COMS W4733, HW3

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PROBLEM 1

(a) The coordinates $(x', y', \theta')^T$ could be expressed in terms of the link length L and configuration variables x, y, θ, and ϕ:

$$\begin{bmatrix} x' \\ y' \\ \theta' \end{bmatrix} = \begin{bmatrix} x + \frac{L}{2}\cos\theta + \frac{L}{2}\cos(\theta+\phi) \\ y + \frac{L}{2}\sin\theta + \frac{L}{2}\sin(\theta+\phi) \\ \theta+\phi \end{bmatrix}$$

Then

$$\begin{aligned} \dot{x}' &= \dot{x} - \frac{L}{2}\dot{\theta}\sin\theta - \frac{L}{2}(\dot{\theta}+\dot{\phi})\sin(\theta+\phi) \\ \dot{y}' &= \dot{y} + \frac{L}{2}\dot{\theta}\cos\theta + \frac{L}{2}(\dot{\theta}+\dot{\phi})\cos(\theta+\phi) \end{aligned}$$

So the second link's no-slip constraints $\dot{x}'\sin\theta' - \dot{y}'\cos\theta' = 0$ could be expressed in terms of these variables and their velocity:

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \dot{\theta}\frac{L}{2}\cos\phi - \frac{L}{2}(\dot{\theta}+\dot{\phi}) = 0.$$

Or:

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \dot{\theta}L\cos^2\frac{\phi}{2} - \dot{\phi}\frac{L}{2} = 0.$$

(b) The robot's three constraints could be expressed in Pfaffian form $A^T(q)\dot{q} = 0$:

$$\begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -1 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -\frac{L}{2}(\cos\phi+1) & 0 & -\frac{L}{2} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ v \\ \dot{\phi} \end{bmatrix} = \mathbf{0}$$

Since there are three constraints and 5 configuration velocities, there are 2 controllable degrees of freedom. Set the two controllable degrees of freedom to v and $\dot{\phi}$, we have the set of all allowed velocities:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ v \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ \frac{2\sin\phi}{L(\cos\phi+1)} & -\frac{1}{\cos\phi+1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \dot{\phi} \end{bmatrix}$$

(c) Given $\dot{x}(t)$ and $\dot{y}(t)$, from result of 1.(b), we have $v = \sqrt{\dot{x}^2 + \dot{y}^2}$.

Since $\dot{\theta} = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$, we have:

$$\begin{aligned} \dot{\phi} &= \frac{2\sin\phi}{L} v - (\cos\phi + 1)\dot{\theta} \\ &= \frac{2\sin\phi}{L}\sqrt{\dot{x}^2 + \dot{y}^2} - (\cos\phi + 1)\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \end{aligned}$$

So the input velocities could be expressed as following:

$$\begin{aligned} v &= \sqrt{\dot{x}^2 + \dot{y}^2} \\ \dot{\phi} &= \frac{2\sin\phi}{L}\sqrt{\dot{x}^2 + \dot{y}^2} - (\cos\phi + 1)\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2} \end{aligned}$$

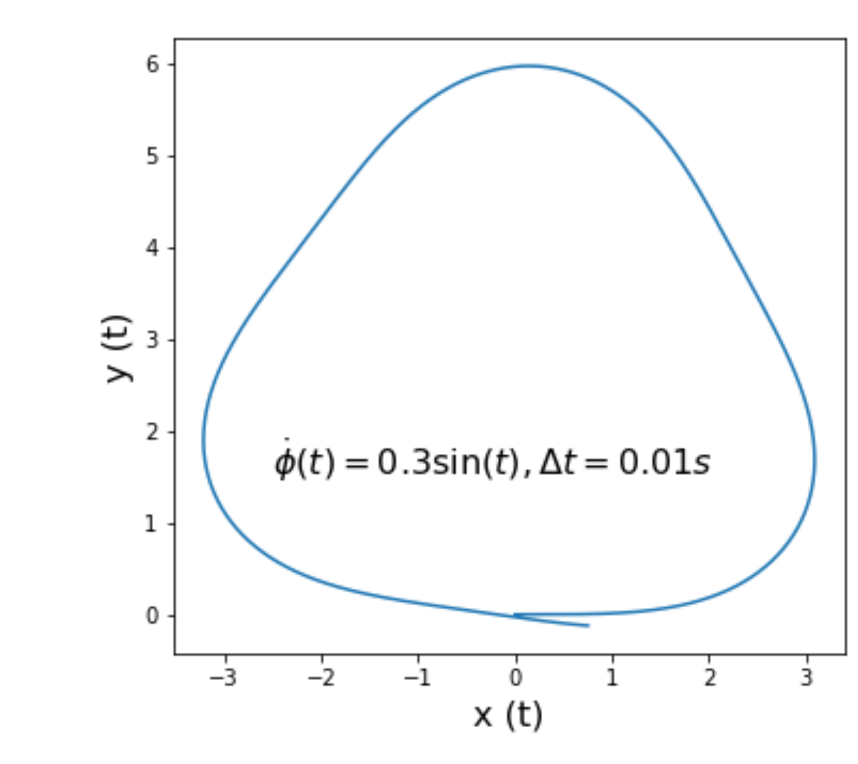
Yes, the robot has singularities. From the forward kinematics, the constraint of $\dot{\theta}$ exsits when $\cos\phi + 1 \neq 0$, which means when $\phi = -\pi$, singularities occur.

PROBLEM 2

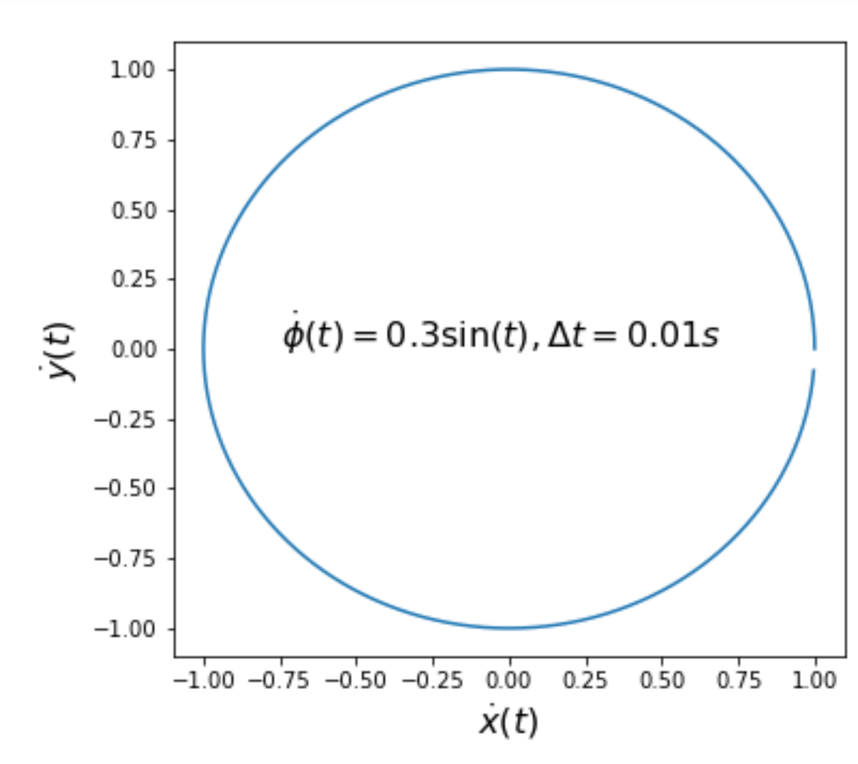
(a) The four ODEs are as following:

$$\begin{aligned} \dot{\phi}(t) &= 0.3\sin(t) \\ \dot{\theta}(t) &= \tan(\phi(t)) \\ \dot{x} &= \cos(\theta(t)) \\ \dot{y} &= \sin(\theta(t)) \end{aligned}$$

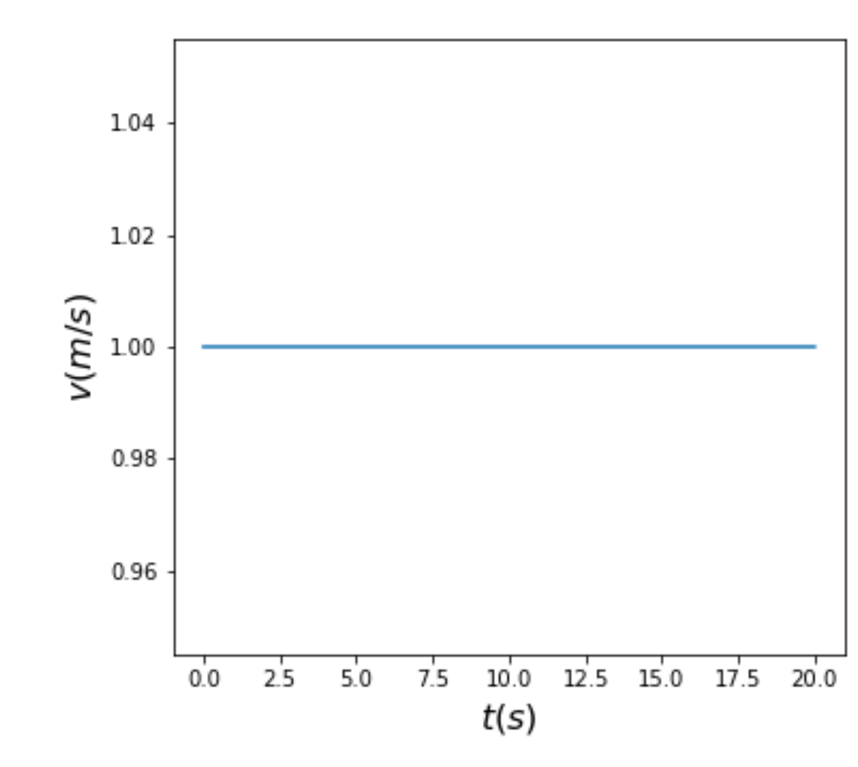
Use Python **odeint** to integrate the above ODES numerically, we could have the $x(t) - y(t)$ plot:



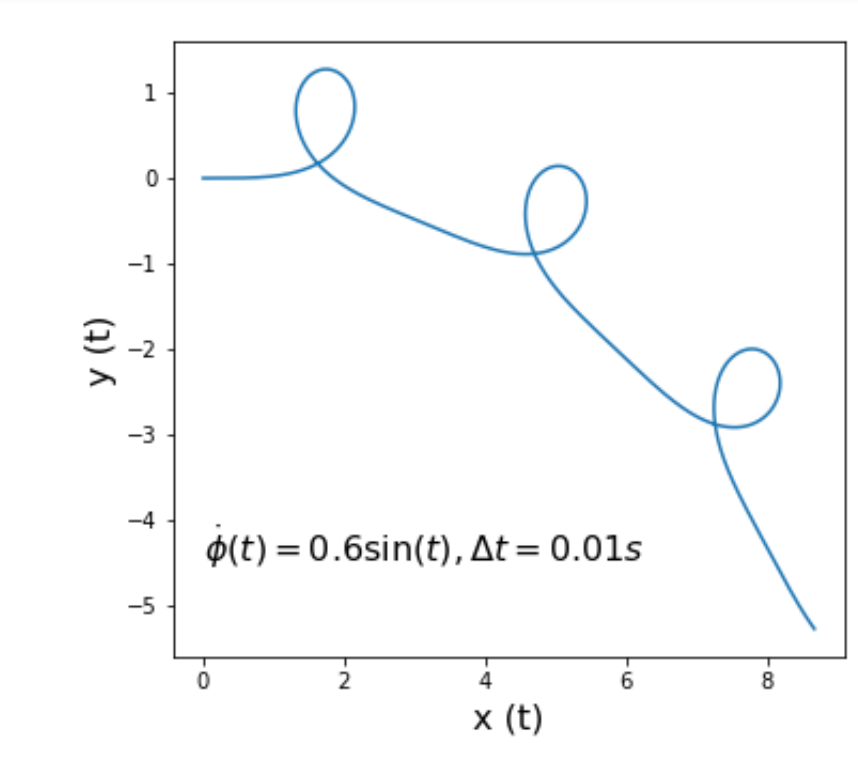
The two velocity inputs, $v = 1$ m/s, we only change the steering rate $\dot{\phi}(t) = 0.3\sin(t)$. So $\phi(t) = 0.3(1 - \cos(t))$, which means the bicycle will always turn left with periodically changing steering rate. Since the rate is 0.3, the bicycle changes steering rate slowly, which generate the $x(t) - y(t)$ plot above.



Above is the $\dot{x}(t) - \dot{y}(t)$ plot, which is a circle with (0, 0) as origin and radius = 1, which verify that $v = 1 = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}$.



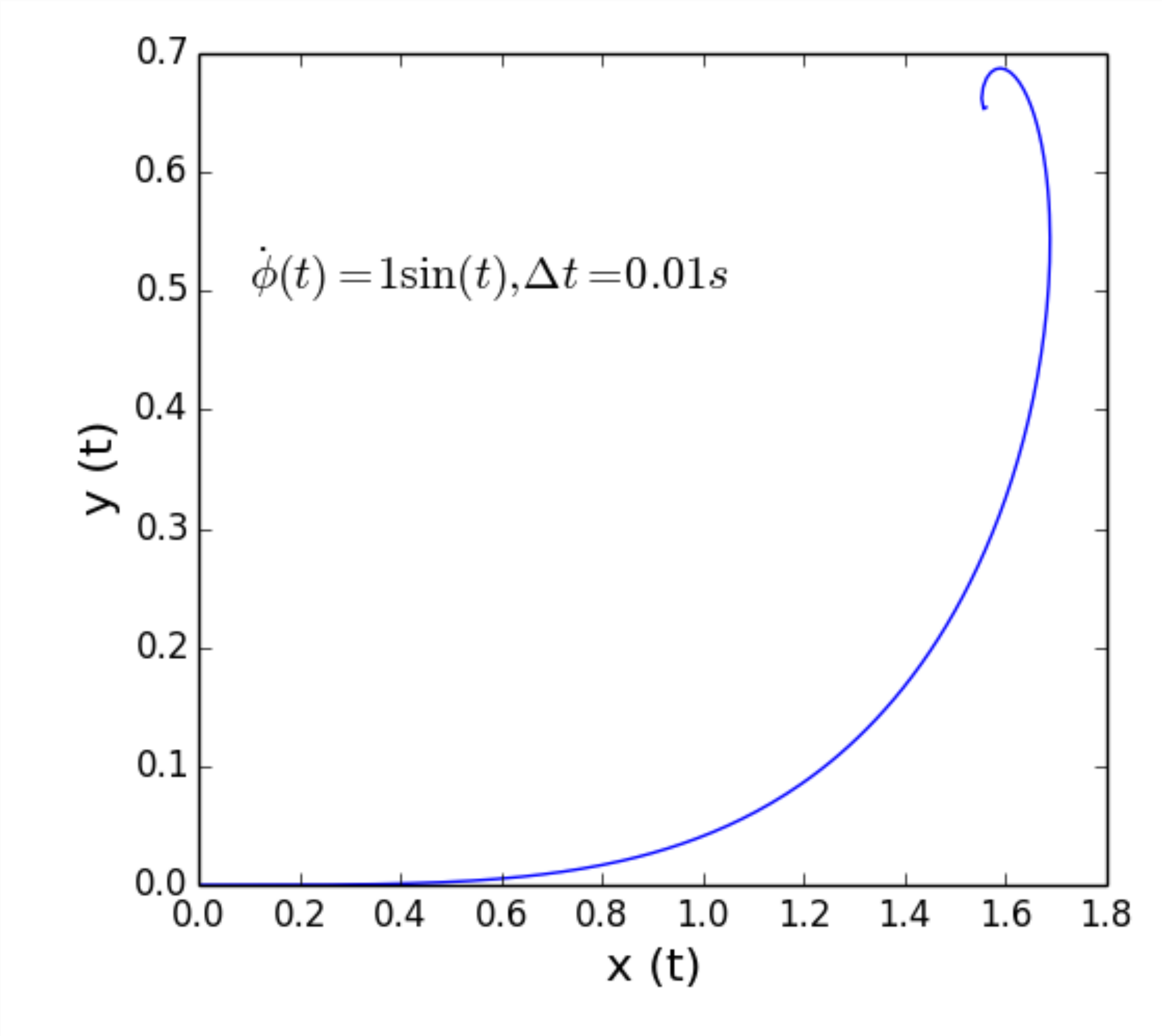
(b)



Above is the $x(t) - y(t)$ plot when the steering rate $\dot{\phi}(t) = 0.6\sin(t)$. So $\phi(t) = 0.6(1 - \cos(t))$, which means the bicycle will always turn left with periodically changing steering rate.

Since the rate is 0.6, the bicycle changes steering rate faster than 2.a., which generates more and smaller loops as the plot above.

(c)



Above is the $x(t) - y(t)$ plot when the steering rate $\dot{\phi}(t) = \sin(t)$. So $\phi(t) = 1 - \cos(t)$, which means if there is no singularities, the bicycle will always turn left with periodically changing steering rate.

However, $\dot{\theta}(t) = \tan(\phi)$ and hence when $\phi = \frac{\pi}{2}$, $\dot{\theta} \to \infty$ and singularity occurs. So when $t = \arccos(1 - \pi/2)$, the singularity occurs and the bicycle stops moving.

To prevent the singularity occuring, for $\dot{\phi}(t) = c * \sin(t)$, we should have $\phi(t) = c(1 - \cos(t)) < \pi/2$. Since $0 \leq 1 - \cos(t) \leq 2$, we should have $c < \pi/4$.