

Alternatives to the t-tests

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The goal of this chapter is

- Distinguish parametric and nonparametric test procedures
- Explain commonly used nonparametric test procedures
- Perform hypothesis testing using nonparametric procedures

Parametric test procedures:

- involve population parameters
- have stringent assumptions

Nonparametric test procedure

- make less stringent demands of the data.
- have stringent assumptions
- results may be as exact as parametric procedures

- This chapter introduces some nonparametric techniques we use when you don't believe the assumptions for the stuff you learned before.
- All procedures that you learned before come in natural triples
 - a hypothesis test
 - a confidence interval obtained by inverting the test,
 - the point estimate obtained by shrinking the confidence level to zero. This is called the Hodges-Lehmann estimator associated with the confidence interval.
- A familiar example is
 - hypothesis test: t test
 - confidence interval: t confidence interval
 - point estimate: sample mean

Now we are going to learn about some competing techniques

- hypothesis test: sign test
- confidence interval: associated confidence interval
- point estimate: sample median

and

- hypothesis test: Wilcoxon signed rank test
- confidence interval: associated confidence interval
- point estimate: associated Hodges-Lehmann estimator

- Y_1, Y_2, \dots, Y_n a random sample from $N(\mu, \sigma^2)$
- \bar{Y} sample mean and s is the sample standard deviation.
- Goal: test $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$
- Test statistics

$$t = \frac{\bar{Y} - \mu_0}{SE(\bar{Y})}$$

where

$$SE(\bar{Y}) = s/\sqrt{n}$$

is the standard error of \bar{Y} .

- Reject H_0 if $|t| > t_{n-1}(\alpha/2)$ or if p-value $< \alpha$
- If $H_a : \mu > \mu_0$, reject H_0 if $t > t_{n-1}(\alpha)$ or if p-value $< \alpha$
- If $H_a : \mu < \mu_0$, reject H_0 if $t < -t_{n-1}(\alpha)$ or if p-value $< \alpha$
- A $100(1 - \alpha)\%$ confidence interval for μ is

$$\bar{Y} \pm t_{n-1}(\alpha/2)SE(\bar{Y})$$

- $Y_{11}, Y_{12}, \dots, Y_{1n_1}$ a random sample from $N(\mu_1, \sigma_1^2)$
- $Y_{21}, Y_{22}, \dots, Y_{2n_2}$ a random sample from $N(\mu_2, \sigma_2^2)$
- \bar{Y}_1 and s_1^2 sample mean and variance of the first sample
- \bar{Y}_2 and s_2^2 sample mean and variance of the first sample
- Goal: test $H_0 : \mu_1 = \mu_2$ against $H_a : \mu_1 \neq \mu_2$
- Two cases to consider
 - $\sigma_1^2 = \sigma_2^2$ (Equal Variances Case)
 - $\sigma_1^2 \neq \sigma_2^2$ (Unequal Variances Case)

- Case 1: $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$t = \frac{\bar{Y}_1 - \bar{Y}_2}{SE(\bar{Y}_1 - \bar{Y}_2)}$$

where

$$SE(\bar{Y}_1 - \bar{Y}_2) = s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

with

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}},$$

the pooled estimate of the standard deviation σ .

- Reject H_0 if $|t| > t_{n_1+n_2-2}(\alpha/2)$ or if p-value $< \alpha$
- If $H_a : \mu > \mu_0$, reject H_0 if $t > t_{n_1+n_2-2}(\alpha)$ or if p-value $< \alpha$
- If $H_a : \mu < \mu_0$, reject H_0 if $t < -t_{n_1+n_2-2}(\alpha)$ or if p-value $< \alpha$
- A $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{n_1+n_2-2}(\alpha/2)SE(\bar{Y}_1 - \bar{Y}_2)$$

- Case 1: $\sigma_1^2 \neq \sigma_2^2$

$$t = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- t does not follow a t-distribution under H_0 . However, its distribution can be well approximated by a t-distribution with degrees of freedom equal to the following complicated formula:

$$d = \frac{(s_1^2/n_1 + s_2^2/n_2)^2}{(s_1^2/n_1)^2/(n_1 - 1) + (s_2^2/n_2)^2/(n_2 - 1)}$$

- approximation is known as Satterthwaite's Method
- Reject H_0 if $|t| > t_d(\alpha/2)$ or if p-value $< \alpha$
- If $H_a : \mu > \mu_0$, reject H_0 if $t > t_d(\alpha)$ or if p-value $< \alpha$
- If $H_a : \mu < \mu_0$, reject H_0 if $t < -t_d(\alpha)$ or if p-value $< \alpha$
- A $100(1 - \alpha)\%$ confidence interval for $\mu_1 - \mu_2$ is

$$\bar{Y}_1 - \bar{Y}_2 \pm t_d(\alpha/2)SE(\bar{Y}_1 - \bar{Y}_2)$$

- Let X_1, X_2, \dots, X_n be a random sample from some continuous distribution F
- The continuity assumption assures that ties are impossible. With probability one we have.
- The continuity assumption is necessary for exact hypothesis tests.
- The continuity assumption is unnecessary for the point estimate and confidence interval.
- The parameter of interest is $\eta = \text{median}(F)$
- Because of the continuity assumption, the median is uniquely defined.

Sign Test for a population median

- Goal test $H_0 : \eta = \eta_0$ against $H_a : \eta \neq \eta_0$
- Let

$$\begin{aligned} S &= \sum_{i=1}^n I(X_i > \eta_0) \\ &= \sum_{i=1}^n s(X_i - \eta_0) \end{aligned}$$

where $s(x) = 1$ if $x > 0$ and $s(x) = 0$ otherwise. S counts the numbers of X_i s that exceed η_0 .

- If H_0 is true then $S \sim \text{Bin}(n, 1/2)$ (this implies that $E(S) = n/2$ and $\text{Var}(S) = n/4$)
- Reject H_0 if S is much less than or much greater than $n/2$. That is if $|S - n/2|$ is large
- The p-value = $2 \min(P(S \leq t), P(S \geq t))$ where t is the realized value of T based on the sample
- To generate a $100(1 - \alpha)\%$ confidence interval for the true median, use the values of η for which we fail to reject H_0 at α .

- In R, install the package BSDA
- To test that median is 1 against the median not zero at 5% use
`SIGN.test(data, md=1, alternative="two.sided", conf.level=0.95)`
- One sided test possible
`SIGN.test(data, md=1, alternative="greater", conf.level=0.95)`
`SIGN.test(data, md=1, alternative="less", conf.level=0.95)`

Example (Income data):

```
income <- c(7, 1110, 7.1, 5.2, 8, 12, 0, 5, 2.1, 2, 46, 7.5)
```

Figure: Histogram and Box Plot

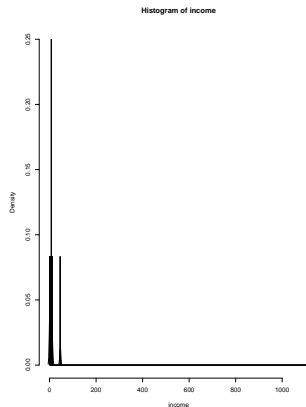


Figure: Box plot

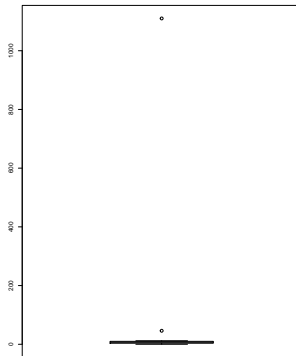
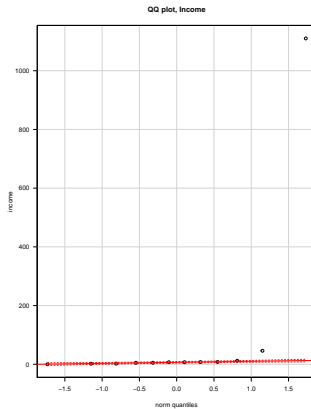


Figure: QQ plot



```
> summary(income)
      Min.   1st Qu.   Median     Mean  3rd Qu.    Max.
 0.000    4.275    7.050   101.000   9.000 1110.000
```

```
> t.test(income, mu=1)
```

One Sample t-test

```
data: income
t = 1.0893, df = 11, p-value = 0.2993
alternative hypothesis: true mean is not equal to 1
95 percent confidence interval:
 -101.0454  303.0287
sample estimates:
mean of x
 100.9917
```


- Presence of outliers has a dramatic effect on a 95% confidence interval for the population mean μ which is $[-101, 303]$
- This t based confidence interval is suspect because the normality assumption is unreasonable.
- A confidence interval for the population median income is more sensible because the median is more likely to be a more reasonable typical value.
- Using the sign procedure, a 95% confidence interval for the population median is $[2.32, 11.57]$.

```
> SIGN.test(income, md=1, alternative="two.sided")
```

One-sample Sign-Test

```
data: income
```

```
s = 11, p-value = 0.006348
```

```
alternative hypothesis: true median is not equal to 1
```

```
95 percent confidence interval:
```

```
2.408455 11.574545
```

```
sample estimates:
```

```
median of x
```

```
7.05
```

| | Conf.Level | L.E.pt | U.E.pt |
|-------------------|------------|--------|---------|
| Lower Achieved CI | 0.8540 | 5.0000 | 8.0000 |
| Interpolated CI | 0.9500 | 2.4085 | 11.5745 |
| Upper Achieved CI | 0.9614 | 2.1000 | 12.0000 |

```
> 2*(1-pbinom(10,12,0.5))  
[1] 0.006347656
```

Example: Age at first heart transplant

```
> age<-c(54, 42, 51, 54, 49, 56, 33, 58, 54, 64, 49)
> sort(age)
[1] 33 42 49 49 51 54 54 54 56 58 64

> summary(age)
  Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
 33.00  49.00   54.00   51.27  55.00   64.00
```

Question of interest: is the typical age at first transplant 50?

Figure: Histogram and Box Plot

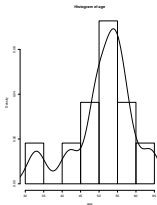


Figure: Box plot

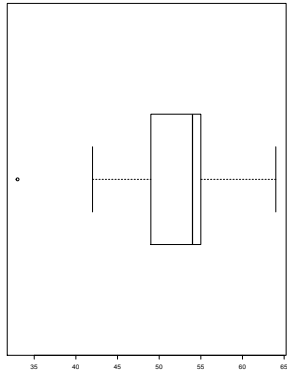
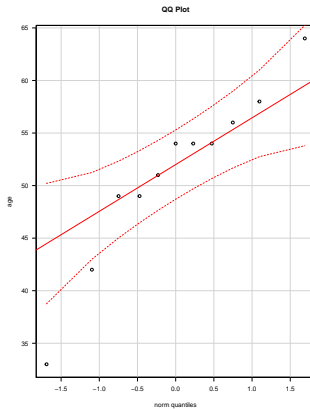


Figure: QQ plot



- The normal QQ-plot of the sample data indicates mild deviation from normality in the left tail
- It is good practice in this case to use the nonparametric test as a double-check of the t-test.

```
> t.test(age, mu=50)
```

One Sample t-test

```
data: age
t = 0.51107, df = 10, p-value = 0.6204
alternative hypothesis: true mean is not equal to 50
95 percent confidence interval:
 45.72397 56.82149
sample estimates:
mean of x
 51.27273
```

A 95% confidence for the mean in [45.72, 56.82]

```
> SIGN.test(age, md=50)
```

One-sample Sign-Test

```
data: age
s = 7, p-value = 0.5488
alternative hypothesis: true median is not equal to 50
95 percent confidence interval:
 46.98909 56.57455
sample estimates:
median of x
      54
```

| | Conf.Level | L.E.pt | U.E.pt |
|-------------------|------------|---------|---------|
| Lower Achieved CI | 0.9346 | 49.0000 | 56.0000 |
| Interpolated CI | 0.9500 | 46.9891 | 56.5745 |
| Upper Achieved CI | 0.9883 | 42.0000 | 58.0000 |

A 95% confidence for the mean in [46.99, 56.59]. Similar conclusion was reached with the t confidence interval and the t-test on μ . You should have less confidence in these results because the normality assumption is tenuous.

- The following result holds. Under H_0

$$\lim_{n \rightarrow \infty} P \left(\frac{S - n/2}{\sqrt{n/4}} \leq x \right) = \Phi(x)$$

- A large sample test reject H_0 if

$$S < n/2 - Z_{\alpha/2} \sqrt{n}/2 \quad \text{or} \quad S > n/2 + Z_{\alpha/2} \sqrt{n}/2$$

- How do you do a one sided test in this case?

- The sign test use simply information in the sign of the observations, no metric on how far the observations are from η_0 is incorporated from into the test
- For a distribution that is symmetric about η_0 , the vector of the absolute value $|X_i - \eta_0|, i = 1, 2, \dots, n$, is a sufficient statistic. So for symmetric distributions, it makes sense to use this information.
- The Wilcoxon procedure assumes you have a random sample X_1, X_2, \dots, X_n from a symmetric distribution (need not be normal) \Rightarrow mean = median ($\eta = \mu$).
- To test $H_0 : \mu = \mu_0$ against $H_a : \mu \neq \mu_0$ requires
 - sign of $X_i - \mu_0$
 - ranks R_1, R_2, \dots, R_n of the $|X_i - \mu_0|$ s
- The test statistic is

$$W = \sum_{i=1}^n s(i)R_i$$

where $s(i) = 1$ if $X_i - \mu_0 > 0$ and 0 otherwise.

Clearly W is also equal to

$$W = \sum_{i=1}^n s(i)R_i = \sum_{j=1}^n jw_j$$

where $w_j = 1$ if the j th ranked absolute value corresponds to a positive difference and 0 otherwise.

- Under $H_0 : \mu = \mu_0$, $s(X_1 - \mu_0), s(X_2 - \mu_0), \dots, s(X_n - \mu_0)$ and R_1, R_2, \dots, R_n are independent.
- Under $H_0 : \mu = \mu_0$, w_1, w_2, \dots, w_n are independent, identically distributed and

$$P(w_1 = 1) = P(w_1 = 0) = 1/2$$

- This implies that

$$E(W) = \sum_{i=1}^n jE(w_j) = \frac{1}{2} \sum_{j=1}^n j = n(n+1)/4$$

and

$$\text{var}(W) = \sum_{j=1}^n j^2 \text{var}(w_j) = \frac{1}{4} (n(n+1)(2n+1))/6 = \frac{n(n+1)(2n+1)}{24}.$$

- Under H_0

$$E(W) = \frac{n(n+1)}{4}$$

and we reject H_0 if $|W - n(n+1)/4|$ is large.

- in R use `wilcox.test(data)`

Wilcoxon Signed Rank Test

Example: here $\mu_0 = 10$.

| X_i | $X_i - 10$ | sign | $ X_i - 10 $ | rank | sign \times rank |
|-------|------------|------|--------------|------|--------------------|
| 20 | 10 | + | 10 | 6 | 6 |
| 18 | 8 | + | 8 | 4.5 | 4.5 |
| 23 | 13 | + | 13 | 8 | 8 |
| 5 | -5 | - | 10 | 3 | -3 |
| 14 | 4 | + | 10 | 4 | 4 |
| 8 | -2 | - | 2 | 1 | -1 |
| 18 | 8 | + | 8 | 4.5 | 4.5 |
| 22 | 12 | + | 12 | 7 | 7 |

$$W = 6 + 4.5 + 8 + 2 + 4.5 + 7 = 32$$

$n = 8$ this implies that under H_0 , $E(W) = 8(9)/2 = 18$

```
> wilcox.test(x,mu=10,conf.int=TRUE,correct=FALSE)
```

Wilcoxon signed rank test

data: x

$V = 32$, p-value = 0.04967 (W =V)

alternative hypothesis: true location is not equal to 10

95 percent confidence interval:

10.99996 21.00005

sample estimates:

(pseudo)median

16.0056

- Example: Two remedies for insomnia and the number of hours of sleep gained was recorded. Same people used in the experiment

```
a<-c(0.7, -1.6, -0.2, -1.2, 0.1, 3.4, 3.7, 0.8, 0.0, 2.0)
```

```
b<- c(1.9, 0.8, 1.1, 0.1, -0.1, 4.4, 5.5, 1.6, 4.6, 3.0)
```

- Goal: test $H_a : \mu_a = \mu_b$ against $H_a : \mu_a \neq \mu_b$

```
d<-a-b
```

```
> sleep<-data.frame(a , b, d)
```

```
> shapiro.test(sleep$d)
```

```
Shapiro-Wilk normality test
```

```
data:  d
```

```
W = 0.83798, p-value = 0.04173
```

p-value < 0.05, we reject the null hypothesis that the differences are from a normal distribution.

```
> wilcox.test(d, mu=0, conf.int=TRUE)
```

Wilcoxon signed rank test with continuity correction

data: d

V = 1, p-value = 0.008004

alternative hypothesis: true location is not equal to 0

95 percent confidence interval:

-2.7999620 -0.7999339

sample estimates:

(pseudo)median

-1.299966

- The following result holds. Under H_0

$$\lim_{n \rightarrow \infty} P \left(\frac{W - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} \leq x \right) = \Phi(x)$$

- A large sample test reject H_0 if

$$S < n(n+1)/4 - Z_{\alpha/2} \sqrt{n(n+1)(2n+1)/24}$$

or if

$$S > n(n+1)/4 + Z_{\alpha/2} \sqrt{n(n+1)(2n+1)/24}$$

- How do you do a one sided test in this case?

- For symmetric distributions, the t, the sign and Wilcoxon procedures are appropriate
- If the underlying distribution is extremely skewed, you can use the sign procedure to get a confidence interval for the population median
- Otherwise, you can transform the data to a scale where the underlying distribution is nearly normal and use t .
- Moderate degrees of skewness will not have a big impact on the t based results.
- Data from heavy-tailed distribution can have a profound impact on the t-test and the t confidence interval
- The sign and the Wilcoxon procedures downweight the influence of outliers by looking at sign and signed ranks instead of the actual values.
- A weakness of nonparametric methods is that they do not easily generalize to complex problems

- The WMW procedure assumes you have independent random samples from two populations
- The distributions are of the $F(x - \eta_i), i = 1, 2$ and interest is in comparing η_1 and η_2 . Here F is assumed to be absolutely continuous.
- The WMW procedure gives a confidence interval for the difference $\Delta = \eta_1 - \eta_2$,

(Wilcoxon) Mann-Whitney two sample procedure

- X_1, X_2, \dots, X_{n_1} and Y_1, Y_2, \dots, Y_{n_2} are independent random samples from two populations
- This test tries to detect location shifts
- To compute the test statistic (see below), combine the two samples and rank the observations.
- Let T_X be the sum of the ranks for the observations in the X -group and T_Y for the Y -group (note that $T_X + T_Y = (n_1 + n_2)(n_1 + n_2 + 1)/2$.)
- Under $H_0 : \eta_1 = \eta_2$, $E(T_X) = n_1(n_1 + n_2 + 1)/2$ and $Var(T_X) = n_1 n_2 (n_1 + n_2 + 1)/12$
- In some applications the Mann-Whitney form U_X of Wilcoxon rank sum test is used:

$$U_X = T_X - \frac{n_1(n_1 + 1)}{2}$$

As such

$$U_X = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} I(X_i > Y_j)$$

- The test is most appropriate when the populations have the same shape and differ only in location (same in dispersion). The distributions do not have to be symmetric.

•

$$\lim_{\min(n_1, n_2) \rightarrow \infty} P \left(\frac{T_X - n_1(n_1 + n_2 + 1)/2}{\sqrt{n_1 n_2 (n_1 + n_2 + 1)/12}} \leq x \right) = \Phi(x)$$

(Wilcoxon) Mann-Whitney two sample procedure

```
group1 (X)<-c(0.8, 2.8, 4.0, 2.4, 1.2, 0.0, 6.2, 1.5, 28.8, 0.7)
group2 (Y) <-c(2.3, 0.3, 5.2, 3.1, 1.1, 0.9, 2.0, 0.7, 1.4, 0.3)
```

Wilcoxon rank sum test with continuity correction

data: x and y

W = 61.5, p-value = 0.4053

alternative hypothesis: true location shift is not equal to 0