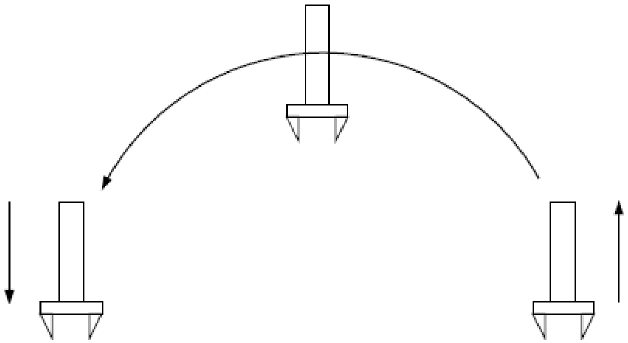
COMS W4733: Computational Aspects of Robotics

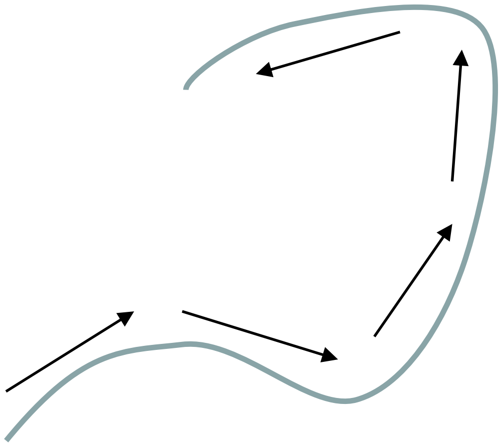
Lecture 9: Trajectory Planning



Instructor: Tony Dear

Trajectory Planning

- So far we've described robot motions in terms of robot velocities and moving joints from one configuration to another configuration nearby ($\Delta\boldsymbol{q}$)
- How do we move between points that are not so close together?

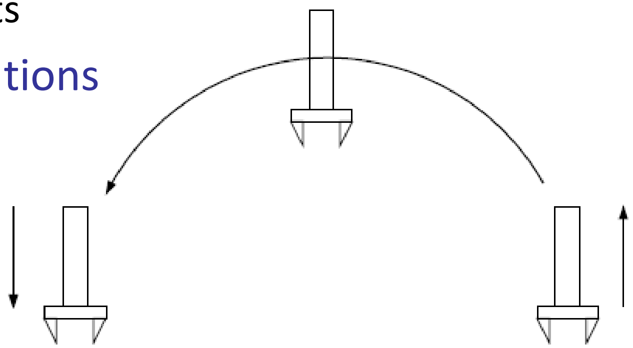


- **Path**: Collection of points that a manipulator follows in sequence
  - $x = \{x_1, x_2, \dots, x_n\}$
- **Trajectory**: Path with a timing law (including velocity, acceleration)
  - $x(t)$ such that $x(t_i) = x_i$ and $x(t_f) = x_f$
- Both can be in either joint space or operational space

- *Trajectory planning*: Given path description and constraints, output a time sequence of positions, velocities, and accelerations

Joint Space Trajectories

- Trajectories in operational space $\boldsymbol{x}_e(t)$ easier to describe, harder to compute
- Often easier to do the following:
  - Specify requirements in operational space (initial, final, intermediate poses)
  - Solve IK to find joint configurations for those requirements
- Find a joint space trajectory $\boldsymbol{q}(t)$ connecting configurations
- Positions and velocities should be continuous
- May need to interpolate between multiple points



Cubic Polynomials

- Suppose we specify initial and final position and velocity conditions (four constraints)
  - $q(t_i) = q_i, \dot{q}(t_i) = \dot{q}_i, q(t_f) = q_f, \dot{q}_f(t_f) = \dot{q}_f$

- Since we have four constraints, we need at least four free parameters
- Cubic polynomial has the form $q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$$q_i = a_0 + a_1 t_i + a_2 t_i^2 + a_3 t_i^3$$
$$\dot{q}_i = a_1 + 2a_2 t_i + 3a_3 t_i^2$$
$$q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$$
$$\dot{q}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2$$

Solve for unknowns a_0, a_1, a_2, a_3

$$\begin{bmatrix} q_i \\ \dot{q}_i \\ q_f \\ \dot{q}_f \end{bmatrix} = \begin{bmatrix} 1 & t_i & t_i^2 & t_i^3 \\ 0 & 1 & 2t_i & 3t_i^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

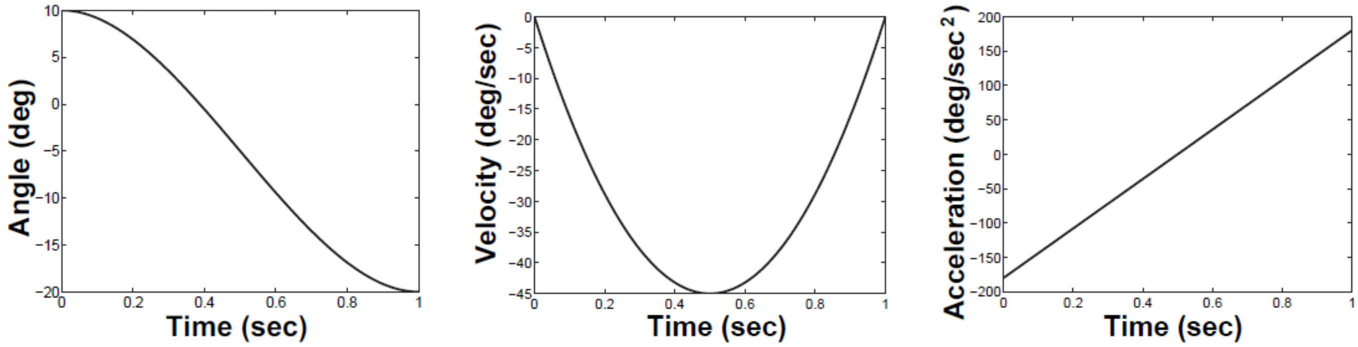
Example: Cubic Polynomial

- Example: $t_i = 0, t_f = 1$

$$\begin{bmatrix} q_i \\ \dot{q}_i \\ q_f \\ \dot{q}_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$q(t) = q_i + \dot{q}_i t + \left(-3q_i - 2\dot{q}_i + 3q_f - \dot{q}_f\right)t^2 + \left(2q_i + \dot{q}_i - 2q_f + \dot{q}_f\right)t^3$$

- $q_i = 10, q_f = -20, \dot{q}_i = \dot{q}_f = 0$



Quintic Polynomials

- Problem: Unspecified accelerations are discontinuous (neither start nor end at 0)
- Jerky motions, may lead to unwanted vibrations and reduced accuracy

- If we specify initial and final accelerations as well, we need six free parameters

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$

$$\begin{bmatrix} q_i \\ \dot{q}_i \\ \ddot{q}_i \\ q_f \\ \dot{q}_f \\ \ddot{q}_f \end{bmatrix} = \begin{bmatrix} 1 & t_i & t_i^2 & t_i^3 & t_i^4 & t_i^5 \\ 0 & 1 & 2t_i & 3t_i^2 & 4t_i^3 & 5t_i^4 \\ 0 & 0 & 2 & 6t_i & 12t_i^2 & 20t_i^3 \\ 1 & t_f & t_f^2 & t_f^3 & t_f^4 & t_f^5 \\ 0 & 1 & 2t_f & 3t_f^2 & 4t_f^3 & 5t_f^4 \\ 0 & 0 & 2 & 6t_f & 12t_f^2 & 20t_f^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

Example: $t_i = 0, t_f = 1$

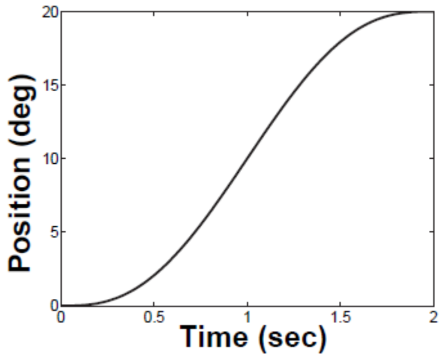
$$\begin{bmatrix} q_i \\ \dot{q}_i \\ \ddot{q}_i \\ q_f \\ \dot{q}_f \\ \ddot{q}_f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 6 & 12 & 20 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

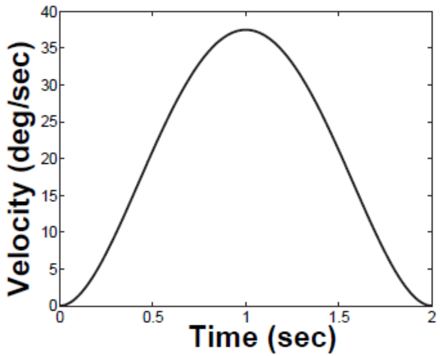
Example: Quintic Polynomial

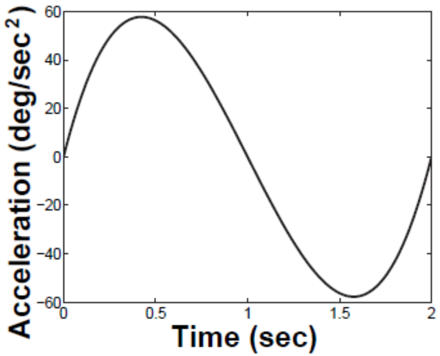
- Problem: Unspecified accelerations are discontinuous (neither start nor end at 0)
- Jerky motions, may lead to unwanted vibrations and reduced accuracy

- If we specify initial and final accelerations as well, we need six free parameters

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$





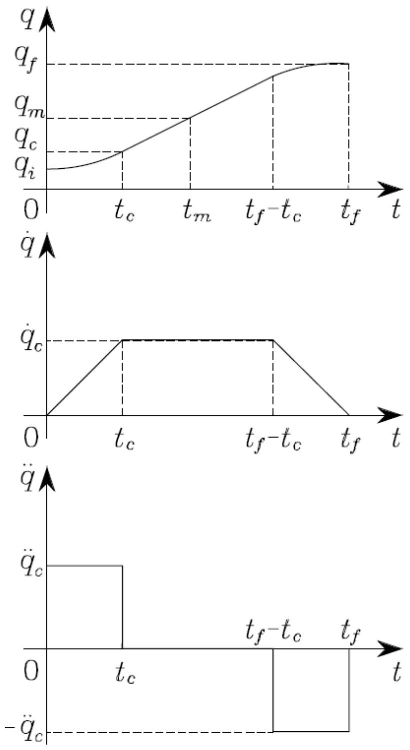


Linear Segment with Parabolic Blends

- Polynomial trajectories result in non-constant velocities
- Manipulator always accelerating, requiring constant force
- May not be ideal for long trajectories

- Another idea: Maintain constant velocity part of the time
- “Ramp up” to desired velocity, then “ramp down” when done

- Trapezoidal velocity profile
- Trajectory assumed to be symmetric from start to end
- Ramp segments are quadratic in position, linear in velocity, and constant in acceleration



LSPB: First Parabolic Segment

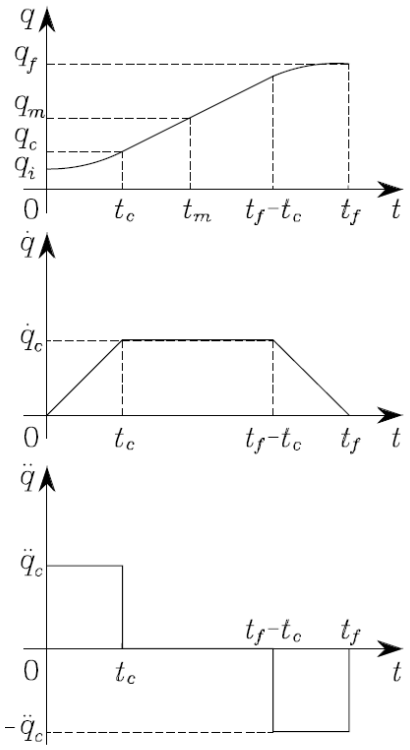
- Linear segment starts at *blend time* t_c, ends at $t_f - t_c$
- Initial and final configurations q_i, q_f and time t_f are specified
- Initial and final velocities $\dot{q}_i = \dot{q}_f = 0$
- Usually either acceleration $\ddot{q}_c$ or *cruise velocity* $\dot{q}_c$ is specified as well

- First parabola has the form $q_{p1}(t) = a_0 + a_1 t + a_2 t^2$

- Initial conditions at $t = 0$: $q_{p1}(0) = q_i = a_0 \quad \dot{q}_{p1}(0) = 0 = a_1$

- Match velocity at t_c: $\dot{q}_{p1}(t_c) = \dot{q}_c = 2a_2 t_c$

- Match acceleration: $\ddot{q}_{p1} = 2a_2 = \ddot{q}_c$



LSPB: Blend Time

- $q_{p1}(t) = q_i + a_2 t^2$ $\quad \dot{q}_c = 2a_2 t_c$, $\ddot{q}_c = 2a_2$ $\Rightarrow$ $\boxed{\dot{q}_c = \ddot{q}_c t_c}$

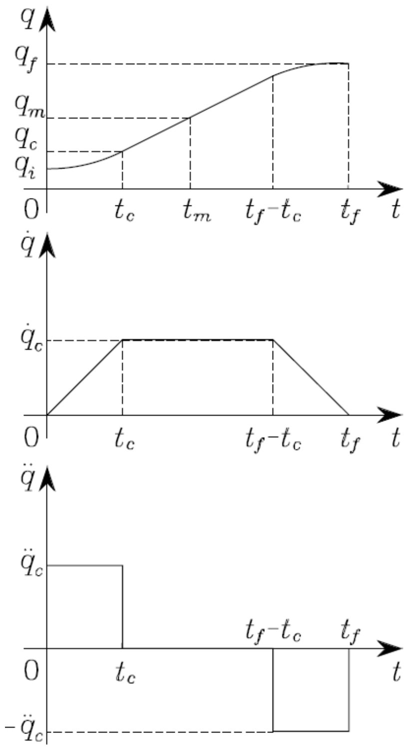
- How does blend time t_c relate to $\dot{q}_c$ and $\ddot{q}_c$?

$$q_m = (q_i + q_f)/2$$
$$t_m = t_f/2$$

- Configuration at t_c: $\quad q_{p1}(t_c) = q_c = q_i + a_2 t_c^2$
- Velocity of linear segment:

$$\dot{q}_c = \frac{q_m - q_c}{t_m - t_c} = \frac{(q_i + q_f)/2 - (q_i + \ddot{q}_c t_c^2/2)}{t_f/2 - t_c} = \frac{(q_i + q_f)/2 - (q_i + \dot{q}_c t_c/2)}{t_f/2 - t_c}$$

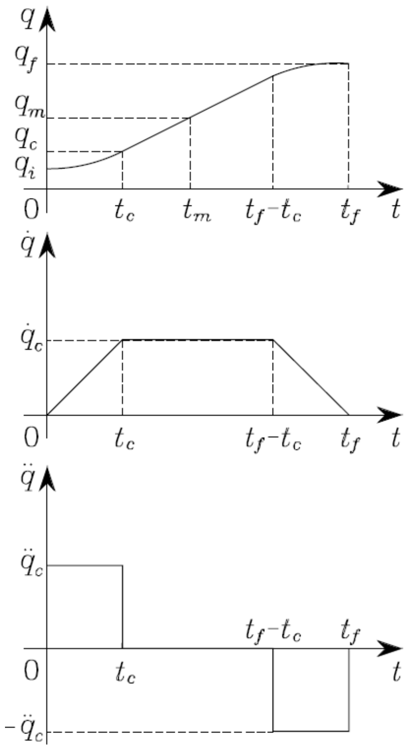
$$\boxed{t_c = \frac{q_i - q_f + \dot{q}_c t_f}{\dot{q}_c} = \frac{t_f}{2} - \frac{1}{2}\sqrt{(t_f^2 \ddot{q}_c - 4(q_f - q_i))/\ddot{q}_c}}$$



LSPB: Linear Segment

$$\boxed{\dot{q}_c = \ddot{q}_c t_c}$$

$$\boxed{t_c = \frac{q_i - q_f + \dot{q}_c t_f}{\dot{q}_c} = \frac{t_f}{2} - \frac{1}{2}\sqrt{(t_f^2 \ddot{q}_c - 4(q_f - q_i))/\ddot{q}_c}}$$



- We have a solution for the first parabolic segment: $q_{p1}(t) = q_i + \frac{1}{2}\ddot{q}_c t^2$

- Linear segment is $q_l(t) = b_0 + b_1 t = b_0 + \dot{q}_c t = b_0 + \ddot{q}_c t_c t$
- Configuration at t_c: $q_l(t_c) = q_c = b_0 + \ddot{q}_c t_c^2$

- From the parabola, $q_{p1}(t_c) = q_c = q_i + \ddot{q}_c t_c^2/2$

- Linear segment solution: $q_l(t) = q_i + \ddot{q}_c t_c (t - t_c/2)$

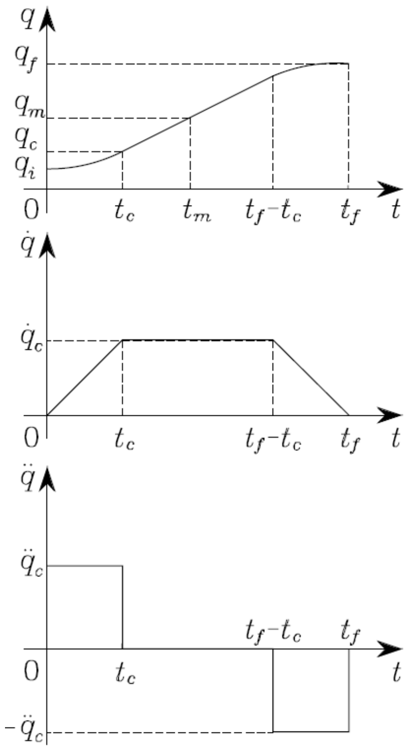
LSPB: Second Parabolic Segment

$$\boxed{\dot{q}_c = \ddot{q}_c t_c}$$

$$\boxed{t_c = \frac{q_i - q_f + \dot{q}_c t_f}{\dot{q}_c} = \frac{t_f}{2} - \frac{1}{2}\sqrt{(t_f^2 \ddot{q}_c - 4(q_f - q_i))/\ddot{q}_c}}$$

$$q_l(t) = q_i + \ddot{q}_c t_c (t - t_c/2)$$

- Second parabolic segment: $q_{p2}(t) = c_0 + c_1 t + c_2 t^2$
- Match acceleration: $\ddot{q}_{p2} = -\ddot{q}_c = 2c_2$
- Match velocity at t_f: $\dot{q}_{p2}(t_f) = 0 = c_1 - \ddot{q}_c t_f$
- Final configuration at t_f: $q_{p2}(t_f) = q_f = c_0 + \ddot{q}_c t_f^2 - \frac{1}{2}\ddot{q}_c t_f^2$
- Second parabolic segment solution: $q_{p2}(t) = q_f - \frac{1}{2}\ddot{q}_c (t_f - t)^2$



LSPB: Summary

$$\boxed{\dot{q}_c = \ddot{q}_c t_c}$$

$$\boxed{t_c = \frac{q_i - q_f + \dot{q}_c t_f}{\dot{q}_c} = \frac{t_f}{2} - \frac{1}{2}\sqrt{(t_f^2 \ddot{q}_c - 4(q_f - q_i))/\ddot{q}_c}}$$

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q}_c t^2 & 0 \le t \le t_c \\ q_i + \ddot{q}_c t_c (t - t_c/2) & t_c < t \le t_f - t_c \\ q_f - \frac{1}{2}\ddot{q}_c (t_f - t)^2 & t_f - t_c < t \le t_f \end{cases}$$

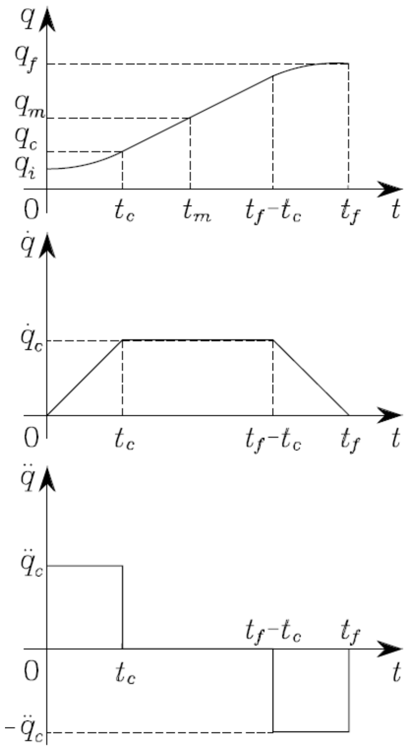
- Initial/final conditions plus either $\dot{q}_c$ or $\ddot{q}_c$ produce a unique solution
- For solution to exist, $\dot{q}_c$ and $\ddot{q}_c$ must satisfy

$$\ddot{q}_c \ge \frac{4|q_f - q_i|}{t_f^2}$$

Parabolas must accelerate and decelerate fast enough

$$\frac{|q_f - q_i|}{t_f} \le |\dot{q}_c| \le \frac{2|q_f - q_i|}{t_f}$$

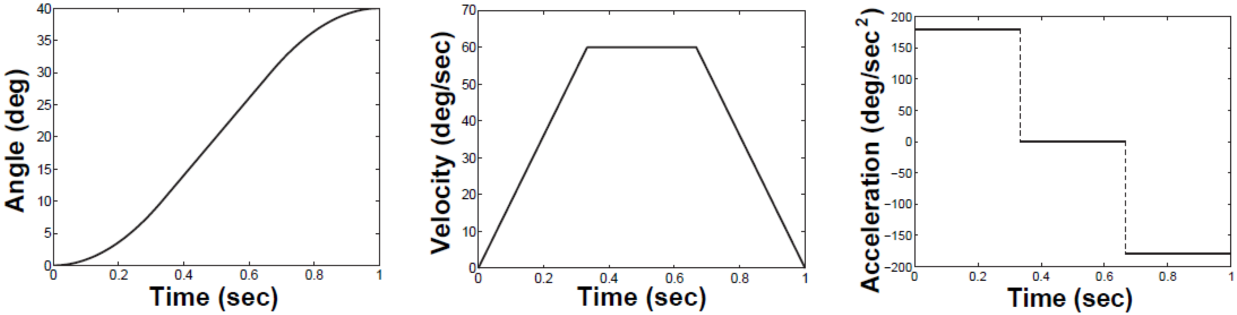
Linear segment cannot be too steep or too shallow



Example: LSPB

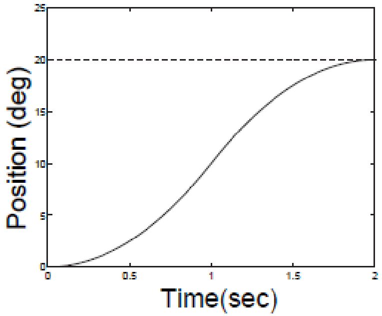
- $t_0 = 0, t_f = 1, q_i = 0, q_f = 40, \dot{q}_c = 60$

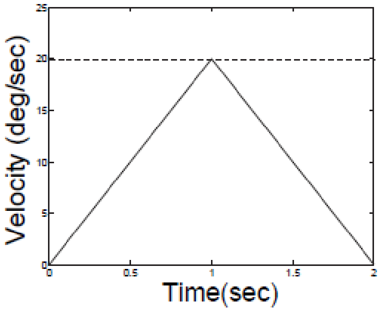
$$t_c = \frac{1}{3}, \qquad q(t) = \begin{cases} 90t^2 & 0 \le t \le 1/3 \\ 60(t - 1/6) & 1/3 < t \le 2/3 \\ 40 - 90(1-t)^2 & 2/3 < t \le 1 \end{cases}$$

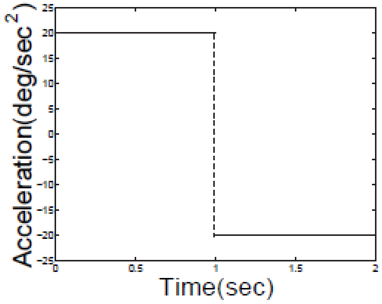


Minimum-Time Trajectories

- Suppose that we want to reach q_f as quickly as possible given acceleration $\ddot{q}_c$
- *Bang-bang trajectory* minimizes t_f and has blend time $t_c = t_f/2$







- Final time is determined by acceleration:

$$t_c = \frac{t_f}{2} - \frac{1}{2}\sqrt{(t_f^2\ddot{q}_c - 4(q_f - q_i))/\ddot{q}_c} = \frac{t_f}{2} - 0$$

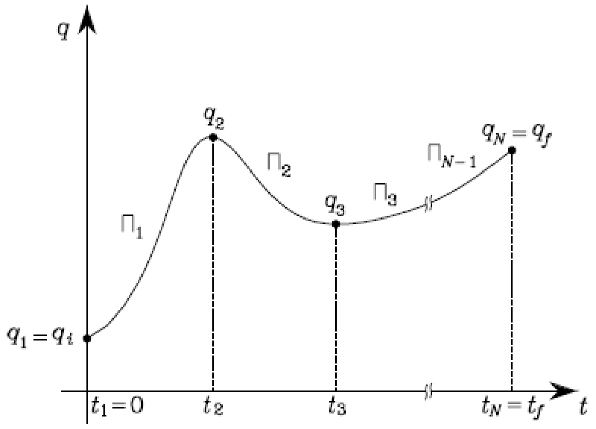
$$t_f = 2\sqrt{|q_f - q_i|/\ddot{q}_c}$$

$$q(t) = \begin{cases} q_i + \frac{1}{2}\ddot{q}_c t^2 & 0 \le t \le t_f/2 \\ q_f - \frac{1}{2}\ddot{q}_c (t_f - t)^2 & t_f/2 < t \le t_f \end{cases}$$

Sequences of Points

- It is often desirable to specify more than two joint configurations for finer control
- For example, higher density of points near obstacles or for tighter turns

- Possible to generate a $n-1$ order polynomial to fit n points, but
  - Computationally expensive as number of points increases
  - Resultant polynomial becomes more and more oscillatory
  - Not robust; changing even one point requires re-solving everything

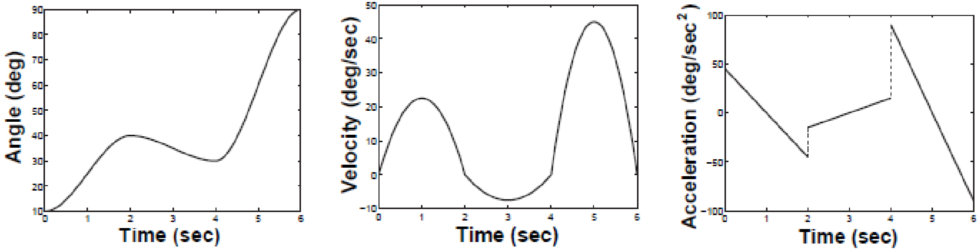


- Better to locally interpolate low-order polynomials
- Place polynomials or LSPBs between pairs of points
- Specify matching conditions for velocities and accelerations

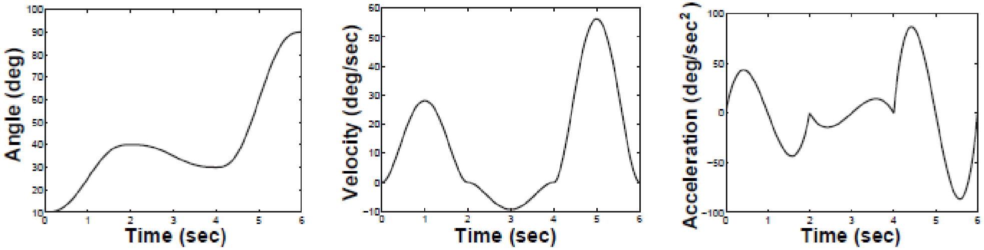
Example: Cubic and Quintic Splines

- 4 points: $q(0) = 10, \dot{q}(0) = 0; q(2) = 40, \dot{q}(2) = 0; q(4) = 30, \dot{q}(4) = 0; q(6) = 90, \dot{q}(6) = 0$

- Cubic spline solution matches all velocity conditions, but accelerations are discontinuous:



- Quintic spline solution allows us to also specify $\ddot{q}(0) = \ddot{q}(2) = \ddot{q}(4) = \ddot{q}(6) = 0$



Summary

- Trajectory planning is the process of generating a time-valued continuous function of configurations that a robot can follow
- Cubic and quintic polynomials allow for specification of velocities and accelerations
- Combining linear segments with parabolic segments allow for constant-velocity trajectories for part of the trajectory
- Sequences of points can be interpolated locally