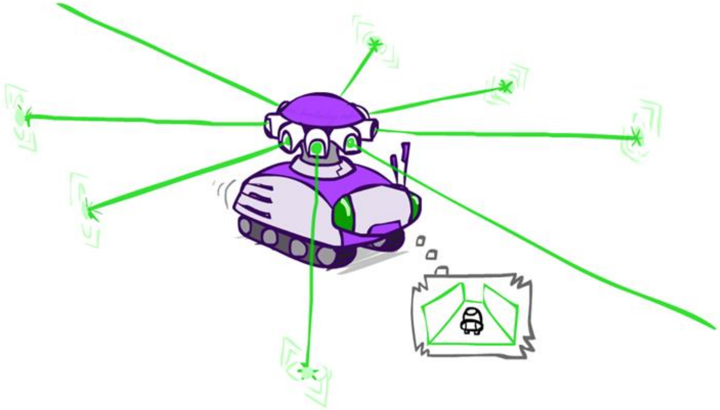
COMS W4733: Computational Aspects of Robotics

Lecture 20: Bayesian Filtering



Instructor: Tony Dear

State Estimation

- So far, many different motion planning algorithms, but only useful for the robot if it can process its percepts!
- E.g., “where is the robot right now”, “what are the current joint angles”?

- **State estimation**: Task of finding an approximation (*belief*) of current state given history of *observations* and *actions*
  - *Observations* comprise information that is derived from the (hidden) state
  - *Actions* comprise inputs that can change the state

- **Localization**: State estimation for robot location in the world

Dead Reckoning

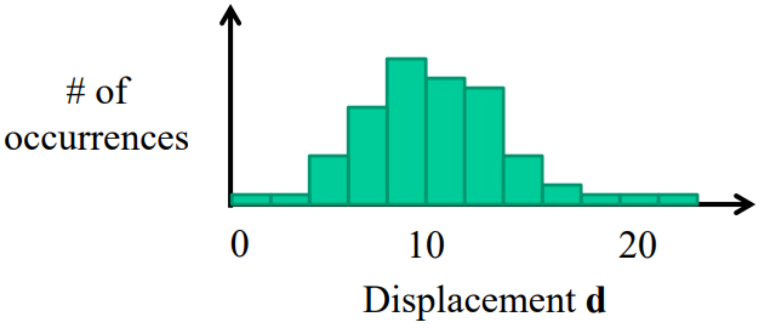
- A robot’s cumulative actions over time provide information about its current state
- **Dead reckoning**: Starting from initial configuration, integrate (velocities) forward in time to obtain new configurations
- **Odometry**: The specific task of integrating wheel rotations for mobile robots

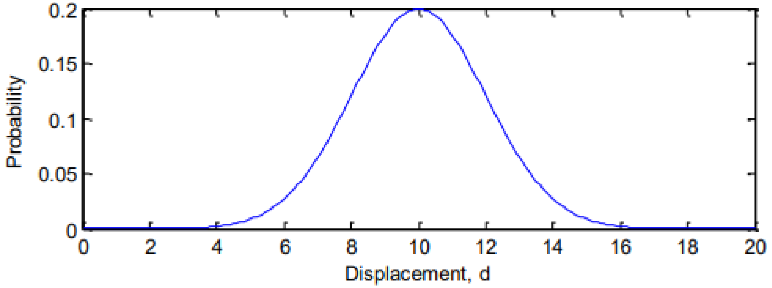
- Problem: Both methods are sensitive to errors!
- Bad data collection and instrument miscalibrations can throw off odometry
- Dead reckoning errors are cumulative and build up over time

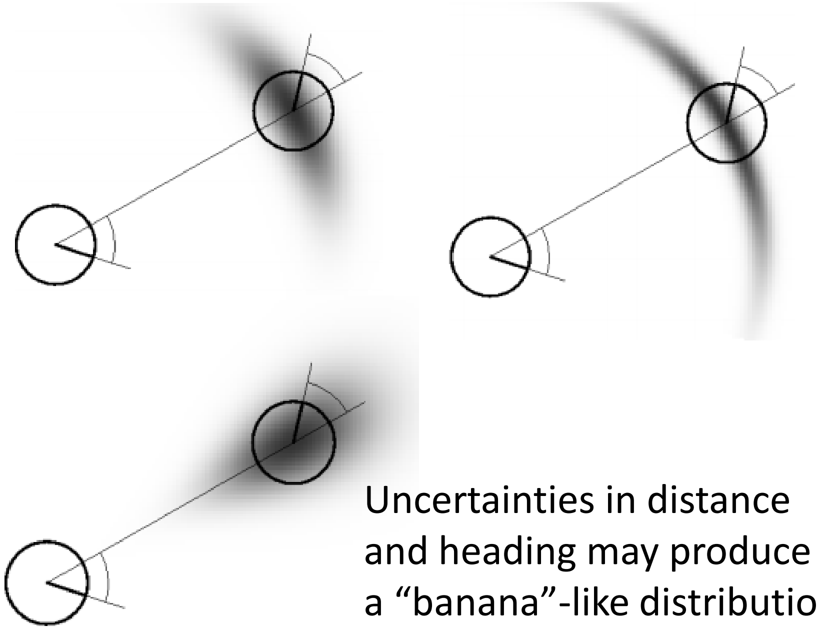
- Observations are needed to “re-calibrate” over time

Example: Robot Odometry

- Suppose we run a 10 cm straight trajectory on a diff-drive robot, but there is noise so that we don't always end up moving 10 cm…







Uncertainties in distance and heading may produce a "banana"-like distribution

Belief Distributions

- Robot's *belief* about its state is really a conditional probability distribution:

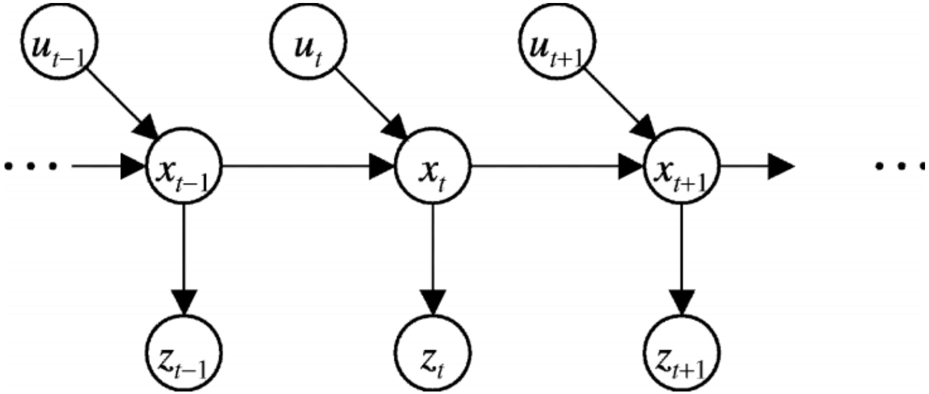
$$B(\boldsymbol{x}_t) = p(\boldsymbol{x}_t \mid \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t})$$

Robot state at time t Actions from 1 to t Observations from 1 to t

- We also have transition and observation models
- Generally assumed to be *Markov* (independent of previous states conditioned on current)
  - *Transition* model: $p(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1}, \boldsymbol{u}_t)$
  - *Observation* model: $p(\boldsymbol{z}_t \mid \boldsymbol{x}_t)$

Hidden Markov Models

- Robot state is follows a **hidden Markov model** that evolves with time



- True states $\boldsymbol{x}_t$ are hidden; we cannot observe them directly
- Markov assumption: $p(\boldsymbol{x}_t \mid \boldsymbol{x}_{1:t-1}, \boldsymbol{z}_{1:t-1}, \boldsymbol{u}_{1:t}) = p(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1}, \boldsymbol{u}_t)$

$$p(\boldsymbol{z}_t \mid \boldsymbol{x}_{1:t}, \boldsymbol{z}_{1:t-1}, \boldsymbol{u}_{1:t}) = p(\boldsymbol{z}_t \mid \boldsymbol{x}_t)$$

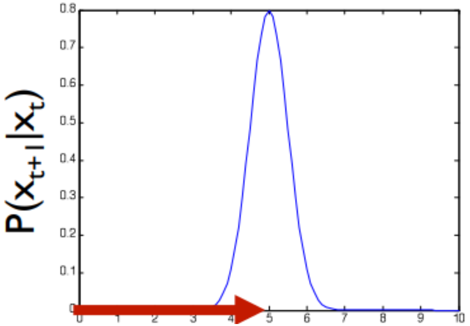
Transition Model

- Transition model generally derived from discretizing robot kinematics
- Ex: Steered unicycle

$$p(\boldsymbol{x}_t \mid \boldsymbol{x}_{t-1}, \boldsymbol{u}_t)$$

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \rho\dot{\psi}\cos\theta \\ \rho\dot{\psi}\sin\theta \end{pmatrix} \qquad \boldsymbol{x}_t = \boldsymbol{f}(\boldsymbol{x}_{t-1}, \boldsymbol{u}_t) + \boldsymbol{v}_t$$

$$\begin{pmatrix} x(t) \\ y(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} x(t-1) + \rho u_1 \Delta t \cos[\theta(t-1)] \\ y(t-1) + \rho u_1 \Delta t \sin[\theta(t-1)] \\ \theta(t-1) + u_2 \Delta t \end{pmatrix} = \boldsymbol{f}(\boldsymbol{x}_{t-1}, \boldsymbol{u}_t)$$



- Noise may occur due to modeling inaccuracy, nonidealities (wheel slip), etc.
- May have different types of noise in each component

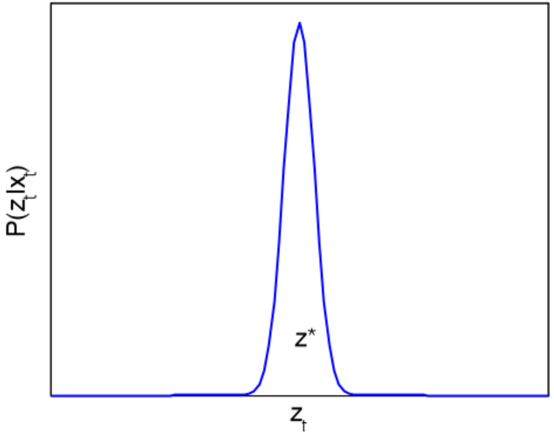
Observation Model

- Depending on sensor type, observations may inherently have uncertainty
- E.g., beam-based sensors: sonar, radar, lidar

- Beams may be reflected by small or moving obstacles or people

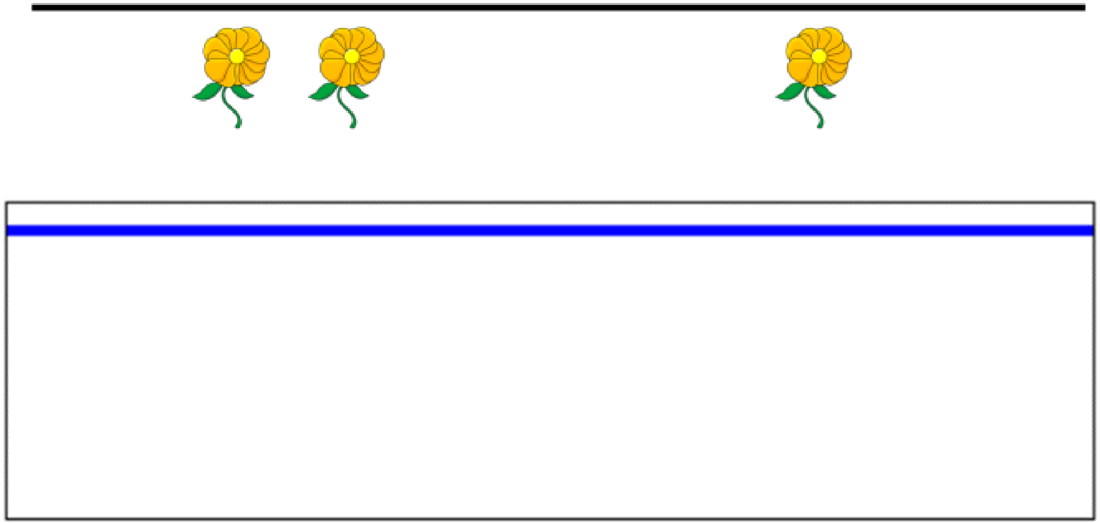
- Simple observation model: $\boldsymbol{z}_t = \boldsymbol{h}(\boldsymbol{x}_t) + \boldsymbol{w}_t$
  - $\boldsymbol{h}$ describes the quantities that we can measure
  - $\boldsymbol{w}_t$ is some form of additive noise

$$p(\boldsymbol{z}_t \mid \boldsymbol{x}_t)$$



Example: Belief Distributions

- Distribution describing where we think we are can be even more complicated when we include observations
- Example: Robot in a flower garden



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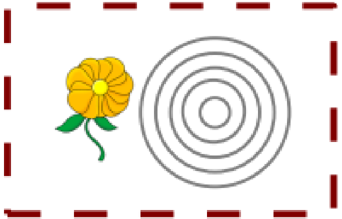
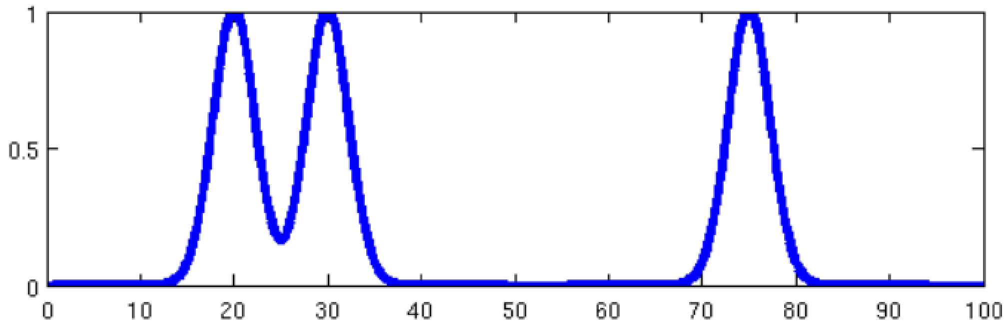


Observe flower



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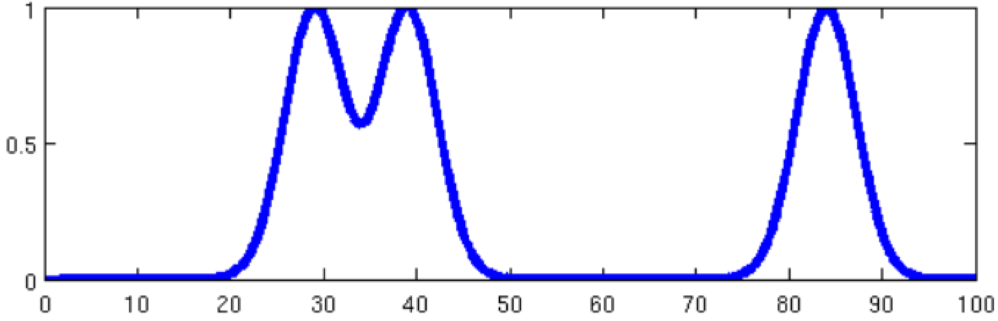

Move forward

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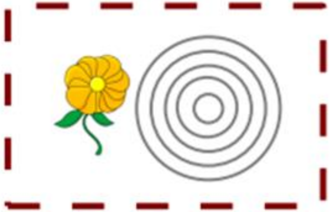

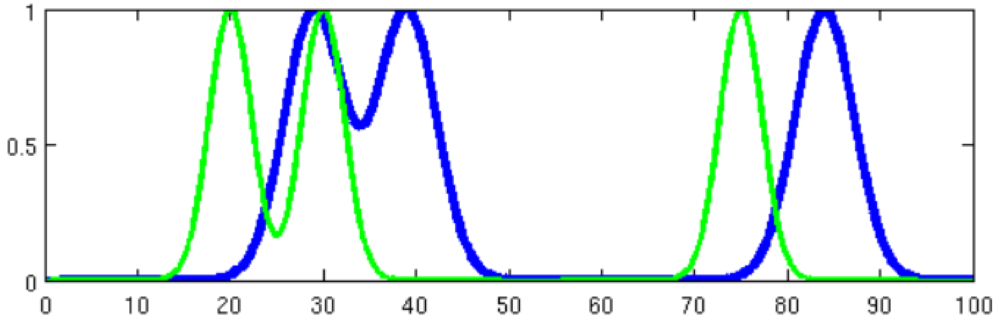

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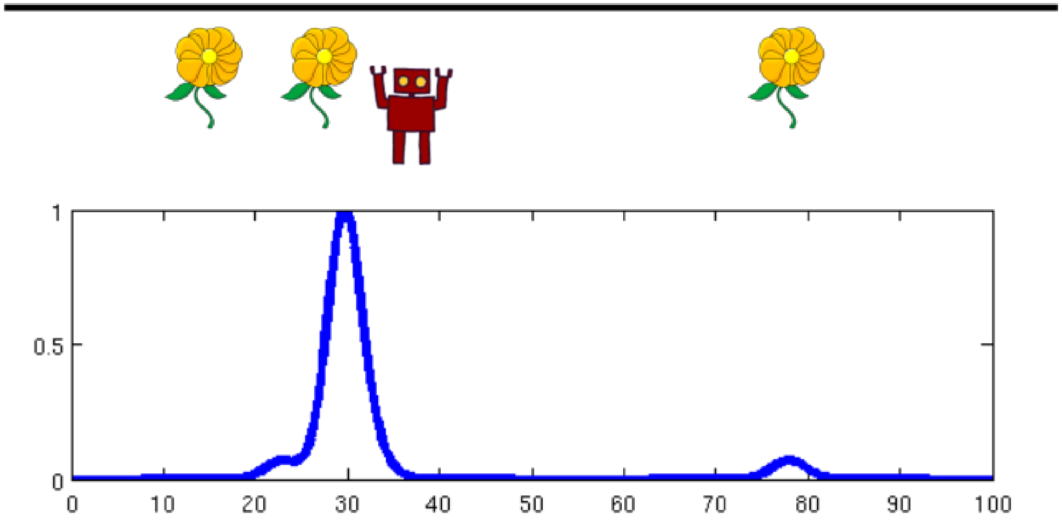


Updated belief distribution about where we are

Probability Review

- Law of total probability (joint distribution followed by marginalization)

$$P(A) = \sum_B P(A|B)P(B) \qquad p(a) = \int_{-\infty}^{\infty} p(a|b)p(b)\,db$$

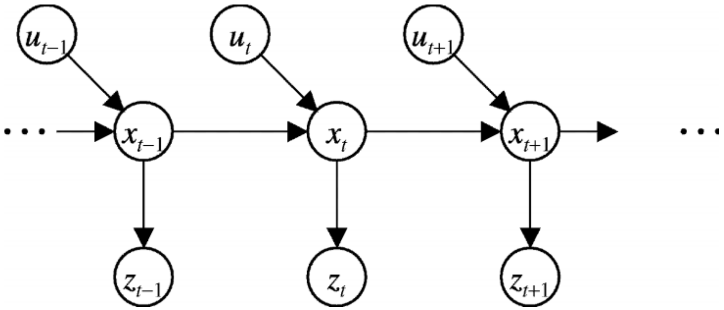
- Bayes' theorem ("reversing" a conditional)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{Likelihood \cdot Prior}{Evidence}$$

- $P(A|B) = \eta\, P(B|A)P(A)$ if $P(B)$ is constant (e.g., B is observed evidence)

Taking an Action: Predict

- Suppose we started with $B(\boldsymbol{x}_{t-1})$
- Now we take an action, but not observation



- Current belief: $B'(\boldsymbol{x}_t) = p(\boldsymbol{x}_t|\boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1})$

$$p(\boldsymbol{x}_t|\boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1}) = \int p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1})p(\boldsymbol{x}_{t-1}|\boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1})\, d\boldsymbol{x}_{t-1}$$

Law of total probability

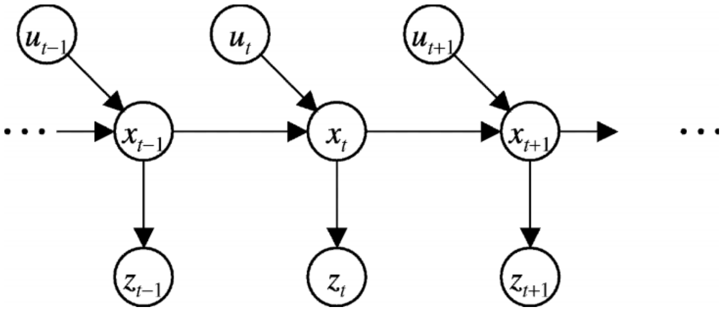
$$p(\boldsymbol{x}_t|\boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1}) = \int p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t)p(\boldsymbol{x}_{t-1}|\boldsymbol{u}_{1:t-1}, \boldsymbol{z}_{1:t-1})\, d\boldsymbol{x}_{t-1}$$

Markov assumption

$$p(\boldsymbol{x}_t|\boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1}) = \int p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t)B(\boldsymbol{x}_{t-1})\, d\boldsymbol{x}_{t-1}$$

Making an Observation: Update

- We currently have the belief $B'(\boldsymbol{x}_t)$
- Suppose we make an observation $\boldsymbol{z}_t$



- We now have the fully updated belief $B(\boldsymbol{x}_t)$
- Consider $B(\boldsymbol{x}_t) = p(\boldsymbol{x}_t | \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t})$:

$$p(\boldsymbol{x}_t | \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t}) = \frac{p(\boldsymbol{z}_t | \boldsymbol{x}_t, \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1}) p(\boldsymbol{x}_t | \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1})}{p(\boldsymbol{z}_t | \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1})}$$

Bayes' theorem

$$p(\boldsymbol{x}_t | \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t}) = \eta \, p(\boldsymbol{z}_t | \boldsymbol{x}_t, \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1}) \, p(\boldsymbol{x}_t | \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t-1})$$

Observation is constant

$$p(\boldsymbol{x}_t | \boldsymbol{u}_{1:t}, \boldsymbol{z}_{1:t}) = \eta \, p(\boldsymbol{z}_t | \boldsymbol{x}_t) B'(\boldsymbol{x}_t)$$

Markov assumption

Bayes Filter Algorithm

Algorithm **Bayes_filter**(*Bel(x)*, $\boldsymbol{d}$):

1. $\eta = 0$
2. **if** $\boldsymbol{d}$ is an *action* data item $\boldsymbol{u}$ **then**
3. **for all** $\boldsymbol{x}$ **do**
4. $B'(\boldsymbol{x}) = \int P(\boldsymbol{x}|\boldsymbol{u}, \boldsymbol{x}')B(\boldsymbol{x}')d\boldsymbol{x}'$
5. **if** $\boldsymbol{d}$ is a *perceptual* data item $\boldsymbol{z}$ **then**
6. **for all** $\boldsymbol{x}$ **do**
7. $B'(\boldsymbol{x}) = P(\mathbf{z}|\boldsymbol{x})B(\boldsymbol{x})$
8. $\eta = \eta + B'(\boldsymbol{x})$
9. **for all** $\boldsymbol{x}$ **do**
10. $B'(\boldsymbol{x}) = \eta^{-1}B'(\boldsymbol{x})$
11. **return** $B'(\boldsymbol{x})$

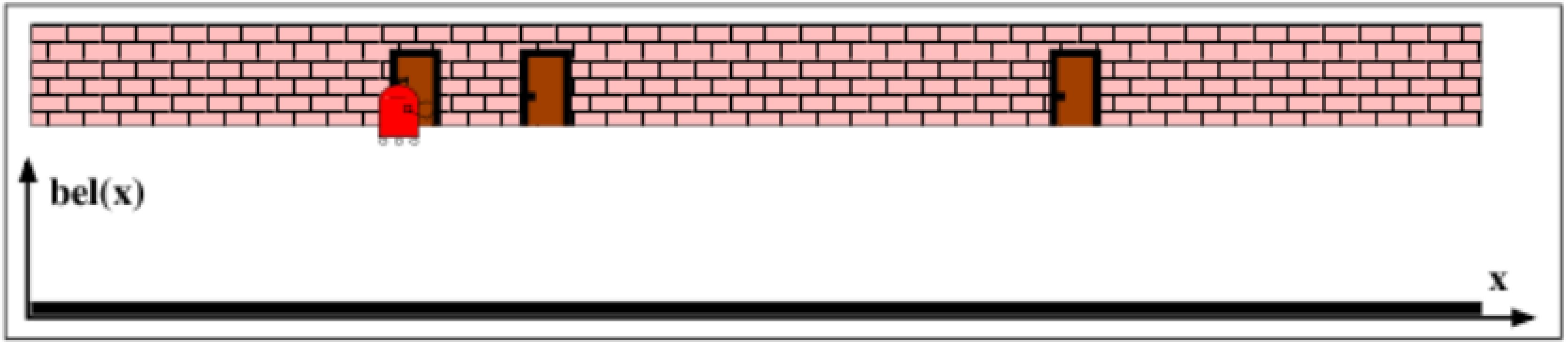
Prediction:

$$B'(\boldsymbol{x}_t) = \int \underbrace{p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1}, \boldsymbol{u}_t)}_{\text{Transition model}} B(\boldsymbol{x}_{t-1})\, d\boldsymbol{x}_{t-1}$$

Observation:

$$B(\boldsymbol{x}_t) = \eta\, \underbrace{p(\mathbf{z}_t|\boldsymbol{x}_t)}_{\text{Observation model}} B'(\boldsymbol{x}_t)$$

Example: Bayes Filter



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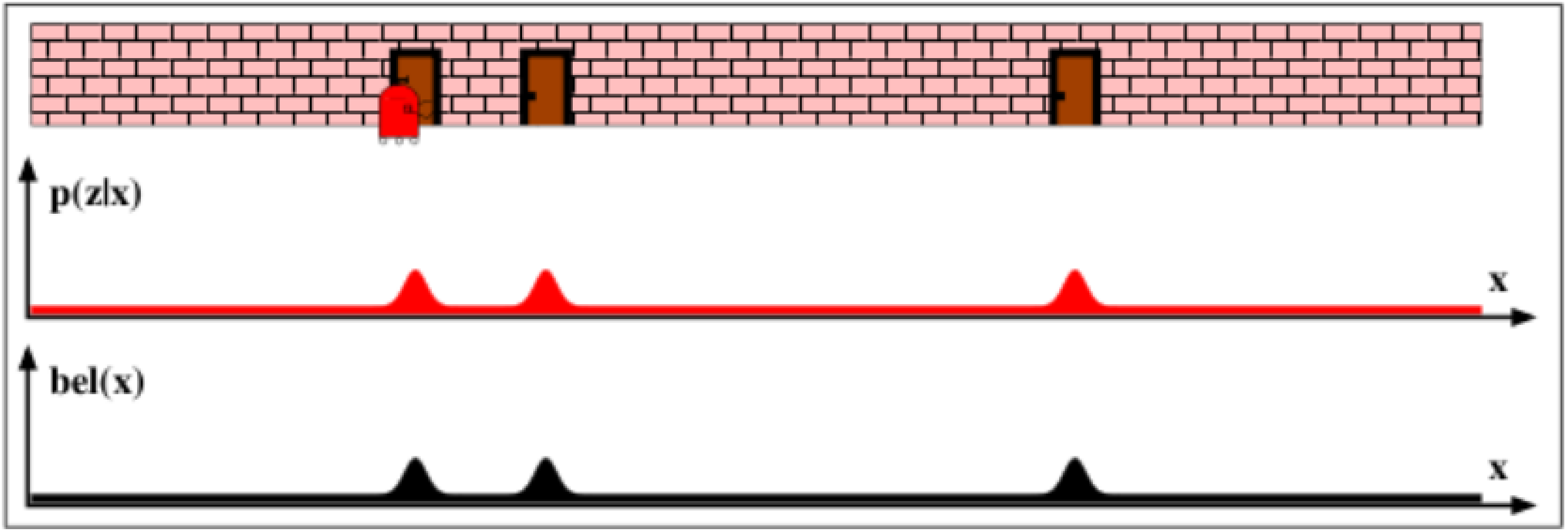

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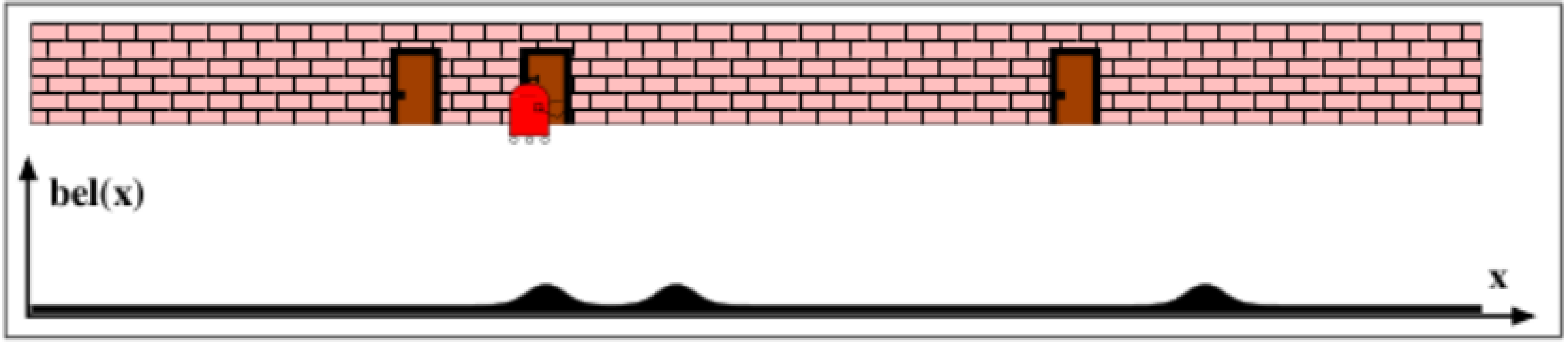

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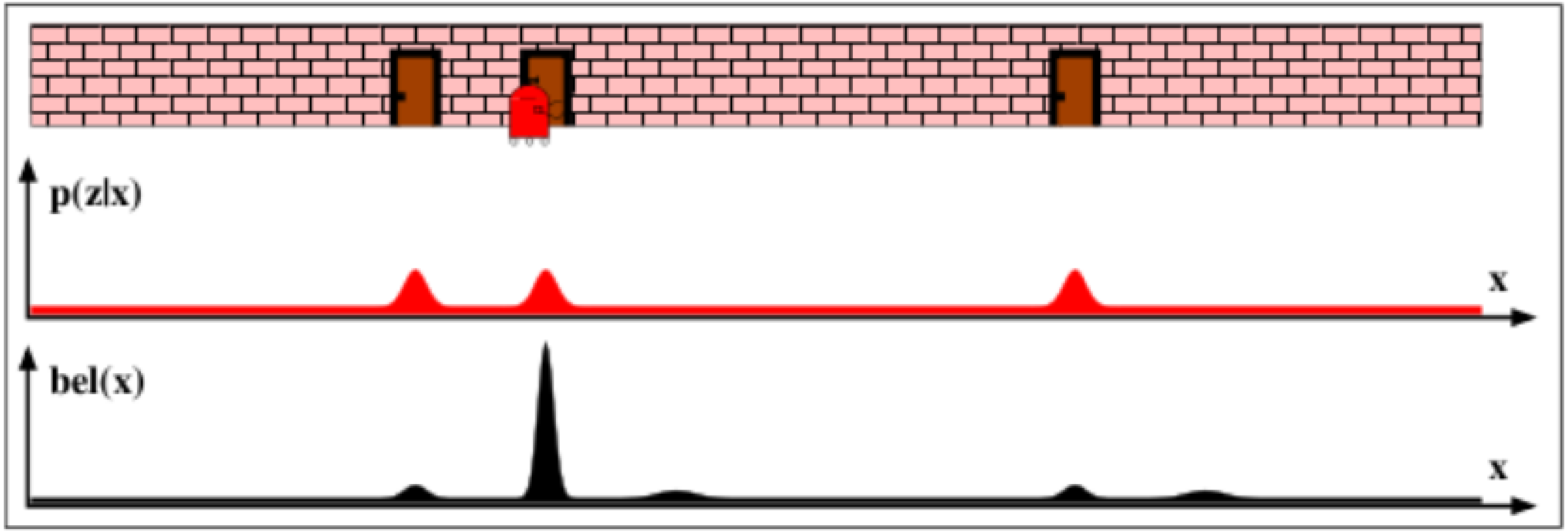

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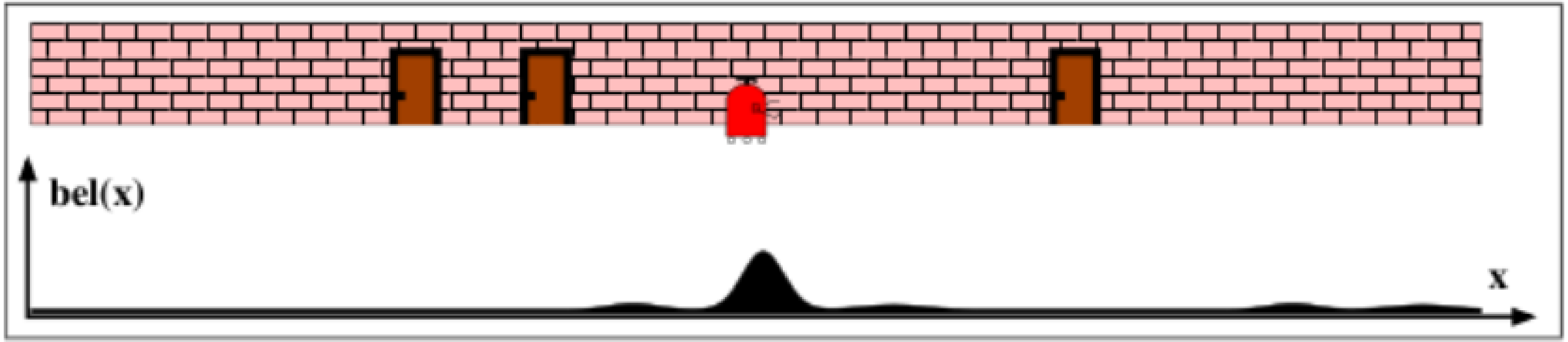

Bayes Filter Considerations

- We have a recursive framework to compute robot's posterior belief given prior belief, new action, and new observation
- Assume we have both transition and observation models

- What's the problem?
- Belief distributions can become arbitrarily complicated
- Summing or integrating may be computationally intractable

- We can try to deal with special cases (Gaussians) or approximations…