# Drafts for sharing

Jianyu MA

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# 1 Files from Teachers

## 1.1 Topology

- In feuille 2.5 Exo 1, we should assume $X$ is a Hausdorff space since this condition is used twice in the provided solution.

# 2 TIM discussion

- Define $v_1(t) := \varphi(t) + \int_0^t \psi(x)\mathrm{d}x$ and $v_2(t) := \varphi(t) + \int_t^0 \psi(x)\mathrm{d}x$, then $u(x,y) = v_1(x+y) + v_2(x-y)$. We have $\partial_{x^2} v_1(x+y) = \partial_{y^2} v_1(x+y) = v_1''(x+y)$ and $\partial_{y^2} v_2(x-y) = \partial_y(-\partial_y v_2'(x-y)) = v_2''(x-y) = \partial_{x^2} v_2(x-y)$
- $AC = AC = 4, \angle BAC = \frac{2\pi}{3}, CD \perp AB, AF =?$

  $AH = 1, DH = \sqrt{3}, DH \parallel FG$, so $FH = \frac{EG}{EH} DH = \dfrac{2}{3}\sqrt{3}, AF = \sqrt{FG^2 + AG^2} = \dfrac{4}{\sqrt{3}}$.

# 3 Facebook discussion

**Question 1.** *If $A = (a_{ij})$, then*

$$\operatorname{Tr}\left(A^k\right) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k} a_{i_k i_1}.$$

*My Solution.* This prove use a tricky mathematical induction. Keep in mind we have $(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$.

a) Prove this proposition for all $k = 2^n, n \in \mathbb{N}$. For $n = 0, 1$, it is obvious. Notice that $\operatorname{Tr}\left(A^{2^{n+1}}\right) = \operatorname{Tr}\left((A^2)^{2^n}\right)$ and $\left(A^2\right)_{ij} = \sum_{k=1}^n a_{ik} a_{kj}$, so by replacing $a_{ij}$ with $\sum_{k=1}^n a_{ik} a_{kj}$ we make the induction step from $n$ to $n+1$.

b) Prove that if this proposition is true for some $k_0 \geq 2$ then it is true for $k_0 - 1$. Since

$$\frac{\partial \operatorname{Tr}(AB)}{\partial a_{ik}} = \frac{\partial}{\partial a_{ik}} \sum_{i_1, i_2 = 1}^{n} a_{i_1 i_2} b_{i_2 i_1} = b_{ki}$$

we have the matrix equality $\frac{\partial \operatorname{Tr}(AB)}{\partial A} := (\frac{\partial \operatorname{Tr}(AB)}{\partial a_{ik}}) = B^{\intercal}$, which is same as saying

$$\operatorname{Tr}\left(\frac{\partial \operatorname{Tr}(AB)}{\partial A}\right) = \operatorname{Tr}(B^{\intercal}) = \operatorname{Tr}(B).$$

In our case, let $B := A^{k_0 - 1}$, we get

$$\begin{aligned}
\operatorname{Tr}(B) = &\operatorname{Tr}(A^{k_0-1}) \\
&= \operatorname{Tr}\left(\frac{\partial \operatorname{Tr}(A^{k_0})}{\partial A}\right) \\
&= \operatorname{Tr}\left(\frac{\partial}{\partial A} \sum_{i_1, i_2, \dots, i_k = 1}^{n} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k} a_{i_k i_1}\right) \\
&= \operatorname{Tr}\left(\sum_{i_3, i_4, \dots, i_k = 1}^{n} a_{j i_3} a_{i_3 i_4} a_{i_4 i_5} \dots a_{i_{k-1} i_k} a_{i_k i_1}\right)_{ij} \\
&= \sum_{i=j=1}^{n} \left(\sum_{i_3, i_4, \dots, i_k = 1}^{n} a_{j i_3} a_{i_3 i_4} a_{i_4 i_5} \dots a_{i_{k-1} i_k} a_{i_k i_1}\right)_{ij} \\
&= \sum_{j, i_3, i_4, \dots, i_k = 1}^{n} a_{j i_3} a_{i_3 i_4} a_{i_4 i_5} \dots a_{i_{k-1} i_k} a_{i_k i_1} \\
&= \sum_{i_1, i_2, \dots, i_{k-1} = 1}^{n} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-2} i_{k-1}} a_{i_{k-1} i_1}.
\end{aligned}$$

c) Any $k \in \mathbb{N}$ is small than a $k_0 = 2^{n_0}$, repeat the second step we see that this proposition is true for $k$.