Thesis Activity Report for 2021-2022

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In my thesis, we are interested in the metric barycenters of some given measure on the Wasserstein space $(\mathcal{W}_2(M), W)$ over some Polish metric space (M, d), usually being a complete Riemannian manifold. These barycenters, for which we shall call Wasserstein barycenters, are themselves probability measures as points in $\mathcal{W}_2(M)$. We aim to discover new properties of Wasserstein barycenters and establish central limit theorems having them serve the role of expectation.

During my master 2 internship under the supervision of my current supervisor, we already have the goal in mind to extend the results on Wasserstein barycenters' absolute continuity for compact manifolds by Kim and Pass [2017] to the non-compact case. At the beginning of this thesis year, I examined carefully their proofs to expect possible generalization but without success. Two months later, I found a breakthrough point by employing the concept of entropy in the assumptions. Our result at this stage was that, for a complete Riemannian manifold, if a probability measure $\mathbb{P}$ on $\mathcal{W}_2(M)$ gives mass to the set of absolutely measures (with respect to the volume measure on M) with entropies uniformly bounded from above, then its unique Wasserstein barycenter is absolutely continuous. This proof was inspired by the displacement convexity in Wasserstein spaces, properly stated as Jensen inequality in Kim and Pass's paper. Being restricted our views to displacement convexity, we were not able to go further until the end of the first year. That is to say, after this small victory, I was blocked for a quite long time.

During this blocked time, I turned to the investigation of central limit theorems. I thought of following the same strategy of proving barycenter's absolute continuity, that is to say, working from simples cases or concrete examples to general cases. For this, I read some current research papers on this topic such as Del Barrio and Loubes [2019] and so on, and they led me to read systematically textbooks on convex analysis and set topologies, such as Rockafellar [1970], Rockafellar and Wets [2009] and Beer [1993]. These books offered me better understanding of non-smooth analysis appeared in some important papers such as Cordero-Erausquin et al. [2001], Gigli [2011], Figalli and Gigli [2011] and McCann [1997]. It took time to propose possible forms of central limit theorems and find out mistakes in those pre-mature or unreliable proofs. Some ideas were not feasible even in Euclidean spaces, so we finally choose put them aside for a while.

We once conjectured that using a new definition of the metric barycenter for the case of hyperbolic plane, i.e., replacing the distance squared d^2 by $-\log\cosh d$, we might resolve our current difficulty or find something new. But we could not go very far and I started to have doubts on this searching path.

Finally, in July, I changed my focus from the geometry side to the measure theory side in the problem of absolute continuity. At that time, I got some inspirations from general functionals on the Wasserstein space, for which one can consult Ambrosio et al. [2021]. Thanks to some classic techniques presented in Bogachev and Ruas [2007], I then found a very promising approach based on measure theory and for which a tentative proof is available.

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Signatures of the PhD student and his supervisor

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