COMS W4733: Computational Aspects of Robotics

Homework 3

Due: March 11, 2019

Problem 1 (20 points)

(a) The forward kinematics are as follows:

$$
\begin{aligned}
x' &= x + \frac{L}{2}\cos\theta + \frac{L}{2}\cos(\theta+\phi) \\
y' &= y + \frac{L}{2}\sin\theta + \frac{L}{2}\sin(\theta+\phi) \\
\theta' &= \theta + \phi
\end{aligned}
$$

Plugging into the no-slip constraint and simplifying, we get

$$
\left(\dot{x} - \frac{L}{2}\dot{\theta}\sin\theta - \frac{L}{2}(\dot{\theta}+\dot{\phi})\sin(\theta+\phi)\right)\sin(\theta+\phi) - \left(\dot{y} + \frac{L}{2}\dot{\theta}\cos\theta + \frac{L}{2}(\dot{\theta}+\dot{\phi})\cos(\theta+\phi)\right)\cos(\theta+\phi) = 0
$$

$$
\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - \frac{L}{2}(1+\cos\phi)\dot{\theta} - \frac{L}{2}\dot{\phi} = 0
$$

(b) The Pfaffian form of the constraints is written as

$$
\begin{pmatrix}
\sin\theta & -\cos\theta & 0 & 0 & 0 \\
\cos\theta & \sin\theta & 0 & -1 & 0 \\
\sin(\theta+\phi) & -\cos(\theta+\phi) & -\frac{L}{2}(1+\cos\phi) & 0 & -\frac{L}{2}
\end{pmatrix}
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ v \\ \dot{\phi} \end{pmatrix} = 0
$$

The set of allowed velocities is given by the null space of $\mathbf{A}^T$:

$$
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ v \\ \dot{\phi} \end{pmatrix} =
\begin{pmatrix}
\cos\theta & 0 \\
\sin\theta & 0 \\
\frac{2}{L}\tan(\phi/2) & -\frac{1}{1+\cos\phi} \\
1 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}
$$

There are two controllable degrees of freedom, corresponding to the forward velocity v and joint velocity $\dot{\phi}$.

(c) We first note that $u_1 = v$ and $u_2 = \dot{\phi}$, so $v = \sqrt{\dot{x}^2 + \dot{y}^2}$, as usual. We also have that $\theta = \text{Atan2}(\dot{y}, \dot{x})$, and so we can write its derivative as $\dot{\theta} = (\ddot{y}\dot{x} - \ddot{x}\dot{y})/(\dot{x}^2 + \dot{y}^2)$. u_2 appears in the kinematics of $\dot{\theta}$, so we can plug in and solve for u_2:

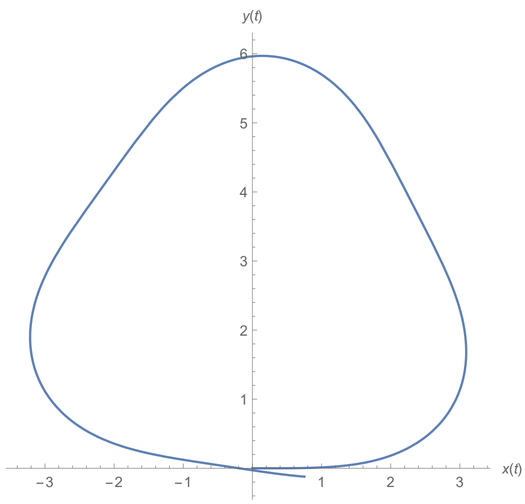
$$\dot{\theta} = \frac{2}{L}\tan(\phi/2)u_1 - \frac{1}{1+\cos\phi}u_2 \implies u_2 = (1+\cos\phi)\left(\frac{2}{L}\tan(\phi/2)u_1 - \dot{\theta}\right)$$

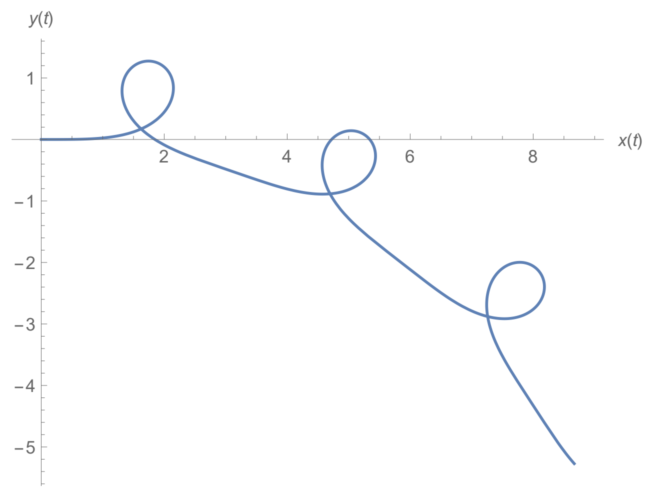
$$\dot{\phi} = \frac{2}{L}(\sin\phi)\sqrt{\dot{x}^2 + \dot{y}^2} - (1+\cos\phi)\frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{x}^2 + \dot{y}^2}$$

A singularity occurs at $\phi = \pi$. Although $\dot{\phi}$ has a solution there, it is confined to $\dot{\phi} = 0$. This corresponds to the rear link swinging all the way around to coincide with the front link. Even if this were physically possible, the robot ends up losing a degree of maneuverability as it can only move along the direction that it is oriented. It cannot steer away from this configuration, since $\dot{\phi}$ is not allowed to be nonzero. Closer inspection of the forward kinematics from above also confirms this observation, as $\phi = 0$ forces $\dot{\theta}$ to become undefined.

Problem 2 (20 points)

The two trajectories resulting from each input to the bike are shown below:





There are a few points to make about these plots. The first is that since we have a constant velocity input $v = 1$, the bike is going to keep moving forward no matter what the steering angle or steering velocity are. With the first input, the bike is changing its steering angle relatively slowly. So as ϕ increases, the bike starts turning left, and as ϕ decreases, the bike starts to turn back to the right. However, the bike's trajectory never actually makes it to turn in a rightward direction; it merely heads in a straight forward direction before the steering angle turns back to the left. This results in a somewhat circular, periodic trajectory.

With a faster change in steering angle, the qualitative behavior of the bike is actually still quite similar, even though the overall trajectory looks very different. Looking at the right plot, we still see segments alternating between left turns and somewhat straight segments. The difference is that the leftward turns are more extreme to the point that loops are completed before each straight segment, resulting in net distance traveled after each cycle.

When the input is too large at $\dot{\phi}(t) = \sin(t)$, the steering angle itself also becomes large from time to time. More specifically, it becomes possible for ϕ to reach and surpass $\pi/2$ radians, making the $\dot{\theta}$ term in the forward kinematics infinite. This is effectively a singularity of the vehicle, since in reality the front wheel of the bike should not be able to turn all the way around.