COMS W4733: Computational Aspects of Robotics

Lecture 10: Mobile Robots

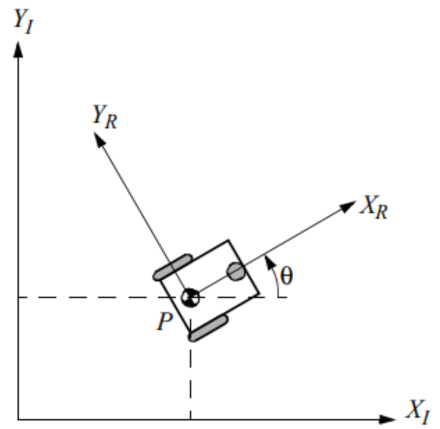
Instructor: Tony Dear

Mobile Robots

- Generally less complex than manipulators
- Still modeled as rigid bodies / coordinate frames moving around

- Crucial difference: Mobile robots are not attached to anything!
- We know everything about a manipulator from its joints
- No such instantaneous info about a mobile robot

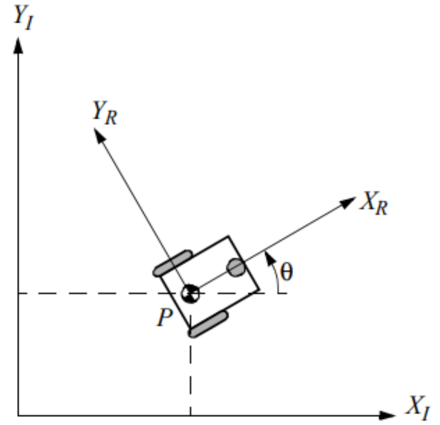
- Key component of mobile robots: Wheels
- Mathematically describe them as constraints



Mobile Robot Kinematics

- Good news: We will generally talk about mobile robots in 2D only
  - I.e., 2 translation DOFs and 1 rotation DOF

- Generally have at least two coordinate frames: a fixed, *inertial* frame o_i-$x_i y_i$ and a moving, *local or body,* frame o_r-$x_r y_r$ attached to the robot



- Configuration variables include $x, y, \theta \in \boldsymbol{Q}$
- Position and orientation of o_r relative to o_i

- Forward kinematics: $\dot{\boldsymbol{q}} = f(\boldsymbol{q}, \boldsymbol{u})$
- Unlike manipulators, inputs $\boldsymbol{u}$ can vary from robot to robot

Kinematic Constraints

- We have implicitly used *holonomic* constraints $h(\boldsymbol{q}) = 0$
  - Restrict the space of valid configurations (e.g. prismatic or revolute joints)

- Wheels give rise to *nonholonomic* constraints $a(\boldsymbol{q}, \dot{\boldsymbol{q}}) = 0$
- Specifically, wheel constraints are *Pfaffian* (linear)
  - Restrict the space of valid velocities but not configurations
  - Can freely move to any position but not with arbitrary velocities

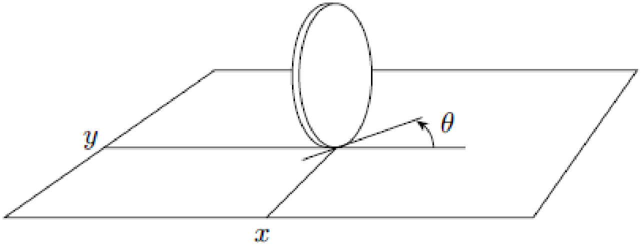
- There may be multiple (k) constraints

$$a_i^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0$$

$1 \times n$ vector,
$n =$ # configuration variables

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0$$

$k \times n$ matrix

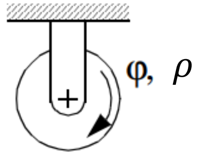


Kinematic Constraints

- No-slip / no-slide constraint: Wheel velocity v_{perp} should be 0 in sideways direction
- If wheel is oriented at angle θ relative to inertial frame:

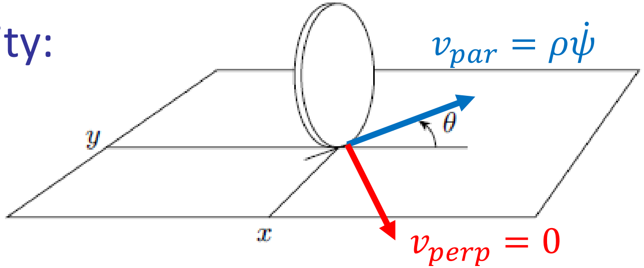
$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$

- What if we can also control wheel's orientation ψ?
- Linear velocity on edge of wheel due to spinning is $v_{par} = \rho\dot{\psi}$



- Rolling constraint: v_{par} should be equal to travel velocity:

$$\dot{x}\cos\theta + \dot{y}\sin\theta = \rho\dot{\psi} = v_{par}$$



Maneuverability

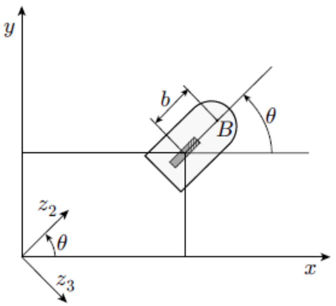
- A robot's wheel must move orthogonal to *zero motion line*
- Multiple no-slip constraints intersect at an *instantaneous center of rotation (ICR)*

- Each independent constraint contributes a row to $\boldsymbol{A}^T(\boldsymbol{q})$
- Valid velocity directions must lie in *null space* of $\boldsymbol{A}^T(\boldsymbol{q})$
- **Degree of maneuverability**: $\dim\left(\mathrm{null}\left(\boldsymbol{A}^T(\boldsymbol{q})\right)\right)$

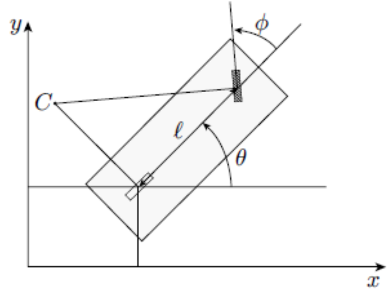
- Unicycle has degree of maneuverability equal to 2
- Bicycle has degree of maneuverability equal to 2

- Maneuverability = 3? No constraining wheels (e.g. casters)
- Maneuverability = 0? Vehicle cannot move at all!

Unicycle

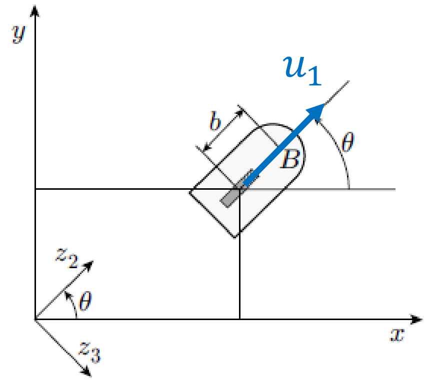


Bicycle



Unicycle

- Suppose we place a rigid body (*chassis*) on top of the single wheel
- Wheel constraints: $$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$ $$\dot{x}\cos\theta + \dot{y}\sin\theta = \rho\dot{\psi}$$ $$\begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & -\rho \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = 0$$
- Null space of $\boldsymbol{A}^T(\boldsymbol{q})$: $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} \rho\cos\theta & 0 \\ \rho\sin\theta & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \rho\dot{\psi}\cos\theta \\ \rho\dot{\psi}\sin\theta \end{bmatrix}$$
- $\dim(\text{null}(\boldsymbol{A}^T)) = \dim(\boldsymbol{q}) - \dim(\boldsymbol{A}^T) = 4 - 2 = 2$
- $u_1 = \dot{\psi}$ is wheel turn rate; $u_2 = \dot{\theta}$ is the unicycle steer rate



Bicycle

- Now suppose we have two wheels; one is steerable with angle denoted by ϕ
  - Robot coordinate frame placed at rear wheel
- Suppose we directly control forward velocity; wheel will roll passively

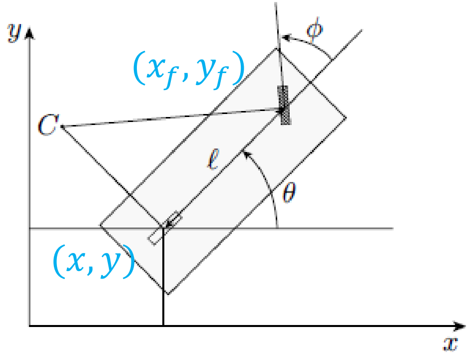
- Two no-slip constraints:

$$\dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) = 0$$
$$\dot{x}\sin(\theta) - \dot{y}\cos(\theta) = 0$$

$$\begin{bmatrix} x_f \\ y_f \end{bmatrix} = \begin{bmatrix} x + l\cos\theta \\ y + l\sin\theta \end{bmatrix}$$

- Find $\dot{x}_f$ and $\dot{y}_f$ in terms of $\dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}$:

$$\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - l\dot{\theta}\cos\phi = 0$$



- Stack the constraints:

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = 0$$

Bicycle

$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -l\cos\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = 0$$

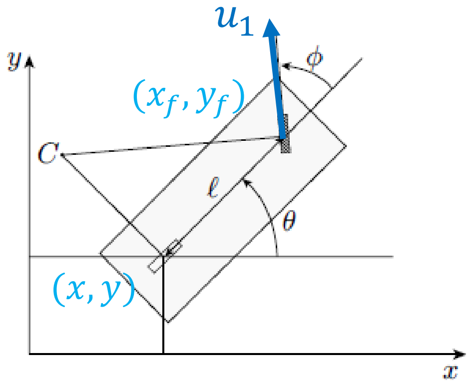
Null space

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\theta\cos\phi & 0 \\ \sin\theta\cos\phi & 0 \\ (\sin\phi)/l & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

- Null space of $\boldsymbol{A}^T(\boldsymbol{q})$ has dimension 2, giving us 2 inputs

- u_2 is the steer rate of the front wheel of the bicycle
- u_1 is the forward driving velocity

- Robot only moves when being driven forward ($u_1 \neq 0$)
- Steering ($u_2 \neq 0$) only changes front wheel direction
- Robot turns ($\dot{\theta} \neq 0$) whenever $\phi \neq 0$



Bicycle

- Equations useful if u_1 corresponds to input at front wheel
- What if the bike is rear-wheel drive?

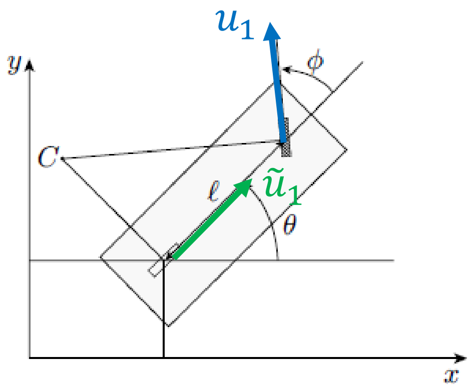
$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}\cos\theta\cos\phi & 0\\ \sin\theta\cos\phi & 0\\ (\sin\phi)/l & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

- Constraints and kinematics do not change!

- Input at rear wheel $\tilde{u}_1$ may be related to front wheel input u_1

$$\tilde{u}_1 = u_1 \cos\phi$$

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ (\tan\phi)/l & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{u}_1\\ u_2\end{bmatrix}$$



Differential-Drive Car

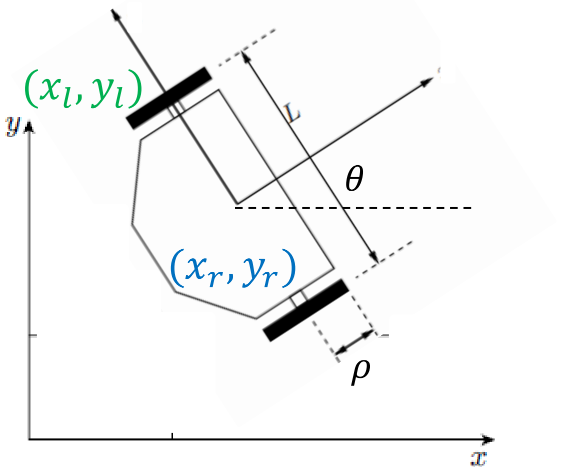
- Most indoor mobile robots do not move like a car
- Differential-drive configuration has two independently driven wheels at (possibly) different speeds

- Kinematics: $(x_l, y_l) = \left(x - \frac{L}{2}\sin\theta\,, y + \frac{L}{2}\cos\theta\right)$ $\quad (x_r, y_r) = \left(x + \frac{L}{2}\sin\theta\,, y - \frac{L}{2}\cos\theta\right)$

- Three constraints: Two rolling, one no-slip (why not two?)

$$\dot{x}_l \cos\theta + \dot{y}_l \sin\theta = \rho\dot{\psi}_l$$
$$\dot{x}_r \cos\theta + \dot{y}_r \sin\theta = \rho\dot{\psi}_r$$

$$\dot{x}_l \sin\theta - \dot{y}_l \cos\theta = 0$$



- Substitute in kinematics, rewrite in matrix form:

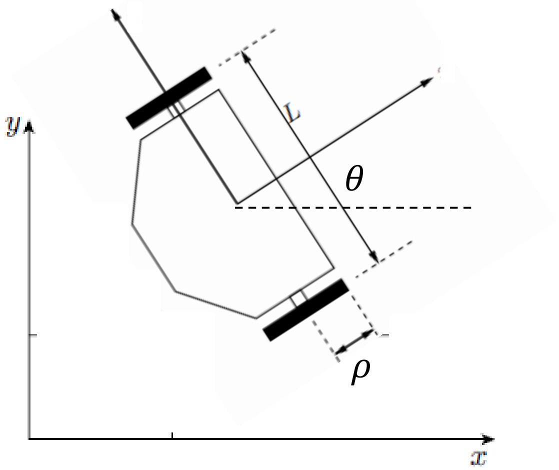
$$\boldsymbol{A}^T(\boldsymbol{q})\dot{\boldsymbol{q}} = \begin{bmatrix} \cos\theta & \sin\theta & -L/2 & -\rho & 0 \\ \cos\theta & \sin\theta & L/2 & 0 & -\rho \\ \sin\theta & -\cos\theta & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi}_l \\ \dot{\psi}_r \end{bmatrix} = 0$$

Differential-Drive Car

- Null space has dimension 2 ($\dim(\boldsymbol{q}) = 5$, minus 3 constraints)

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\psi}_l \\ \dot{\psi}_r \end{bmatrix} = \begin{bmatrix} \rho/2\,(\cos\theta) & \rho/2\,(\cos\theta) \\ \rho/2\,(\sin\theta) & \rho/2\,(\sin\theta) \\ -\rho/L & \rho/L \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \quad \dot{x} = \frac{\rho}{2}\cos\theta\,(\dot{\psi}_l + \dot{\psi}_r) \quad \dot{y} = \frac{\rho}{2}\sin\theta\,(\dot{\psi}_l + \dot{\psi}_r) \quad \dot{\theta} = \frac{\rho}{L}\,(\dot{\psi}_r - \dot{\psi}_l)$$

- $\dot{x}$ and $\dot{y}$ velocities are similar to the unicycle case
  - Here we're *averaging* contributions from each side
  - If $\dot{\psi}_l = \dot{\psi}_r$ then the car moves identically to the unicycle



- Difference in the wheel rates allows car to turn
  - If $\dot{\psi}_l = -\dot{\psi}_r$, then the car turns in place ($\dot{x} = \dot{y} = 0$)

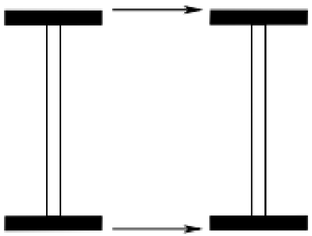
Differential-Drive Car

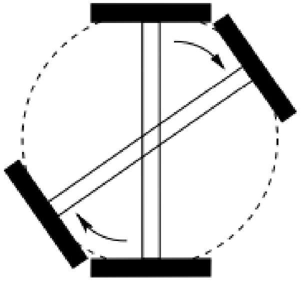
$$\dot{x} = \frac{\rho}{2}\cos\theta\,(\dot{\psi}_l + \dot{\psi}_r)$$

$$\dot{y} = \frac{\rho}{2}\sin\theta\,(\dot{\psi}_l + \dot{\psi}_r)$$

$$\dot{\theta} = \frac{\rho}{L}\,(\dot{\psi}_r - \dot{\psi}_l)$$

- If $\dot{\psi}_l = 0$, the car turns about the left wheel
- If $\dot{\psi}_r = 0$, the car turns about the right wheel
- When $\dot{\psi}_l \neq \dot{\psi}_r \neq 0$, the car both translates and rotates
- $\dot{x}$ and $\dot{y}$ velocities are similar to the unicycle case
  - Here we're *averaging* contributions from each side
  - If $\dot{\psi}_l = \dot{\psi}_r$ then the car moves identically to the unicycle
- Difference in the wheel rates allows car to turn
  - If $\dot{\psi}_l = -\dot{\psi}_r$, then the car turns in place ($\dot{x} = \dot{y} = 0$)





Inverse Kinematics

- For planar robots, possible DOFs in the workspace is always 3: x, y, θ
- But we've seen robots with degree of maneuverability less than 3
- I.e., robots can *achieve* more DOFs than they can control
- Contrast to manipulators, whose holonomic constraints physically limit their workspace

- **Inverse kinematics**: Given workspace velocities, what the required inputs?
- Suppose we specify $\dot{x}(t)$ and $\dot{y}(t)$
- This automatically restricts $\theta(t)$ and, by extension, $\dot{\theta}(t)$

$$\Rightarrow \quad \theta(t) = \mathrm{Atan2}(\dot{y}(t), \dot{x}(t)) \qquad \dot{\theta}(t) = \frac{\ddot{y}(t)\dot{x}(t) - \ddot{x}(t)\dot{y}(t)}{\dot{x}(t)^2 + \dot{y}(t)^2}$$

Inverse Kinematics

- Unicycle:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\psi}\end{bmatrix} = \begin{bmatrix}\rho\cos\theta & 0\\ \rho\sin\theta & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \Rightarrow \begin{aligned} u_1 &= \dot{\psi} = \frac{1}{\rho}\sqrt{\dot{x}^2+\dot{y}^2}\\ u_2 &= \dot{\theta} = \frac{\ddot{y}\dot{x}-\ddot{x}\dot{y}}{\dot{x}^2+\dot{y}^2}\end{aligned}$$

- Bicycle:

$$\begin{bmatrix}\dot{x}\\ \dot{y}\\ \dot{\theta}\\ \dot{\phi}\end{bmatrix} = \begin{bmatrix}\cos\theta & 0\\ \sin\theta & 0\\ (\tan\phi)/l & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix} \Rightarrow \begin{aligned} u_1 &= \sqrt{\dot{x}^2+\dot{y}^2}\\ u_2 &= \dot{\phi} = \frac{l\left(u_1\ddot{\theta}-\dot{u}_1\dot{\theta}\right)}{\dot{x}^2+\dot{y}^2+\left(l\dot{\theta}\right)^2}\end{aligned}$$

- Diff drive car:

$$\begin{aligned}\dot{x} &= \frac{\rho}{2}\cos\theta\,(\dot{\psi}_l+\dot{\psi}_r)\\ \dot{y} &= \frac{\rho}{2}\sin\theta\,(\dot{\psi}_l+\dot{\psi}_r)\\ \dot{\theta} &= \frac{\rho}{L}\,(\dot{\psi}_r-\dot{\psi}_l)\end{aligned} \Rightarrow \begin{aligned}\dot{\psi}_l &= \frac{1}{\rho}(\sqrt{\dot{x}^2+\dot{y}^2}-L\dot{\theta}/2)\\ \dot{\psi}_r &= \frac{1}{\rho}(\sqrt{\dot{x}^2+\dot{y}^2}+L\dot{\theta}/2)\end{aligned}$$

Trajectory Planning

- We can plan trajectories for mobile robots just as we did for manipulators
- Assume no obstacles, boundary conditions specified
- Any trajectory $\boldsymbol{q}(t)$ must satisfy the mobile robot's constraints

- Suppose initial / final configurations are specified: $\boldsymbol{q}_i = (x_i, y_i, \theta_i)^T, \boldsymbol{q}_f = \left(x_f, y_f, \theta_f\right)^T$
- Kinematics requires the velocity conditions: $$\begin{aligned} \dot{x}_i &= k_i \cos\theta_i & \dot{x}_f &= k_f \cos\theta_i \\ \dot{y}_i &= k_i \sin\theta_i & \dot{y}_f &= k_f \sin\theta_i \end{aligned}$$
- k_i and k_f are free parameters
- Unlike with manipulators (holonomic constraints), velocities are pre-determined; workspace is not restricted, but our trajectories are!

Polynomial Interpolation

- Since we have both position and velocity boundary conditions, we can use interpolation techniques to find a trajectory in the workspace

- Ex: Cubic polynomials with $t_i = 0, t_f = 1$

$$x(t) = t^3 x_f - (t-1)^3 x_i + \alpha_x t^2 (t-1) + \beta_x t (t-1)^2$$
$$y(t) = t^3 y_f - (t-1)^3 y_i + \alpha_y t^2 (t-1) + \beta_y t (t-1)^2$$

- Coefficients α and β can be solved when k_i and k_f are chosen
- Ex: $k_i = k_f = k > 0$

$$\alpha_x = k\cos\theta_f - 3x_f \qquad \alpha_y = k\sin\theta_f - 3y_f$$
$$\beta_x = k\cos\theta_i + 3x_i \qquad \beta_y = k\sin\theta_i + 3y_i$$

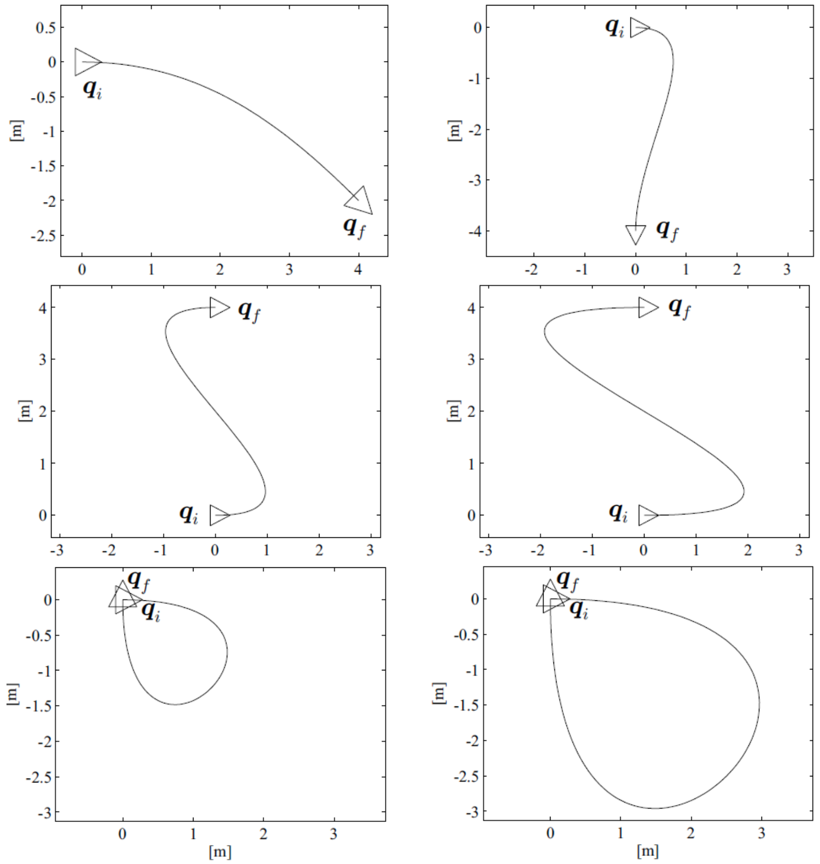
- Once we have $x(t)$ and $y(t)$, plug into inverse kinematics to find robot inputs!

Unicycle Examples

- $k = 5$ (starting forward speed)

- Parallel parking with $k = 10$ and $k = 20$

- Reorientation with $k = 10$ and $k = 20$



Summary

- Wheeled mobile robots are subject to nonholonomic constraints

- Robots may have fewer controllable DOFs (maneuverability) than workspace DOFs
- May be able to achieve arbitrary 2D poses but not with arbitrary trajectories

- Constraints provide the FK models for configuration velocities
- Can solve for IK for simple systems to get expressions for controlled velocities

- Planned trajectories must also satisfy any constraints