COMS W4733: Computational Aspects of Robotics

Homework 2

Solutions

Problem 1 (15 points)

(a) This problem is equivalent to a RRP arm, where a prismatic joint at the end effector of the original RR arm models the laser coming out of it. In general there are infinitely many solutions if we only specify the desired position of the point p, since there are three joint variables but only two specified coordinates. The relative link lengths do not affect this; the position workspace of the RRP equivalent is the entire plane with no holes (unlike the annulus workspaces of the RR arm with unequal link lengths).

(b) The forward kinematics of the RRP equivalent arm are similar to those of the original RR arm:

$$\begin{aligned} x &= l_1 c_1 + (l_2 + d_3) c_{12} \\ y &= l_1 s_1 + (l_2 + d_3) s_{12} \\ \phi &= \theta_1 + \theta_2 \end{aligned}$$

Substituting in $x = p_x$, $y = p_y$, and ϕ into the first two equations,

$$\begin{aligned} p_x &= l_1 c_1 + (l_2 + d_3) c_\phi \\ p_y &= l_1 s_1 + (l_2 + d_3) s_\phi \end{aligned}$$

We can square the two equations and add them, which would eliminate θ_1:

$$l_1^2 = (p_x - (l_2 + d_3)c_\phi)^2 + (p_y - (l_2 + d_3)s_\phi)^2$$

This is a quadratic equation that would produce zero, one, or two solutions for d_3. With this in hand, finding θ_1 is a straightforward application of Atan2:

$$\theta_1 = \text{Atan2}(p_y - (l_2 + d_3)s_\phi, p_x - (l_2 + d_3)c_\phi)$$

And of course we can then find $\theta_2 = \phi - \theta_1$.

(c) No solutions exist when the quadratic equation above has no solutions for d_3. While we can use the algebraic equation to get a feel for what this means, we can also think about this geometrically. Since the total orientation ϕ is specified, the second link of the RR arm must lie along the line that passes through p at the angle ϕ. This means that the position of joint 2 must lie somewhere on this line. The position of joint 2 is given by the forward kinematics of the first joint, whose workspace is just a circle (not a disk!) of radius l_1 centered around the robot's base. So as long as the line intersects this circle somewhere, either in one or two places, there will be a solution. Otherwise, no solution exists.

Problem 2 (15 points)

(a) The linear velocity Jacobian can be found by taking partial derivatives of the FK equations.

$$\mathbf{J}_P = \begin{pmatrix} -(L_1 + L_2c_2 + L_3c_{23})s_1 & -(L_2s_2 + L_3s_{23})c_1 & -L_3s_{23}c_1 \\ (L_1 + L_2c_2 + L_3c_{23})c_1 & -(L_2s_2 + L_3s_{23})s_1 & -L_3s_{23}s_1 \\ 0 & L_2c_2 + L_3c_{23} & L_3c_{23} \end{pmatrix}$$

The angular velocity Jacobian can be built column by column, each of which is the z axis of the corresponding joint. The first revolute joint has its z_0^0 axis as $(0,0,1)^T$, since the axis never rotates. The z axes of the other two joints can be found from the intermediate DH transforms between the joints. From θ_1 to θ_2 there is a θ_1 rotation about z_0^0, a rotation of 90 degrees about x_1^0, and a translation of L_1 along x_1^0. Plugging the parameters into the third column of the transformation A_1^0 gives us $(s_1, -c_1, 0)^T$ for z_1^0. We can do the same thing to find z_2^0, but we don't even have to do that—note that z_2^0 and z_1^0 are always parallel. So the angular velocity Jacobian is

$$\mathbf{J}_P = \begin{pmatrix} 0 & s_1 & s_1 \\ 0 & -c_1 & -c_1 \\ 1 & 0 & 0 \end{pmatrix}$$

The full Jacobian is then these two matrices stacked together.

(b) The simplified form of the determinant is

$$\det(\mathbf{J}_P) = -L_2L_3(L_1 + L_2c_2 + L_3c_{23})s_3$$

Singularities occur when either $\theta_3 = 0$ or π, or when $L_1 + L_2c_2 + L_3c_{23} = 0$.

(c) When θ_3 is either 0 or π, the last two links of the arm are stretched out; this is exactly what happens at an elbow singularity. The second condition leads to a shoulder singularity, because the end effector ends up being located on the first joint's rotation axis. If that happens, then rotating the first joint produces no net end effector movement. See the lecture slides on "Arm Singularities" from Lecture 5 for illustrations of these examples.

Problem 3 (25 points)

(a) First, since the manipulator is planar, there is no motion in the z direction of the base frame, which comes out of the plane. So that row is all zeros. For the angular velocities, the first and third joints are revolute with parallel z axes, again coming out of the plane. The second joint is prismatic. So the angular velocity Jacobian is

$$\mathbf{J}_O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

The first two rows of the linear velocity Jacobian can be found by differentiation of the FK equations.

$$\mathbf{J}_P = \begin{pmatrix} -(l_1 + d_2)s_1 - l_2c_1 - l_3s_{13} & c_1 & -l_3s_{13} \\ (l_1 + d_2)c_1 - l_2s_1 + l_3c_{13} & s_1 & l_3c_{13} \\ 0 & 0 & 0 \end{pmatrix}$$

(b) We're interested in computing

$$\det \begin{pmatrix} -(l_1+d_2)s_1 - l_2c_1 - l_3s_{13} & c_1 & -l_3s_{13} \\ (l_1+d_2)c_1 - l_2s_1 + l_3c_{13} & s_1 & l_3c_{13} \\ 1 & 0 & 1 \end{pmatrix}.$$

One way to do so is to sum the determinants of the upper-left 2×2 matrix and the upper-right 2×2 matrix (each scaled by the 1s in the bottom row). The simplified expression is $l_1 + d_2$, which is obviously 0 when $d_2 = -l_1$. This occurs if the prismatic joint slides backward and causes the two revolute joints to lie on the same axis.

(c) This is underconstrained, since we have only two specifications but three DOFs. We therefore use the right pseudoinverse, which produces a solution that minimizes the resultant joint velocities.

$$\mathbf{J} = \begin{pmatrix} -4.598 & 0.866 & -1.732 \\ 3.9641 & 0.5 & 1 \end{pmatrix}$$

$$\dot{\mathbf{q}}^* = \mathbf{J}^+ \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \mathbf{J}^T(\mathbf{J}\mathbf{J}^T)^{-1} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.448 \\ 0.836 \\ -0.195 \end{pmatrix}$$

(d) The homogeneous solution is given by $(\mathbf{I} - \mathbf{J}^+\mathbf{J})\dot{\mathbf{q}}_0$:

$$\dot{\mathbf{q}} = \dot{\mathbf{q}}^* + \begin{pmatrix} 0.0732 & -0.0958 & -0.242 \\ -0.0958 & 0.126 & 0.317 \\ -0.242 & 0.317 & 0.801 \end{pmatrix} \dot{\mathbf{q}}_0$$

(e) The problem is now overconstrained. We have to use the left pseudoinverse, which minimizes the error between the desired and actual end effector velocity.

$$\mathbf{J} = \begin{pmatrix} \mathbf{J}_P \\ \mathbf{J}_O \end{pmatrix}$$

$$\dot{\mathbf{q}} = \mathbf{J}^+(-1, 2, 1, -3, 0, -2)^T = (\mathbf{J}^T\mathbf{J})^{-1}\mathbf{J}^T(-1, 2, 1, -3, 0, -2)^T = \begin{pmatrix} 1.424 \\ -0.442 \\ -3.424 \end{pmatrix}$$

(f) The manipulator is unable to achieve the velocity components $\dot{z}$, ω_x, and ω_y, since it cannot move in those directions at all. The actual velocities achieved are therefore

$$\mathbf{v}_e = \mathbf{J}\dot{\mathbf{q}} = (-1, 2, 0, 0, 0, -2)^T.$$

Problem 4 (15 points)

(a) This problem is almost the same as the standard symmetric LSPD problem, except that the final velocity is not 0, so it is not quite symmetric. The provided t_f is not the final time in the symmetric case, since we want a velocity of 1 rad/s at t_f. The acceleration magnitude is 2 rad/s^2, which means the velocity reaches 0 rad/s at $t_f' = 2.5$ s if it decelerates from 1 rad/s at $t_f = 2$ s.

The second quantity we need is the configuration that the joint will end up at if it continued moving until 2.5 s. To do this we match the full form of the second parabolic segment, given by $q(t) = q_f' - \frac{1}{2}\ddot{q}_c(t_f' - t)^2$, against the boundary condition $q(2) = 2$. This gives us $q_f' = 2.25$ rad/s. This gives us all the information we need to derive the full trajectory. The blend time is given by

$$t_c = \frac{t_f'}{2} - \frac{1}{2}\sqrt{\frac{(t_f')^2\ddot{q}_c - 4(q_f' - q_i)}{\ddot{q}_c}} = 0.589.$$

We then have

$$q(t) = \begin{cases} t^2, & 0 \le t \le 0.589 \\ 1.178(t - 0.295), & 0.589 < t \le 1.911 \\ 2.25 - (2.5 - t)^2, & 1.911 < t \le 2.5 \end{cases}$$

(b) Since the parabolic acceleration $\ddot{q}_c$ is not specified, the symmetric final time t_f' also remains an unknown. We can use the parabolic equation $q(t) = q_f' - \frac{1}{2}\ddot{q}_c(t_f' - t)^2$ with the two boundary conditions of 2 rad and 1 rad/s at $t_f = 2$ and obtain the following equations:

$$\begin{aligned} 2 &= q_f' - \frac{1}{2}\ddot{q}_c(t_f' - 2)^2 \\ 1 &= \ddot{q}_c(t_f' - 2) \end{aligned}$$

We also have the relationship between cruise velocity and acceleration:

$$\ddot{q}_c = \frac{\dot{q}_c^2}{q_i - q_f' + \dot{q}_c t_f'} = \frac{1.5^2}{-q_f' + 1.5t_f'}$$

Solving the three equations above simultaneously gives us $t_f' = 2.8$, $q_f' = 2.4$, and $\ddot{q}_c = 1.25$. Plugging into the expressions for t_c and the overall trajectory, we obtain

$$t_c = \frac{t_f'}{2} - \frac{1}{2}\sqrt{\frac{(t_f')^2\ddot{q}_c - 4(q_f' - q_i)}{\ddot{q}_c}} = 1.2$$

$$q(t) = \begin{cases} 0.625t^2, & 0 \le t \le 1.2 \\ 1.5(t - 0.6), & 1.2 < t \le 1.6 \\ 2.4 - 0.625(2.8 - t)^2, & 1.6 < t \le 2.8 \end{cases}$$