -generic. Find a sufficiently large Mahlo cardinal $\alpha < \lambda$ such that $\mathbb{Q}, f \in V[H \restriction \alpha]$.

Consider the two-step iteration after the α th stage. By Theorem 2.14, we see that in $V[H \restriction \alpha]$ these two steps are equivalent to $\text{Add}(\omega, \omega_1) * \mathbb{J}_{\omega_1}$. Applying Lemma 2.15 we can find in $V[H \restriction \alpha + 2]$ a \mathbb{Q} -name \mathbb{R} of a *ccc* forcing notion which adds a countable partition of ω_1 into f -rainbow sets. Since $\mathbb{S}_{[\alpha+2, \lambda]}$ is equivalent to \mathbb{S} we conclude by Claim 2.18 that $\mathbb{Q} * \mathbb{R}$ is *ccc* in $V[H]$, thus proving (c). $\square_{2.16}$

We conclude this subsection by analyzing the term forcing associated with \mathbb{S} . Let \mathbb{R}_λ be a forcing notion consisting of functions on $\lambda \cap O$. For each $f \in \mathbb{R}_\lambda$, the following are satisfied:

- (1) for any $\alpha \in \lambda \cap O$, $\Vdash_{\mathbb{S}_\alpha} f(\alpha) \in \text{Coll}(\omega_1, \omega_2)$;
- (2) the support of a condition $f \in \mathbb{R}_\lambda$, namely $\text{supp}(f) =_{\text{def}} \{\alpha \in \lambda \cap O : f(\alpha) \neq \emptyset\}$, is an Easton set;
- (3) there exists $\delta \in \omega_1$ such that $\Vdash_{\mathbb{S}_\alpha} \text{ht}(f(\alpha)) < \delta$ for every $\alpha \in \text{supp}(f)$.
We denote by $\text{ht}(f)$ the minimal $\delta \in \omega_1$ which satisfies this property.

If $f, g \in \mathbb{R}_\lambda$ then $g \leq_{\mathbb{R}_\lambda} f$ iff $\text{supp}(f) \subseteq \text{supp}(g)$ and if $\alpha \in \text{supp}(f)$ then $\Vdash_{\mathbb{S}_\alpha} f(\alpha) \leq g(\alpha)$.

By the nature of the collapse and the maximality principle of forcing, if $f, g \in \mathbb{R}_\lambda$ are compatible then there is a greatest lower bound $h \in \mathbb{R}_\lambda$ for f, g .

Claim 2.19. *Let $\mathbb{D}_\lambda = \text{Add}(\omega, \omega_1 \times (\lambda \cap E)) \times \mathbb{R}_\lambda$. Then \mathbb{D}_λ projects onto \mathbb{S} .*

Proof.

Suppose that $(s, f) \in \mathbb{D}_\lambda$. Define $p = p_{s,f} \in \mathbb{S}$ as follows. Let $\text{supp}(p) = \text{supp}(s) \cup \text{supp}(f)$. If $\gamma \in \text{supp}(s)$ let $p(\gamma) = s(\gamma)$ and if $\gamma \in \text{supp}(f)$ let $p(\gamma) = f(\gamma)$. Define $\pi : \mathbb{D}_\lambda \rightarrow \mathbb{S}$ by $\pi((s, f)) = p_{s,f}$. We claim that π projects \mathbb{D}_λ to \mathbb{S} .

Clearly, π sends $0_{\mathbb{D}_\lambda}$ to $0_{\mathbb{S}}$ and preserves order. Suppose that $(s, f) \in \mathbb{D}_\lambda$ and $q \in \mathbb{S}$ satisfies $p_{s,f} \geq_{\mathbb{S}} q$. We try to find $(t, g) \in \mathbb{D}_\lambda$ such that $(s, f) \geq_{\mathbb{D}_\lambda} (t, g)$ and $q \geq_{\mathbb{S}} \pi(t, g) = p_{t,g}$. Let $t = \sigma(q)$. Define $\text{supp}(g) = O \cap \text{supp}(q)$. If $\alpha \in \text{supp}(g)$ then let $g(\alpha)$ be an \mathbb{S}_α -term for which if $q \restriction \alpha$ belongs to the generic subset of \mathbb{S}_α , then $g(\alpha) = q(\alpha)$ and if not then $g(\alpha) = f(\alpha)$.

One can verify that g is a well-defined condition, in particular if $\nu = \max\{ht(f), ht(q)\} \in \omega_1$ then $ht(g) \leq \nu$ and hence $g \in \mathbb{R}_\lambda$. Since $q \restriction \alpha \Vdash_{\mathbb{S}_\alpha} f(\alpha) \geq q(\alpha)$ for every $\alpha \in \text{supp}(g)$, we have $f \geq_{\mathbb{R}_\lambda} g$. To see that $q \geq_{\mathbb{S}} p_{t,g}$, induct on $\alpha \geq \lambda$. Arriving at α , assume that $q \restriction \alpha \geq_{\mathbb{S}_\alpha} p_{t,g} \restriction \alpha$. Now if $\alpha \in E$, use the fact that $t = \sigma(q)$ for the α -th stage. If $\alpha \in O$, by the induction hypothesis, $p_{t,g} \leq q \restriction \alpha$ and the fact that $p_{t,g}(\alpha) = g(\alpha)$, we know that $q \restriction \alpha \Vdash_{\mathbb{S}_\alpha} g(\alpha) = q(\alpha)$. In all other cases use the definition of the iteration.

□_{2.19}

Clearly $\text{Add}(\omega, \omega_1 \times \lambda)$ is isomorphic to $\text{Add}(\omega, \omega_1 \times (\lambda \cap E))$. In what follows, they are identified to be the same. Notice that \mathbb{S} and \mathbb{R} have uniform definitions with the parameter λ and the ω_1 in the ground model. Therefore, in any model of set theory with ν being a 2-Mahlo cardinal, we can make sense of \mathbb{S}_ν and \mathbb{R}_ν respectively.

2.3. Getting the stationary Chang's Conjecture. Let $pr : [\text{Ord}]^2 \leftrightarrow \text{Ord}$ be a pairing function which satisfies the requirement that if $\delta = pr(\beta, \gamma)$ then $\beta \leq \gamma \leq \delta$. Recall that a cardinal κ is *huge* if there exists an elementary embedding $j : V \rightarrow M$ with critical point κ and $j^{(\kappa)}M \subset M$.

Lemma 2.20. *Assume that:*

- (a) κ is a huge cardinal, j is a huge embedding for κ and $\lambda = j(\kappa)$.
- (b) $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \kappa, \beta < \kappa \rangle$ is a finite support iteration of length κ .
- (c) $\mathbb{P}_0 = \text{Coll}(\omega, < \kappa)$.
- (d) If $\alpha \in (0, \kappa)$, $pr^{-1}(\alpha) = (\alpha_0, \alpha_1)$ and $\mathbb{P}_{\alpha_1} \cap V_{\alpha_0}$ is a regular suborder of \mathbb{P}_{α_1} then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{Q}_\alpha$ where \mathbb{Q}_α is a \mathbb{P}_α -name of $(\mathbb{R}_\kappa)^{V^{\mathbb{P}_{\alpha_1} \cap V_{\alpha_0}}}$.
- (e) In all other cases \mathbb{Q}_α is a name of the trivial forcing.

Then \mathbb{P} satisfies κ -cc and $V^{\mathbb{P}*\mathbb{S}}$ satisfies the stationary Chang's Conjecture, where \mathbb{S} is the \mathbb{P} -name for \mathbb{S}_λ , as defined in Lemma 2.16.

Proof.

To see \mathbb{P} satisfies κ -cc, work by induction on $\alpha \leq \kappa$. Assume that (α_0, α_1) satisfy $\alpha = pr(\alpha_0, \alpha_1)$. By the induction hypothesis we know that \mathbb{P}_α is κ -cc. In the trivial stages it is clear that \mathbb{P}_α forces that \mathbb{Q}_α is κ -cc, so assume that we arrived at a non-trivial stage, namely $\mathbb{P}_{\alpha_1} \cap \tilde{V}_{\alpha_0}$ is a regular suborder of \mathbb{P}_{α_1} and hence of \mathbb{P}_α . From the fact that \mathbb{P}_α is κ -cc we infer that $\Vdash_{\mathbb{P}_{\alpha_1} \cap \tilde{V}_{\alpha_0}} \mathbb{P}_\alpha / \mathbb{P}_{\alpha_1} \cap \tilde{V}_{\alpha_0}$ is κ -cc, and that $\Vdash_{\mathbb{P}_{\alpha_1} \cap \tilde{V}_{\alpha_0}} (\mathbb{R}_\kappa)^{V^{\mathbb{P}_{\alpha_1} \cap \tilde{V}_{\alpha_0}}}$ is κ -cc. As $\mathbb{P}_{\alpha_1} \cap \tilde{V}_{\alpha_0}$ is a small forcing, it preserves the weak compactness of κ . Hence in $V^{\mathbb{P}_{\alpha_1} \cap \tilde{V}_{\alpha_0}}$ we see that κ -cc forcings are κ -Knaster and we are done.

Next we show that $V^{\mathbb{P}*\mathbb{S}}$ is a model of the stationary Chang's Conjecture. Observe that by the construction, $\mathbb{P} * \mathbb{R}_\lambda$ is a regular suborder of $j(\mathbb{P})$. Let $G * H \subseteq \mathbb{P} * \mathbb{R}_\lambda$ be V -generic, and let $\tilde{G} \subseteq j(\mathbb{P}) / G * H$ be generic over $V[G * H]$. Apply Silver's criterion to lift j and to get $j_0 : V[G] \rightarrow M[\tilde{G}]$. Let $h \subseteq \text{Add}(\omega, \kappa \times \lambda)$ be generic over $V[\tilde{G}]$ and let $L \subseteq \mathbb{S}$ be generic over $V[G]$, as given by $h \times H$ and Claim 2.19. All of the above happens in $V[\tilde{G}][h]$.

Note that in $V[\tilde{G}][h]$, $j_0 \restriction \text{Add}(\omega, \kappa \times \lambda) : \text{Add}(\omega, \kappa \times \lambda) \rightarrow \text{Add}(\omega, \lambda \times j(\lambda))$ is a complete embedding, with the isomorphic image being $\text{Add}(\omega, \kappa \times j''\lambda)$. As a result, $j_0''h \subseteq \text{Add}(\omega, \kappa \times j''\lambda)$ is generic over $V[\tilde{G}]$.

We shall argue that by going to a further forcing extension, there is some $\tilde{L} \subseteq j_0(\mathbb{S})$ generic over $M[\tilde{G}]$, so that $j_0''L \subseteq \tilde{L}$. To this end, notice that $j_0''H \in M[\tilde{G}]$. Now in $M[\tilde{G}]$, we find a *master condition* for $j_0''H$. More precisely, we define a lower bound $r \in j_0(\mathbb{R}_\lambda)$ for all the conditions in $j_0''H$. Let

$$\text{supp}(r) = \bigcup \{j_0(\text{supp}(q)) : q \in H\}.$$

Each set of the form $j_0(\text{supp}(q))$ is Easton in $M[\tilde{G}]$ by the elementarity of j_0 and hence $\text{supp}(r)$ is also an Easton set in $M[\tilde{G}]$. Indeed, $M[\tilde{G}] \models \lambda = \omega_1$, so $\text{supp}(r)$ is a union of ω_1 many Easton sets in $M[\tilde{G}]$ and hence Easton.

For every $\alpha \in \text{supp}(r)$ let $r(\alpha)$ be a $j_0(\mathbb{S}) \restriction \alpha$ -term for the greatest lower bound for $\{j_0(q)(\alpha) : q \in H\}$. To verify that r is a condition in $j_0(\mathbb{R}_\lambda)$, let $\gamma_q = ht(q)$ for each $q \in H$, so $\gamma_q \in \kappa$. Since $\kappa = \text{crit}(j_0)$ we see that $ht(j_0(q)) = \gamma_q$ as well. Hence $ht(r) \leq \kappa$. As $M[\tilde{G}] \models \kappa < \lambda = \omega_1$, we see that r is a condition in $j_0(\mathbb{R}_\lambda)$ indeed.

Let $\tilde{h} \times \tilde{H} \subseteq j_0(\text{Add}(\omega, \kappa \times \lambda)) / j_0''h \times j_0(\mathbb{R}_\lambda) / r$ be generic over $V[\tilde{G} * h]$. In $V[\tilde{G} * h][\tilde{h} \times \tilde{H}]$, one can lift j_0 to $j_1 : V[G][h \times H] \rightarrow M[\tilde{G}][\tilde{h} \times \tilde{H}]$. Let $\tilde{L} \subseteq j_0(\mathbb{S})$ be $M[\tilde{G}]$ -generic as given by $\tilde{h} \times \tilde{H}$ and Claim 2.19. Let $\iota = j_1 \restriction V[G * L]$. It is clear that $\iota : V[G * L] \rightarrow M[\tilde{G} * \tilde{L}]$ is an elementary embedding as desired. This elementary embedding lives in $V[\tilde{G} * h][\tilde{h} \times \tilde{H}]$.

We claim that $V[G * L]$ satisfies stationary Chang's conjecture. In order to prove this, suppose that $e : [\lambda]^{<\omega} \rightarrow \kappa, e \in V[G * L]$ and recall that $\lambda = \omega_2, \kappa = \omega_1$ in $V[G * L]$. Our goal is to find $A \subseteq \lambda, \text{otp}(A) = \omega_1$, A is stationary in its supremum such that $e''[A]^{<\omega}$ is countable.

In V , let $A = j''(\lambda \cap \text{cof}^V(\omega))$, namely the pointwise image of j over the set of $\alpha \in \lambda$ with countable cofinality in V . Let $\beta = \sup(A)$. Since $j : V \rightarrow M$ is a huge embedding with target λ , we see that M is λ -closed so $A \in M$. Moreover, as j is continuous at points of countable cofinality, A is stationary in β .

Since $j(\mathbb{P})$ satisfies λ -cc, we conclude that A is stationary in β in $M[\tilde{G}]$. Recall that \mathbb{S} is proper, so by elementarity $j_0(\mathbb{S})$ is proper in $M[\tilde{G}]$. Recall that $\lambda = \omega_1$ in $M[\tilde{G}]$, so A remains stationary in β in $M[\tilde{G} * \tilde{L}]$.

Now $\kappa < \lambda = \omega_1$ in $M[\tilde{G} * \tilde{L}]$, and $\iota(e)''[A]^{<\omega} \subseteq \kappa$, so $M[\tilde{G} * \tilde{L}]$ knows that there exists a subset of $\iota(\lambda)$ of order type ω_1 , stationary in its supremum, with countable image under $\iota(e)$. By elementarity, this statement is true also in $V[G * L]$, as required.

□_{2.20}

Now we can finish the proof of Theorem 0.3.

Proof of Theorem 0.3.

First, we force with $\mathbb{P} * \mathbb{S}$ as in Lemma 2.20 where κ is huge witnessed by $j : V \rightarrow M$, $\lambda = j(\kappa)$ and $\mathbb{S} = (\mathbb{S}_\lambda)^{V^\mathbb{P}}$. Let $G \subseteq \mathbb{P} * \mathbb{S}$ be V -generic. Notice that in $V[G]$, $\kappa = \omega_1$ and $2^\omega = 2^{\omega_1} = \omega_2 = \lambda$. From Lemma 2.20 we infer that $V[G]$ satisfies the stationary Chang's Conjecture.

Working in $V[G]$, we define a finite support iteration \mathbb{Q} of *ccc* forcings of length ω_2 , forcing $\omega_1 \rightarrow^{\text{poly}} (\text{par}(<\omega_1))^2_{<\omega_1-t-bdd}$. This is possible, since by Lemma 2.16(c) $\omega_1 \rightarrow^{\text{poly}}_{\aleph_1-ccc} (\text{par}(<\omega_1))^2_{<\omega_1-t-bdd}$ holds in $V[G]$. Using the standard book-keeping trick, one can define an ω_2 -length finite support iteration of \aleph_1 -sized *ccc* posets to force $\omega_1 \rightarrow^{\text{poly}} (\text{par}(<\omega_1))^2_{<\omega_1-t-bdd}$.

Let $H \subseteq \mathbb{Q}$ be $V[G]$ -generic. Since \mathbb{Q} is *ccc*, we deduce from Claim 2.11 that $V[G][H]$ satisfies the stationary Chang's Conjecture. By Remark 2.12, $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$ (even $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{<\omega_2-t-bdd}^2$) holds in $V[G][H]$.

□_{0.3}

3. PFA DOES NOT IMPLY $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$

Recall Theorem 2.3 that PFA implies $\omega_1 \rightarrow^{\text{poly}} (\omega_1 - st)_{<\omega_1-t-bdd}^2$, hence in particular, $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{<\omega_1-t-bdd}^2$. We have seen that by Theorem 0.3 that $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$ is consistent relative to the existence of a huge cardinal. A natural question is whether PFA already implies $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$. In this section, we show that the answer is no.

Let $(*)$ be the statement which says that there exists a sequence of sets $(S_\alpha \in [\alpha]^{\leq \aleph_0} : \alpha \in S_{\omega_1}^{\omega_2})$ such that for every $\beta \in S_{\omega_1}^{\omega_2}$ there is a club $c_\beta \subseteq \beta \cap \text{cof}(\omega)$ so that if $\alpha \in c_\beta$ then $c_\beta \cap \alpha \subseteq S_\alpha$.

Claim 3.1. *If $(*)$ holds then $\omega_2 \not\rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$.*

Proof.

Given the sequence $(S_\alpha : \alpha \in S_{\omega_1}^{\omega_2})$ we define a normal ω -bounded coloring $f : [\omega_2]^2 \rightarrow \omega_2$ such that for any $\alpha \in S_{\omega_1}^{\omega_2}$, $f(\cdot, \alpha) \upharpoonright S_\alpha$ is constant. This is possible since each S_α is countable.

Suppose we are given $A \subseteq \omega_2$ such that $\text{otp}(A) = \omega_1, \beta = \sup(A)$ and A is stationary in β . Choose $\alpha \in A \cap c_\beta \cap S_{\omega_2}^{\omega_2}$ such that $|A \cap c_\beta \cap \alpha| \geq 2$. By (*), $A \cap c_\beta \cap \alpha \subseteq c_\beta \cap \alpha \subseteq S_\alpha$, so A is not f -rainbow.

□_{3.1}

Remark 3.2. (*) is a consequence of $\square_{\omega_1, \omega}$. Therefore, unlike $\omega_2 \rightarrow^{\text{poly}} (\omega_1)_{\omega-bdd}^2$, the stationary variation $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$ has non-trivial large cardinal strength.

Our objective is to show that (*) is consistent with PFA. We begin with a ground model which satisfies PFA, we force (*) and we would like to argue that PFA is preserved in the generic extension. The proof follows Beaudoin [4] and Magidor, and we refer to [2, Section 7] for a similar argument and a comprehensive discussion concerning the essential idea of [4].

Theorem 3.3. *PFA is consistent with (*), and hence does not imply the positive relation $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$.*

Proof.

We define a forcing notion \mathbb{P} to add witness for (*) with initial segments. A condition $p \in \mathbb{P}$ iff

- $\text{dom}(p)$ is a proper initial segment of $S_{\omega_2}^{\omega_2}$,
- if $\alpha \in \text{dom}(p)$, then $p(\alpha)$ is a countable subset of α ,
- for every $\beta \in S_{\omega_1}^{\omega_2}$ and $\beta \leq \sup \text{dom}(p)$, there exists a club $c_\beta \subset \beta$ such that for any $\alpha \in c_\beta$, $c_\beta \cap \alpha \subset g(\alpha)$.

The order $\leq_{\mathbb{P}}$ is end extension, namely, $q \leq_{\mathbb{P}} p$ iff $\text{dom}(q) \supseteq \text{dom}(p)$ and $q \restriction \text{dom}(p) = p$.

Observe that \mathbb{P} is countably closed and $(\omega_1 + 1)$ -strategically closed. Therefore, ω_1 and ω_2 are preserved after forcing with \mathbb{P} .

Choose a V -generic set $G \subseteq \mathbb{P}$. For every $\alpha \in S_{\omega_2}^{\omega_2}$ let $S_\alpha = p(\alpha)$ for some (any) $p \in G$ with $\alpha \in \text{dom}(p)$. Observe that $(S_\alpha : \alpha \in S_{\omega_2}^{\omega_2})$ exemplifies (*) in $V[G]$. We claim that $V[G]$ is a model of PFA.

To see this, suppose that \mathbb{Q} is a \mathbb{P} -name of a proper forcing notion and $\tilde{D} = (\tilde{D}_i : i \in \omega)$ is a \mathbb{P} -name for a family of ω_1 dense subsets of \mathbb{Q} . Given $p \in \mathbb{P}$ we have to find a condition $t \leq p$ which forces the existence of a filter on \mathbb{Q} meeting every \tilde{D}_i . We will define a \mathbb{P} -name for an auxiliary forcing notion \mathbb{R} and we will show that $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ is proper. This \mathbb{R} will help us define t .

In $V^{\mathbb{P}}$, a condition $f \in \mathbb{R}$ is an increasing and continuous function $f : \alpha \rightarrow \omega_2^V$, where $\alpha \in \omega_1$, satisfying that if $\delta \in \text{dom}(f)$ and δ is a limit ordinal then $f''\delta \subseteq S_{f(\delta)}$. The order of \mathbb{R} is reverse inclusion.

Claim 3.4. $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ is proper.

Proof.

Suppose that χ is a sufficiently large regular cardinal, $(p, q, r) \in \mathbb{P} * \mathbb{Q} * \mathbb{R}$, $M \prec \mathcal{H}(\chi)$ is countable and $(p, q, r), \mathbb{P} * \mathbb{Q} * \mathbb{R} \in M$. We want to extend (p, q, r) to an $(M, \mathbb{P} * \mathbb{Q} * \mathbb{R})$ -generic condition.

Let $\delta = M \cap \omega_1 \in \omega_1$ and let $\gamma = \sup(M \cap \omega_2)$ so $\text{cf}(\gamma) = \omega$. Since \mathbb{P} is countably closed we can choose an M -generic sequence $(p_n : n \in \omega)$ such that $p_0 = p$. Let $p^* = \bigcup_{n \in \omega} p_n \cup \langle \gamma, M \cap \omega_2^V \rangle \in \mathbb{P}$. Using the fact that \mathbb{Q} is a \mathbb{P} -name of a proper forcing, we can choose a \mathbb{P} -name q^* so that p^* forces that q^* extends q and q^* is $(M[G], \mathbb{Q})$ -generic.

By the choice of p^*, q^* we see that $(p^*, q^*) \Vdash M[G * H] \cap \text{Ord} = M \cap \text{Ord}$. Let $G * H$ be generic for $\mathbb{P} * \mathbb{Q}$ over V , containing (p^*, q^*) . Since $M[G * H]$ is countable, we can build a decreasing generic sequence $\bar{r} = (r_i : i \in \omega)$ of conditions in \mathbb{R} over $M[G * H]$. We define $r^* = \bigcup_{i \in \omega} r_i \cup \langle \delta, \gamma \rangle$. We claim that $r^* \in \mathbb{R}$. Indeed, r^* is continuous because \bar{r} is generic over $M[G * H]$ and $r^{**}\delta = \bigcup_{i \in \omega} r_i''\delta \subset M[G * H] \cap \omega_2^V = M \cap \omega_2^V = p^*(\gamma) = S_\gamma = S_{r^*(\delta)}$.

Let r^* be a $\mathbb{P} * \mathbb{Q}$ -name for a condition in \mathbb{R} such that (p^*, q^*) forces that r^* has the properties as above. Then condition $(p^*, q^*, r^*) \leq_{\mathbb{P} * \mathbb{Q} * \mathbb{R}} (p, q, r)$ is $(M, \mathbb{P} * \mathbb{Q} * \mathbb{R})$ -generic.

□_{3.4}

Fix a condition $p_0 \in \mathbb{P}$. Let $N \prec \mathcal{H}(\chi)$ be of size \aleph_1 such that $\omega_1 \subseteq N$, so $N \cap \omega_2 = \delta \in S_{\omega_1}^{\omega_2}$, and such that $p_0, D \in N$ and $\mathbb{P} * \mathbb{Q} * \mathbb{R} \in N$.

We apply PFA in the ground model to $\mathbb{P} * \mathbb{Q} * \mathbb{R}$ and obtain a generic filter $F \subseteq \mathbb{P} / p_0 * \mathbb{Q} * \mathbb{R}$ over N . Define $t = \bigcup \{p \in \mathbb{P} : \exists q \exists r, (p, q, r) \in F\}$. Notice that $t \in \mathbb{P}$ by virtue of the third coordinate of elements in F , and $t \leq_{\mathbb{P}} p_0$. More precisely, let $d'_\delta =_{\text{def}} \bigcup \{r : \exists (p, q, \check{r}) \in F\}$ and $d_\delta = d'_\delta[\omega_1]$. There is no generality lost here since $\{(p, r) \in \mathbb{P} * \mathbb{R} : \exists r \in V \ p \Vdash_{\mathbb{P}} r = \check{r}\}$ is a dense subset of $\mathbb{P} * \mathbb{R}$. Then it is easy to see that d_δ is a club in δ of order type ω_1 and for each $\alpha \in \lim d_\delta$, it is the case that $d_\delta \cap \alpha \subset t(\alpha)$. Therefore, $\lim d_\delta$ is a witness to the fact that $t \in \mathbb{P}$.

Therefore, $t \Vdash_{\mathbb{P}} \{q \in \mathbb{Q} : \exists (p, q, r) \in F\}$ is a generic filter for \mathbb{Q} meeting all of $(D_i : i \in \omega_1)$.

□_{3.3}

As we will see in Remark 4.4, MM implies the failure of $(*)$.

4. A POSITIVE RESULT FOR $< \omega_1$ -TYPE BOUNDED COLORINGS UNDER MARTIN'S MAXIMUM

Recall that MM implies $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{< \omega - bdd}^2$ as shown in [2]. We improve this result by relaxing the boundedness assumption.

Theorem 4.1. *MM implies $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{< \omega_1 - t - bdd}^2$.*

Proof.

Suppose that $f : [\omega_2]^2 \rightarrow \omega_2$ is $< \omega_1$ -type bounded. Fix an ordinal $\gamma \in \omega_1$ such that $\text{otp}(t_{\alpha\beta}) \leq \gamma$ for every $\alpha < \beta < \omega_2$. We shall define a stationary-preserving forcing $\mathbb{Q} = \mathbb{Q}_f$ which forces a closed f -rainbow set of order type ω_1 .

Let M be a countable elementary submodel of $\mathcal{H}(\chi)$, where χ is a sufficiently large regular cardinal. We denote $\sup(M \cap \omega_2)$ by δ_M . Let K be the structure $(\mathcal{H}(\chi), \in, <^*, f, \gamma)$, where $<^*$ is a fixed well-ordering of $\mathcal{H}(\chi)$. Let

\mathcal{S} be the collection of countable elementary submodels of K . Hence for any $N \in \mathcal{S}$, $\gamma < N \cap \omega_1$. We define a forcing notion \mathbb{P} as follows. A condition $p \in \mathbb{P}$ is a pair (ℓ_p, \mathcal{N}_p) such that:

- (a) $\mathcal{N}_p = (N_i : i < k)$, $N_i \in \mathcal{S}$ for every $i < k$, $N_i \in N_{i+1}$ for every $i < k - 1$.
- (b) $\ell_p : \mathcal{N}_p \rightarrow \omega$, and for each $n \in \omega$, $a_n^p =_{\text{def}} \{\delta_N : N \in \text{dom}(\ell_p), \ell_p(N) = n\}$ is f -rainbow.

If $p, q \in \mathbb{P}$ then $q \leq_{\mathbb{P}} p$ iff $\mathcal{N}_p \subseteq \mathcal{N}_q$ and $\ell_q \upharpoonright \mathcal{N}_p = \ell_p$.

We shall use the following terminology for describing a sort of a one-point extension. If $p \in \mathbb{P}$ then the pair $(M, n) \in \mathcal{S} \times \omega$ is *addable* to p iff $p \in M$ and $a_n^p \cup \{\delta_M\}$ is f -rainbow. If (M, n) is addable to p then $q = p + (M, n)$ will be the condition (ℓ_q, \mathcal{N}_q) where $\mathcal{N}_q = \mathcal{N}_p \cup \{M\}$, $\ell_q \upharpoonright \mathcal{N}_p = \ell_p$ and $\ell_q(M) = n$.

Claim 4.2. \mathbb{P} is proper and if $G \subset \mathbb{P}$ is generic over V , then $S_G =_{\text{def}} \{\delta_N : \exists p \in G, N \in \mathcal{N}_p\}$ is a stationary subset of ω_2^V .

Proof.

The proof is essentially the same as that in [15] and [2]. But we include a short proof adapted to the present context for completeness. To see that \mathbb{P} is proper, let $\theta > \chi$ be a large enough regular cardinal and let $M^* \prec H(\theta)$ be a countable elementary submodel containing \mathbb{P}, K . Given $p \in M^* \cap \mathbb{P}$, since p is finite, there exists $m_0 \in \omega$ larger than $\ell_p'' \mathcal{N}_p$. Let $M = M^* \cap K \in \mathcal{S}$. We claim that $q = p + (M, m_0)$ is (M^*, \mathbb{P}) -generic. To see this, let $r \leq q$ and a dense open set $D \in M^* \cap \mathcal{P}(\mathbb{P})$ be given. We may assume $r \in D$. The goal is to find $t \in D \cap M^*$ such that t is compatible with r . Note that r can be decomposed into the part inside M , denoted as $r \upharpoonright M$, and the part outside M . To simplify presentations, we will assume the part of r outside M consists of \in -increasing $\{M_0, M_1, M_2\}$ with $M_0 = M$ and $\ell_r(M_i) = m_i$ for $i < 3$. Consider

$$B =_{\text{def}} \{(\alpha_0, \alpha_1, \alpha_2) \in [\omega_1]^3 : \exists t \in D, \mathcal{N}_t = (\mathcal{N}_r \cap M) \cup \{M'_0, M'_1, M'_2\},$$

$$\ell_t \upharpoonright M = \ell_r \upharpoonright M, \forall i < 3 \ell_t(M'_i) = m_i, \alpha_i = \delta_{M'_i}\}.$$

Clearly B is definable from D and $r \upharpoonright M$. So $B \in M^*$. As $B \in H(\chi)$ as well, we know that $B \in M$, hence $B \in M_i$ for $i < 3$. Let Qx be the Keisler's quantifier asserting "there exist uncountably many x ". Note that $Q\alpha_0 Q\alpha_1 Q\alpha_2 (\alpha_0, \alpha_1, \alpha_2) \in B$ holds. To see this, let $\delta_i = \delta_{M_i}$ for $i < 3$, then we know $(\delta_0, \delta_1, \delta_2) \in B$. Work in M_2 , we know by elementarity that $Q\alpha_2 (\delta_0, \delta_1, \alpha_2) \in B$. Repeat the same argument with M_1 and M_0 , we will get $Q\alpha_0 Q\alpha_1 Q\alpha_2 (\alpha_0, \alpha_1, \alpha_2) \in B$.

By elementarity of M , $Q\alpha_0 Q\alpha_1 Q\alpha_2 (\alpha_0, \alpha_1, \alpha_2) \in B$ holds in M . We say ν is *good* for A if for any $\{\xi, \zeta\}_{<} \in [A]^2$, if $\nu < \zeta$ then $\nu \notin t_{\xi\zeta} =_{\text{def}} \{\eta : f(\eta, \zeta) = f(\xi, \zeta)\}$. Consider $C = \{\delta_N : N \in \mathcal{N}_r\}$. Since C is finite and all the color classes have order type bounded above by γ , the order type of $\bigcup_{\{\xi, \zeta\} \in [C]^2} t_{\xi\zeta}$ is bounded above by $(\gamma + \omega)^\omega$ (this is the ordinal exponentiation), which is $< M \cap \omega_1$. Therefore, we can find $\alpha_0 \in M$ that

is good for C such that $Q\alpha_1 \ Q\alpha_2 \ (\alpha_0, \alpha_1, \alpha_2) \in B$ holds. Repeat the same argument, we can continue to find $\alpha_1 \in M$ good for $C \cup \{\alpha_0\}$, and $\alpha_2 \in M$ good for $C \cup \{\alpha_0, \alpha_1\}$. Finally, let $t \in D \cap M$ witness $\{\alpha_0, \alpha_1, \alpha_2\} \in B$. It is easy to see that t and r are compatible.

To see S_G is a stationary subset of ω_2^V , let \dot{C} be a \mathbb{P} -name for a club in ω_2^V and $p \in \mathbb{P}$ be given. Let θ be a large enough regular cardinal, and $M^* \prec H(\theta)$ be a countable elementary submodel containing $\mathbb{P}, K, \dot{C}, p$. Let $q \leq_{\mathbb{P}} p$ be an (M^*, \mathbb{P}) -generic condition, with $M^* \cap K \in \mathcal{N}_q$. Then $q \Vdash_{\mathbb{P}} \sup(M \cap \omega_2^V) = \sup(M^* \cap \omega_2^V) = \sup(M^*[G] \cap \omega_2^V) \in \dot{C}$.

□_{4.3}

Let $G \subset \mathbb{P}$ be generic over V . In $V[G]$, ω_2^V is collapsed and has cofinality ω_1 . Let $\ell = \bigcup \{\ell_p : p \in G\}$, so ℓ partitions S_G into countably many f -rainbow subsets. Therefore, at least one of them is stationary in ω_2^V . Let \bar{n} be a name of the least natural number for which the set $S = \ell^{-1}(\{\bar{n}\})$ is stationary in ω_2^V . In $V[G]$, let \mathbb{R} be a forcing notion shooting a club of order type ω_1 into S . More precisely, \mathbb{R} consists of countable closed subsets of S , ordered by end extension. In V , let \mathbb{R} be the corresponding \mathbb{P} -name. Let $\mathbb{Q} = \mathbb{P} * \mathbb{R}$.

We shall prove that \mathbb{Q} preserves stationary subsets of ω_1 . This will conclude the proof since the generic set for \mathbb{Q} gives a closed f -rainbow subset of order type ω_1 . Suppose that $(p, r) \in \mathbb{Q}$ and by extending p if needed we may assume that $p \Vdash_{\mathbb{P}} \bar{n} = n$. Define $S_p = \{\delta \in \omega_2 : \exists M \in \mathcal{S}, \delta_M = \delta \text{ and } (M, n) \text{ is addable to } p\}$. Notice that $S_p \subseteq S^{\omega_2}$.

Claim 4.3. S_p is stationary.

Proof.

Suppose for the sake of contradiction that $C \subseteq \omega_2$ is a club of ω_2 disjoint from S_p . Let $G \subset \mathbb{P}$ containing p be generic over V . By an easy density argument, we know for a tail $N \in S_G$, $p \in N$. Let $M \in S_G$ containing p such that $\delta_M \in C \cap \ell^{-1}(n)$. By definition, (M, n) is addable to p and hence $\delta_M \in S_p$. This contradicts with our hypothesis.

□_{4.2}

Let T be a stationary subset of ω_1 in the ground model and suppose that $(p, r) \in \mathbb{Q}$. Let \dot{C} be a \mathbb{Q} -name for a club in ω_1 . Let θ be a sufficiently large regular cardinal. Choose $M^* \prec H(\theta)$ containing $(p, r), \mathbb{Q}, K, \dot{C}$ such that letting $M =_{\text{def}} M^* \cap K$, we have (M, n) is addable to p and $M \cap \omega_1 \in T$. This is possible by Claim 4.3 and the observation that if $M_0, M_1 \in \mathcal{S}$ with $p \in M_0 \cap M_1$ and $\delta_{M_0} = \delta_{M_1}$, then (M_0, n) is addable to p iff (M_1, n) is addable to p .

Let $q = p + (M, n)$. Notice that q is (M^*, \mathbb{P}) -generic by Claim 4.2 and therefore $q \Vdash_{\mathbb{P}} M^*[G] \cap \text{Ord} = M^* \cap \text{Ord}$. In particular, $\sup(M^*[G] \cap \omega_2^V) = \sup(M^* \cap \omega_2^V) \in \ell^{-1}(n)$. It follows that q forces that there is a decreasing generic sequence over $M^*[G]$, say $\{r_n \in \mathbb{R} : n \in \omega\}$ with $r_0 = r$ and a

master condition t which satisfies $t \leq_{\mathbb{R}} r_n$ for every $n \in \omega$. Since $(q, \underline{t}) \Vdash_{\mathbb{Q}} M[G * H] \cap \omega_1 = M \cap \omega_1 \in \mathcal{C} \cap T$, we are done.

□_{4.1}

Remark 4.4. Adapting the proof of Theorem 4.1, one can also show the following: for any $f : [\omega_2]^2 \rightarrow \omega_2$ such that there exists a stationary $S \subset S_0^2$ and $\gamma < \omega_1$ with the order type of $t_{\eta\alpha} = \{\eta' < \alpha : f(\eta', \alpha) = f(\eta, \alpha)\}$ being bounded above by γ for any $\alpha \in S$ and $\eta < \alpha$, then there exists a closed f -rainbow set $C \subset S$ of order type ω_1 . In particular, this shows that MM implies $\neg(*)$. To see this, suppose a putative $(*)$ -sequence $\langle S_\alpha : \alpha \in S_0^2 \rangle$ is given. We may find a stationary $S \subset S_0^2$ and $\gamma < \omega_1$ such that for all $\alpha \in S$, $\text{otp}(S_\alpha) \leq \gamma$. Define a coloring f on $[\omega_2]^2$ such that for each $\alpha \in S$,

- $f(\cdot, \alpha) \upharpoonright S_\alpha$ is constant and
- for any $\alpha \in S$ and any $\eta < \alpha$, $\text{otp}(t_{\eta\alpha}) \leq \gamma$.

Thus f satisfies the modified hypothesis above as witnessed by S and γ . Therefore, MM implies there is a closed f -rainbow $C \subset S$ of order type ω_1 . Let $\beta = \sup C$ and c_β be the witness to the property of the $(*)$ -sequence (recall Definition 3.1). Letting $D = C \cap c_\beta$, we see that on one hand, for any $\alpha \in D$, $D \cap \alpha \subset S_\alpha$, but on the other hand, for any $\alpha \in D$, $|D \cap S_\alpha| \leq 1$. We have reached a contradiction.

5. SOME OPEN PROBLEMS

We have seen that $\omega_2 \not\rightarrow^{\text{poly}} (\omega_2 - cl)_{2-bdd}^2$ in ZFC. On the other hand, it is consistent to get a closed rainbow subset of type strictly greater than ω_1 .

Question 5.1. Is it consistent that $\omega_2 \rightarrow^{\text{poly}} (\alpha - cl)_{2-bdd}^2$ for every ordinal $\alpha \in \omega_2$?

Note $\alpha = \omega_1 + \omega_1$ is the first point which challenges our method.

Question 5.2. Is it consistent that $\omega_2 \rightarrow^{\text{poly}} (\omega_2 - st)_{2-bdd}^2$?

One may wonder if Corollary 2.8 can be improved:

Question 5.3. Is it consistent that $\aleph_{\omega+1} \rightarrow^{\text{poly}} (\omega_1 - cl)_{\aleph_{\omega+1}, (<\aleph_\omega)-bdd}^2$?

We have only dealt with successors of singular cardinals of countable cofinality. Our method is limited because the consistency of the higher analogue of Theorem 2.3 is not clear (see Question 5.2).

Question 5.4. Is it consistent that κ is a singular cardinal with $cf(\kappa) > \omega$ and $\kappa^+ \rightarrow^{\text{poly}} (cf(\kappa)^+)_{<\kappa-bdd}^2$?

We saw that $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{<\omega_1-t-bdd}^2$ and $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - st)_{\omega-bdd}^2$ are respectively consistent. A natural question is whether the following joint strengthening is possible.

Question 5.5. Is it consistent that $\omega_2 \rightarrow^{\text{poly}} (\omega_1 - cl)_{\omega-bdd}^2$ under any assumption?

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