COMS W4733: Computational Aspects of Robotics

Homework 1

Solutions

Problem 1 (15 points)

(a) The first two transformations only involve translations. The last transformation involves both a translation and a rotation. The rotation may be found in one of several ways. For example, we can first rotate frame 0 about z_0 by +90 degrees, aligning x_0 with x_3, followed by a rotation about x_0 by 180 degrees. Equivalently, we can rotate about z_0 by -90 degrees, aligning y_0 with y_3, followed by a rotation about y_0 by 180 degrees. Either way, we get the same result:

$$R_3^0 = R_z(+90)R_x(180) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$R_3^0 = R_z(-90)R_y(180) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The rotation matrix R_3^0 can be substituted into the transformation A_3^0. The other two have no rotation displacements, so their rotation parts are the identity matrix. Finally, the translation components can be found by simply reading off measurements in the figure.

$$A_1^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2^0 = \begin{pmatrix} 1 & 0 & 0 & -0.5 \\ 0 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_3^0 = \begin{pmatrix} 0 & 1 & 0 & -0.5 \\ 1 & 0 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(b) If the camera is rotated, that only changes frame 3. So the transformations A_1^0 and A_2^0 aren't affected. After this change, the camera frame is displaced from the base frame by a rotation of 180 degrees about x_3. So the rotation is simply

$$R_3^0 = R_x(180) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and the overall homogeneous transformation is

$$A_3^0 = \begin{pmatrix} 1 & 0 & 0 & -0.5 \\ 0 & -1 & 0 & 1.5 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(c) First, the transformation A_2^0 is modified so that the rotation part R_2^0 includes the new rotation about z as $R_z(+90)$. Secondly, the translation is modified to be $(0,1,1)^T + (-0.2, 0.8, 0.2)^T = (-0.2, 1.8, 1.2)^T$. The overall transformation becomes

$$A_3^0 = \begin{pmatrix} 0 & -1 & 0 & -0.2 \\ 1 & 0 & 0 & 1.8 \\ 0 & 0 & 1 & 1.2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since we want to find A_2^3, we need to consider how frame 3 is transformed into frame 2 (instead of the reverse ordering). The translation component is $o_2^3 = (0.3, -0.3, 1.8)^T$ (note that this is relative to frame 3! The z component is positive because z_3 points downward). As in part 1, the rotation component can be found in many different ways. One way would be to first rotate frame 3 about z_3 by +90 degrees and align y_2 and y_3. Then rotate 180 degrees about y_3.

$$R_3^0 = R_z(+90)R_y(180) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

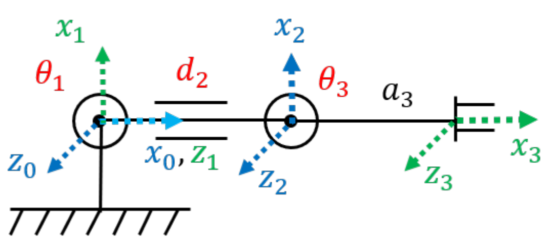
Putting the components together, we have

$$A_2^3 = \begin{pmatrix} 0 & -1 & 0 & 0.3 \\ -1 & 0 & 0 & -0.3 \\ 0 & 0 & -1 & 1.8 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

As a sanity check, you can also find A_3^2 and invert that transformation to find A_2^3.

Problem 2 (20 points)

(a) First note that all z axes point along the axis of actuation for each joint. The requirement that x_i intersects z_{i-1} specifies x_1, x_2, and x_3. It is possible for you to have chosen the negative directions for these axes. Finally, the tool frame has z_3 parallel to z_2.



(b) The DH table is given as follows. If you chose opposite directions for any of the x axes, simply add 180 degrees to the corresponding θ_i entry.

Link	a_i	α_i	d_i	θ_i
1	0	90	0	$\theta_1 + 90$
2	0	-90	d_2	0
3	a_3	0	0	$\theta_3 - 90$

(c) Each row corresponds to a homogeneous transformation:

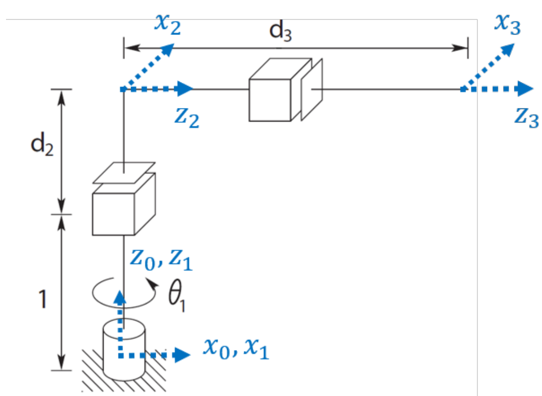
$$A_1^0 = \begin{pmatrix} -s_1 & 0 & c_1 & 0 \\ c_1 & 0 & s_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_3^2 = \begin{pmatrix} s_3 & c_3 & 0 & a_3 s_3 \\ -c_3 & s_3 & 0 & -a_3 c_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3^0 = A_1^0 A_2^1 A_3^2 = \begin{pmatrix} c_{13} & -s_{13} & 0 & a_3 c_{13} + d_2 c_1 \\ s_{13} & c_{13} & 0 & a_3 s_{13} + d_2 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) The workspace is an annulus with outer radius 3 and inner radius 1.

Problem 3 (20 points)

(a) Note that because z_0 and z_1 are the same axes, we have complete freedom in deciding where to place O_1. For simplicity, here we've chosen the entire frame 1 to coincide with frame 0, but you could have chosen to place it anywhere along the first link. z_2 and z_3 should point along the second link, while x_2 and x_3 can point either into (as we've shown here) or out of the page.



(b) The DH table is given as follows. If you chose opposite directions for any of the x axes, simply add 180 degrees to the corresponding θ_i entry.

Link	a_i	α_i	d_i	θ_i
1	0	0	0	θ_1
2	0	90	$d_2 + 1$	90
3	0	0	d_3	0

(c) Each row corresponds to a homogeneous transformation:

$$A_1^0 = \begin{pmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_2^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & d_2 + 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A_3^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_3^0 = A_1^0 A_2^1 A_3^2 = \begin{pmatrix} -s_1 & 0 & c_1 & d_3 c_1 \\ c_1 & 0 & s_1 & d_3 s_1 \\ 0 & 1 & 0 & d_2 + 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(d) The workspace is just a quarter slice of a cylinder with height equal to 2 and radius equal to 2.

Problem 4 (15 points)

(a) Given only a position, the robot is underconstrained and in general has infinite solutions to the inverse kinematics problem. This is true as long as the desired position is in the robot's workspace, which covers the entire plane as long as there are no joint limits. If orientation is also specified, then there is exactly one solution.

(b) From the forward kinematics, we have the following equations:

$$\begin{aligned} p_x &= a_3 c_{13} + d_2 c_1 \\ p_y &= a_3 s_{13} + d_2 s_1 \\ \phi &= \theta_1 + \theta_3 \end{aligned}$$

Substitute ϕ into the first two equations and use Atan2 to find θ_1 (d_2 will cancel out):

$$\begin{aligned} &p_x = a_3 \cos\phi + d_2 \cos\theta_1 \to \cos\theta_1 = \frac{1}{d_2}(p_x - a_3 \cos\phi) \\ &p_y = a_3 \sin\phi + d_2 \sin\theta_1 \to \sin\theta_1 = \frac{1}{d_2}(p_y - a_3 \sin\phi) \\ &\theta_1 = \text{Atan2}(p_y - a_3 \sin\phi, p_x - a_3 \cos\phi) \end{aligned}$$

Now we can find $\theta_3 = \phi - \theta_1$ quite easily. Finally, d_2 comes from solving either of the first two equations after plugging in θ_1 and θ_3.

Problem 5 (30 points)

(a) The DH table is given as follows. Note that there is no ambiguity in any of the parameters since all frames are given.

Link	a_i	α_i	d_i	θ_i
1	0	-90	0	θ_1
2	0	90	0	θ_2
3	45	-90	550	θ_3
4	-45	90	0	θ_4
5	0	-90	300	θ_5
6	0	90	0	θ_6
7	0	0	60	θ_7