Jianyu MA's DM Algebra

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Question 1 (Changement de base). *Soient*

$$f : A \longrightarrow B$$

un morphisme d'anneaux commutatifs, M un A-module. Montrez que la règle

$$b \cdot (b' \otimes x) = bb' \otimes x$$

$b, b' \in B, x \in M$ *définit une structure d'un B-module sur $B \otimes_A M$.*

My Solution. $B \otimes_A M$ is a A-module hence a Abel group. By the definition of ring B action over $B \otimes_A M$, $\forall b_1, b_2, b' \in B, x \in M$ and $1 \in B$ the multiplicative identity, we can check following properties:

- distributive law, this is included in how we define this action, $b \cdot (u + v) := b \cdot u + b \cdot v$, $\forall u, v \in B \otimes_A M$. And due to this property, we can check other properties only for tensor product of two elements in B and M.
- a monoid action, $(b_1 b_2) \cdot (b' \otimes x) = b_1 b_2 b' \otimes x = b_1(b_2 b') \otimes x = b_1 \cdot (b_2 \cdot (b' \otimes x))$, $1 \cdot (b' \otimes x) = 1b' \otimes x = b' \otimes x$
- linearity, $(b_1 + b_2) \cdot (b' \otimes x) = (b_1 + b_2)b' \otimes x = b_1 b' \otimes x + b_2 b' \otimes x = b_1 \cdot (b' \otimes x) + b_2 \cdot (b' \otimes x)$

Question 2. *Soient A un anneau commutatif, $S \subset A$ une partie multiplicative, d'où le morphisme d'anneaux canonique*

$$i : A \longrightarrow A_S$$

Soit M un A-module, d'où un A_S-module $A_S \otimes_A M$, d'après Ex. 1. *Définir un isomorphisme de A_S-modules*

$$A_S \otimes_A M \xrightarrow{\sim} M_S$$

My Solution. Consider a bilinear map $\tilde{f}$ from $A_S \times_A M$ to M_S,

$$\begin{aligned} \tilde{f} : A_S \times_A M &\to M_S \\ (\frac{a}{t}, x) &\mapsto \frac{a \cdot x}{t} \end{aligned}$$

we then get a uniquely determined map f from $A_S \otimes_A M$ to M_S which sends $\frac{a}{t} \otimes x$ to $\frac{a \cdot x}{t}$.

f is surjective since $f(\frac{1}{t} \otimes x) = \frac{x}{t}$. Assume that $\sum_j \frac{a_j}{t_j} \otimes x_j$ is an element in the kernel of f, where $a_j \in A, t_j \in S, x_j \in M$, then $\sum_j \frac{a_j \cdot x_j}{t_j} = 0$ in $S^{-1}M$ which means $\exists t \in S, t(\sum_j a_j x_j \prod_{i \neq j} t_i) = 0$, where i, j ranges over a given finite index set. Thus,

$$\sum_j \frac{a_j}{t_j} \otimes x_j = \sum_j \frac{1}{t_j} \otimes a_j \cdot x_j = \sum_j \frac{1}{t \prod_i t_i} \otimes t(a_j x_j \prod_{i \neq j} t_i)$$
$$= \frac{1}{t \prod_i t_i} \otimes t(\sum_j a_j x_j \prod_{i \neq j} t_i) = 0$$

we prove that f is injective since the kernel of f is trivial.

Hence f is a isomorphism from $A_S \otimes_A M$ onto M_S

Question 3. *Soient A un anneau commutatif, M, N, L des A -modules. Définissons un morphisme des A -modules*

$$\phi : \mathrm{Hom}_A (M \otimes_A N, L) \longrightarrow \mathrm{Hom}_A (M, \mathrm{Hom}_A(N, L))$$

par la régle suivante. Si $f \in \mathrm{Hom}_A (M \otimes_A N, L)$ alors

$$\phi(f)(x)(y) = f(x \otimes y), x \in M, y \in N$$

Montrer que ϕ est un isomorphisme (définir le morphisme inverse).

My Solution. We prove it by defining an inverse morphism

$$\psi : \mathrm{Hom}_A (M, \mathrm{Hom}_A(N, L)) \longrightarrow \mathrm{Hom}_A (M \otimes_A N, L) .$$

First, we define a $\tilde{\psi} : \mathrm{Hom}_A (M, \mathrm{Hom}_A(N, L)) \to \mathrm{Hom}_A (M \times N, L)$, if $g \in \mathrm{Hom}_A (M, \mathrm{Hom}_A(N, L))$ then $\tilde{\psi}(g)(x, y) := g(x)(y)$. $\tilde{\psi}(g)$ is a bilinear morphism, so we can define $\psi(g)$ as the unique morphism in $\mathrm{Hom}_A (M \times N, L)$ induced by $\tilde{\psi}$. Let's check that ψ and ϕ are inverse to each other.

If $f \in \mathrm{Hom}_A (M \otimes_A N, L)$, for $x \in M, y \in N$,

$$\psi(\phi(f))(x \otimes y) = \tilde{\psi}(\phi(f))(x, y) = \phi(f)(x)(y) = f(x \otimes y)$$

so $\psi(\phi(f)) = f$ since both sides coincide with all $x \otimes y \in M \otimes N$.

If $g \in \mathrm{Hom}_A (M, \mathrm{Hom}_A(N, L))$, for $x \in M, y \in N$,

$$\phi(\psi(g))(x)(y) = \psi(g)(x \otimes y) = \tilde{\psi(g)}(x, y) = g(x)(y)$$

so $\phi(\psi(g)) = g$.

Question 4 (Fonctorialité de Hom)**.** *Soient A un anneau commutatif, M, M', N, N' des A -modules. Un morphisme des A-modules*

$$f : N \longrightarrow N'$$

induit le morphisme des A-modules

$$f_* : \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, N')$$

Où

$$f_*(h) := fh, \quad h \in \operatorname{Hom}_A(M, N)$$

De même, un morphisme des A -modules

$$g : M \longrightarrow M'$$

induit le morphisme des A -modules

$$g^* : \operatorname{Hom}_A(M', N) \longrightarrow \operatorname{Hom}_A(M, N)$$

Où

$$g^*(h') = h'g, h' \in \operatorname{Hom}_A(M', N)$$

(i) Montrez que

$$\operatorname{Id}_{N*} = \operatorname{Id}_{\operatorname{Hom}_A(M,N)} = {\operatorname{Id}_M}^*$$

(ii) Montrer que si

$$N \xrightarrow{g} N' \xrightarrow{f} N''$$

sont des morphismes des A-modules alors

$$(fg)_* = f_* g_* : \operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, N'')$$

(iii) Montrer que si

$$M \xrightarrow{g} M' \xrightarrow{f} M''$$

sont des morphismes des A-modules alors

$$(fg)^* = g^* f^* : \operatorname{Hom}_A(M'', N) \longrightarrow \operatorname{Hom}_A(M, N)$$

My Solution. (i) If $f \in \operatorname{Hom}_A(M, N)$, then by definition

$$\operatorname{Id}_{N*}(f) = \operatorname{Id}_N f = f = f \operatorname{Id}_M = {\operatorname{Id}_M}^*(f).$$

(ii) If $h \in \operatorname{Hom}_A(M, N)$, then by definition

$$(fg)_*(h) = fgh = f(gh) = f_* g_*(h)$$

(iii) If $h \in \operatorname{Hom}_A(M'', N)$, then by definition

$$(fg)^*(h) = hfg = (hf)g = g^* f^*(h)$$

Question 5. *Montrer que*

(a) *une suite de morphismes de A-modules*

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N''$$

est exacte ssi pour tout A-module M la suite

$$0 \longrightarrow \mathrm{Hom}_A(M, N') \xrightarrow{f_*} \mathrm{Hom}_A\,(M, N) \xrightarrow{g_*} \mathrm{Hom}_A\,(M, N'')$$

est exacte (la partie "seulement" si a étée fait déjà);

(b) *une suite de morphismes de A-modules*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

est exacte ssi pour tout A-module N la suite

$$0 \longrightarrow \mathrm{Hom}_A\,(M'', N) \xrightarrow{g^*} \mathrm{Hom}_A\,(M, N) \xrightarrow{f^*} \mathrm{Hom}_A(M', N)$$

est exacte.

My Solution. (a) If we have an exact sequence

$$0 \longrightarrow N' \xrightarrow{f} N \xrightarrow{g} N'',$$

then for $h \in \mathrm{Hom}_A(M, N')$, $fh = 0$ is equivalent to $h = 0$

$\mathrm{kernel}(f_*) = \{h \in \mathrm{Hom}_A(M, N') | f_* h = fh = 0\} = \{h \in \mathrm{Hom}_A(M, N') | h = 0\}$;

as for $\mathrm{kernel}(g_*) = \{k \in \mathrm{Hom}_A(M, N') | g_* k = gk = 0\}$ it is obvious that $\mathrm{image}(f_*) \subset \mathrm{kernel}(g_*)$ and inversely for $k \in \mathrm{kernel}(g_*)$ and $x \in N'$ we define $f_k(x)$ as the only element in set $f^{-1}\{k(x)\}$. $f_k \in \mathrm{Hom}_A(M, N')$ since f and k are morphisms and $f_*(f_k) = k$, so $\mathrm{kernel}(g_*) \subset \mathrm{image}(f_*)$. Hence we prove the existence of the other exact sequence.

If we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_A(M, N') \xrightarrow{f_*} \mathrm{Hom}_A\,(M, N) \xrightarrow{g_*} \mathrm{Hom}_A\,(M, N'')\,,$$

since $\mathrm{Hom}_A(M, \mathrm{kernel}(f)) \subset \mathrm{kernel}(f_*)$ we have $\mathrm{Hom}_A(M, \mathrm{kernel}(f)) = 0$ and hence $\mathrm{kernel}(f) = 0$; from $g_* f_* = (gf)_* = 0$ we know $gf = 0$, so $\mathrm{image}(f) = \mathrm{kernel}(g)$. Then we prove the other exact sequence.

(b) If we have an exact sequence

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0,$$

then $\mathrm{image}(g) = M''$ and $hg = 0$ implies $h = 0$

$$\mathrm{kernel}(g_*) = \{h \in \mathrm{Hom}_A(M'', N) | g^* h = hg = 0\} = 0;$$

moreover $\text{kernel}(f^*) = \{k \in \text{Hom}_A(M,N) | f^*k = kf = 0\}$, for $k \in \text{kernel}(f^*)$ and $y \in M$ we define a g_k from $M'' = \text{image}(f)$ to N such that $g_k(g(y)) = k(y)$. This is well-defined because if $g(y_1) = g(y_2)$ for $y1, y2 \in M$ then $y_1 - y_2 \in \text{kernel}(g) = \text{image}(f)$ and $k(y_1 - y_2) = k(y_1) - k(y_2) = 0$ which gives $g_k(g(y_1)) = g_k(g(y_2))$. By definition, we can check that $g_k \in \text{Hom}_A(M, N')$ and $g^*(g_k) = k$. Therefore, $\text{image}(g^*) = \text{kernel}(f^*)$ and we get the other exact sequence.

Assume now we have an exact sequence

$$0 \longrightarrow \text{Hom}_A(M'', N) \xrightarrow{g^*} \text{Hom}_A(M, N) \xrightarrow{f^*} \text{Hom}_A(M', N).$$

Consider the canonical projection $\pi : M'' \to M''/g(M)$, from $\pi g = 0$ we have $g^*\pi^* = (\pi g)^* = 0$ so $\pi^*(\text{Hom}_A(M''/g(M), N)) \subset \text{kernel}(g^*) = 0$. Then either $\text{Hom}_A(M''/g(M), N) = 0$ or π is trivial, but both imply $M''/g(M) = 0$. In addition, $(fg)^* = g^*f^* = 0$ we have $fg = 0$ and hence get the other exact sequence.

Question 6. *Soient*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

une suite exacte de A -modules et N un A -module. Mq la suite

$$M' \otimes_A N \xrightarrow{f \otimes \text{Id}_N} M \otimes_A N \xrightarrow{g \otimes \text{Id}_N} M'' \otimes_A N \longrightarrow 0$$

est exacte.

My Solution. For two A-module N, L, $\text{Hom}_A(N, L)$ is another A-module N, so by functor property of Hom we have an exact sequence:

$$0 \longrightarrow \text{Hom}_A(M'', \text{Hom}_A(N, L)) \xrightarrow{g^*} \text{Hom}_A(M, \text{Hom}_A(N, L)) \xrightarrow{f^*} \text{Hom}_A(M', \text{Hom}_A(N, L)).$$

Since $\text{Hom}_A(M, \text{Hom}_A(N, L))$ is isomorphic to $\text{Hom}_A(M \otimes_A N, L)$ we get:

$$0 \longrightarrow \text{Hom}_A(M'' \otimes_A N, L) \xrightarrow{\psi g^*} \text{Hom}_A(M \otimes_A N, L) \xrightarrow{\psi f^*} \text{Hom}_A(M' \otimes_A N, L).$$

Use this functor property again, we finish the proof,

$$M' \otimes_A N \xrightarrow{f \otimes \text{Id}_N} M \otimes_A N \xrightarrow{g \otimes \text{Id}_N} M'' \otimes_A N \longrightarrow 0.$$

Question 7. *Soit A un anneau commutatif, $\mathfrak{p} \subset A$ un idéal premier. Rappelons que $A_\mathfrak{p}$ désigne l'anneau de fractions A_S pour $S = A \backslash \mathfrak{p}$. De même, pour un A -module $M, M_\mathfrak{p}$ désigne M_S. Montrez que $M = 0$ ssi $M_\mathfrak{p} = 0$ pour tout* $\mathfrak{p} \in \text{Spec}(A)$.

My Solution. If $M = 0$, then obviously $M_\mathfrak{p} = 0$ for all $\mathfrak{p} \in \text{Spec}(A)$.

On the other hand, if $M_{\mathfrak{p}} = 0$ then $A_{\mathfrak{p}} \otimes_A M = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$ by the first question. Consider a morphism between two A-module A and M

$$\begin{aligned} v : A &\to M \\ a &\mapsto a \cdot 1_M \end{aligned}$$

if $M \neq 0$ then kernel(v) is a proper ideal of A. Let $\mathfrak{q}$ be the maximal ideal contains kernel(v). We have $\mathfrak{q} \in \mathrm{Spec}(A)$ and $A_{\mathfrak{q}} \otimes_A M \neq 0$ since $\frac{1}{1} \otimes_A 1_M \neq 0$ as its isomorphic image $\frac{1_M}{1}$ is not zero in $M_{\mathfrak{q}}$.