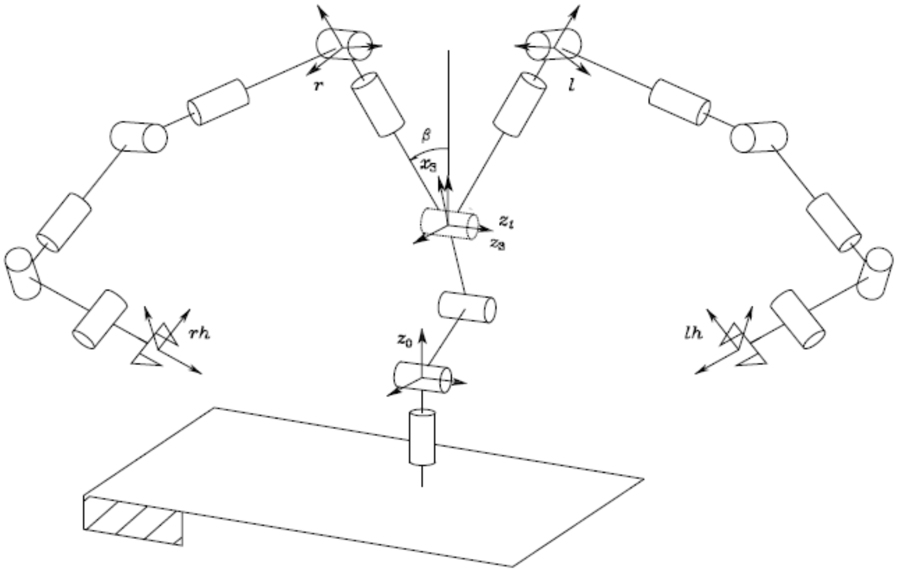
COMS W4733: Computational Aspects of Robotics

Lecture 4: FK Examples and Inverse Kinematics



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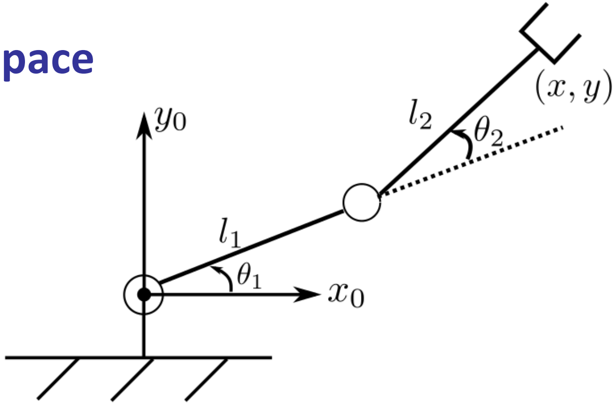
Review: Forward Kinematics

- A robot manipulator can be arbitrarily complicated

- Joint variables $\boldsymbol{q} = (q_1, \dots, q_n)^T \in$ **joint/configuration space**
- End effector pose $\boldsymbol{x}_e \in$ **operational space**
  - Generally position and orientation

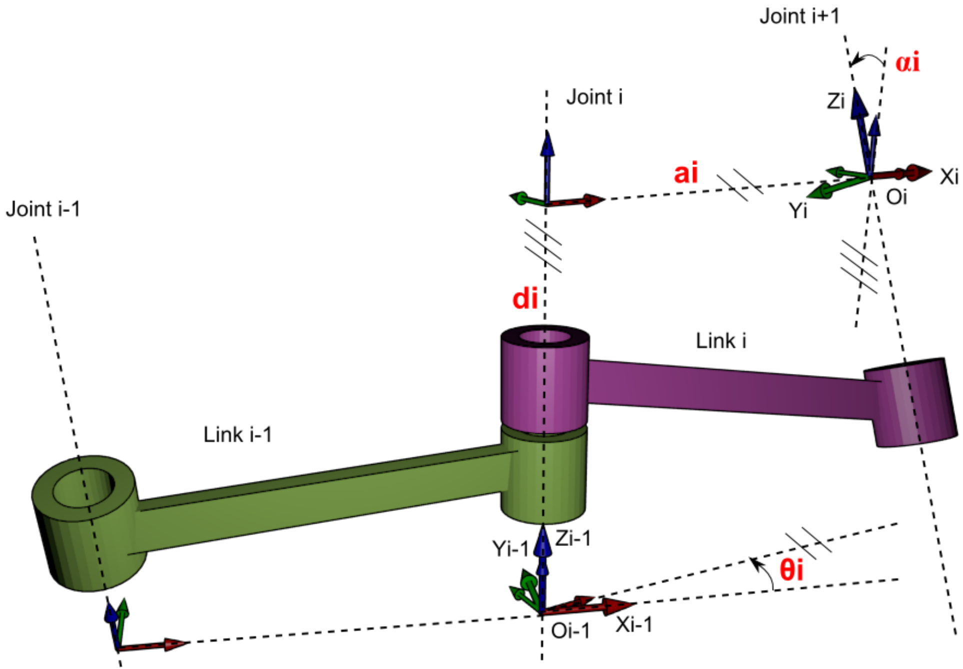
- Forward kinematics finds a mapping $\boldsymbol{x}_e = \boldsymbol{k}(\boldsymbol{q})$



$$\boldsymbol{A}_4^0 = \begin{bmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) & l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) & l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \\ 0 & 0 & 1 \end{bmatrix}$$

Review: DH Parameters

- Rotate about z_{i-1} by θ_i (**joint angle**) and translate along z_{i-1} by d_i (**link offset**)
- Translate along x_i by a_i (**link length**) and rotate about x_i by α_i (**link twist**)



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Review: DH Parameters

- Rotate about z_{i-1} by θ_i (**joint angle**) and translate along z_{i-1} by d_i (**link offset**)
- Translate along x_i by a_i (**link length**) and rotate about x_i by α_i (**link twist**)

- Each set of 4 DH parameters provides the following homogeneous transformation

$$\boldsymbol{A}_i^{i-1} = \underbrace{\begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate about and translate along } z_{i-1}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\text{Rotate about and translate along } x_i} = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Overall forward kinematic map found as $\boldsymbol{T}_n^0 = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 \cdots \boldsymbol{A}_{n-1}^{n-2} \boldsymbol{A}_n^{n-1}$

Popular Configurations

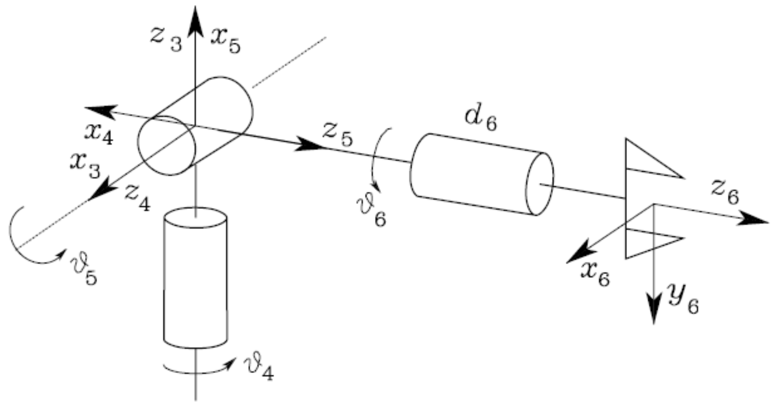
- Stanford arm (1969)
  - Creator: Victor Scheinman
  - One of first arms controlled by computer
  - Spherical arm plus spherical wrist

- SCARA arm (1981)
  - Selective Compliance Assembly Robot Arm
  - Popular for assembly, pick-and-place

Example: Spherical Wrist

- From the textbook:

Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6



- What's wrong here?
- In reference configuration, the joint angle θ_i is offset by another ± 90 degrees between frames 3 and 4, as well as 4 and 5
- "More correct" DH parameters:

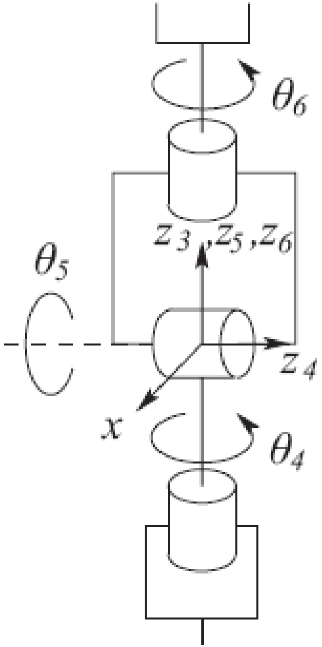
Link	a_i	α_i	d_i	θ_i
4	0	-90	0	$\theta_4 - 90$
5	0	90	0	$\theta_5 - 90$
6	0	0	d_6	$\theta_6 + 90$

Example: Spherical Wrist

- DH parameters given in the textbook correspond to a wrist reference configuration that is "stretched out"

- No additional joint angle rotations other than θ_i
- All x axes are aligned

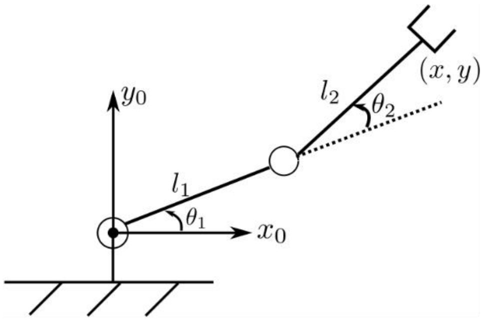
Link	a_i	α_i	d_i	θ_i
4	0	-90	0	θ_4
5	0	90	0	θ_5
6	0	0	d_6	θ_6

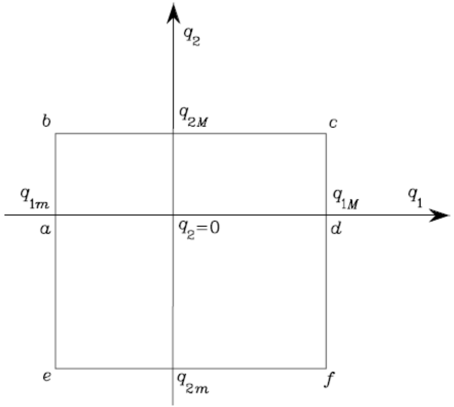


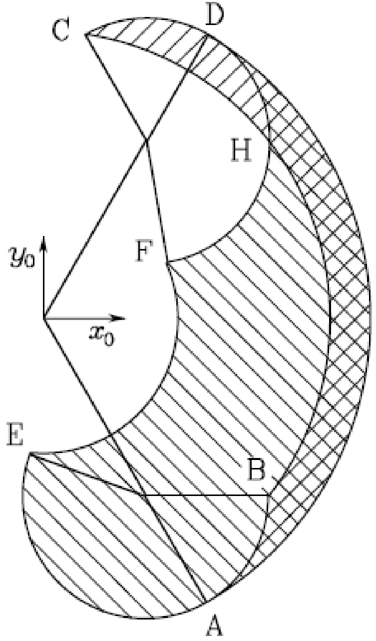
Workspace

- Often useful to characterize all the positions that a manipulator can reach
- How do the position FKs map joint space to operational space?
- *Joint limits* may cut down on the actual workspace

- Ex: 2-link RR arm with limits on both θ_1 and θ_2







Inverse Kinematics

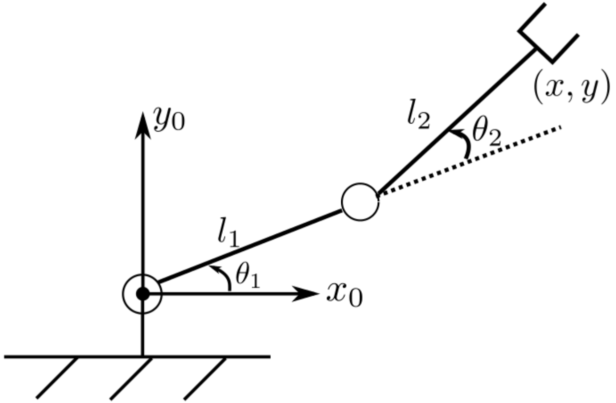
- How do we get a robot to achieve a desired pose or trajectory?
- **Inverse kinematics**: Given end effector pose, solve for the joint variables
- Much more difficult than forward kinematics!

- Nonlinear equations may not have closed-form solution
- Multiple or infinite solutions for *redundant* robots
- No solutions if trying to reach outside workspace

- Analytical vs numerical solutions

RR Arm

- How many joint variables does the RR arm have?
- Can we fully specify 2D pose (x, y, ϕ)?
- How many solutions if we just specify position?



- Suppose we want to reach $p_W = \left(p_{Wx}, p_{Wy}\right)^T$
- Compute: $p_{Wx}^2 + p_{Wy}^2 = l_1^2 + l_2^2 + 2l_1 l_2 \cos\theta_2$
- Note that since $-1 \leq \cos\theta_2 \leq 1$,

$$-1 \leq \frac{p_{Wx}^2 + p_{Wy}^2 - l_1^2 - l_2^2}{2l_1 l_2} \leq 1$$

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$
$$\phi = \theta_1 + \theta_2$$

$$(l_1 - l_2)^2 \leq p_{Wx}^2 + p_{Wy}^2 \leq (l_1 + l_2)^2$$

What does this mean?

RR Arm θ_2 Solution

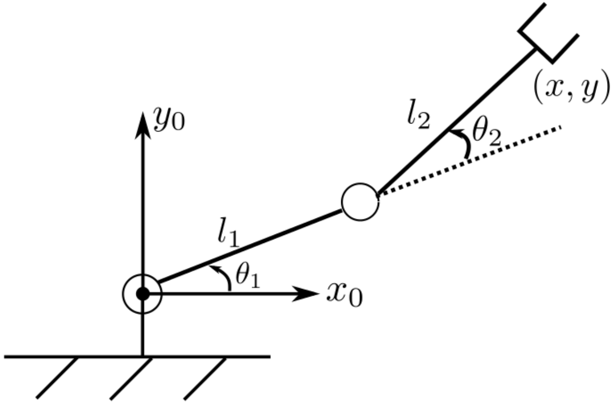
$$\boxed{\theta_2 = \operatorname{acos}\frac{p_{Wx}^2 + p_{Wy}^2 - l_1^2 - l_2^2}{2l_1 l_2}}$$

- This yields two possible solutions for θ_2
  - "Elbow up" vs "elbow down"

$$p_{Wx} = l_1 c_1 + l_2(c_1 c_2 - s_1 s_2)$$
$$p_{Wy} = l_1 s_1 + l_2(s_1 c_2 + c_1 s_2)$$

$$\sin\theta_1 = \frac{p_{Wy}(l_1 + l_2 c_2) - p_{Wx} l_2 s_2}{p_{Wx}^2 + p_{Wy}^2}$$

$$\cos\theta_1 = \frac{p_{Wx}(l_1 + l_2 c_2) + p_{Wy} l_2 s_2}{p_{Wx}^2 + p_{Wy}^2}$$



$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$
$$\phi = \theta_1 + \theta_2$$

How to find solution to satisfy both equations?

RR Arm θ_1 Solution

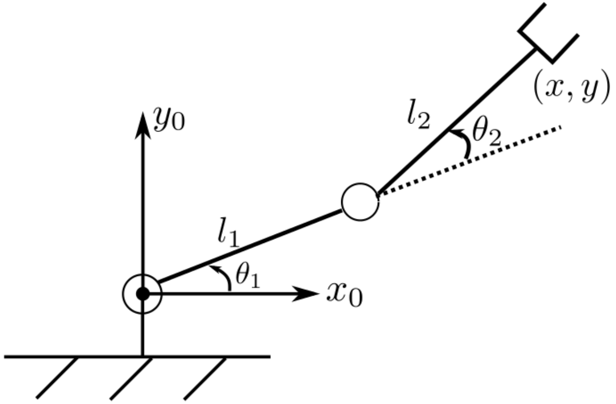
$$\boxed{\theta_2 = \text{acos}\frac{p_{Wx}^2 + p_{Wy}^2 - l_1^2 - l_2^2}{2l_1 l_2}}$$

$$\boxed{\theta_1 = \text{Atan2}(\sin\theta_1, \cos\theta_1)}$$

- We can use Atan2 to keep sign information
- Otherwise, $\text{atan}\left(\frac{y}{x}\right) = \text{atan}\left(\frac{-y}{-x}\right)$ (ambiguous!)

$$\sin\theta_1 = \frac{p_{Wy}(l_1 + l_2 c_2) - p_{Wx} l_2 s_2}{p_{Wx}^2 + p_{Wy}^2}$$

$$\cos\theta_1 = \frac{p_{Wx}(l_1 + l_2 c_2) + p_{Wy} l_2 s_2}{p_{Wx}^2 + p_{Wy}^2}$$

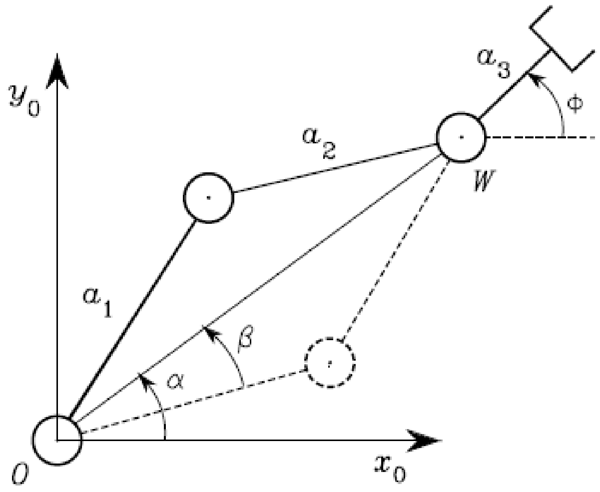


$$x = l_1\cos\theta_1 + l_2\cos(\theta_1 + \theta_2)$$
$$y = l_1\sin\theta_1 + l_2\sin(\theta_1 + \theta_2)$$
$$\phi = \theta_1 + \theta_2$$

RRR Arm

- What if we now add a third link (and third joint)?
- If we only specify position again, we still have two equations but now three unknowns
  - Infinitely many solutions!
- Suppose we now want both position and orientation: (p_x, p_y, ϕ)

- The total orientation is $\phi = \theta_1 + \theta_2 + \theta_3$
- Note that p_W is completely determined:
  - $p_W = (p_x - a_3 \cos \phi \,, p_y - a_3 \sin \phi)$
- Solve for θ_1 and θ_2 using RR arm solution
- $\theta_3 = \phi - \theta_1 - \theta_2$



General Strategies for IK

- Determine how many solutions to expect
  - How many joint DOFs? How many workspace DOFs?
- If robot is complex, try to *decouple* into independent components
  - Try to rely on previously solved subproblems
- Unravel algebraic equations from the forward kinematics
- Apply trigonometric identities, Atan2 function