PhD Application for CIMI & EUR-MINT

Jianyu MA

September 29, 2022

About Jianyu MA

Institution	Internship Topic
Wuhan University BS UPS Toulouse III M1	Stochastic calculus
UPS Toulouse III M2	(Riemannian) geometry

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Why IMT? Common research interest with supervisor J.Bertrand
Enjoyable environment, wonderful library, nice profs...

Existence and uniqueness for barycenter of probability measure

Heuristic definition

In mechanics, for a system of particles m_ν with positional vectors $\mathbf{r}_\nu$, the barycenter c has positional vector

$$\mathbf{r}_c := \sum_\nu \frac{m_\nu}{M}\mathbf{r}_\nu \equiv \arg\min_{\mathbf{r}} \sum_\nu \frac{m_\nu}{M}\|\mathbf{r}_\nu - \mathbf{r}\|^2$$

where $M := \sum_\nu m_\nu$, arg min: "argument that achieves the minimum".

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Definition

For a probability measure λ on a metric space (E, d), barycenter $x \in E$ is

$$x \in \arg\min_{z\in E} \int_E d^2(z,y)\,\mathrm{d}\,\lambda(y)$$

Ongoing work, part I

Example

For two points x, y in a length space E, consider measure $\frac{1}{2}(\delta_x + \delta_y)$, if a barycenter z exists,

$$\text{barycenter} \equiv \text{midpoint}: \quad d(x,z) = d(z,y) = \frac{1}{2} d(x,y).$$

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Remark

- Points on equator are midpoints for north and south poles of a sphere.

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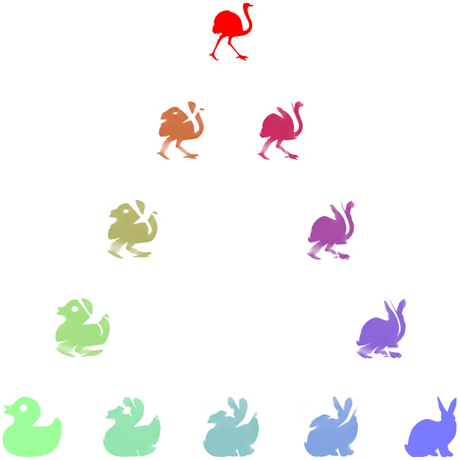
  $\Longrightarrow$ Barycenter: not always exists

Metricized space of measures on $\mathbb{R}^2$

Images represent uniform probability measures on them.
We regard images as points in our metric space.



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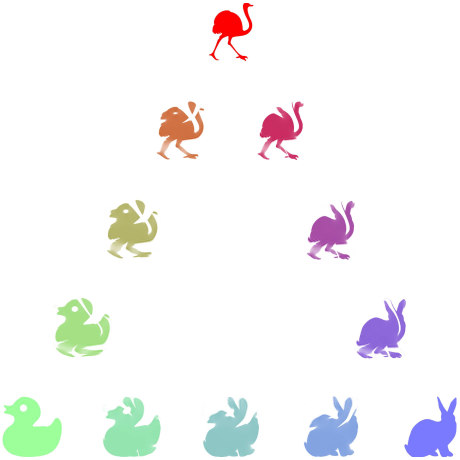
Space	Probability measures on $\mathbb{R}^2$
Metric	Wasserstein metric
Measure	$\frac{1}{2}(\delta_{\text{img1}} + \delta_{\text{img2}})$
Barycenter	Middle image of two images



Ongoing work, part II

Motivations from Statistical Inference

barycenter (of discrete measures) $\equiv$ Fréchet mean

Results in the field

Bounded sets are compact	$\Rightarrow$	Existence of barycenter
Non-positive curvature	$\Rightarrow$	Uniqueness of barycenter
Optimal transportation theory	inspires	Barycenter on Wasserstein space

Ongoing work, part II

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Current progress: Convex analysis and Optimal transportation

- Investigate the case of $\mathbb{R}^n$

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Current progress: Convex analysis and Optimal transportation

- Investigate the case of $\mathbb{R}^n$
- Generalize to Riemannian manifolds

Barycenter and Central Limit Theorem (abbr. as CLT)

Recall classic CLT

Central Limit Theorem

Draw n independent random samples $X_1, X_2, \ldots, X_n$ from a population with overall mean μ and finite variance σ^2, denote by

$$\bar{X}_n := \frac{X_1 + X_2 + \cdots + X_n}{n}$$

the sample mean, we have

$$\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{\text{law}} N(0,1),$$

where $N(0,1)$ is the standard normal distribution.

CLT in terms of barycenter

Barycenter = Mean of measure on a metric space

Crucial properties of barycenter to get CLT

- **Uniqueness** Converges to a unique mean
- **Consistency** Convergence of measures $\Longrightarrow$ Convergence of their barycenters

Some metric spaces to investigate

1. Riemannian manifolds
2. Wasserstein space: measures on a metric space with finite second moments

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