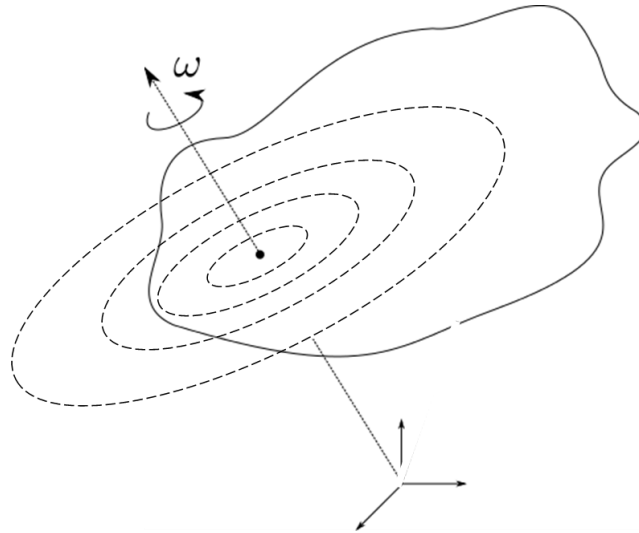


COMS W4733: Computational Aspects of Robotics

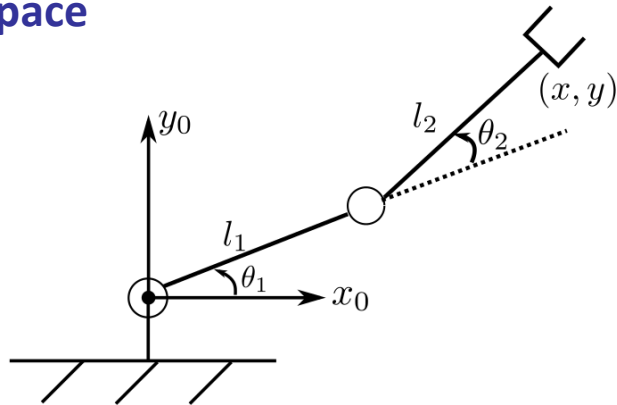
Lecture 5: Differential Kinematics



Instructor: Tony Dear

Review: Kinematics So Far

- Joint variables $\mathbf{q} = (q_1, \dots, q_n)^T \in \text{joint/configuration space}$
 - Ex: $\mathbf{q} = (\theta_1, \theta_2)^T$ for the RR arm
- End effector pose $\mathbf{x}_e \in \text{operational space}$
 - $\mathbf{x}_e = (x, y, \phi)^T$ in 2D space
 - $\mathbf{x}_e = \{(x, y, z)^T, \mathbf{R}_{3 \times 3}\}$ in 3D space
- Forward kinematics: $\mathbf{x}_e = \mathbf{k}(\mathbf{q})$
 - I move the joints; what is the pose of the end effector?
- Inverse kinematics: $\mathbf{q} = \mathbf{k}^{-1}(\mathbf{x}_e)$
 - How do I control the joints to reach a desired pose?



Frame Velocities

- Our manipulators are not staying still—they will generally be moving around
- How are joint velocities $\dot{\mathbf{q}}$ and operational velocities $\dot{\mathbf{x}}_e$ related?
- We'll show that $\dot{\mathbf{x}}_e$ is a sum of velocity contributions from individual joints
- From forward kinematics:
$$\mathbf{T}_e(\mathbf{q}) = \begin{bmatrix} \mathbf{R}_e(\mathbf{q}) & \mathbf{p}_e(\mathbf{q}) \\ \mathbf{0}^T & 1 \end{bmatrix}$$
- If we take the time derivative of $\mathbf{p}_e(\mathbf{q})$, we obtain end effector's *linear velocity*
- What about $\mathbf{R}_e(\mathbf{q})$?

Derivative of a Rotation Matrix

- $\mathbf{R}(t)$ is orthogonal:

$$\mathbf{R}(t)\mathbf{R}^T(t) = \mathbf{I}$$

- Differentiate both sides wrt time:

$$\dot{\mathbf{R}}(t)\mathbf{R}^T(t) + \mathbf{R}(t)\dot{\mathbf{R}}^T(t) = \mathbf{0}$$

- Define $\mathbf{S}(t) = \dot{\mathbf{R}}(t)\mathbf{R}^T(t)$:

$$\mathbf{S}(t) + \mathbf{S}^T(t) = \mathbf{0}$$

- $\mathbf{S}(t)$ is skew-symmetric:

$$\mathbf{S}(t) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$

- Post-multiply $\mathbf{S}(t)$ by $\mathbf{R}(t)$:

$$\mathbf{S}(t)\mathbf{R}(t) = \dot{\mathbf{R}}(t)$$

- Cross product property:

- (arbitrary vector \mathbf{p})

$$\dot{\mathbf{R}}(t)\mathbf{p} = \boldsymbol{\omega}(t) \times \mathbf{R}(t)\mathbf{p} \quad \boldsymbol{\omega}(t) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Angular Velocity

- $\dot{R}(t)p = S(t)R(t)p = \omega(t) \times R(t)p$

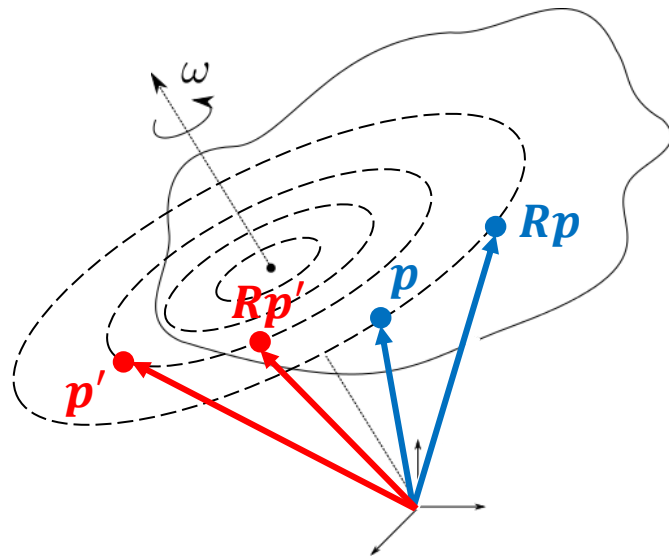
- $S(t) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \omega(t) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$

- $\dot{p} = \frac{d}{dt}(R(t)p) = \dot{R}(t)p$ is the linear velocity of $R(t)p$

- From mechanics, $\omega(t)$ is the *angular velocity* of $R(t)p$

- Since p is arbitrary, we can associate ω with the rotation specified by R

- Compute $\dot{R}(t) = S(\omega(t))R(t)$ and extract the elements of S



Angular Velocity Jacobian

- Going back to general form of forward kinematics:

$$T_e(q) = \begin{bmatrix} R_e(q) & p_e(q) \\ \mathbf{0}^T & 1 \end{bmatrix}$$

- Can show that $\dot{R}_e(q) = S(\omega_e)R_e(q)$ leads to

$$\omega_e = \sum_{i=1}^n \omega_{e,i}^0 = \sum_{i=1}^n \rho_i \dot{q}_i z_{i-1}^0$$

\uparrow
 Angular velocity contribution of i th joint in frame 0

$\rho_i = \mathbf{I}$ if joint i is revolute
 $\rho_i = \mathbf{0}$ if joint i is prismatic (no angular velocity from these)

- z_{i-1}^0 is the unit vector describing z axis of frame $i - 1$ relative to frame 0
- We can factor the above and write the *angular velocity Jacobian* $J_o(q)$:

$$\omega_e = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = (\rho_1 z_0^0 \quad \cdots \quad \rho_n z_{n-1}^0) \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = J_o(q) \dot{q}$$

$3 \times n$ matrix relating joint velocities \dot{q}
 to end effector angular velocities ω_e

Linear Velocity Jacobian

- Linear velocity from position forward kinematics: $\dot{\mathbf{p}}_e(\mathbf{q}) = (\dot{x}, \dot{y}, \dot{z})^T$

$$\dot{\mathbf{p}}_e(\mathbf{q}) = \frac{d}{dt} \begin{pmatrix} p_{e,x}(\mathbf{q}) \\ p_{e,y}(\mathbf{q}) \\ p_{e,z}(\mathbf{q}) \end{pmatrix} = \begin{pmatrix} \frac{\partial p_{e,x}}{\partial q_1} \dot{q}_1 + \frac{\partial p_{e,x}}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial p_{e,x}}{\partial q_n} \dot{q}_n \\ \frac{\partial p_{e,y}}{\partial q_1} \dot{q}_1 + \frac{\partial p_{e,y}}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial p_{e,y}}{\partial q_n} \dot{q}_n \\ \frac{\partial p_{e,z}}{\partial q_1} \dot{q}_1 + \frac{\partial p_{e,z}}{\partial q_2} \dot{q}_2 + \dots + \frac{\partial p_{e,z}}{\partial q_n} \dot{q}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial p_{e,x}}{\partial q_1} & \dots & \frac{\partial p_{e,x}}{\partial q_n} \\ \frac{\partial p_{e,y}}{\partial q_1} & \dots & \frac{\partial p_{e,y}}{\partial q_n} \\ \frac{\partial p_{e,z}}{\partial q_1} & \dots & \frac{\partial p_{e,z}}{\partial q_n} \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \mathbf{J}_P(\mathbf{q}) \dot{\mathbf{q}}$$

- $\mathbf{J}_P(\mathbf{q})$: *linear velocity Jacobian* relating joint velocities $\dot{\mathbf{q}}$ to end effector velocities $\dot{\mathbf{p}}_e$
 - $3 \times n$ matrix in 3D ($2 \times n$ in planar case)
- Linear* mapping, dependent on joint configuration \mathbf{q}

Linear Velocity Jacobian

- Linear velocity from position forward kinematics: $\dot{\mathbf{p}}_e(\mathbf{q}) = (\dot{x}, \dot{y}, \dot{z})^T$
- As with \mathbf{J}_O , we can also break down \mathbf{J}_P as sum of joint contributions

$$\dot{\mathbf{p}}_e = \sum_{i=1}^n \dot{\mathbf{p}}_{e,i}^0$$

- Linear velocity contribution from prismatic joint:

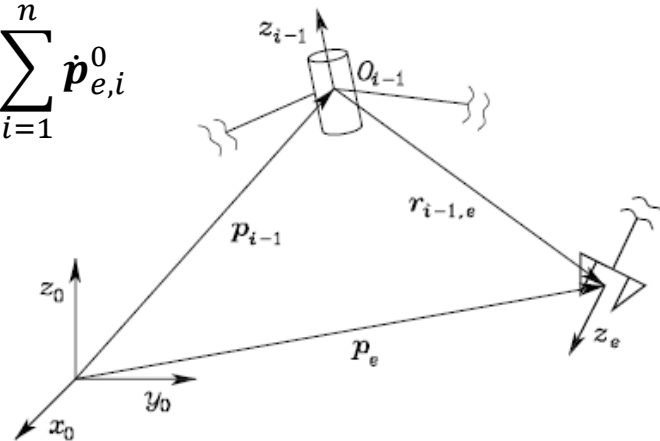
$$\dot{\mathbf{p}}_{e,i}^0 = \dot{d}_i \mathbf{z}_{i-1}^0 \quad [J_{Pi}] = [\mathbf{z}_{i-1}^0]$$

- Linear velocity contribution from revolute joint:

- Recall that $\dot{\mathbf{p}} = \boldsymbol{\omega} \times \mathbf{r}$

$$\dot{\mathbf{p}}_{e,i}^0 = \dot{\theta}_i \mathbf{z}_{i-1}^0 \times (\mathbf{p}_e - \mathbf{p}_{i-1}) \quad [J_{Pi}] = [\mathbf{z}_{i-1}^0 \times (\mathbf{p}_e - \mathbf{p}_{i-1})]$$

\uparrow
 Vector from revolute joint to end effector



The Full Jacobian

- Together, the linear and angular velocity Jacobians fully describe the mapping between joint velocities $\dot{\mathbf{q}}$ and end effector velocities $\mathbf{v}_e = (\dot{\mathbf{p}}_e, \boldsymbol{\omega}_e)^T$:

$$\mathbf{v}_e = \begin{pmatrix} \dot{\mathbf{p}}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \begin{pmatrix} J_P(\mathbf{q}) \\ J_O(\mathbf{q}) \end{pmatrix} \dot{\mathbf{q}} = \mathbf{J}(\mathbf{q}) \dot{\mathbf{q}}$$

- For general 3D manipulators, J_P is $3 \times n$, J_O is $3 \times n$, and \mathbf{J} is $6 \times n$
- For planar 2D manipulators, J_P is $2 \times n$, J_O is $1 \times n$, and \mathbf{J} is $3 \times n$
- Unlike position kinematics, this mapping is linear(!!!)
- \mathbf{J} depends on \mathbf{q} ; velocity relationship changes depending on configuration

Example: RR Arm

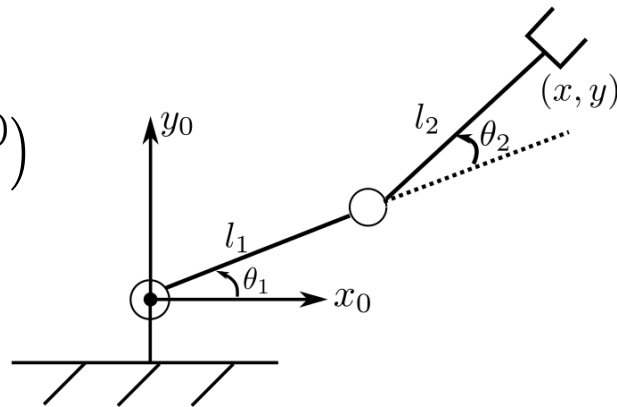
- Planar robot, 2 joints— 2×2 linear velocity Jacobian J_P
- Partial derivatives of end effector position wrt q :

$$J_P = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{pmatrix} = \begin{pmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

- Angular velocity Jacobian: $J_O = (z_0^0 \quad z_1^0)$
- Both joints have constant, parallel z axes $(0,0,1)^T$
- Planar robot, so only need ω_z component:

$$J_O = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Also equal to $\left(\frac{\partial \phi}{\partial q_1} \quad \frac{\partial \phi}{\partial q_2} \right)!$



$$x = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2)$$

$$y = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

$$\phi = \theta_1 + \theta_2$$

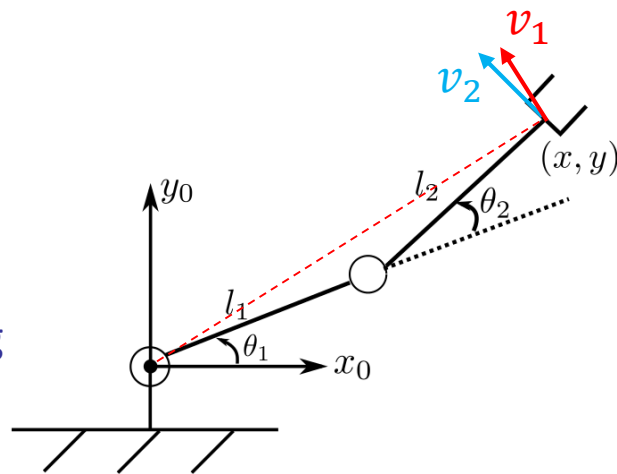
Example: RR Arm

- Putting the entire Jacobian together:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

v_1 v_2

- Again, this is a *linear, configuration-dependent* mapping
- Each *row* shows the contribution of each joint to a component of the end effector velocity
- Each *column* shows the end effector's velocity due to unit velocity of a single joint



$$\begin{aligned} x &= l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y &= l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ \phi &= \theta_1 + \theta_2 \end{aligned}$$

Example: RP Arm

- Non-planar robot, 2 joints— 3×2 linear velocity Jacobian

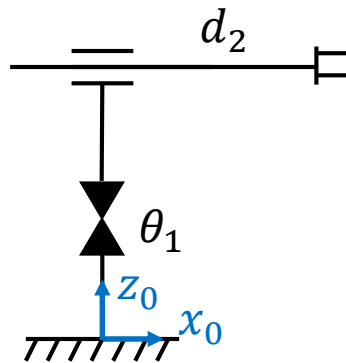
$$J_P = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} \end{pmatrix} = \begin{pmatrix} -d_2 \sin \theta_1 & \cos \theta_1 \\ d_2 \cos \theta_1 & \sin \theta_1 \\ 0 & 0 \end{pmatrix}$$

No velocity can be achieved in z direction!

- 3×2 angular velocity Jacobian
- Joint 1 is revolute, z axis given by $(0,0,1)^T$ in frame 0
- Joint 2 is prismatic, no angular velocity contribution

$$J_O = (\mathbf{z}_0^0 \quad \mathbf{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

No angular velocity can be achieved in ω_x, ω_y !



$$\begin{aligned} x &= d_2 \cos \theta_1 \\ y &= d_2 \sin \theta_1 \\ z &= l_1 \end{aligned}$$

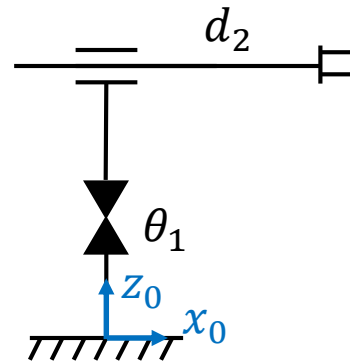
Example: RP Arm

- Full Jacobian is 6×2 :

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} -d_2 \sin \theta_1 & \cos \theta_1 \\ d_2 \cos \theta_1 & \sin \theta_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{pmatrix}$$

- This manipulator is effectively planar, since the end effector is limited to sweeping out a planar workspace

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{pmatrix} = \begin{pmatrix} -d_2 \sin \theta_1 & \cos \theta_1 \\ d_2 \cos \theta_1 & \sin \theta_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{pmatrix}$$



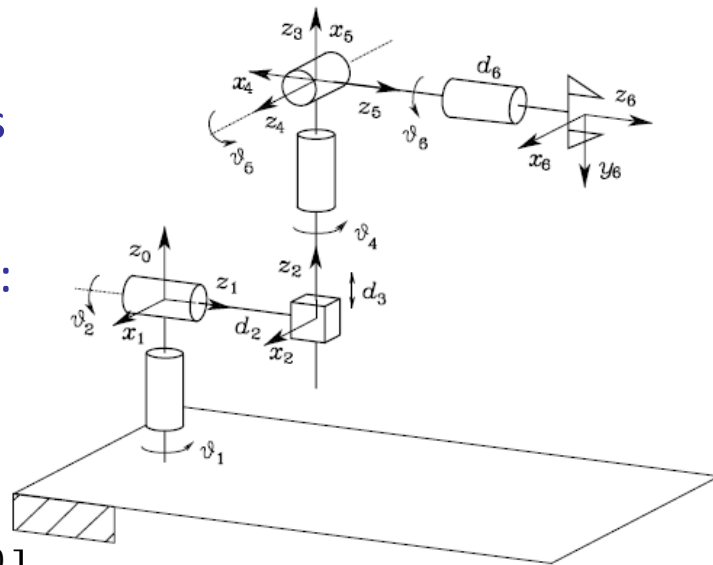
$$\begin{aligned} x &= d_2 \cos \theta_1 \\ y &= d_2 \sin \theta_1 \\ z &= l_1 \end{aligned}$$

Example: Stanford Manipulator

- Recall: Spherical arm plus spherical wrist
- 2 revolute joints, 1 prismatic, 3 revolute joints
- DH gives us the intermediate transformations:

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Example: Stanford Manipulator

- Building up consecutive homogeneous transforms between frames 0 and i :

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^0 = T_1^0 A_2 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & -d_2 s_1 \\ s_1 c_2 & c_1 & s_1 s_2 & d_2 c_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_3^0 = T_2^0 A_3 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 & c_1 & s_1 s_2 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 & 0 & c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^0 = T_3^0 A_4 = \begin{bmatrix} c_1 c_2 c_4 - s_1 s_4 & -c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 c_4 + c_1 s_4 & -s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 & -c_2 & s_2 s_4 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_5^0 = T_4^0 A_5 = \begin{bmatrix} (c_1 c_2 c_4 - s_1 s_4) c_5 - c_1 s_2 s_5 & c_1 c_2 s_4 + s_1 c_4 & (c_1 c_2 c_4 - s_1 s_4) s_5 + c_1 s_2 c_5 & d_3 c_1 s_2 - d_2 s_1 \\ (s_1 c_2 c_4 + c_1 s_4) c_5 - s_1 s_2 s_5 & s_1 c_2 s_4 - c_1 c_4 & (s_1 c_2 c_4 + c_1 s_4) s_5 + s_1 s_2 c_5 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 c_5 - c_2 s_5 & -s_2 s_4 & -s_2 c_4 s_5 + c_2 c_5 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = T_5^0 A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ r_{21} & r_{22} & r_{23} & s_1 s_2 d_3 - c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ r_{31} & r_{32} & r_{33} & c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

p_e

Linear velocity Jacobian J_P can be found via partial derivatives here

Example: Stanford Manipulator

- Building up consecutive homogeneous transforms between frames 0 and i :

$$\begin{aligned}
 T_1^0 = A_1 &= \begin{bmatrix} c_1 & 0 & -s_1 & \boxed{0} \\ s_1 & 0 & c_1 & \boxed{0} \\ 0 & -1 & 0 & \boxed{0} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad p_1 & T_2^0 = T_1^0 A_2 &= \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & -d_2 s_1 \\ s_1 c_2 & c_1 & s_1 s_2 & d_2 c_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} & T_3^0 = T_2^0 A_3 &= \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & \boxed{d_3 c_1 s_2 - d_2 s_1} \\ s_1 c_2 & c_1 & s_1 s_2 & \boxed{d_3 s_1 s_2 + d_2 c_1} \\ -s_2 & 0 & c_2 & \boxed{d_3 c_2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad p_3 \\
 T_4^0 = T_3^0 A_4 &= \begin{bmatrix} c_1 c_2 c_4 - s_1 s_4 & -c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & \boxed{d_3 c_1 s_2 - d_2 s_1} \\ s_1 c_2 c_4 + c_1 s_4 & -s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & \boxed{d_3 s_1 s_2 + d_2 c_1} \\ -s_2 c_4 & -c_2 & s_2 s_4 & \boxed{d_3 c_2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad p_4 \\
 T_5^0 = T_4^0 A_5 &= \begin{bmatrix} (c_1 c_2 c_4 - s_1 s_4) c_5 - c_1 s_2 s_5 & c_1 c_2 s_4 + s_1 c_4 & (c_1 c_2 c_4 - s_1 s_4) s_5 + c_1 s_2 c_5 & \boxed{d_3 c_1 s_2 - d_2 s_1} \\ (s_1 c_2 c_4 + c_1 s_4) c_5 - s_1 s_2 s_5 & s_1 c_2 s_4 - c_1 c_4 & (s_1 c_2 c_4 + c_1 s_4) s_5 + s_1 s_2 c_5 & \boxed{d_3 s_1 s_2 + d_2 c_1} \\ -s_2 c_4 c_5 - c_2 s_5 & -s_2 s_4 & -s_2 c_4 s_5 + c_2 c_5 & \boxed{d_3 c_2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad p_5 \\
 T_6^0 = T_5^0 A_6 &= \begin{bmatrix} r_{11} & r_{12} & r_{13} & \boxed{c_1 s_2 d_3 - s_1 d_2 + d_6 (c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5)} \\ r_{21} & r_{22} & r_{23} & \boxed{s_1 s_2 d_3 - c_1 d_2 + d_6 (c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2)} \\ r_{31} & r_{32} & r_{33} & \boxed{c_2 d_3 + d_6 (c_2 c_5 - c_4 s_2 s_5)} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad p_e
 \end{aligned}$$

J_P can also be found column by column using $[\mathbf{z}_{i-1}^0 \times (\mathbf{p}_e - \mathbf{p}_{i-1})]$ for revolute joints and $[\mathbf{z}_{i-1}^0]$ for prismatic

Example: Stanford Manipulator

- Building up consecutive homogeneous transforms between frames 0 and i :

$$\begin{aligned}
 T_1^0 = A_1 &= \begin{bmatrix} c_1 & 0 & \boxed{-s_1} & 0 \\ s_1 & 0 & \boxed{c_1} & 0 \\ 0 & -1 & \boxed{0} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_0^0 & T_2^0 = T_1^0 A_2 &= \begin{bmatrix} c_1 c_2 & -s_1 & \boxed{c_1 s_2} & -d_2 s_1 \\ s_1 c_2 & c_1 & \boxed{s_1 s_2} & d_2 c_1 \\ -s_2 & 0 & \boxed{c_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_1^0 & T_3^0 = T_2^0 A_3 &= \begin{bmatrix} c_1 c_2 & -s_1 & \boxed{c_1 s_2} & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 & c_1 & \boxed{s_1 s_2} & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 & 0 & \boxed{c_2} & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_2^0 \\
 T_4^0 = T_3^0 A_4 &= \begin{bmatrix} c_1 c_2 c_4 - s_1 s_4 & -c_1 s_2 & \boxed{-c_1 c_2 s_4 - s_1 c_4} & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 c_4 + c_1 s_4 & -s_1 s_2 & \boxed{-s_1 c_2 s_4 + c_1 c_4} & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 & -c_2 & \boxed{s_2 s_4} & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_3^0 & z_4^0 \\
 T_5^0 = T_4^0 A_5 &= \begin{bmatrix} (c_1 c_2 c_4 - s_1 s_4) c_5 - c_1 s_2 s_5 & c_1 c_2 s_4 + s_1 c_4 & \boxed{(c_1 c_2 c_4 - s_1 s_4) s_5 + c_1 s_2 c_5} & d_3 c_1 s_2 - d_2 s_1 \\ (s_1 c_2 c_4 + c_1 s_4) c_5 - s_1 s_2 s_5 & s_1 c_2 s_4 - c_1 c_4 & \boxed{(s_1 c_2 c_4 + c_1 s_4) s_5 + s_1 s_2 c_5} & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 c_5 - c_2 s_5 & -s_2 s_4 & \boxed{-s_2 c_4 s_5 + c_2 c_5} & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_5^0 \\
 T_6^0 = T_5^0 A_6 &= \begin{bmatrix} r_{11} & r_{12} & \boxed{r_{13}} & c_1 s_2 d_3 - s_1 d_2 + d_6(c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ r_{21} & r_{22} & \boxed{r_{23}} & s_1 s_2 d_3 - c_1 d_2 + d_6(c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ r_{31} & r_{32} & \boxed{r_{33}} & c_2 d_3 + d_6(c_2 c_5 - c_4 s_2 s_5) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_5^0
 \end{aligned}$$

Angular velocity Jacobian J_O can be built as $[\rho_i z_{i-1}^0]$ by extracting third column of each transform's rotation

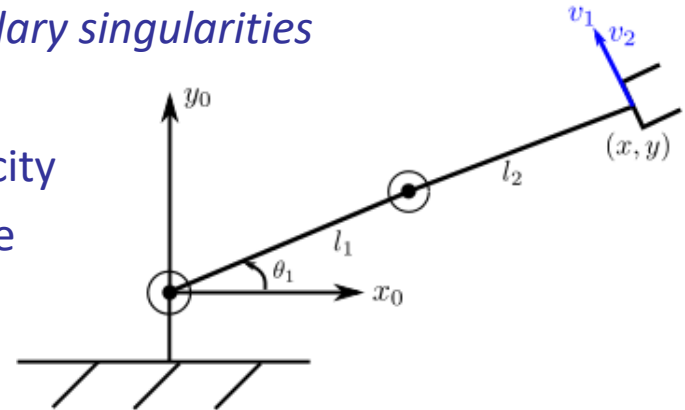
Kinematic Singularities

- Since $\mathbf{v}_e = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$ is a linear mapping, we may be tempted to invert it
- Even if $\mathbf{J}(\mathbf{q})$ is square, certain configurations may not be “invertible”
- **Kinematic singularity:** Configuration \mathbf{q}_s for which $\mathbf{J}(\mathbf{q})$ is not invertible
 - Algebraically, $\mathbf{J}(\mathbf{q})$ loses rank and $\det \mathbf{J}(\mathbf{q}) = 0$; rows and columns become linearly dependent and $\mathbf{J}(\mathbf{q})$ gains eigenvalues equal to 0
- Physically, mobility of the robot is reduced; it may be very difficult or require large velocities in the joints to produce end effector movement

Example: RR Arm

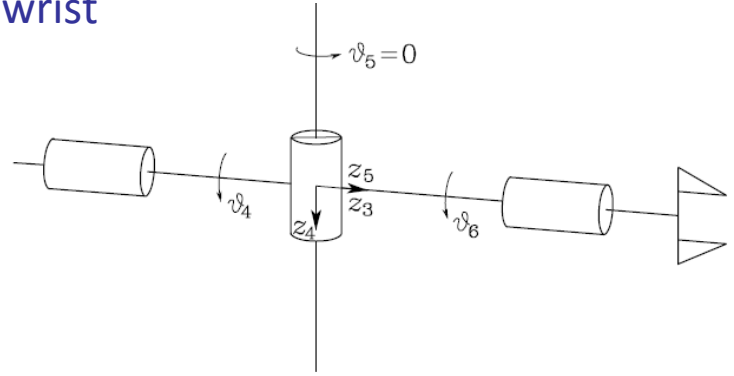
$$\mathbf{J}_P = \begin{pmatrix} -l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

- Consider the linear velocity Jacobian of the RR arm, since \mathbf{J}_O is constant
- \mathbf{J}_P loses rank when $0 = \det \mathbf{J}_P = -l_2 c_{12}(l_1 s_1 + l_2 s_{12}) + l_2 s_{12}(l_1 c_1 + l_2 c_{12}) = l_1 l_2 s_2$
- In other words, $\theta_2 = 0$ or $\theta_2 = \pi$: examples of *boundary singularities*
- Moving either joint produces same end effector velocity
- Allowable velocity directions no longer span the plane

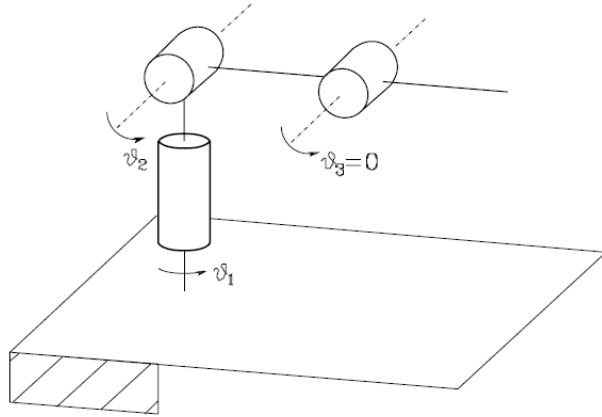


Wrist Singularities

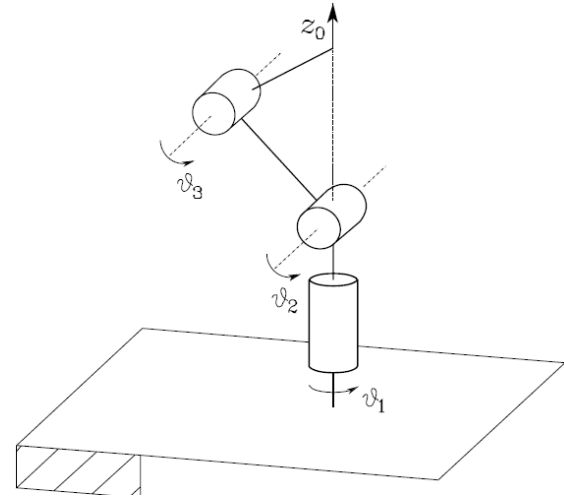
- Consider angular velocity Jacobian J_O of spherical wrist
- Singularities occur when $\theta_5 = 0, \theta_5 = \pi$
- z_3 and z_5 rotation axes are aligned
- Loss of mobility since rotations in θ_4 and θ_6 produce same end effector movements
- Attempting to solve inverse problem will not yield unique solutions



Arm Singularities



- *Elbow singularity:* $\theta_3 = 0, \theta_3 = \pi$
- Outstretched arm at workspace boundary



- *Shoulder singularity:* $a_2 c_2 + a_3 c_{23} = 0$
- Arm end point intersects z_0 rotation axis
- θ_1 rotations produce no end effector movement

Summary

- Differential kinematics provides a linear, configuration-dependent mapping between joint velocities and operational velocities
- Jacobian derivation can be done separately for linear and angular velocities
- Jacobian shows individual joint contributions to overall end effector velocity
- Manipulators singularities can occur at workspace boundaries or in configurations where the arm loses operational degrees of freedom