Divide & Conquer

Did you mean recursion?

Nattee Niparnan

Department of Computer Engineering
Chulalongkorn University

What is?

- D&C is an algorithm design framework
  - Some problem can be efficiently solve by this design, some are not
- Key Idea:
  - Solve a problem instance by divide it into smaller instances of the same type
    - A smaller instances is called a subproblem
  - Use recursive to solve the subproblems
    - The subproblem is also solved in the same way: recursive
    - The subproblems are divided repeatedly until the instance can no longer be divide, which should be very small instance that can be solved directly
  - Conquer (Combine) the result of subproblem into a result of the original instance

D&C and Recursive Programming

- Divide and Conquer extensively use recursion
- It is much easier to do recursion according to the definition of D&C
- Analysis of D&C usually be done by Master Method

Example

- We will discuss several example
  - Binary Search
  - Merge Sort
  - Quicksort
  - Modulo exponential
  - Maximum Contiguous Sum of Subsequence
  - Strassen’s Matrix Multiplication
  - Closest Pair

Binary Search

Binary Search Problem

- Input:
  - Array A[1..n], sorted, n >= 1
  - A key k
- Output:
  - If k exists in A, return its first position
  - If k does not, return -1
- Example
  - Input: A = [1, 2, 4, 5, 6, 7, 8, 10], k = 4 output: 3
  - Input: A = [1, 2, 4, 5, 6, 6, 6, 10], k = 6 output: 5 (not 6 nor 7)
  - Input: A = [1, 2, 4, 5, 6, 6, 6, 10], k = 100 output: -1

A method to design D&C algorithm

- Consider a generic problem instance
- Ask: if we know output of smaller instance, how can we use it to construct the output of the original instance
- Try:
  - Dividing problem into a subproblem with n-1 input, the idea might help solve larger division into n-k inputs
  - Dividing problem into k non-overlapping subproblems, usually 2 problem of the same size

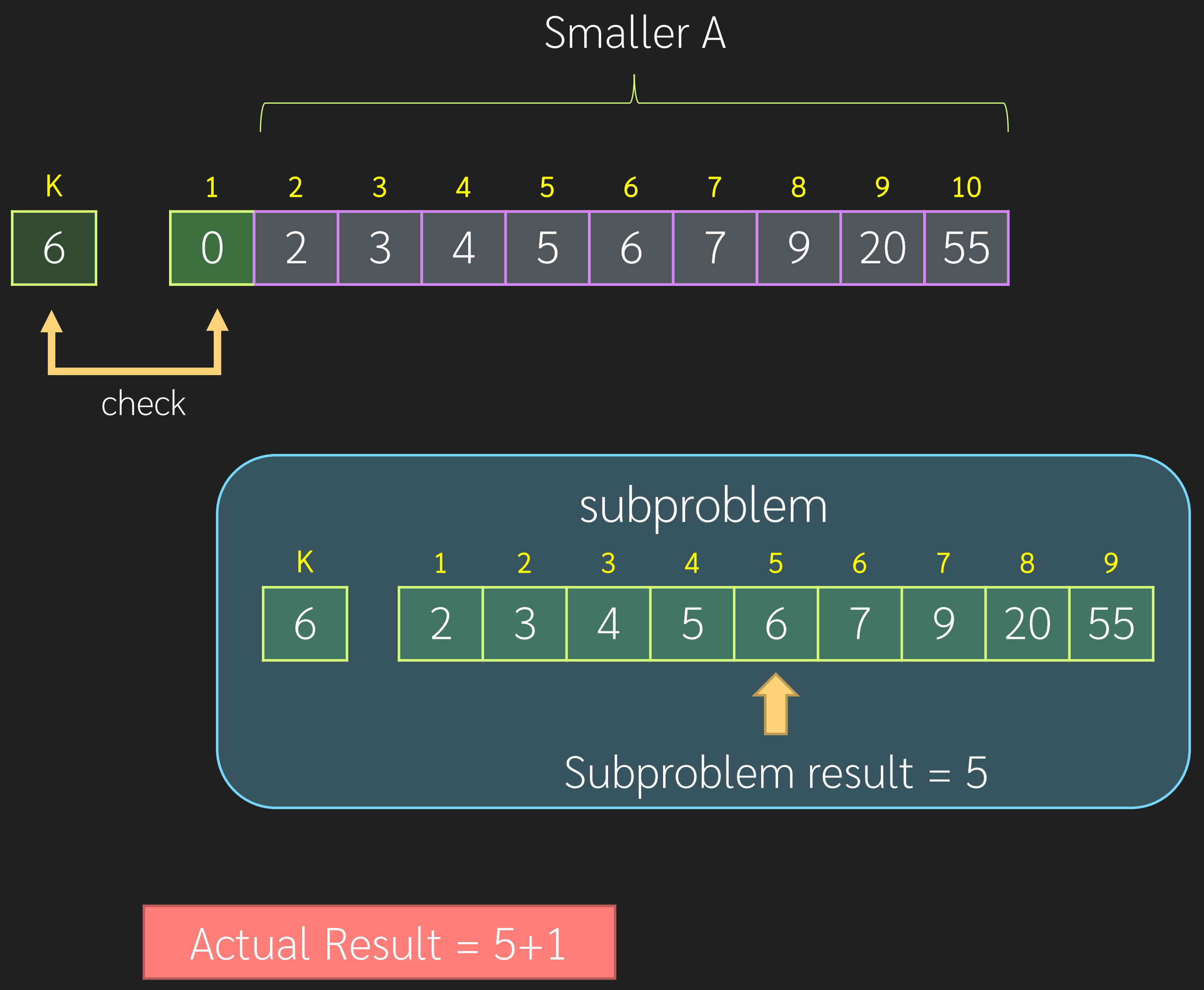
Binary Search, Naïve Approach

```
def bsearch_naive(A[1..n],k)
  for i from 1 to n
    if A[i] == k
      return i
  return -1
end
```

- Normal linear search
- O(N)
- Does not utilize the fact that A[1..n] is sorted

Binary Search, D&C v 0.1

- Key Idea
  - A subproblem is exactly another instance of a Binary Search
  - Divide: A[1..n], k into A[2..n],k
  - Conquer: if A[1] is K, return 1, if not return the result of the sub problem
    - Need to adjust position
  - Trivial case: when n == 1, we check only A[1] without doing any subproblem



Pseudo-code and its analysis

```
def bsearch_slow(A[1..n],k)
  if n == 1
    if A[1] == k
      return 1
    return -1
  if A[1] == k
    return 1
  else
    B = A[2..n]
    r = bsearch_slow(B,k)
    if r != -1
      return r+1
    else
      return r
end
```

- Trivial case becomes initial condition of the recurrence relation $T(1) = O(1)$
- $T(n) = T(n-1) + a + b$
  - a = time to divide the problem
    - $O(n)$ in this case (because we need to create another array)
  - b = time to conquer the problem
    - $O(1)$
  - $T(n) = T(n-1) + O(n)$

Solving

$$T(n) = \begin{cases} T(n-1) + O(n) & ; n > 1 \\ 1 & ; n = 1 \end{cases}$$

- By substitution

$$\left.\begin{aligned} T(n) &= T(n-1) + n \\ T(n-1) &= T(n-2) + n - 1 \\ T(n-2) &= T(n-3) + n - 2 \\ &\dots \\ T(n-i) &= T(n-i-1) + n - i \\ &\dots \\ T(n-(n-2)) &= T(n-(n-1)) + 2 \\ T(1) &= 1 \end{aligned}\right\}$$

Sum both sides of the equation,
Recursive terms cancel out

Result is $T(n) = \sum i = \frac{n(n+1)}{2} = O(n^2)$

This is slower than naïve!
Because our division is too slow

Improving v0.1

```
template <typename T>
int bsearch_slow(queue<T> &v, T k) {
  if (v.size() == 1) {
    if (v.front() == k)
      return 1;
    return -1
  } else {
    if (v.front() == k)
      return 1;
    v.pop();
    int r = bsearch_slow(v,k);
    return (r == -1) ? r : r+1;
  }
}

int bsearch_slow(vector<T> &v, T k) {
  queue<T> q;
  for (auto &x : v) q.push(v);
  return bsearch_slow(q,k);
}
```

- Better subproblem division using queue
  - T(n) = T(n-1)+1
    - Solve into T(N) = O(N)
  - We also need O(N) (only one time) to convert a given array into a queue
    - Still result in O(N) total time
- We can do better!

Much better v0.1

```
template <typename T>
int bsearch_slow_2(vector<T> &v, T k,int start) {
  if (start == v.size() - 1) {
    if (v[start] == k) return start;
    return -1
  } else {
    if (v[start] == k) return start;
    return bsearch_slow_2(v,k,start+1);
  }
}

template <typename T>
int bsearch_slow_2(vector<T> &v, T k) {
  return bsearch_slow_2(v,k,0);
}
```

```
template <typename T>
int bsearch_slow_3(vector<T> &v, T k,int start) {
  if (v[start] == k) return start;
  if (start == v.size()-1) return -1;
  return bsearch_slow_3(v,k,start+1);
}

template <typename T>
int bsearch_slow_3(vector<T> &v, T k) {
  return bsearch_slow_3(v,k,0);
}
```

- We can DRY the code further
- Better code, but still $T(n) = O(n)$, not faster than naïve method

V0.1 Summary

- Use one smaller subproblem (less by 1) to help solve the original problem
- Need to divide
- Need to conquer
- Using correct data structure (or better indexing) helps getting better performance
- Pseudo-code can tell the idea more clearly but implementation details depends on expertise in programming

V0.2: Different Division

- V0.1 divides n into n-1, and solve the remaining 1
- Can we try dividing A of size n into 2 arrays of size n/2?

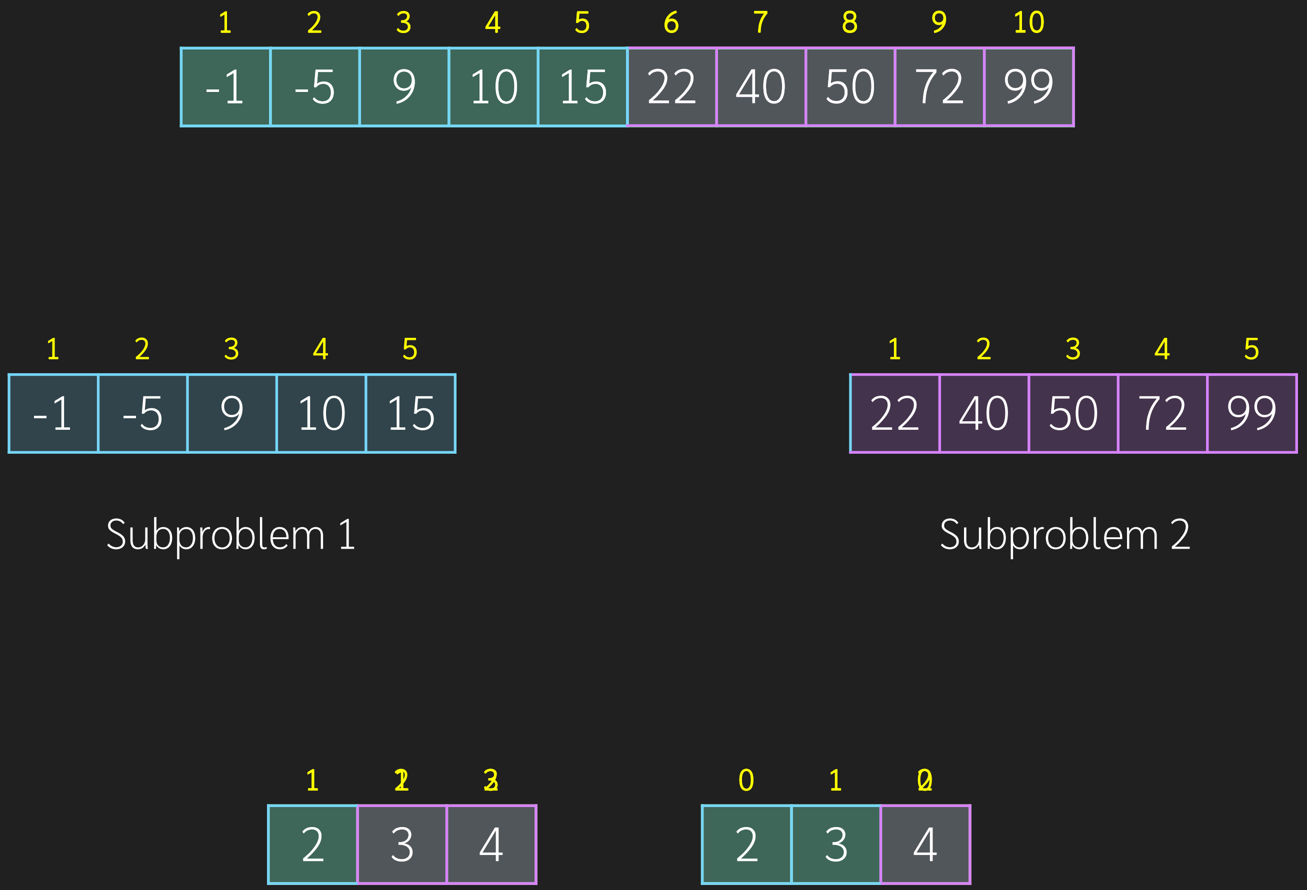
$$T(n) = \begin{cases} 2T(n/2) + O(1) & ; n > 1 \\ 1 & ; n = 1 \end{cases}$$

Assume that we use indexing technique to get B, C efficiently

Can we use Master Method to solve this into O(n)?

```
def bsearch_half(A[1..n],k)
  if n == 1
    if A[1] == k
      return 1
    return -1
  m = n/2
  B=A[1..m]
  r = bsearch_half(B,k);
  if r != -1
    return r
  C=A[m+1..n]
  r = bsearch_half(C,k);
  if r != -1
    return r + m
  return -1
end
```

Divide into two subproblems of similar size



- Divide by m = n/2
  - Integer division
- Subproblem 1 = 1..m
- Subproblem 2 = m+1..n
- For odd number of n, starting index is relevant to the size of each subproblem

Recap

- Naïve (non-)Binary Search is a linear search using O(N)
- V0.1 is just a linear search written as a divide and conquer
  - The cost of divide has to be managed so that the recurrence relation is $T(n) = T(n-1) + 1 = O(n)$
- V0.2 tries to divide into 2 equal subproblems
  - $T(n) = 2T(n/2) + 1 = O(n)$, different recurrence relation but same performance
- To get better performance, we need to reduce the number of subproblems

Actual Binary Search

- Utilize the fact that the data is sorted
- We will divide the problem into 2 parts: left and right
  - Check the middle point
  - only one part is solved, the other is discarded, depends on the value of the middle point

$$T(n) = \begin{cases} T(n/2) + O(1) & ; n > 1 \\ 1 & ; n = 1 \end{cases}$$

The result is T(n) = O(log n)

```
def bsearch(A[1..n],k)
  if n == 1
    if A[1] == k
      return 1
    return -1
  else
    m = n/2
    if A[m] => k
      r = bsearch(A[1..m],k)
    else
      r = bsearch(A[m+1..n],k)
      if r != -1
        r = r + m
    return r;
end
```

Actual Code

```
template <typename T>
int bsearch(vector<T> &v, T k,int start, int stop) {
  if (start == stop) return v[start] == k ? start : -1;
  int m = (start+stop) >> 1;  //bitwise shift right
  if (v[m] >= k) return bsearch(v,k,start,m);
  else return bsearch(v,k,m+1,stop);
}

template <typename T>
int bsearch(vector<T> &v, T k) {
  return bsearch(v,k,0,v.size()-1);
}
```

- Using moving index
- Non-recursive version exists

What’s wrong with these code?

```cpp
template <typename T>
int bsearch(vector<T> &v, T k,int start, int stop) {
  if (start == stop) return v[start] == k ? start : -1;
  int m = (start+stop) >> 1;  //bitwise shift right
  if (v[m] == k) return m;
  if (v[m] <= k) return bsearch(v,k,m+1,stop);
  if (v[m] > k) return bsearch(v,k,start,m-1);
}
```

```cpp
template <typename T>
int bsearch(vector<T> &v, T k,int start, int stop) {
  if (start > stop) return -1;
  int m = (start+stop) >> 1;  //bitwise shift right
  if (v[m] >= k) return bsearch(v,k,start,m);
  else  return bsearch(v,k,m+1,stop);
}
```

```cpp
template <typename T>
int bsearch(vector<T> &v, T k) {
  return bsearch(v,k,0,v.size()-1);
}
```

- Does it work correctly according to the problem definition?

Non recursive version

- It is possible to write Binary Search as a loop
  - Maintain two variable, start and stop
  - Each iteration, start and stop move closer to each other
- When should be stop?
  - What happen when start = stop ?
  - Also when start+1 = stop?
- Try it yourself

Summary for Binary Search

- First example of divide and conquer
- Different dividing strategies may give different performance
  - Not necessarily better than naïve method
- Pseudo-code is better in providing idea and insight
- Actual code is better in analysis
- Care should be taken when converting pseudo-code into a code
- D&C Benefit can be achieved when we divide by half

Merge sort

D&C Sorting, easy divide

The Sorting Problem

- Input:
  - Array A[1..n] of n data (we must be able to compare a pair of them)
- Output:
  - The same array but re-arranged such that A[i] <= A[i+1] for every i from 1 to n-1
- Example instance
  - Input: [1,5,3,2,7,1]
  - Output: [1,1,2,3,5,7]

Sorting by Divide & Conquer

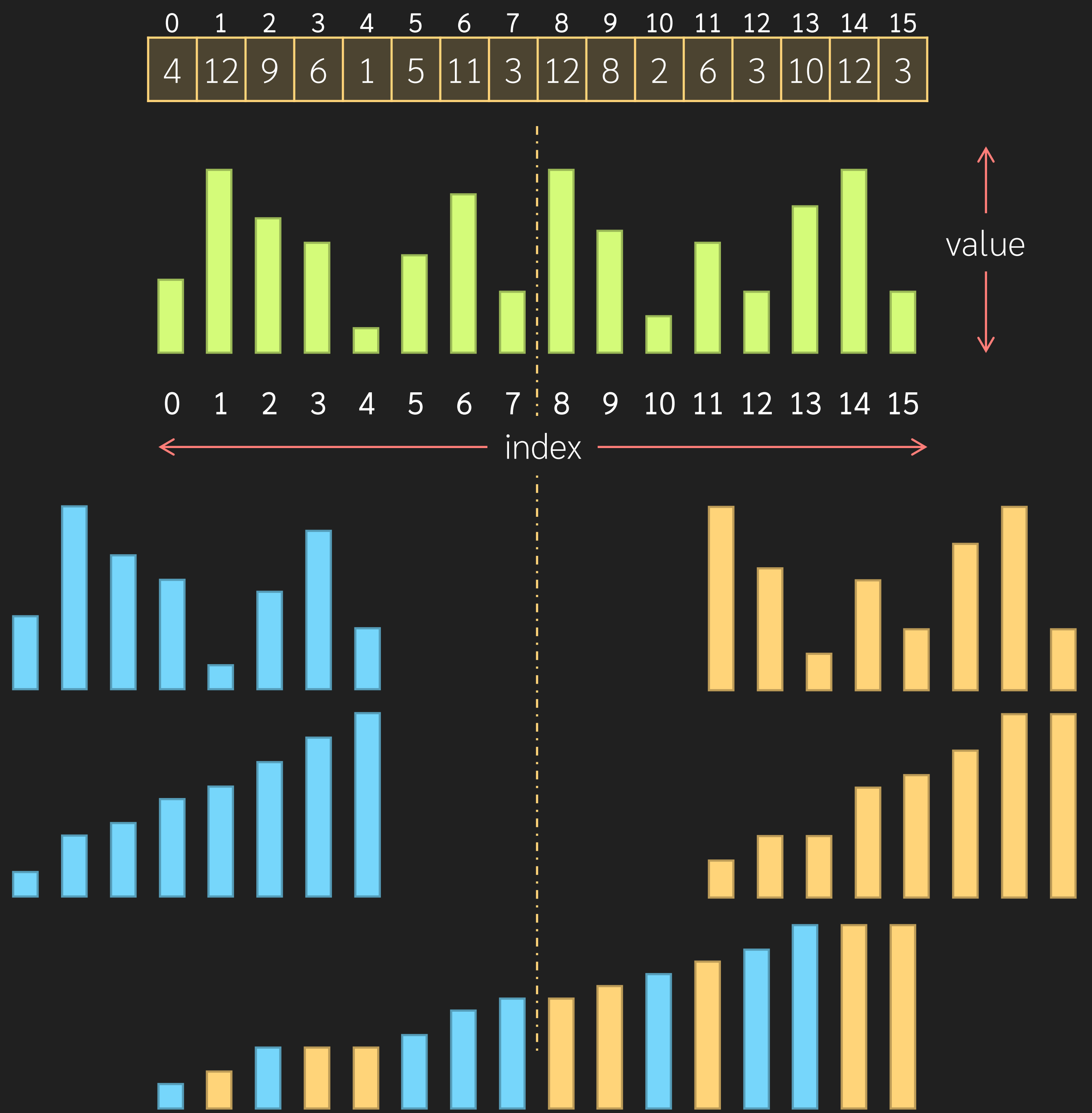
- We have iterative sorting that is $O(n^2)$ such as insertion sort or selection sort
- If, for D&C we get $T(n) = T(n-1) + O(n)$
- To go better than that, we need
  - $T(n) = T(n-1) + O(\log n)$
  - $T(n) = 2T(n/2) + O(n)$
- Either reduce by one and conquer in $O(\log n)$
  - Can we re-write our heapsort into a divide & conquer with $T(n) = T(n-1) + O(\log n)$
- For Merge sort, it is $T(n) = 2T(n/2) + O(n)$

Merge Sort

- Invented by John von Neumann in 1945

Study under
David Hilbert

- Divide:
  - At middle point, into 2 non-overlapping subproblems
- Conquer:
  - Merge 2 sorted array into one



Example

- Divide at middle point
- Solve each part recursively
- Merge two sorted array

Pseudocode

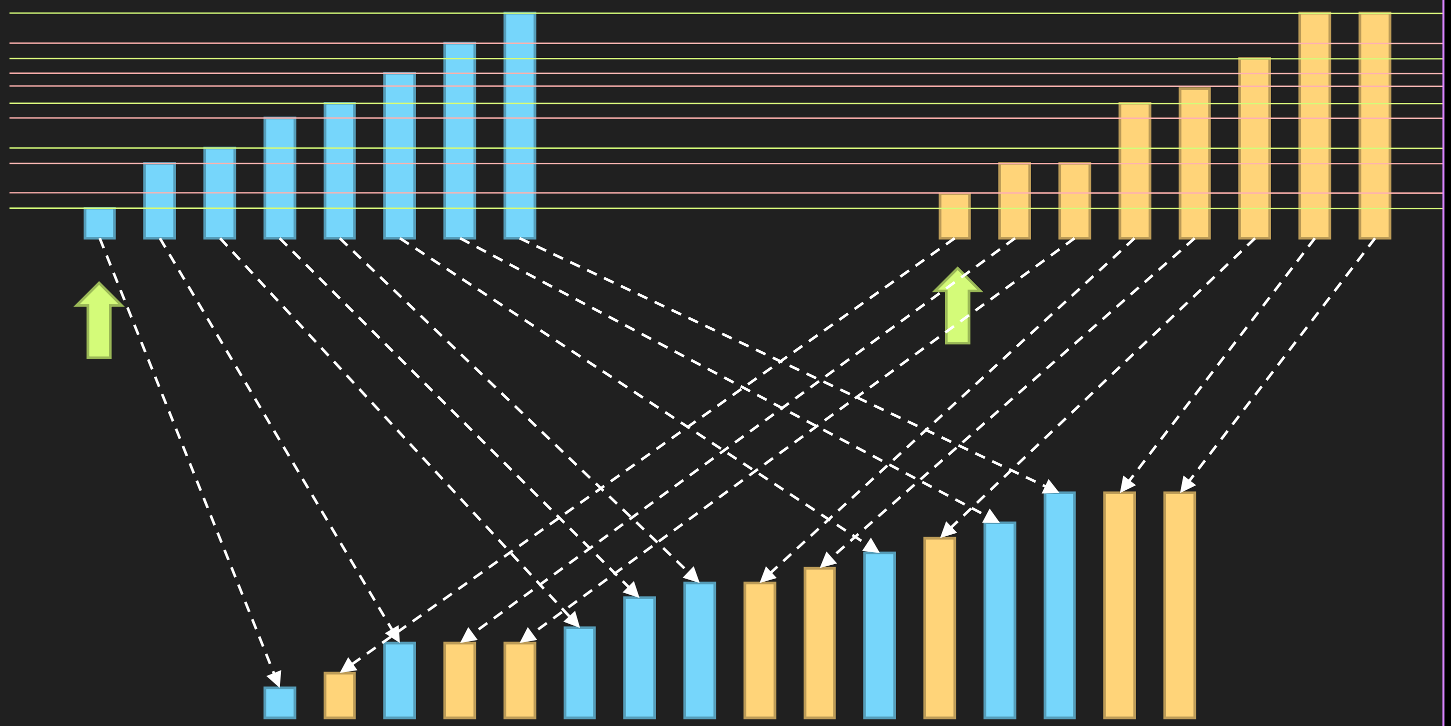
- How to merge?
- Think of selection sort, we pick the maximum of the unsorted and move to front of sorted
  - For merge, our unsorted is actually two parts, each is already sorted
  - Picking maximum of the two sorted array is very simple, just compare the max of each array
  - Can also start from minimum

```
def merge_sort(A[1..n])
  if n == 1
    return A
  m = n/2
  B = merge_sort(A[1..m])
  C = merge_sort(A[m+1..n])
  A = merge(B,C)
end
```

$$T(n) = \begin{cases} 2T(n/2) + T\_merge & ; n > 1 \\ 1 & ; n = 1 \end{cases}$$

The result is T(n) = O(n log n)
if *T_merge* is O(n)

Merge



```
#B and C is sorted
def merge(B[1..m],C[1..n])
  A = []
  while (B is not empty || C is not empty)
    if (B is not empty && C is not empty)
      if B[1] < C[1]
        move front of B to the end of A
      else
        move front of C to the end of A
    else
      if (C is empty)
        move front of B to the end of A
      else
        move front of C to the end of A
  end
  return A
end
```

Merge is θ(n)

Actual Code

```
template <typename T>
void merge(vector<T> &v,int start, int m, int stop,vector<T> &tmp) {
  bi = start; //index of B
  ci = m+1;   //index of C
  for (int i = start; i<= stop;i++) {
    if (ci > stop) { tmp[i] = v[bi++]; continue; }
    if (bi > m)    { tmp[i] = v[ci++]; continue; }
    tmp[i] = (v[bi] < v[ci]) ? v[bi++] : v[ci++];
  }
  for (int i = start; i<= stop;i++) v[i] = tmp[i];
}

template <typename T>
void merge_sort(vector<T> &v,int start, int stop,vector<T> &tmp) {
  if (start < stop) {
    int m = (start + stop) >> 1;
    merge_sort(v,start,m,tmp);
    merge_sort(v,m+1,stop,tmp);
    merge(v,start,m,stop,tmp);
  }
}
```

More on Merge Sort

- Comparing to heap sort, merge sort requires more temporary space
  - Both is O(n log n)
- Merge sort is better for sorting linked list where random access is slow
  - Because it consider adjacent element
  - Used in external sorting and parallel sorting
- Sort in python is Timsort which is a hybrid of Insertion sort and merge sort
  - when sort small amount of data, use insertion, for larger, use merge

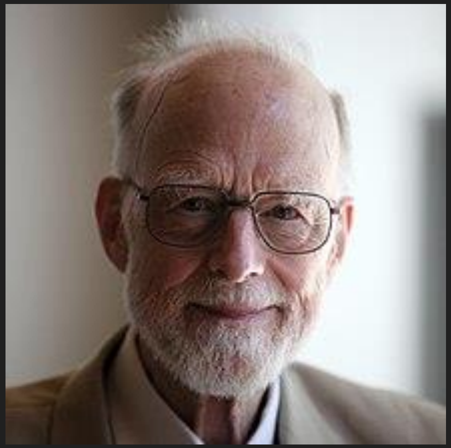
Question

- Instead of divide by half, can we divide into 4 sub problem of same size?
  - $T(n) = 4T(n/4) + O(n)$
  - How to merge?
  - Better performance?
- Can we divide into n/2 subproblem?
  - What is the complexity?

Quick Sort

D&C Sorting, easy merge

Quicksort



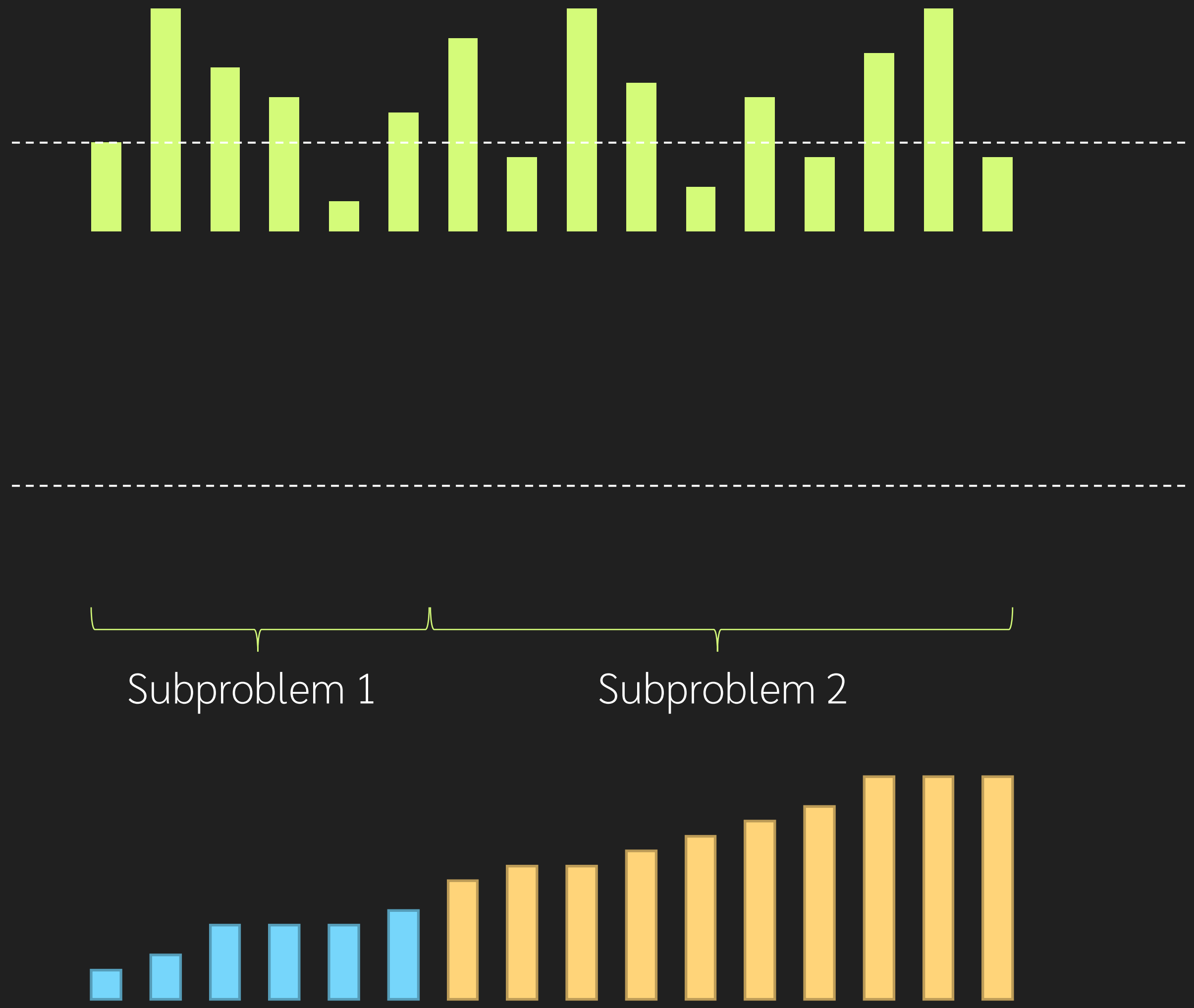
- Invented in 1959 by Sir Charles Antony Richard Hoare
- Merge sort need large temporary space, can we do better?
  - Instead of merge, can we use simplest possible conquer
- Quicksort achieves this by require additional condition for subproblem
  - We divide the problem smartly so that the subproblem satisfy this condition

Study under
Andrey Kolmogorov

Quick Sort D&C

- Divide:
  - two subproblems, maybe different size
  - Additional condition
    - Subproblem 1 must contains only A[i] such that A[i] <= pivot
    - Subproblem 2 must contains only A[i] such that A[i] >= pivot
    - Pivot can either be on either sub1 or sub2
- Conquer:
  - Because every data in the sub1 is less than elements of sub2, we can conquer by just append sub2 to sub1

Example



- Pick a partition
- Divide into <= partition part and >= partition part
- Sort recursively
- Just attach result together

Pseudocode

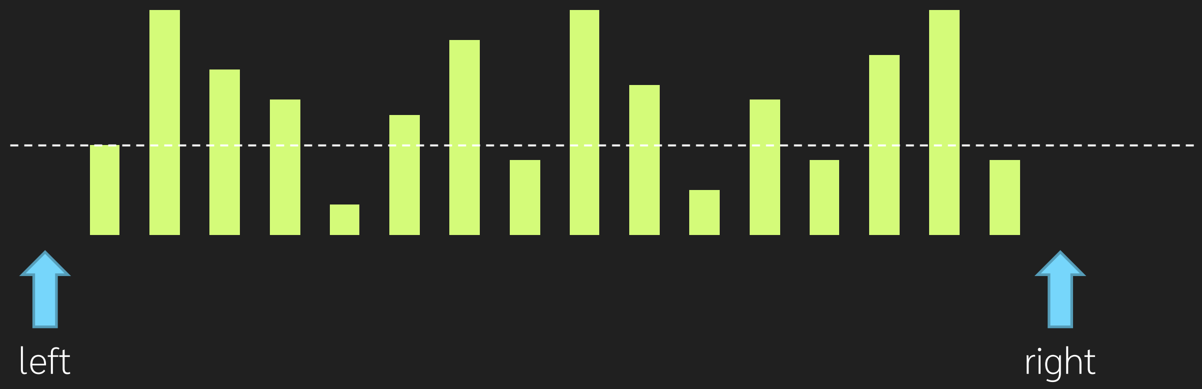
```
def quick_sort(A[1..n],start,stop)
  if start < stop
    p = partition(A,start,stop)
    quick_sort(A,start,p)
    quick_sort(A,p+1,stop)
end
```

- Partition by Hoare's algorithm
- There is a simpler partitioning algorithm by Nico Lomuto
- Both is O(n)

```
def partition(A[1..n],start,stop)
  #can use anything except stop
  pivot = A[start];

  left = start-1
  right = stop+1
  while (left < right)
    do
      left = left + 1
    until (A[left] >= pivot)
    do
      right = right - 1
    until (A[right] <= pivot)
    if (left < right)
      swap(A[left],A[right])
    else
      return right
  end
  return right
end
```

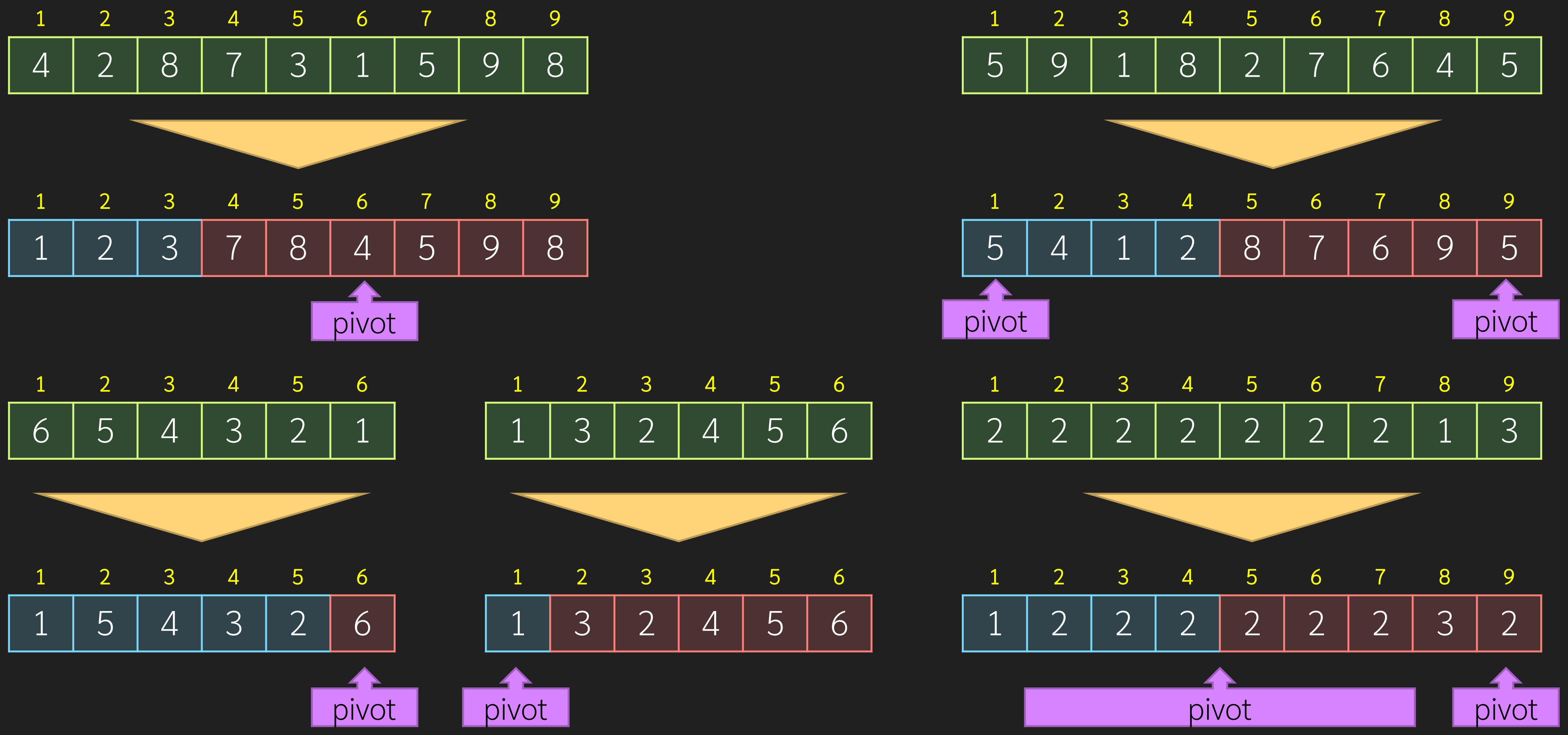
Partitioning




```
def partition(A[1..n],start,stop)
  #can use anything except stop
  pivot = A[start];

  left = start-1
  right = stop+1
  while (left < right)
    do
      left = left + 1
    until (A[left] >= pivot)
    do
      right = right - 1
    until (A[right] <= pivot)
    if (left < right)
      swap(A[left],A[right])
    else
      return right
  end
  return right
end
```

Partitioning



Analysis

- We cannot directly use Master Method
  - Because we don't know that the subproblems actually equally divide
- Consider worst case and best case
- Worst case, each partition result in very imbalance subproblem
  - $T(n) = T(1) + T(n-1) + O(n)$
  - This is $O(n^2)$
- Best case, each partition is balanced
  - $T(n) = 2T(n/2) + O(n)$
  - This is $O(n \log n)$

Average Case

Let $f(n) = T_{avg}(n)$

$$T(n) = T(k) + T(n-k) + n$$

$$T_{avg}(n) = \frac{1}{n}\sum_{i=1}^{n-1}\left(T_{avg}(i) + T_{avg}(n-i)\right) + n$$

$$= \frac{2}{n}\sum_{i=1}^{n-1} T_{avg}(i) + n$$

$$nf(n) = 2(f(1) + f(2) + \cdots + f(n-2) + f(n-1)) + n^2 \quad (1)$$

$$(n-1)f(n-1) = 2\left(f(1) + f(2) + \cdots + f(n-2)\right) + (n-1)^2 \quad (2)$$

$$nf(n) - (n-1)f(n-1) = 2f(n-1) + n^2 - (n-1)^2 \quad (1)\text{-}(2)$$

$$nf(n) = (n+1)f(n-1) + 2n - 1$$

$$\frac{nf(n)}{n(n+1)} = \frac{(n+1)f(n-1)}{n(n+1)} + \frac{2n-1}{n(n+1)}$$

$$\frac{f(n)}{(n+1)} = \frac{f(n-1)}{n} + \frac{2n-1}{n(n+1)}$$

Average Case

$$\frac{f(n)}{(n+1)} = \frac{f(n-1)}{n} + \frac{2n-1}{n(n+1)}$$

$$\frac{f(n-1)}{(n)} = \frac{f(n-2)}{n-1} + \frac{2(n-1)-1}{(n-1)((n-1)+1)}$$

$$\cdots = \cdots$$

$$\frac{f(2)}{3} = \frac{f(1)}{2} + \frac{2(2-1)}{2(2+1)}$$

$$\frac{f(n)}{(n+1)} = \sum_{i=2}^{n} \frac{2i-1}{i(i+1)}$$

$$= 2\sum_{i=2}^{n} \frac{i}{i(i+1)} - \sum_{i=2}^{n} \frac{1}{i(i+1)}$$

$$= 2\sum_{i=2}^{n} \frac{1}{(i+1)} - \sum_{i=2}^{n} \frac{1}{i(i+1)}$$

$2(\ln(n) + c)$

Partial sum harmonic series less than ln(k) + 1

something less than 0.5 because $\sum(\frac{1}{i(i+1)}) = 1$

$$\frac{f(n)}{(n+1)} = \ln(n) + c$$

$$f(n) = n\ln(n) + n + c'$$

$$= \theta(n\log n)$$

Quicksort's average case is θ(n log n)

How likely is the worst case

- If we pick pivot by the first element (or the last element) and the data is sorted (either ascending or descending), it’s worst case
- Can be practically avoided by picking pivot at random
  - With Hoare’s algorithm, do not pick a pivot at the last element

Dealing with quicksort uncertainty

- Sort of Gnu C++ use hybrid sort (similar to Timsort in Python)
  - Hybrid of Introsort follow by Insertion Sort
  - Introsort is a hybrid of Quicksort + Heapsort (when depth of recursion exceed some limit, we convert to heapsort)
- Better pivot selection
  - Median-of-median-of-five

Modular Exponentiation

Calculating $a^n \bmod k$

Problem

- Calculate a^n mod k
- Input:
  - Three positive integers a, n and k
- Output:
  - The value of a^n mod k
- Example instance
  - $a = 2$, $n = 92$, $k = 10$ output: 4

Usage

- Public key cryptography, especially RSA algorithm
- In RSA, we need 3 things, m, e and d
  - m is calculated from p*q where p and q are large prime number
  - e is any integer from 1 to LCM(p-1,q-1)
  - d is a modular inverse of e (in mod LCM(p-1,q-1))
    - Very easy to calculate d if we know e, p,q
    - Very hard to calculate d if we know only e and m (but not p, q)
- Encrypt a value t as $c = t^e \bmod m$
- Decrypt a value c by $t = c^d \bmod m$

Divide & Conquer a^n mod k

- Observe that $(a \cdot b) \bmod k = [(a \bmod k) \cdot (b \bmod k)] \bmod k$
- Lets n = x + y
  - Then, $a^n \bmod k = a^{(x+y)} \bmod k = (a^x \bmod k) + (a^y \bmod k) \bmod k$
- In other words, to calculate a^n mod k, we need to calculate
  - a^x mod k and
  - a^y mod k
  - Now, let x = n/2

$$a^n = \begin{cases} a^x * a^x, & x \text{ is even} \\ a^x * a^x * a, & x \text{ is odd} \end{cases}$$

Pseudocode

```
def mod_expo(a,n,k)
  if n == 1
    return a mod k
  if n mod 2 == 0
    tmp = mod_expo(a,n/2,k)
    return (tmp * tmp) mod k
  else
    tmp = mod_expo(a,n/2,k)
    tmp = (tmp * tmp) mod k
    return (tmp * (a mod k)) mod k
  end
end
```

- $T(n) = T(n/2) + O(1)$
- $T(n) = O(\log n)$

Example $2^{92} \bmod 10$

$2^{92} = 4951760157141521099596496896$

$2^{92} = 2^{46} \times 2^{46}$ $= 4 \times 4 \bmod 10 = 6$

$2^{46} = 2^{23} \times 2^{23}$ $= 8 \times 8 \bmod 10 = 4$

$2^{23} = 2^{11} \times 2^{11} \times 2$ $= 8 \times 8 \times 2 \bmod 10 = 8$

$2^{11} = 2^{5} \times 2^{5} \times 2$ $= 2 \times 2 \times 2 \bmod 10 = 8$

$2^{5} = 2^{2} \times 2^{2} \times 2$ $= 4 \times 4 \times 2 \bmod 10 = 2$

$2^{2} = 2^{1} \times 2^{1}$ $= 2 \times 2 \bmod 10 = 4$

Maximum Sum of Subarray

The problem

- Given array A[1..n] of numbers, may contain negative number
  - Find a non-empty subarray A[p..q] such that the summation of the values in the subarray is maximum
- Input:
  - A[1..n]
- Output:
  - p and q, where 1 <= p <= q <= n and summation of A[p..q] is maximum
- Example:
  - A = [1, 4, 2, 3] output: 1 and 4
  - A = [-2, -1, -3, -5] output: 2 and 2
  - A = [2, 3, -6, 4, -2, 3, -5, -4, 3] output: 4 and 6

Naïve $O(n^3)$

```
def mss_naive(A[1..n])
  max = A[1]
  for p from 1 to n
    for q from p to n
      sum = 0
      for j from p to q
        sum += A[j]
      if (sum > max)
        max = sum
  return max
end
```

- Try all possible $O(n^2)$ subarray
- Need $O(n)$ per subarray to calculate the summation

Using prefix sum to reduce innermost loop

```
def mss_prefix_sum(A[1..n])
  let S be an array [0..n]
  S[0] = 0;
  sum = 0;
  for i from 1 to n
    sum = sum + A[i]
    S[i] = sum


  max = A[1]
  for p from 1 to n
    for q from p to n
      sum = S[q] - S[p-1]
      if (sum > max)
        max = sum
  return max
end
```

- Let s[k] = $\sum_{i=1}^{k} A[i]$
  - Also S[0] = 0
- Calculating $\sum_{i=p}^{q} A[i]$ is just S[q] – S[p-1]

	1	2	3	4	5	6	7	8
A	3	-4	1	2	-1	-1	5	-4

1+2+-1-1+5 = 6

	1	2	3	4	5	6	7	8
S	3	-1	0	2	1	0	5	1

5 - -1 = 6

D&C

- Try divide at the middle point
- Divide:
  - $m = n/2$
  - $A[1..n]$ into $A[1..m]$ and $A[m+1..n]$
- Conquer:
  - The answer from $A[1..m]$ is p_1, q_1 in range $[1..m]$
  - The answer from $A[m+1..n]$ is p_2, q_2 in range $[m+1..n]$
  - Bu we need to consider p in range $[1..m]$ while q is in range $[m+1..n]$

D&C v0.1

```
def mss1(A[1..n],start,stop,S)
  if (start == stop)
    return A[start]
  m = (start+stop) / 2

  r1 = mss1(A,start,m)
  r2 = mss1(A,m+1,stop)

  r3 = A[start]
  for p from 1 to m
    for q from m+1 to n
      sum = S[q] - S[p-1]
      if (sum > r3)
        r3 = sum
  return max(r1,r2,r3)
end
```

- Notice that $\sum_1^n A[i] = \sum_1^{n-1} A[i] + A[n]$
- Hence, $S[i] = S[i-1] + A[i]$

```
def mss1(A[1..n])
  let S be an array [0..n]
  S[0] = 0;
  for i from 1 to n
    S[i] =  S[i-1] + A[i]

  mss1(A,1,n,S)
end
```

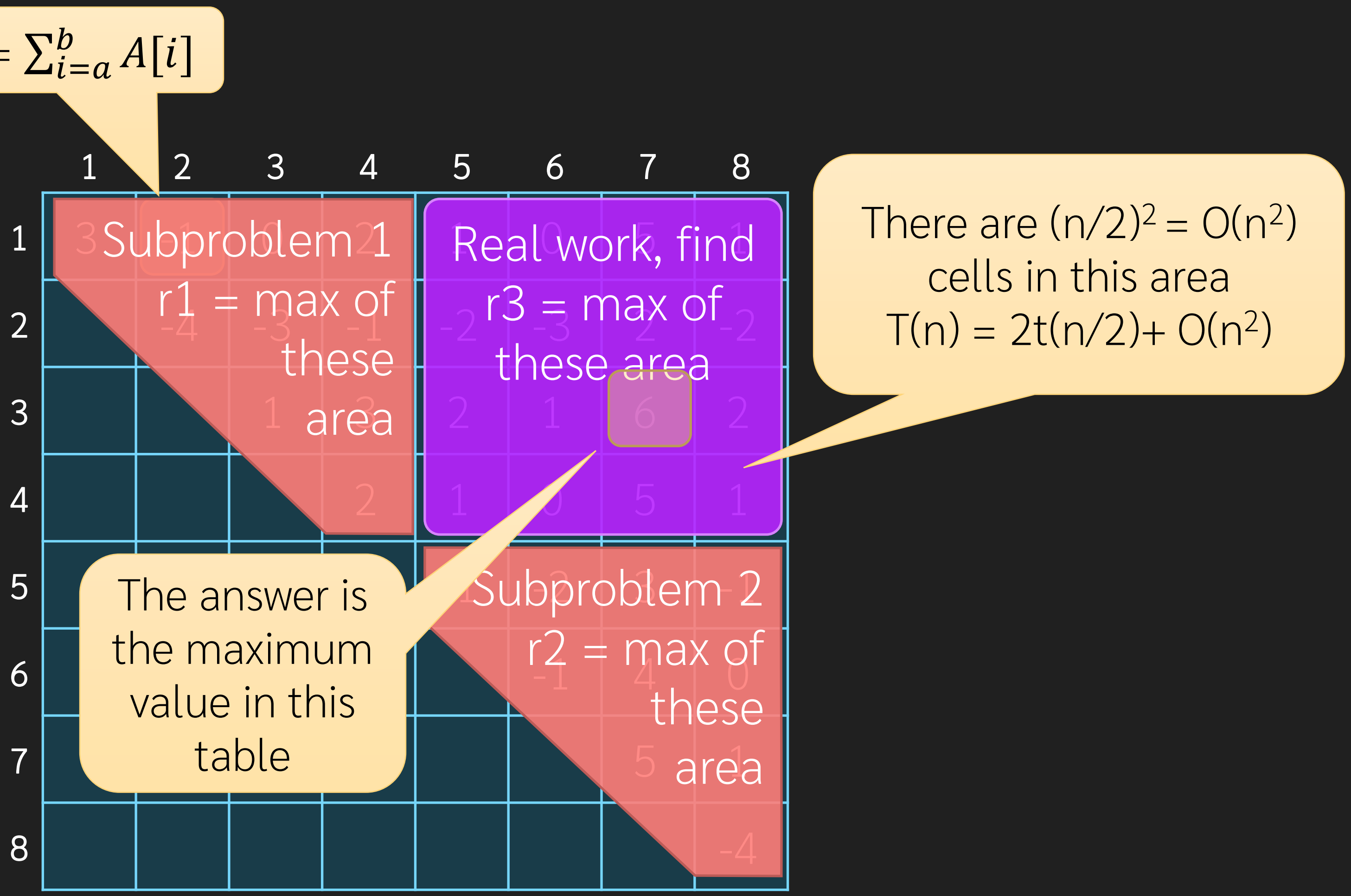
Visualization

1	2	3	4	5	6	7	8
3	-4	1	2	-1	-1	5	-4

```
def mss1(A[1..n],start,stop,S)
  if (start == stop)
    return A[start]
  m = (start+stop) / 2

  r1 = mss1(A,start,m)
  r2 = mss1(A,m+1,stop)

  r3 = A[start]
  for p from 1 to m
    for q from m+1 to n
      sum = S[q] - S[p-1]
      if (sum > r3)
        r3 = sum
  return max(r1,r2,r3)
end
```



Actual Version

There are only n/2 B[*][m] and B[m+1][*]

We can find max of them in O(n)
Hence T(n) = 2T(n/2)+o(n) = O(n log n)

- Reduce real work from $O(n^2)$ to $O(n)$
- Notice that B[a][b] = B[a][k] + B[k+1][b]
- Real work find max when a is from 1..m and b = m+1..n
    - If we calculate B[*][m] and B[m+1][*]
    - Any B[a][b] in these range can be calculated from B[a][m] + B[m+1][b]

	1	2		4	5	6	7	8
1	3	-1	0	2	1	0	5	1
2		-4	-3	-1	-2	-3	2	-2
3			1	3	2	1	6	2
4				2	1	0	5	1
5					-1	-2	3	-1
6						-1	4	0
7							5	1
8								-4

- Hence $\max_{\substack{1\le a\le m\\ m+1\le b\le n}} B[a][b] = \max_{1\le a\le m} B[a][m] + \max_{m+1\le b\le n} B[m+1][b]$

Pseudocode

```
def mss(A,start,stop,S)
  if (start == stop)
    return A[start]
  m = (start+stop) / 2

  r1 = mss(A,start,m,S)
  r2 = mss(A,m+1,stop,S)

  #find max of B[start..m][m]
  max_sum_left = get_sum(S,m,m)
  for i in start to m-1
    max_sum_left = max(max_sum_left,get_sum(S,i,m))

  #find max of B[m+1..stop][stop]
  max_sum_right = get_sum(S,m+1,m+1)
  for j in m+2 to stop
    max_sum_right = max(max_sum_right,get_sum(S,m+1,j))

  r3 = max_sum_left + max_sum_right
  return max(r1,r2,r3)
end
```

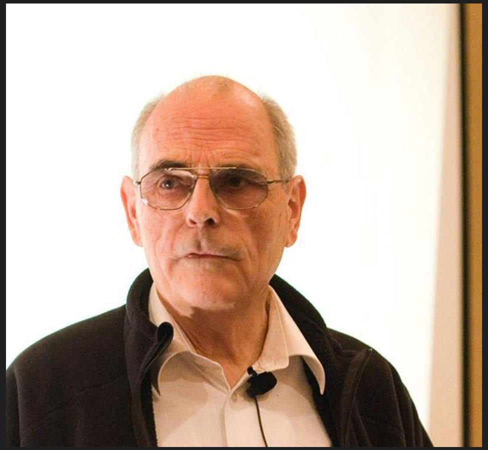
```
#this return sum A[a..b] in O(1)
def get_sum(S,a,b)
  return S[b] - S[a-1]
end
```

- Real work part is O(n)

Strassen’s Matrix Multiplication

The Problem

- Invented by Volker Strassen
- Input:
  - two square matrices of the same size
  - A[1..n][1..n] and B[1..n][1..n]
- Output:
  - The multiplication of A and B
  - C = AB



$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

Square matrix size n

Naïve Method

- basic $\theta(n^3)$

```
def multi(A[1..n][1..n],B[1..n][1..n])
  let C[1..n][1..n] be a matrix of size n * n
  for i in 1 to n
    for j in 1 to n
      sum = 0
      for k in 1 to n
        sum += a[i][k] * b[k][j];
      C[i][j] = sum
end
```

Block Multiplication

- Divide Matrix into blocks
  - $A = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$
  - $B = \begin{bmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{bmatrix}$
  - $C = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix}$
- Calculate multiplications of blocks
  - $C_{1,1} = A_{1,1}B_{1,1} + A_{1,2}B_{2,1}$
  - $C_{1,2} = A_{1,1}B_{1,2} + A_{1,2}B_{2,2}$
  - $C_{2,1} = A_{2,1}B_{1,1} + A_{2,2}B_{2,1}$
  - $C_{2,2} = A_{2,1}B_{1,2} + A_{2,2}B_{2,2}$

$$A = \left[\begin{array}{cccc|cccc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ \hline a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} \end{array}\right]$$

$A_{1,1}$ $A_{1,2}$ $A_{2,1}$ $A_{2,2}$

Matrix addition is $O(n^2)$

$T(n) = 8T(n/2)+O(n^2)$

$= O(n^3)$

Strassen's Algorithm

- Compute M_1 … M_7
  - $M_1 = (A_{1,1} + A_{2,2})(B_{1,1} + B_{2,2})$
  - $M_2 = (A_{2,1} + A_{2,2})(B_{1,1})$
  - $M_3 = (A_{1,1})(B_{1,2} - B_{2,2})$
  - $M_4 = (A_{2,2})(B_{2,1} - B_{1,1})$
  - $M_5 = (A_{1,1} + A_{1,2})(B_{2,2})$
  - $M_6 = (A_{2,1} - A_{1,1})(B_{1,1} + B_{1,2})$
  - $M_7 = (A_{1,2} - A_{2,2})(B_{2,1} + B_{2,2})$
- Note that each M can be computed by one single multiplication of n/2 * n/2 matrices, with 1 or 2 addition of a matrix

- The result can be computed as
  - $C_{1,1} = M_1 + M_4 - M_5 + M_7$
  - $C_{1,2} = M_3 + M_5$
  - $C_{2,1} = M_2 + M_4$
  - $C_{2,2} = M_1 + M_2 + M_3 + M_6$
- Total of 7 matrix multiplications and 18 matrix addition

Analysis

- $T(n) = 7T(n/2) + 18(n/2)^2$
  - Using Master's method
  - $a = 7$, $b = 2$, $f(n) = O(n^2)$
    - $c = \log_2 7 \approx 2.807$
  - Hence, $T(n) = O(n^{2.807})$

Year	Algorithm	Complexity
1969	Strassen	$O(n^{2.807})$
1987	Coppersmith–Winograd	$O(n^{2.375})$
2010	Improved CW (by Andrew Stothers)	$O(n^{2.374})$
2011	Further improvement of CW (by Virginia Vassilevska Williams)	$O(n^{2.3728642})$
2014	Improve over Williams' (by François Le Gall)	$O(n^{2.3728639})$
2022	Duan, Wu, Zhou	$O(n^{2.37188})$

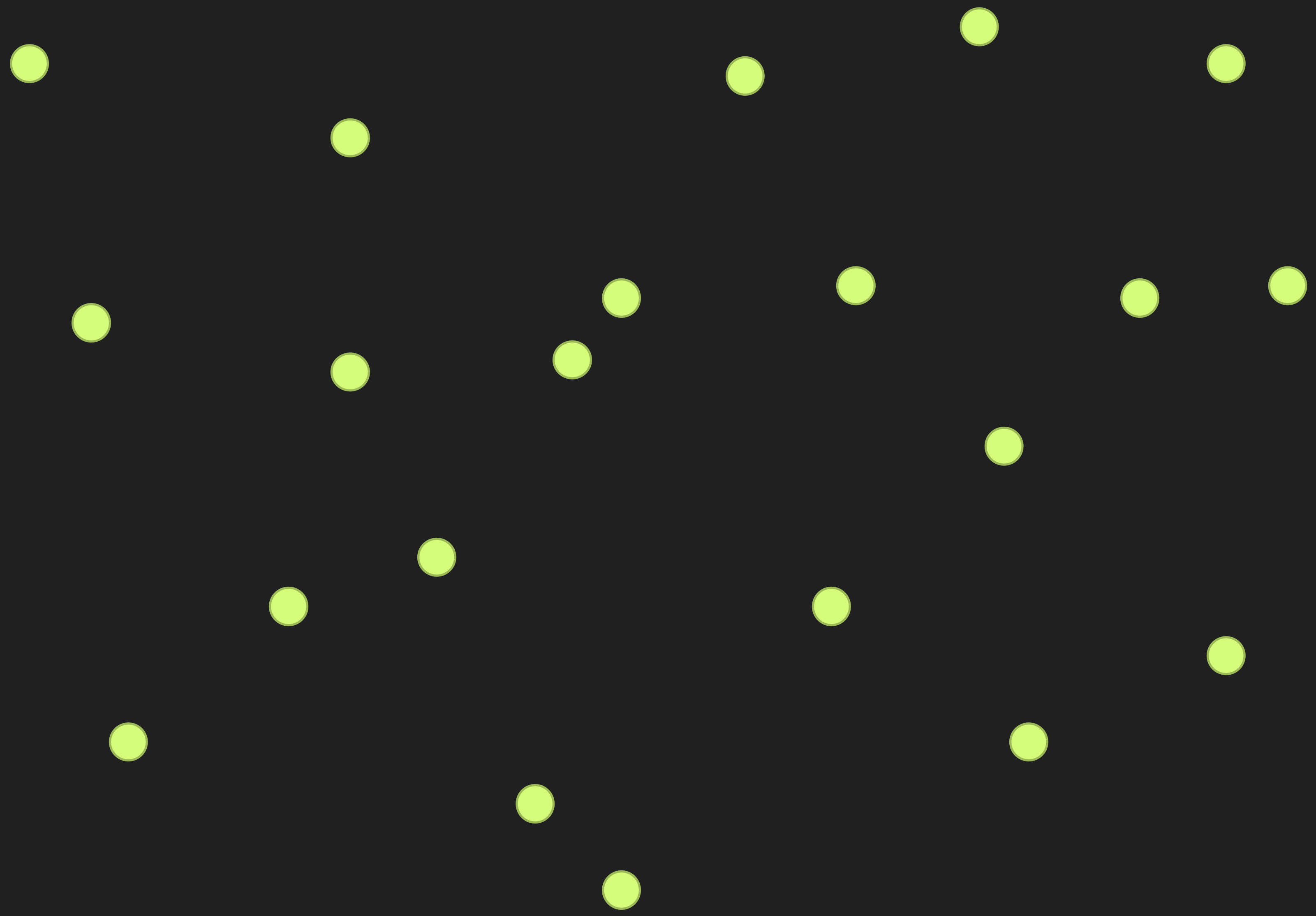
- Most are not used in practice
  - these are galactic algorithms
  - N must be galactically large before it is faster compared to Strassen's

Closest Pair

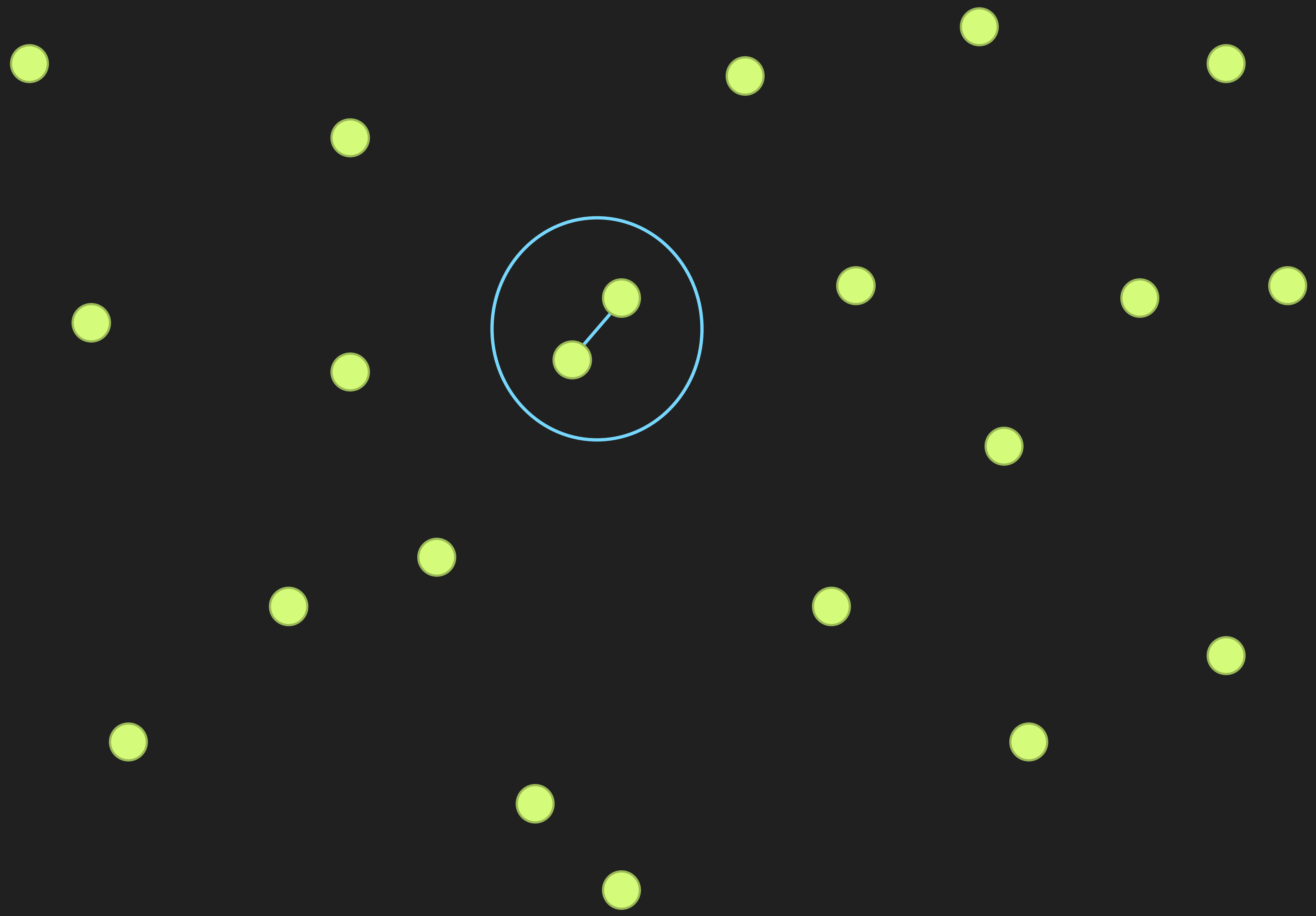
The Problem

- Input:
  - Coordinates of n points in 2D
    - $(x_1,y_1), (x_2,y_2), \ldots ,(x_n,y_n)$
- Output
  - A pair of points from the given set
    - $(x_a,y_a), (x_b,y_b)$ $1 <= a,b <= n$
    - Such that the distance between the points is minimal

Input Example



Output Example



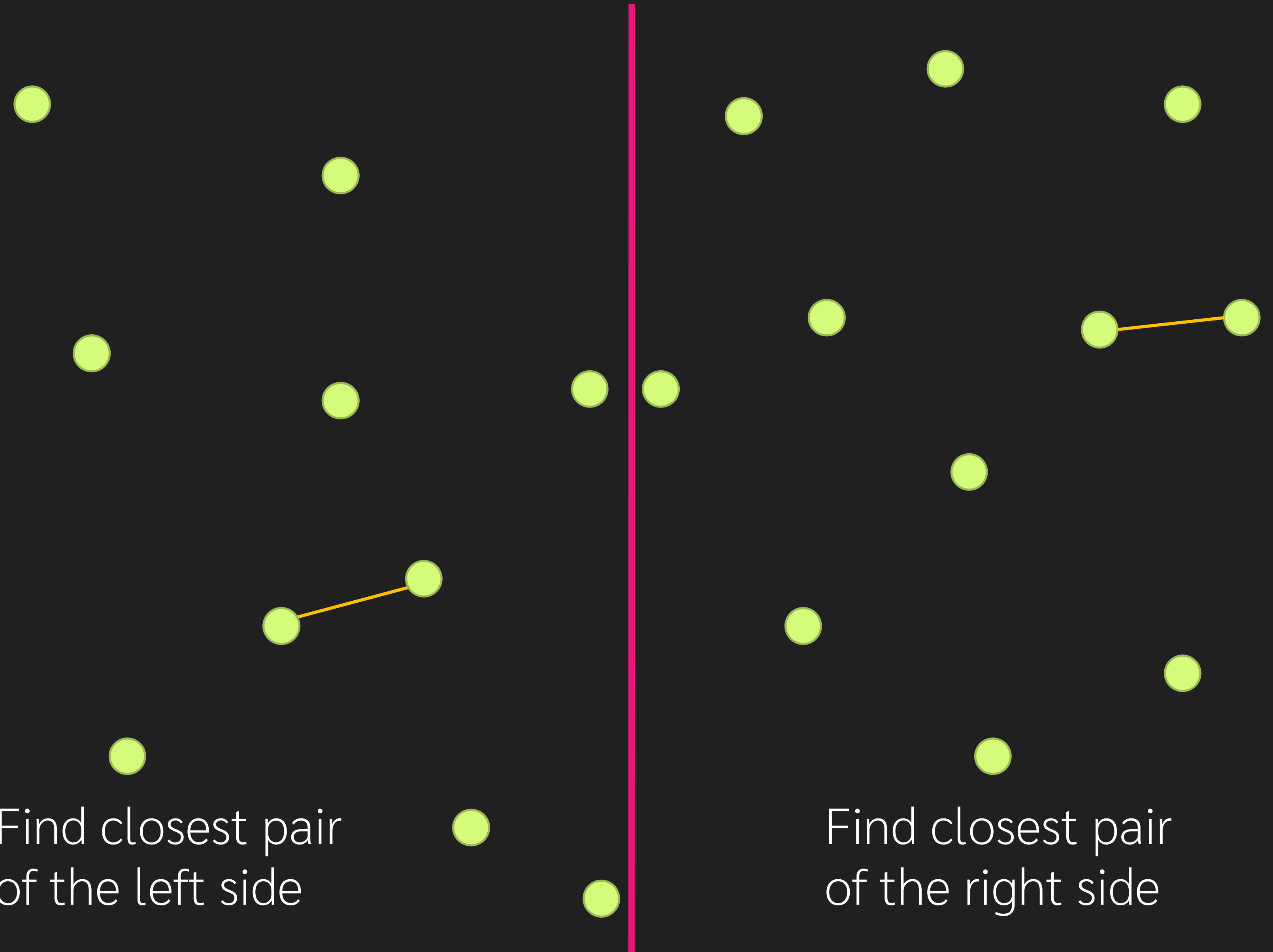
The Naïve Approach

- Try all possible pairs of points
  - There are $n(n+1)/2$ pairs
  - Compute the distance of each pair
    - Takes $\Theta(1)$ for each pair
- In total, it is $\Theta(n^2)$

DC approach

- What if we know the solution of the smaller problem
    - What if we know the Closest Pair of half of the points?
        - Which half?

Divide by X axis



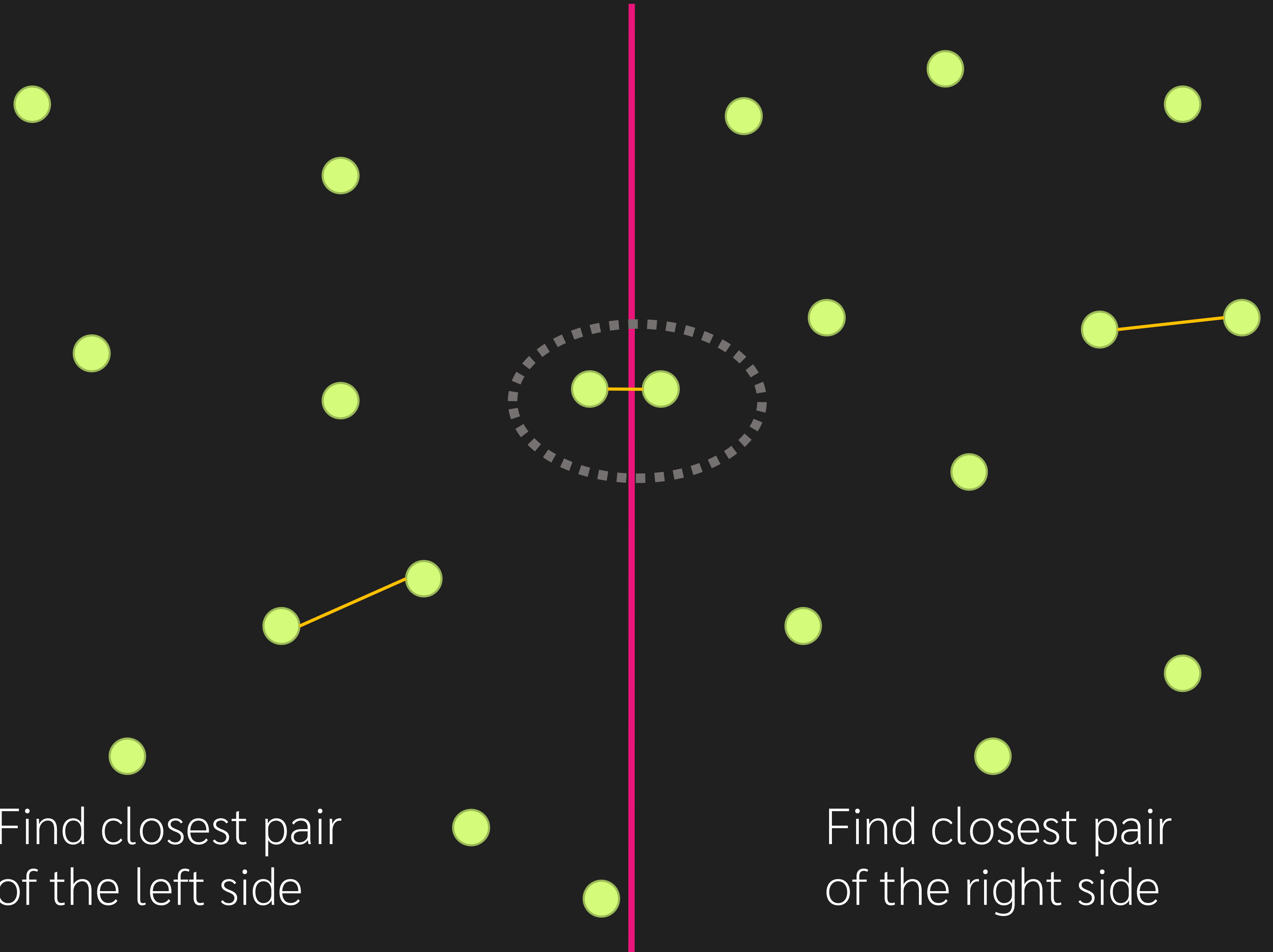
Find closest pair
of the left side

Find closest pair
of the right side

Conquer

- Similar to the MSS problem, solutions of the subproblems do not cover every possible pair of points
  - Missing the pairs that “span” over the boundary
  - There are $(n/2)^2$ such pairs, if we simply consider everything, it would be $O(n^2)$, still quadratic running time
  - To get better complexity, we something better than $O(n^2)$ for the real work

Divide by X axis



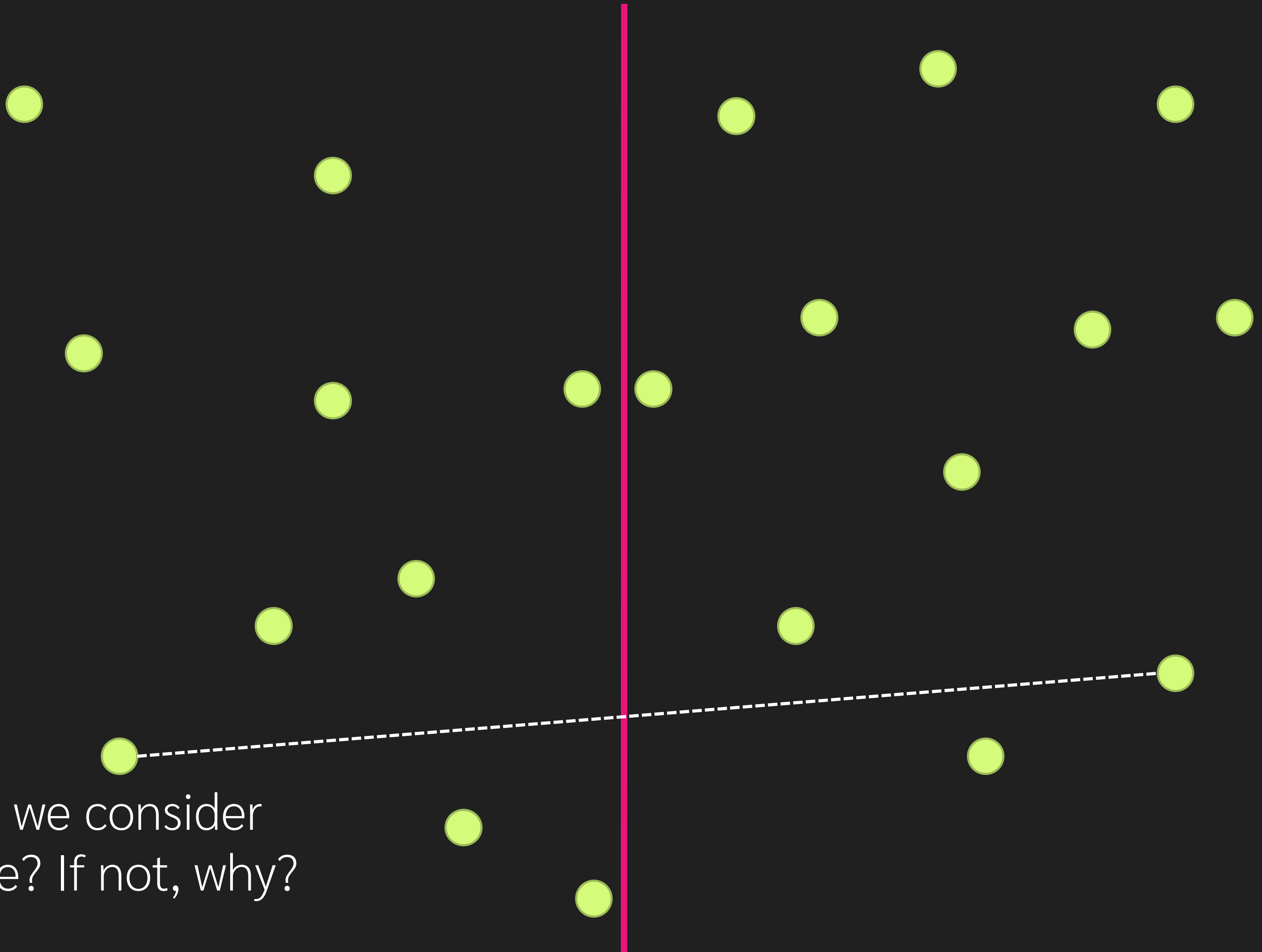
Find closest pair
of the left side

Find closest pair
of the right side

Find Closest Spanning Pair

- Should consider only the nearer pairs
  - Skip pairs whose distance is definitely larger than b
  - Should not consider the pair on the far left with that on the far right

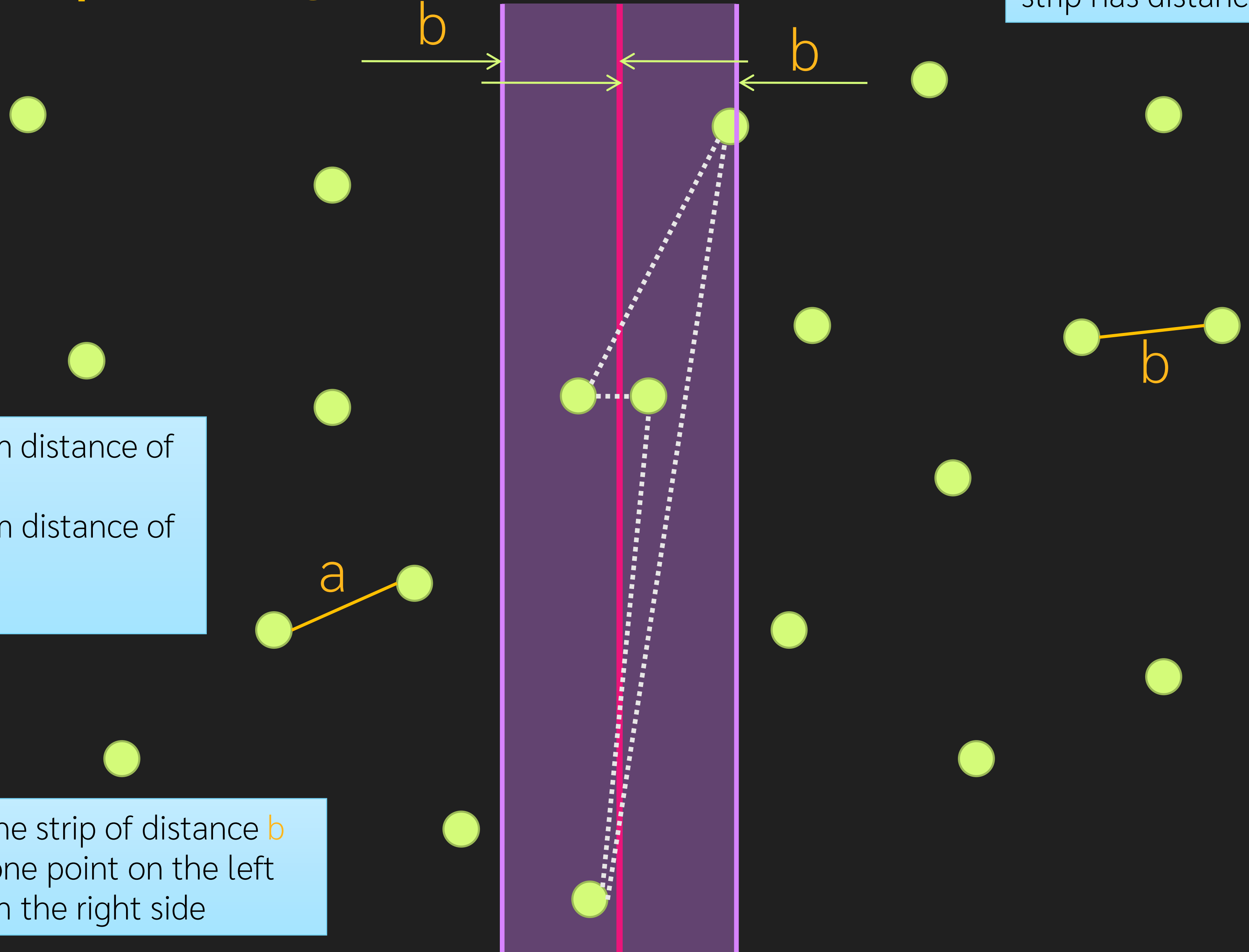
Find Closest Spanning Pair



Should we consider
this one? If not, why?

Possible Spanning Pair

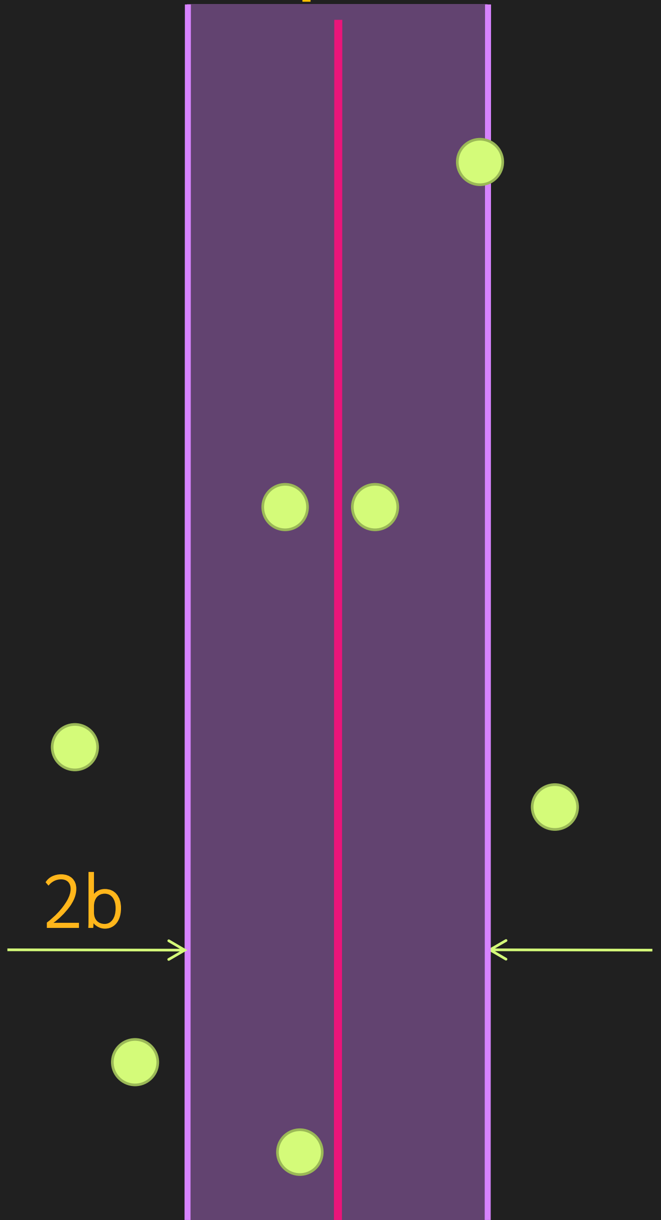
Any pair of points outside the strip has distance more than b



- Let a be the minimum distance of the left subproblem
- Let b be the minimum distance of the right subproblem
- Assume $b < a$

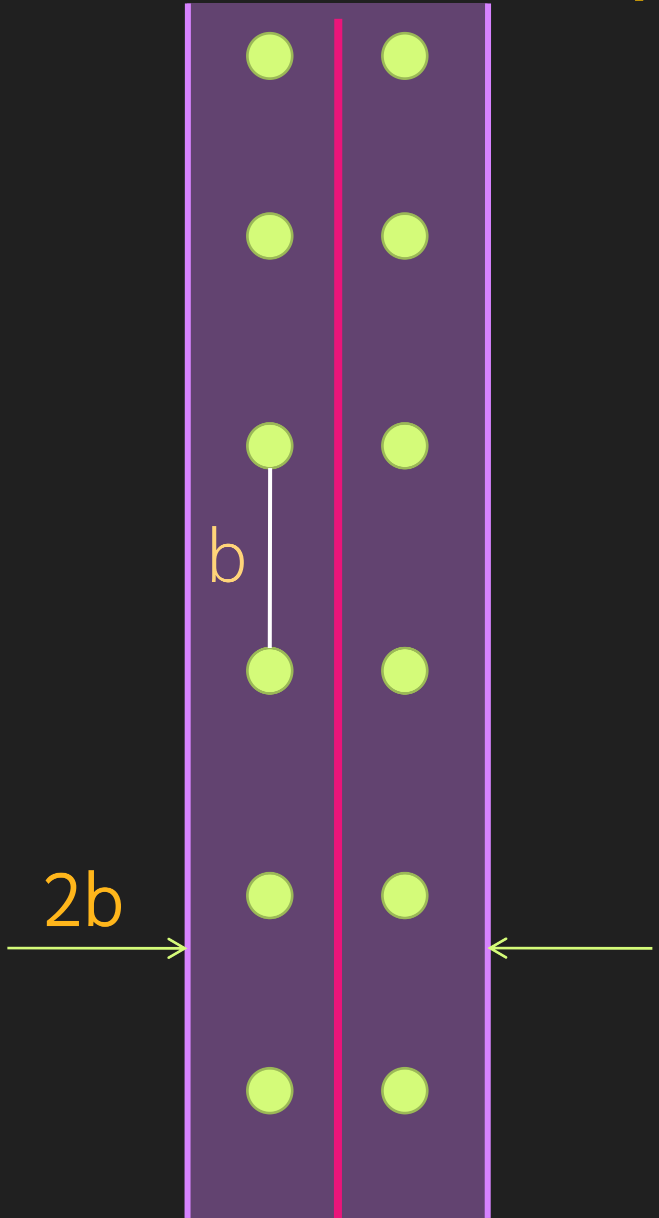
Consider only pairs in the strip of distance b from the division line, one point on the left side, another point from the right side

Point in Strips



- How many pairs in the strip?
    - Can it be more than n?

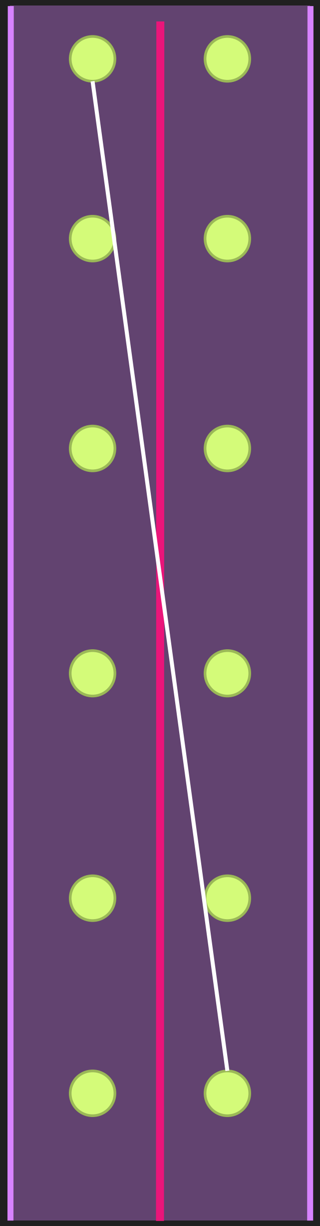
Pairs of Point in Strips can be is $O(N^2)$



- Bad news
  - There can be as much as n/2 points on each side
  - Consider a set of vertically aligned point
    - Each are b unit apart
    - So, all points will be in the strip of 2b width
- The problem
  - If we check every pair of points, we still stuck with $O(n^2)$ time

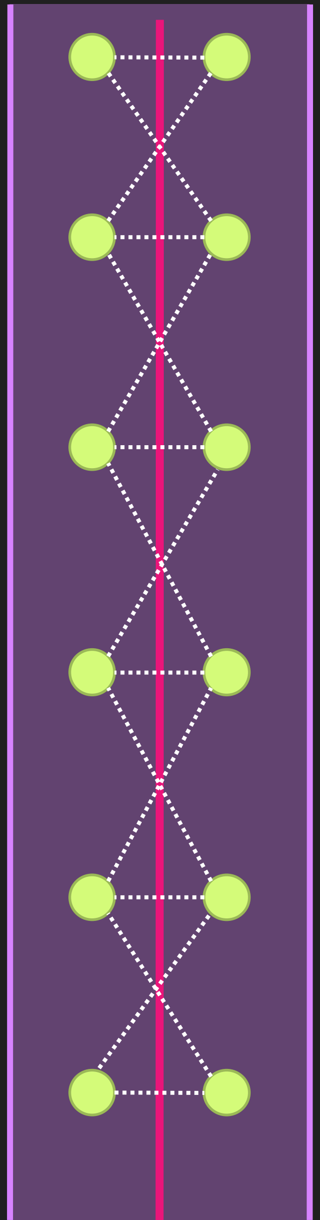
The Solution

- Think Vertically
- Do we have to check for every pair in the strip?
  - No, just like the case of X-axis
  - Don't consider pairs that is surely further than b



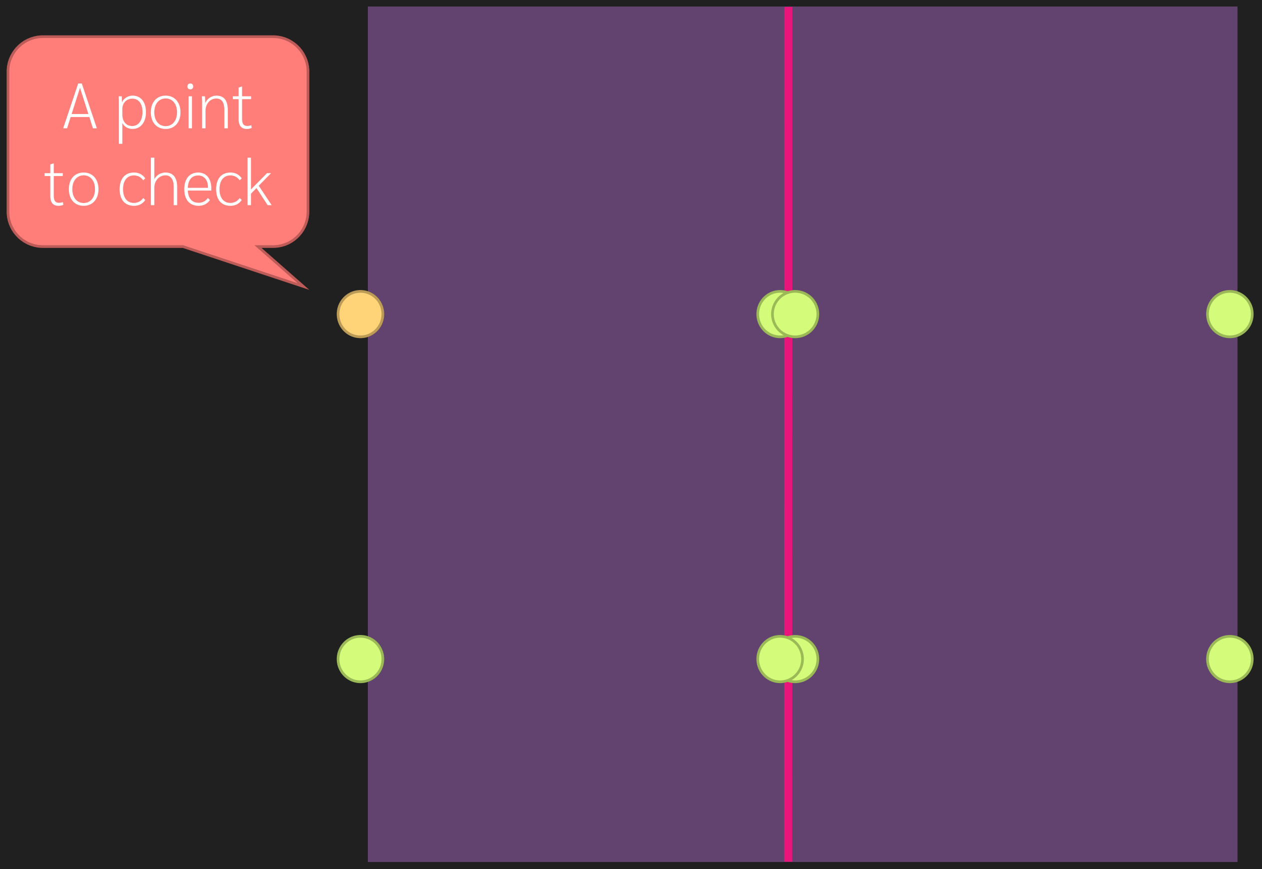
Spanning Pair to be considered

- X-value must be in the strip
  - Check only point in the left side to point in the right side
- Y-value
  - For points in the strip
    - Check only point whose y-value is not more than b unit apart



Question is still remains

- How many pair to be checked?



There are, at most, 4 points for each starting point

Implementation Detail

- For each point on one side, if we have to loops over every points on the other side to test whether the Y-value falls within range, it will still take O(n)
  - However, if the points are sorted by Y-value, we can scan from top to bottom and stop when difference of Y-value exceeds b

```
Let L[1..p] be points on the left strip, sorted by y-value
Let R[1..q] be points on the right strip, sorted by y-value
Let b be the width of the strip

result = (L[1],R[1])
min = distance(L[1],R[1])
right_idx = 1
for left_idx = 1 to p
  while right_idx < q && L[left_idx].y+b < R[right_idx].y
    d = distance(L[left_idx],R[left_idx])
    if d < min
      result = (left_idx,right_idx)
    right_idx  = right_idx + 1
#dit again, starting with right side
```

If we sort every time we do recursive call,
Real work will be O(n lg n)

That would result in $O(n \lg^2 n)$

Better Approach

- Point must be sorted in x-value so that dividing can be done in $O(1)$
- Point must also be sorted in y-value, so that we can stop checking points in the strip
- Sorting points only at the beginning
  - Both sorting can be done in $O(n \lg n)$ at the preprocess step
- Data is passed to the function in two separated list, one is x-sorted another one is y-sorted
  - When divide, both list are separated
  - Can be done in $O(n)$