Alternatives to the t-tests

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The goal of this chapter is

- Distinguish parametric and nonparametric test procedures
- Explain commonly used nonparametric test procedures
- Perform hypothesis testing using nonparametric procedures

Introduction

Parametric test procedures:

- involve population parameters
- have stringent assumptions

Nonparametric test procedure

- make less stringent demands of the data.
- have stringent assumptions
- results may be as exact as parametric procedures

- This chapter introduces some nonparametric techniques we use when you don't believe the assumptions for the stuff you learned before.
- All procedures that you learned before come in natural triples
  - a hypothesis test
  - a confidence interval obtained by inverting the test,
  - the point estimate obtained by shrinking the confidence level to zero. This is called the Hodges-Lehmann estimator associated with the confidence interval.
- A familiar example is
  - hypothesis test: t test
  - confidence interval: t confidence interval
  - point estimate: sample mean

Now we are going to learn about some competing techniques

- hypothesis test: sign test
- confidence interval: associated confidence interval
- point estimate: sample median

and

- hypothesis test: Wilcoxon signed rank test
- confidence interval: associated confidence interval
- point estimate: associated Hodges-Lehmann estimator

- $Y_1, Y_2, \ldots, Y_n$ a random sample from $N(\mu, \sigma^2)$
- $\bar{Y}$ sample mean and is the sample standard deviation.
- Goal: test $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$
- Test statistics

  $$t = \frac{\bar{Y} - \mu_0}{SE(\bar{Y})}$$

  where

  $$SE(\bar{Y}) = s/\sqrt{n}$$

  is the standard error of $\bar{Y}$.
- Reject H_0 if $|t| > t_{n-1}(\alpha/2)$ or if p-value $< \alpha$
- If $H_a : \mu > \mu_0$, reject H_0 if $t > t_{n-1}(\alpha)$ or if p-value $< \alpha$
- If $H_a : \mu < \mu_0$, reject H_0 if $t < -t_{n-1}(\alpha)$ or if p-value $< \alpha$
- A $100(1-\alpha)\%$ confidence interval for μ is

  $$\bar{Y} \pm t_{n-1}(\alpha/2) SE(\bar{Y})$$

- $Y_{11}, Y_{12}, \ldots, Y_{1n_1}$ a random sample from $N(\mu_1, \sigma_1^2)$
- $Y_{21}, Y_{22}, \ldots, Y_{2n_2}$ a random sample from $N(\mu_2, \sigma_2^2)$
- $\bar{Y}_1$ and s_1^2 sample mean and variance of the first sample
- $\bar{Y}_2$ and s_2^2 sample mean and variance of the first sample
- Goal: test $H_0 : \mu_1 = \mu_2$ against $H_a : \mu_1 \neq \mu_2$
- Two cases to consider
  - $\sigma_1^2 = \sigma_2^2$ (Equal Variances Case)
  - $\sigma_1^2 \neq \sigma_2^2$ (Unequal Variances Case)

- Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$t = \frac{\bar{Y}_1 - \bar{Y}_2}{SE(\bar{Y}_1 - \bar{Y}_2)}$$

where

$$SE(\bar{Y}_1 - \bar{Y}_2) = s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

with

$$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}},$$

the pooled estimate of the standard deviation σ.

- Reject H_0 if $|t| > t_{n_1+n_2-2}(\alpha/2)$ or if p-value $< \alpha$
- If $H_a : \mu > \mu_0$, reject H_0 if $t > t_{n_1+n_2-2}(\alpha)$ or if p-value $< \alpha$
- If $H_a : \mu < \mu_0$, reject H_0 if $t < -t_{n_1+n_2-2}(\alpha)$ or if p-value $< \alpha$
- A $100(1-\alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{n_1+n_2-2}(\alpha/2)SE(\bar{Y}_1 - \bar{Y}_2)$$

- Case 1: $\sigma_1^2 \neq \sigma_2^2$

$$t = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- t does not follow a t-distribution under H_0. However, its distribution can be well approximated by a t-distribution with degrees of freedom equal to the following complicated formula:

$$d = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1 - 1) + (s_2^2/n_2)^2/(n_2 - 1)}$$

- approximation is known as Satterthwaite's Method
- Reject H_0 if $|t| > t_d(\alpha/2)$ or if p-value $< \alpha$
- If $H_a : \mu > \mu_0$, reject H_0 if $t > t_d(\alpha)$ or if p-value $< \alpha$
- If $H_a : \mu < \mu_0$, reject H_0 if $t < -t_d(\alpha)$ or if p-value $< \alpha$
- A $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{Y}_1 - \bar{Y}_2 \pm t_d(\alpha/2) SE(\bar{Y}_1 - \bar{Y}_2)$$

- Let $X_1, X_2, \ldots, X_n$ be a random sample from some continuous distribution F
- The continuity assumption assures that ties are impossible. With probability one we have no ties.
- The continuity assumption is unnecessary for the point estimate and confidence interval.
- The parameter of interest is $\eta = \text{median(F)}$
- Because of the continuity assumption, the median is uniquely defined.

- Goal test $H_0 : \eta = \eta_0$ against $H_a : \eta \neq \eta_0$
- Let

$$
\begin{aligned}
S &= \sum_{i=1}^{n} I(X_i > \eta_0) \\
&= \sum_{i=1}^{n} s(X_i - \eta_0)
\end{aligned}
$$

  where $s(x) = 1$ if $x > 0$ and $s(x) = 0$ otherwise. S counts the numbers of X_is that exceed η_0.
- If H_0 is true then $S \sim Bin(n, 1/2)$ (this implies that $E(S) = n/2$ and $Var(S) = n/4$)
- Reject H_0 is S is much less than or much greater than $n/2$. That is if $|S - n/2|$ is large
- The p-value $= 2\min(P(S \leq t), P(S \geq t))$ where t is the realized value of T based on the sample
- To generate a $100(1-\alpha)\%$ confidence interval for the true median, use the values of η for which we fail to reject H_0 at α.

- In R, install the package BSDA
- To test that median is 1 against the median not zero at 5% use

```
SIGN.test(data, md=1,alternative="two.sided",conf.level=0.95)
```

- One sided test possible

```
SIGN.test(data, md=1,alternative="greater",conf.level=0.95)
SIGN.test(data, md=1,alternative="less",conf.level=0.95)
```

```
Example (Income data):
income <- c(7, 1110, 7.1, 5.2, 8, 12, 0, 5, 2.1, 2, 46, 7.5)
```

Figure: Histogram and Box Plot

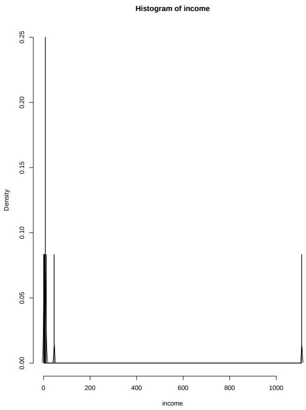


Figure: Box plot

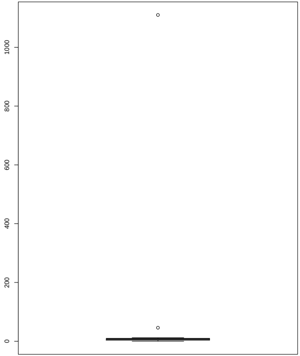
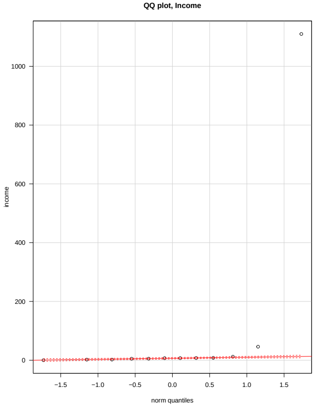


Figure: QQ plot



```
> summary(income)
   Min.  1st Qu.   Median     Mean  3rd Qu.     Max.
  0.000    4.275    7.050  101.000    9.000 1110.000

> t.test(income, mu=1)

One Sample t-test

data:  income
t = 1.0893, df = 11, p-value = 0.2993
alternative hypothesis: true mean is not equal to 1
95 percent confidence interval:
 -101.0454  303.0287
sample estimates:
mean of x
 100.9917
```

- Presence of outliers has a dramatic effect on a 95% confidence interval for the population mean μ which is $[-101, 303]$
- This t based confidence interval is suspect because the normality assumption is unreasonable.
- A confidence interval for the population median income is more sensible because the median is more likely to be a more reasonable typical value.
- Using the sign procedure, a 95% confidence interval for the population median is $[2.32, 11.57]$.

```
> SIGN.test(income, md=1, alternative="two.sided")

One-sample Sign-Test

data:  income
s = 11, p-value = 0.006348
alternative hypothesis: true median is not equal to 1
95 percent confidence interval:
  2.408455 11.574545
sample estimates:
median of x
       7.05

                  Conf.Level L.E.pt  U.E.pt
Lower Achieved CI     0.8540 5.0000  8.0000
Interpolated CI       0.9500 2.4085 11.5745
Upper Achieved CI     0.9614 2.1000 12.0000
> 2*(1-pbinom(10,12,0.5))
[1] 0.006347656
```

Example: Age at first heart transplant

```
> age<-c(54, 42, 51, 54, 49, 56, 33, 58, 54, 64, 49)
> sort(age)
 [1] 33 42 49 49 51 54 54 54 56 58 64

> summary(age)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  33.00   49.00   54.00   51.27   55.00   64.00
```

Question of interest: is the typical age at first transplant 50?

Figure: Histogram and Box Plot

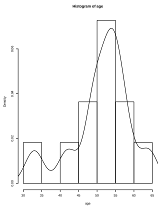


Figure: Box plot

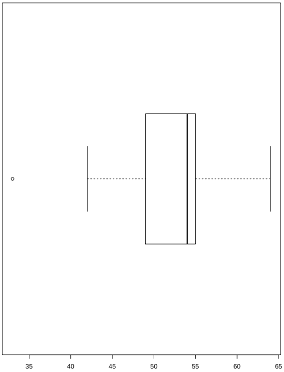
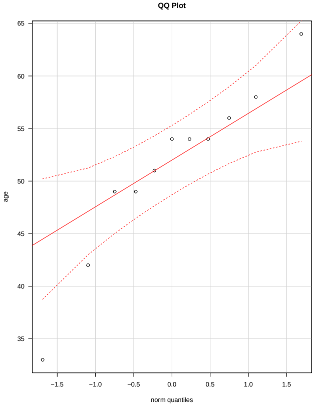


Figure: QQ plot



- The normal QQ-plot of the sample data indicates mild deviation from normality in the left tail
- It is good practice in this case to use the nonparametric test as a double-check of the t-test.

```
> t.test(age, mu=50)

One Sample t-test

data:  age
t = 0.51107, df = 10, p-value = 0.6204
alternative hypothesis: true mean is not equal to 50
95 percent confidence interval:
 45.72397 56.82149
sample estimates:
mean of x
 51.27273
```

A 95% confidence for the mean in [45.72, 56.82]

```
> SIGN.test(age, md=50)

One-sample Sign-Test

data:  age
s = 7, p-value = 0.5488
alternative hypothesis: true median is not equal to 50
95 percent confidence interval:
 46.98909 56.57455
sample estimates:
median of x
         54

                  Conf.Level  L.E.pt  U.E.pt
Lower Achieved CI     0.9346 49.0000 56.0000
Interpolated CI       0.9500 46.9891 56.5745
Upper Achieved CI     0.9883 42.0000 58.0000
```

A 95% confidence for the mean in [46.99, 56.59]. Similar conclusion was reached with the t confidence interval and the t-test on μ. You should have less confidence in these results because the normality assumption is tenous.

- The following result holds. Under H_0

$$\lim_{n\to\infty} P\left(\frac{S-n/2}{\sqrt{n/4}} \leq x\right) = \Phi(x)$$

- A large sample test reject H_0 if

$$S < n/2 - Z_{\alpha/2}\sqrt{n}/2 \quad \text{or} \quad S < n/2 + Z_{\alpha/2}\sqrt{n}/2$$

- How do you do a one sided test in this case?

Wilcoxon Signed Rank Test

- The sign test use simply information in the sign of the observations, no metric on how far the observations are from η_0 is incorporated from into the test
- For a distribution that is symmetric about η_0, the vector of the absolute value $|X_i - \eta_0|, i = 1, 2, \ldots, n$, is a sufficient statistic. So for symmetric distributions, it makes sense to use this information.
- The Wilcoxon procedure assumes you have a random sample $X_1, X_2, \ldots, X_n$ from a symmetric distribution (need not be normal) $\Rightarrow$ mean $=$ median $(\eta = \mu)$.
- To test $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$ requires
    - sign of $X_i - \mu_0$
    - ranks $R_1, R_2, \ldots, R_n$ of the $|X_i - \mu_0|$s
- The test statistic is

$$W = \sum_{i=1}^{n} s(i) R_i$$

where $s(i) = 1$ if $X_i - \mu_0 > 0$ and 0 otherwise.

Clearly W is also equal to

$$W = \sum_{i=1}^{n} s(i) R_i = \sum_{j=1}^{n} j w_j$$

where $w_j = 1$ if the jth ranked absolute value corresponds to a positive difference and 0 otherwise.

- Under $H_0 : \mu = \mu_0, s(X_1 - \mu_0), s(X_2 - \mu_0), \ldots, s(X_n - \mu_0)$ and $R_1, R_2, \ldots, R_n$ are independent.
- Under $H_0 : \mu = \mu_0, w_1, w_2, \ldots, w_n$ are independent, identically distributed and

$$P(w_1 = 1) = P(w_1 = 0) = 1/2$$

- This implies that

$$E(W) = \sum_{i=1}^{n} jE(w_j) = \frac{1}{2}\sum_{j=1}^{n} j = n(n+1)/4$$

and

$$var(W) = \sum_{j=1}^{n} j^2 var(w_j) = \frac{1}{4}(n(n+1)(2n+1))/6 = \frac{n(n+1)(2n+1)}{24}.$$

- Under H_0

$$E(W) = \frac{n(n+1)}{4}$$

  and we reject H_0 if $|W - n(n+1)/4|$ is large.
- in R use wilcox.test(data)

Example: here $\mu_0 = 10$.

X_i	X_i-10	sign	$\lvert X_i - 10\rvert$	rank	sign $\times$ rank
20	10	+	10	6	6
18	8	+	8	4.5	4.5
23	13	+	13	8	8
5	-5	-	5	3	-3
14	4	+	4	2	2
8	-2	-	2	1	-1
18	8	+	8	4.5	4.5
22	12	+	12	7	7

$$W = 6 + 4.5 + 8 + 2 + 4.5 + 7 = 32$$

$n = 8$ this implies that under $H_0, E(W) = 8(9)/4 = 18$

```
> wilcox.test(x,mu=10,conf.int=TRUE,correct=FALSE)

Wilcoxon signed rank test

data:  x
V = 32, p-value = 0.04967     (W =V)
alternative hypothesis: true location is not equal to 10
95 percent confidence interval:
 10.99996 21.00005
sample estimates:
(pseudo)median
       16.0056
```

- Example: Two remedies for insomnia and the number of hours of sleep gained was recorded. Same people used in the experiment

```
a<-c(0.7, -1.6, -0.2, -1.2, 0.1, 3.4, 3.7, 0.8, 0.0, 2.0)
b<- c(1.9, 0.8, 1.1, 0.1,  -0.1, 4.4, 5.5, 1.6, 4.6, 3.0)
```

- Goal: test $H_a : \mu_a = \mu_b$ against $H_a : \mu_a \neq \mu_b$

```
d<-a-b
> sleep<-data.frame(a , b, d)
> shapiro.test(sleep$d)
Shapiro-Wilk normality test

data:  d
W = 0.83798, p-value = 0.04173

p-value< 0.05, we reject the null hypothesis that the differences are
from a normal distribution.
```

```
> wilcox.test(d, mu=0, conf.int=TRUE)

Wilcoxon signed rank test with continuity correction

data:  d
V = 1, p-value = 0.008004
alternative hypothesis: true location is not equal to 0
95 percent confidence interval:
 -2.7999620 -0.7999339
sample estimates:
(pseudo)median
     -1.299966
```

- The following result holds. Under H_0

$$\lim_{n\to\infty} P\left(\frac{W - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} \le x\right) = \Phi(x)$$

- A large sample test reject H_0 if

$$S < n(n+1)/4 - Z_{\alpha/2}\sqrt{n(n+1)(2n+1)/24}$$

or if

$$S > n(n+1)/4 + Z_{\alpha/2}\sqrt{n(n+1)(2n+1)/24}$$

- How do you do a one sided test in this case?

Nonparametric Analysis of Paired Data

- For symmetric distributions, the t, the sign and Wilcoxon procedures are appropriate
- It the underlying distribution is extremely skewed, you can use the sign procedure to get a confidence interval for the population median
- Otherwise, you can transform the data to a scale where the underlying distribution is nearly normal and use t .
- Moderate degrees of skewness will not have a big impact on the t based results.
- Data from heavy-tailed distribution can have a profound impact on the t-test and the t confidence interval
- The sign and the Wilcoxon procedures downweight the influence of outliers by looking at sign and signed ranks instead of the actual values.
- A weakness of nonprametric methods is that they do not easily generalize to complex problems

- The WMW procedure assumes you have independent random samples from two populations
- The distributions are of the $F(x - \eta_i), i = 1, 2$ and interest is in comparing η_1 and η_2. Here F is assumed to be absolutely continuous.
- The WMW procedure gives a confidence interval for the difference $\Delta = \eta_1 - \eta_2$,

(Wilcoxon) Mann-Whitney two sample procedure

- $X_1, X_2, \ldots, X_{n_1}$ and $Y_1, Y_2, \ldots, Y_{n_2}$ are independent random samples from two populations
- This test tries to detect location shifts
- To compute the test statistic (see below), combine the two samples and rank the observations.
- Let T_X be the sum of the ranks for the observations in the X-group and T_Y for the Y-group (note that $T_X + T_y = (n_1 + n_2)(n_1 + n_2 + 1)/2.$)
- Under $H_0 : \eta_1 = \eta_2, E(T_X) = n_1(n_1 + n_2 + 1)/2$ and $Var(T_X) = n_1 n_2(n_1 + n_2 + 1)/12$
- In some applications the Mann-Whitney form U_X of Wilcoxon rank sum test is used:

$$U_X = T_X - \frac{n_1(n_1+1)}{2}$$

  As such

$$U_X = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} I(X_i > Y_j)$$

- The test is most appropriate when the populations have the same shape and differ only in location (same in dispersion). The distributions do not have to be symmetric.
- $$\lim_{\min(n_1,n_2)\to\infty} P\left(\frac{T_X - n_1(n_1+n_2+1)/2)}{\sqrt{n_1 n_2(n_1+n_2+1)/12}} \leq x\right) = \Phi(x)$$

```
group1 (X)<-c(0.8, 2.8, 4.0, 2.4, 1.2, 0.0, 6.2, 1.5, 28.8, 0.7)
group2 (Y) <-c(2.3, 0.3, 5.2, 3.1, 1.1, 0.9, 2.0, 0.7, 1.4, 0.3)

Wilcoxon rank sum test with continuity correction

data:  x and y
W = 61.5, p-value = 0.4053
alternative hypothesis: true location shift is not equal to 0
```