THE SECOND-ORDER VERSION OF MORLEY'S THEOREM ON THE NUMBER OF COUNTABLE MODELS DOES NOT REQUIRE LARGE CARDINALS

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Abstract. The consistency of a second-order version of Morley's Theorem on the number of countable models was proved in [EHMT23] with the aid of large cardinals. We here dispense with them.

1. Introduction

Vaught's Conjecture [10], which asserts that a countable first-order theory must have either at most countably many or exactly $2^{\aleph_0}$ many non-isomorphic countable models, is one of the most important problems in Model Theory. A strong positive result about Vaught's Conjecture is a result of the late Michael Morley [8] which states that the number of isomorphism classes of countable models of a countable first-order theory is always at most $\aleph_1$ or exactly $2^{\aleph_0}$. Under this formulation, the result follows trivially from the continuum hypothesis. To avoid this artifact, one can identify countable models with members of the Cantor set (see [6] or [4]) and prove:

Theorem 1.1 (Absolute Morley). *Let T be a first-order theory (or more generally, a sentence of $L_{\omega_1,\omega}$) in a countable signature. Then either T has at most $\aleph_1$ isomorphism classes of countable models, or there is a perfect set of non-isomorphic countable models of T.*

The isomorphism relation among countable models can be formulated as a Σ^1_1 equivalence relation. It is then easy to see that the following result [2] (see [6]) is a strengthening of the Absolute Morley Theorem:

Theorem 1.2. *Let E be a Σ^1_1 equivalence relation on $\mathbb{R}$. If there is no perfect set of pairwise inequivalent reals, then there are at most $\aleph_1$ equivalence classes.*

Second-order logic is the natural generalization of first-order logic to a two-sorted language with variables for relations as well as for individuals. For a precise formulation of its syntax and semantics, see e.g. [4]. We then can formulate:

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Second-order Absolute Morley: If T is a second-order theory in a countable signature, then either T has at most $\aleph_1$ isomorphism classes of countable models, or there are $2^{\aleph_0}$ many (there is a perfect set of) non-isomorphic countable models of T.

The *σ-projective* hierarchy is obtained by extending the projective hierarchy up through the countable ordinals. See [7] and [1]. We can then formulate:

Second-order Absolute Morley for σ-projective equivalence relations Let E be a σ-projective equivalence relation on $\mathbb{R}$. If there is no perfect set of pairwise E-inequivalent reals, then E has at most $\aleph_1$ equivalence classes.

A slightly weaker assertion is

Second-order Absolute Morley for countable intersections of projective sets Let E be an equivalence relation on $\mathbb{R}$ which is a countable intersection of projective sets. If there is no perfect set of pairwise E-inequivalent reals, then E has at most $\aleph_1$ equivalence classes.

That Second-order Absolute Morley for countable intersections of projective sets implies Second-order Absolute Morley is shown in [4]. It is a straightforward generalization of Burgess' proof. In [4], the following results are established:

Theorem (Theorem A)**.** *Force over L by first adding $\aleph_2$ Cohen reals and then $\aleph_3$ random reals. In the resulting universe of set theory, $2^{\aleph_0} = \aleph_3$ but there is a second-order theory T in a countable signature such that the number of non-isomorphic models of T is exactly $\aleph_2$.*

Theorem (Theorem C)**.** *If there are infinitely many Woodin cardinals, then there is a model of set theory in which Second-order Absolute Morley for countable intersections of projective sets holds.*

The authors of [4] state as their first problem:

Prove that large cardinals are necessary to prove the consistency of Second-order Absolute Morley.

We shall refute that conjecture here by proving:

Theorem 1.3. *Adjoin at least $\aleph_2$ Cohen reals to a model of* CH. *Then Second-order Absolute Morley for σ-projective equivalence relations holds in the resulting model.*

The main idea of both 1.3 and Theorem C occurs in the earlier work [5], in which Foreman and Magidor prove:

Theorem (Theorem B)**.** *In the usual iterated forcing model of* PFA *(thus assuming the existence of a supercompact cardinal), if E is an equivalence relation on $\mathbb{R}$ such that E is a member of $L(\mathbb{R})$, then E has either no more than $\aleph_1$ equivalence classes or else perfectly many equivalence classes.*

The key is *generic absoluteness.* Call an equivalence relation on sets of reals *thin* if it does not have a perfect set of equivalence classes. Simplifying the argument by considering E's definable from a real r, we note they prove (using the large cardinal) that the formula ϕ that defines E defines a thin equivalence relation E' in the intermediate model in which r first appears. This is *downwards generic absoluteness*. In that intermediate model, CH holds and hence E' has $\leq \aleph_1$ equivalence classes. Next they show that the rest of the forcing (after which the formula ϕ defines E) cannot add a new equivalence class to E' unless E has perfectly many equivalence classes. This is *upwards generic absoluteness*. [4] follows the same approach but with a weaker large cardinal hypothesis. We follow the same approach here, but it turns out that because Cohen real forcing is so simple and homogeneous and because adding one Cohen real by forcing adds perfectly many, we don't need the large cardinal. This latter observation substitutes for upwards generic absoluteness, and hence no large cardinal is needed for that. If we add $\aleph_2$ many Cohen reals, then the real that codes the σ-projective set appears at an initial stage at which CH holds. If we add more than $\aleph_2$ Cohen reals, we need to apply an automorphism argument to get that without loss of generality, we may assume r appears in the first ω_1 stages. The required downwards generic absoluteness is proved by induction on the complexity of the σ-projective formulas that define our equivalence relations. A version of our generic absoluteness theorem was proved by Joan Bagaria many years ago with essentially the same proof, but he never published it. We rediscovered it and are not aware of any published reference. Now for the details.

Lemma 1.4. *For any cardinals λ, κ, the following is true in $V^{Add(\omega,\omega_1)\times Add(\omega,\lambda)}$: for any σ-projective formula $\varphi(\bar{x})$ and any $\bar{a} \in \mathbb{R}^{|\bar{x}|}$*

$$H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \varphi(\bar{a}) \text{ if and only if } \Vdash_{Add(\omega,\kappa)} H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \varphi(\bar{a}).$$

Remark 1.5. Since all the $\mathcal{L}_{\omega_1,\omega}$-formulas we consider use real parameters and all the quantifiers are bounded by the set of real numbers, they are absolute between V and $H(\omega_1)$, due to the fact that the latter contains all reals and is closed under countable sequences.

Proof. We induct on the complexity of the formulas. Observe the homogeneity of Cohen forcing implies that if for some p, $p \Vdash_{Add(\omega,\kappa)} H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \varphi(\bar{a})$, then $\Vdash_{Add(\omega,\kappa)} H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \varphi(\bar{a})$. In particular, this means if we have proved the result for $\mathbf{\Sigma}^1_\xi$ formulas, then we get the result for $\mathbf{\Pi}^1_\xi$ formulas immediately.

Suppose we have proved the theorem for all formulas that are $\mathbf{\Sigma}^1_\nu$ for $\nu < \xi < \omega_1$ and ξ is a limit, then $\varphi(\bar{x})$ is of the form $\bigvee_n \{\psi_n(\bar{x}) : \psi_n \in \mathbf{\Sigma}^1_{\xi_n}\}$ for a sequence $\langle \xi_n < \xi : n \in \omega \rangle$. Let $G \times H \subseteq Add(\omega,\omega_1) \times Add(\omega,\lambda)$. In $V[G \times H]$, suppose $H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \varphi(\bar{a})$, equivalently, there is $n \in \omega$, $H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \psi_n(\bar{a})$. By the induction hypothesis, we know that, $\Vdash_{Add(\omega,\kappa)} H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \psi_n(\bar{a})$. As a result, $\Vdash_{Add(\omega,\kappa)} H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \varphi(\bar{a})$. A similar argument shows that if $H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \neg\varphi(\bar{a})$, then $\Vdash_{Add(\omega,\kappa)} H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \neg\varphi(\bar{a})$.

Suppose we have proved the theorem for $\mathbf{\Sigma}^1_\xi$-formulas for $\xi < \omega_1$. Let $\exists x \psi(x, \bar{y})$ be a $\mathbf{\Sigma}^1_{\xi+1}$-formula. Suppose

$$V[G \times H]^{Add(\omega,\kappa)} \models H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \varphi(\bar{a}),$$

where $\bar{a} \in V[G \times H]$. Given $(p, q) \in P =_{def} Add(\omega,\omega_1) \times Add(\omega,\lambda)$ and $\dot{x}$ a $P \times Add(\omega,\kappa)$-name and $\dot{\bar{a}}, \dot{\varphi}$ P-names such that (p, q) forces the above holds for the respective names, find $\alpha < \omega_1, A \subseteq \lambda, B \subseteq \kappa$ countable such that these conditions and names are in $Add(\omega,\alpha) \times Add(\omega, A) \times Add(\omega, B)$ or are $Add(\omega,\alpha) \times Add(\omega, A) \times Add(\omega, B)$-names.

Define an automorphism π on $P \times Add(\omega,\kappa)$ as follows: $\pi(a,b,c) = (a^*, b^*, c^*)$ if and only if

(1) $a \restriction \omega_1 - [\alpha, \alpha + otp(B)) = a^* \restriction \omega_1 - [\alpha, \alpha + otp(B))$, $b = b^*$,
(2) $a^*(\alpha + i) = c(i)$, $c^*(i) = a(\alpha + i)$ for all $i \in B$,
(3) $c^* \restriction \kappa - B = c \restriction \kappa - B$.

In particular, π fixes (p, q), $\dot{\bar{a}}$ and $\dot{\varphi}$. Therefore, $(p, q, \emptyset) \Vdash_{P \times Add(\omega,\kappa)} \psi(\pi(\dot{x}), \dot{\bar{a}})$.

Let $G \times H \times R \subseteq P \times Add(\omega,\kappa)$ be generic containing (p, q). Then we know that $V[G \times H \times R] \models H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \psi((\pi(\dot{x})^{G\times H\times R}), \bar{a})$. Let $x^* = \pi(\dot{x})^{G\times H\times R}$. By the definition of π, we know that $\pi(\dot{x})$ is a $Add(\omega,\omega_1) \times Add(\omega,\lambda)$-name, in particular, $x^* \in V[G \times H]$. By the induction hypothesis, we know that $V[G \times H \times R] \models H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \psi(x^*, \bar{a})$ if and only if $V[G\times H] \models H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \psi(x^*, \bar{a})$. Therefore, $V[G \times H] \models H(\omega_1) \models_{\mathcal{L}_{\omega_1,\omega}} \exists x\psi(x, \bar{a})$, as desired. □

Remark 1.6. The result holds in a more general context, namely, after adding $\aleph_1$ many Cohen reals, then $L(\mathbb{R}) \prec (L(\mathbb{R}))^{V[G]}$ where G is any further Cohen extension. This was known to Bagaria and probably others. The proof we supply is only for what we need.

Lemma 1.7. *In $V^{Add(\omega,\lambda)}$ for any $\lambda \geq \omega_1$, the following holds: let E be a σ-projective equivalence relation. If $\Vdash_{Add(\omega,1)}$ "$\exists\sigma \in \mathbb{R}$ such that for all $r \in V$, $\neg(\sigma E r)$", then $\Vdash_{Add(\omega,1)}$ there exist perfectly many E-classes.*

Proof. Let $\dot{\sigma}$ be an $Add(\omega, 1)$-name such that $\Vdash_{Add(\omega,1)}$ "$\neg\dot{\sigma}Er$ for any $r \in V$".

Claim 1. *$\Vdash_{Add(\omega,1)\times Add(\omega,1)} \neg\dot{\sigma}_{\text{left}} E \dot{\sigma}_{\text{right}}$, where $\dot{\sigma}_{\text{left}}$ ($\dot{\sigma}_{\text{right}}$) is the name $\dot{\sigma}$ produced from the left (right) generic.*

Proof of the Claim. For notational simplicity, let $\dot{\sigma}_0 = \dot{\sigma}_{\text{left}}$ and $\dot{\sigma}_1 = \dot{\sigma}_{\text{right}}$. Suppose otherwise, let $(p, s, t) \in Add(\omega,\lambda) \times Add(\omega, 1) \times Add(\omega, 1)$ force that $\dot{\sigma}_0 E \dot{\sigma}_1$ and $\neg\dot{\sigma}_i Er$ for any $r \in V^{Add(\omega,\lambda)}$, where $i = 0, 1$. Let $A \subseteq \lambda$ be countable such that $p \in Add(\omega, A)$ and $\dot{E}$ is an $Add(\omega, A)$. More precisely, the defining formula and the parameters of E are $Add(\omega, A)$-names. Fix some $\gamma \in \lambda - A$. Consider the following automorphism π on $Add(\omega,\lambda) \times Add(\omega, 1) \times Add(\omega, 1)$: $\pi(a, b, c) = (a^*, b^*, c^*)$ where

- $a \restriction \lambda - \{\gamma\} = a^* \restriction \lambda - \{\gamma\}$,
- $a^*(\gamma) = b$,
- $b^* = a(\gamma)$,
- $c = c^*$.

Hence, $\pi(p, s, r) \Vdash \pi(\dot{\sigma}_0)\pi(\dot{E})\pi(\dot{\sigma}_1)$. By the definition of π, we know that $\pi(p, s, r)$ is compatible with (p, s, r), $\pi(\dot{E}) = \dot{E}$, $\pi(\dot{\sigma}_1) = \dot{\sigma}_1$ and $\pi(\dot{\sigma}_0)$ is an $Add(\omega,\lambda)$-name. Let $G \times g_0 \times g_1 \subseteq Add(\omega,\lambda) \times Add(\omega, 1) \times Add(\omega, 1)$ be generic containing both (p, s, t) and $\pi(p, s, t)$, then in the generic extension, we have $\sigma^* E \sigma_1$ where

$\sigma^* = (\pi(\dot\sigma_0))^{G\times g_0\times g_1} \in V[G]$, as $\pi(\dot\sigma_0)$ is an $Add(\omega,\lambda)$-name. However, as $(p,s,t) \in G \times g_0 \times g_1$, we have that $\neg(\sigma^* E \sigma_1)$, which is a contradiction. □

Let $V^* = V[G]$ where $G \subseteq Add(\omega,\lambda)$ is generic over V. Work in V^*.

Consider the following forcing Q: $p \in Q$ if and only if $p : 2^n \to Add(\omega,1)$ for some $n \in \omega$ such that

- for each $s \in 2^{\leq n}$, there is some $k_s \in \omega$ such that $p(s) \Vdash \dot\sigma \restriction k_s = \tau_s$,
- for $s \neq s' \in 2^m$ and $m \leq n$, $\tau_s \perp \tau_{s'}$, namely $\tau_s \cup \tau_{s'}$ is not a function.
- for $s \sqsubseteq s' \in 2^{\leq n}$, $p(s') \leq p(s)$ and $\tau_{s'} \sqsupset \tau_s$, namely τ_s is a proper initial segment of $\tau_{s'}$.

The order of Q is inclusion. Since Q is a non-trivial countable forcing, Q is forcing-equivalent to $Add(\omega,1)$. Let $T \subseteq Q$ be generic over V^*. In $V^*[T]$, an easy density argument shows that T is a perfect subtree of $2^{<\omega}$ and the branches of T are mutually generic Cohen reals over V^*. Consider $\{\sigma_b : b \in [T]\}$. By Claim 1, we know that if $b \neq b' \in [T]$, then in $V^*[b,b']$, $\neg\sigma_b E \sigma_{b'}$. By Lemma 1.4, in $V^*[T]$, $\neg\sigma_b E\sigma_{b'}$. It remains to see that $\{\sigma_b : b \in [T]\}$ is a perfect set. This is the case since $\sigma_b = \bigcup_{s\sqsubseteq b} \tau_s$ for any $b \in [T]$. □

Theorem 1.8. *Fix a regular cardinal $\kappa \geq \omega_2$. Over a model of CH, $\Vdash_{Add(\omega,\kappa)}$ every σ-projective equivalence relation either has $\leq \aleph_1$ or perfectly many equivalence classes.*

Proof. Let $p \in Add(\omega,\kappa)$ and let $\dot E$ be a name for a thin σ-projective equivalence relation, namely, the σ-projective formula along with the parameters that define it. Since the σ-projective formula and the parameters are essentially countable, there exists $A \subseteq \kappa$ such that $\omega_1 \subseteq A$ and $|A| = \aleph_1$ such that $p \in Add(\omega, A)$ and $\dot E$ is an $Add(\omega,A)$-name. By Lemma 1.4, we know that $p \Vdash_{Add(\omega,A)} \dot E$ is thin.

Claim 1. *In $V^{Add(\omega,A)}$, $\Vdash_{Add(\omega,1)}$ for any $\sigma \in \mathbb{R}$, there is an $r \in V$ such that $\sigma E r$.*

Proof of the Claim. Otherwise, there is a $p \Vdash \exists \sigma \in \mathbb{R}$ such that $\neg(\sigma E r)$ for any $r \in V$. By the homogeneity of Cohen forcing, we have $\Vdash \exists\sigma \in \mathbb{R}$ such that $\neg(\sigma E r)$ for any $r \in V$. By Lemme 1.7, we know that $\Vdash_{Add(\omega,1)}$ there exist perfectly many E-classes. By Lemma 1.4, $\Vdash_{Add(\omega,\kappa)}$ there exist perfectly many E-classes, contradicting the assumption on the thinness of E. □

Consequently, Claim 1 implies that in $V^{Add(\omega,A)}$, $\Vdash_{Add(\omega,\kappa-A)}$ "for any $\sigma \in \mathbb{R}$, there is an $r \in V$ such that $\sigma E r$". Since $V^{Add(\omega,A)}$ is a model of CH, we have that the number of E-classes in $V^{Add(\omega,\kappa)}$ is $\leq \aleph_1$. □

If $2^{\aleph_0} = \aleph_2$, Second-order Morley trivially holds. Although Second-order Absolute Morley holds in the Cohen model, Second-order Morley does not imply Second-order Absolute Morley in general.

Lemma 1.9. *Second-order Absolute Morley implies that for any light-faced projective set A, either A has size $\leq \aleph_1$ or there exists a continuous injection from 2^ω to A.*

Proof. The proof is similar to that of Lemma 3.1 in [4], adapted to the "absolute" scenario. Therefore, we will only sketch the proof. The second-order theory T considered here is the second-order Peano Arithmetic with an addition unary predicate X. Let $\mathcal{L}$ be language. Intuitively speaking, this predicate is coding a real that belongs to A. Namely, our model will be in the form of $\mathcal{A} = (\omega, \dot{X}, \cdots)$ and $\{i \in \omega : \mathcal{A} \models \dot{X}(i)\} \in A$. There is a natural translation of a projective formula ψ to a second-order $\mathcal{L}$-formula $\psi^{\mathcal{L}}$, that is truth-preserving (see [9, 8B.15] for more information). In particular, if $\varphi(x)$ is the projective definition of A, then the requirement that $\{i \in \omega : \mathcal{A} \models X(i)\} \in A$ can be expressed as $\mathcal{A} \models$ "$\exists Y, \forall n, n \in Y$ if and only if $n \in \dot{X}$ and $\varphi^{\mathcal{L}}(Y)$". If $|A| > \aleph_1$, then the application of the hypothesis will give us perfectly many non-isomorphic models, which implies A includes a perfect subset.

□

Remark 1.10. The proof above still works if A is bold-faced. In this case we just need to add more unary predicates for each one of the real parameters in the definition of A.

Theorem 1.11. *There is a model of* ZFC *in which $2^{\aleph_0} = \aleph_2$ and in which there is a projective equivalence relation on $\mathbb{R}$ which has $2^{\aleph_0}$-many equivalence classes but does not have a perfect set of equivalence classes.*

Proof. It is well-known that it is consistent that there is a light-faced projective well order of the reals and $2^{\aleph_0} = \aleph_2$ (see [3]). We use a projective well order of the reals to produce a projective set witnessing the conclusion of Lemma 1.9 fails. More precisely, we will construct a projective *Bernstein set* B, i.e. such that neither B nor $\mathbb{R} \setminus B$ includes a perfect set. In detail, let $\preceq$ be the projective well order. Recursively coding pairs of reals as reals, $\preceq$ induces a projective well order on pairs of reals that we will also call $\preceq$. For each real, we can ask if it codes a perfect subtree of $2^{<\omega}$ by some recursive bijection between ω and $2^{<\omega}$. Define $f : \mathbb{R} \to \mathbb{R}^2$ such that $f(x) = \langle y_x, z_x \rangle$ if:

1) for any $x' \prec x$, y_x and z_x are not equal to either $y_{x'}$ or $z_{x'}$,
2) if x codes a perfect subtree T_x of $2^{<\omega}$, then y_x and z_x are different branches through T_x, the tree determined by x,
3) $\langle y_x, z_x \rangle$ is the $\preceq$-least pair satisfying 1) and 2).

Since $\preceq$ is projective, so is f. Let

$$B = \{y : \exists x \exists z (f(x) = \langle y, z \rangle)\}.$$

Then $y = y_x \in B$ implies $z_x \notin B$.

Then B is projective since f is. However we claim that neither B nor $\mathbb{R} \setminus B$ includes a perfect set, for if such a perfect set P were realized as a perfect T_x, then y_x would be in B and z_x would be in $\mathbb{R} \setminus B$, yet both are in P. □

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