Spring 2019

COMS 4733 Computational Aspects of Robotics Midterm

INSTRUCTIONS

- You have 75 minutes.
- Do not turn the page until the official start time.
- You may use any source of paper-based notes for this exam.
- Write your UNI on the top of each page where indicated.
- Write all your answers in the space provided for each question. We will not look at any work outside the provided exam pages.
- Questions are not sequenced in order of difficulty. Make sure to look ahead if stuck on a particular question.
- If you finish early, you may turn in your exam if you can minimize disturbances to other students. Please wait in your seat if there are fewer than 10 minutes left.

Last Name	
First Name	
UNI	
All the work on this exam is my own. (please sign)	

Q. 1	Q. 2	Q. 3	Q. 4	Total
24	24	28	24	100

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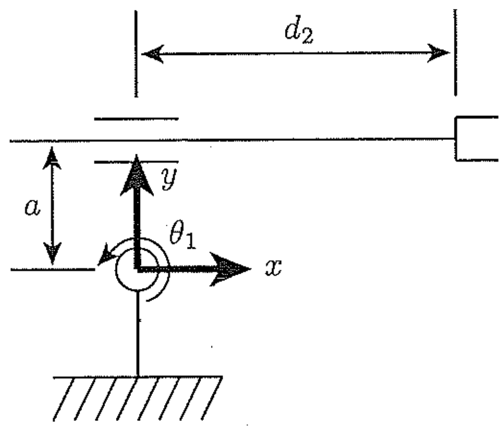
1. (24 points) **Inverse Kinematics**

Consider the RP manipulator below, shown in its reference configuration.



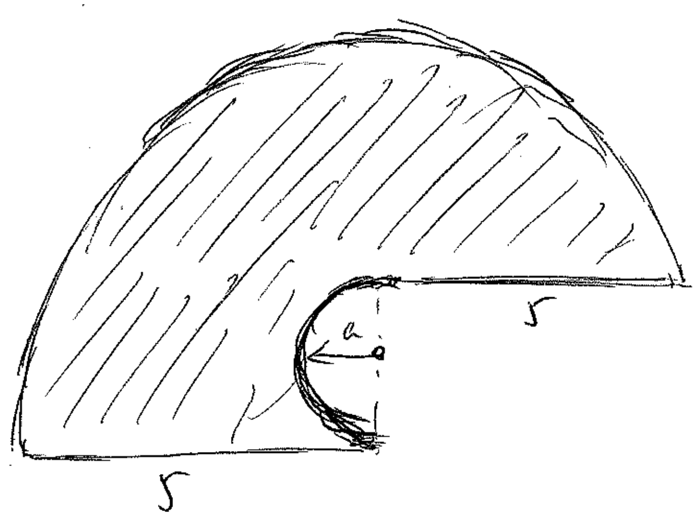
The forward kinematics of the end effector are given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -a\sin\theta_1 + d_2\cos\theta_1 \\ a\cos\theta_1 + d_2\sin\theta_1 \end{bmatrix}.$$

(a) (8 pt) Suppose the manipulator has the following joint limits:

$$0 \le \theta_1 \le \pi, \quad 0 \le d_2 \le 5$$

Describe the workspace of the end effector. You may ignore the ground and the base fixed link. You may draw pictures, but be sure to clearly label any lengths or radii.



(b) (12 pt) Consider the inverse position kinematics problem with the sole joint limit $d_2 \geq 0$. Find the analytical solution for the joint variables θ_1 and d_2, given x and y; no need to simplify your expressions as long as you can express the unknowns in terms of known quantities. Describe when we have no solution, one solution, or multiple solutions. You may find it helpful to recall the inverse of a 2 by 2 matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$x = -as_1 + d_2 c_1$

$y = ac_1 + d_2 s_1$

$x^2 + y^2 = a^2 + d_2^2 \rightarrow \underline{d_2 = \sqrt{x^2 + y^2 - a^2}}.$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a & d_2 \\ d_2 & a \end{pmatrix}\begin{pmatrix} s_1 \\ c_1 \end{pmatrix} \rightarrow \begin{pmatrix} s_1 \\ c_1 \end{pmatrix} = \frac{1}{-a^2 - d_2^2}\begin{pmatrix} a & -d_2 \\ -d_2 & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\underline{\theta_1 = \text{Atan2}\left(-ax + d_2 y,\ d_2 x + ay\right)}.$$

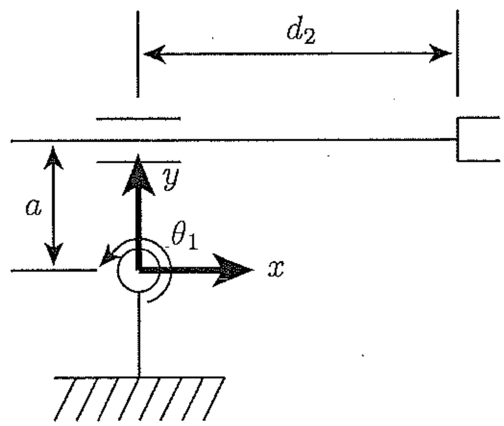
One solution if in workspace, i.e. $a^2 \leq x^2 + y^2$.
No solutions otherwise.

(c) (4 pt) Explain what changes if we want to solve the inverse orientation kinematics as well. In addition to specifying x and y, we also require that the end effector be oriented at an angle θ with respect to the base frame. How does that affect the feasibility of the solution?

3 constraints with 2 DOFs — in general no solutions, unless we're lucky and given that θ_1 is the desired orientation.

2. **(24 points) Differential Kinematics**

Consider again the RP manipulator from the last problem.



The forward kinematics of the end effector are given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -a\sin\theta_1 + d_2\cos\theta_1 \\ a\cos\theta_1 + d_2\sin\theta_1 \end{bmatrix}.$$

(a) **(12 pt)** Find the full Jacobian for the manipulator given that we are operating in SE(3). In other words, find J(q) such that

$$\begin{bmatrix} \mathrm{v} \\ \omega \end{bmatrix} = \mathrm{J}(q)\begin{bmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{bmatrix},$$

where $\mathrm{v} = (\dot{x}, \dot{y}, \dot{z})^T$ is the linear velocity and $\omega = (\omega_x, \omega_y, \omega_z)^T$ is the angular velocity of the end effector.

$$J_p = \begin{pmatrix} -aC_1 - d_2 S_1 & C_1 \\ -aS_1 + d_2 C_1 & S_1 \\ 0 & 0 \end{pmatrix} \qquad J_o = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

$$J = \begin{pmatrix} J_p \\ J_o \end{pmatrix}.$$

(b) **(6 pt)** Suppose that we want to solve the inverse velocity kinematics. Given configuration velocities $\mathbf{v}$ and ω, we want to solve for the joint velocities $\dot{\theta}_1$ and $\dot{d}_2$. Explain the procedure you would follow to do so. State whether the problem is overconstrained, underconstrained, or neither. In addition, state whether the solution is unique, non-unique, or non-exact.

Overconstrained, no exact solution exists.

We can use the left pseudo-inverse.

$$\dot{q} = (J^T J)^{-1} J^T \begin{bmatrix} v \\ \omega \end{bmatrix}.$$

(c) **(6 pt)** Show that when $d_2 = 0$, the manipulator is in a singular configuration if we consider the linear velocity Jacobian $\mathbf{J}_P$ only.

When $d_2 = 0$:

$$\cancel{J = \begin{pmatrix} -ac_1 \\ -as_1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \qquad J_P = \begin{pmatrix} -ac_1 & c_1 \\ -as_1 & s_1 \\ 0 & 0 \end{pmatrix}$$

Rank drops from 2 to 1.

3. **(28 points) Mobile Robots**

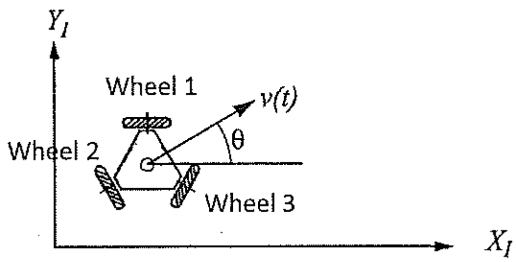
Consider the three-wheeled mobile robot below. $q = (x, y, \theta)^T$ is the pose of the center of the robot relative to a fixed inertial frame. Each wheel is located a distance of l from the center and is subject to a no-slip constraint. Suppose the robot is currently oriented at the angle $\theta = \frac{\pi}{6}$ radians. The instantaneous velocities and orientations of each of the wheels (in the inertial frame) are given as follows:

$$(\dot{x}_1, \dot{y}_1, \theta_1) = \left(\dot{x} - l\dot{\theta}, \dot{y}, 0\right)$$

$$(\dot{x}_2, \dot{y}_2, \theta_2) = \left(\dot{x} + \frac{1}{2}l\dot{\theta}, \dot{y} - \frac{\sqrt{3}}{2}l\dot{\theta}, \frac{2\pi}{3}\right)$$

$$(\dot{x}_3, \dot{y}_3, \theta_3) = \left(\dot{x} + \frac{1}{2}l\dot{\theta}, \dot{y} + \frac{\sqrt{3}}{2}l\dot{\theta}, \frac{\pi}{3}\right)$$

$\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0$



(a) **(12 pt)** At the orientation $\theta = \frac{\pi}{6}$ radians, find the Pfaffian form of the constraints written as $A^T\dot{q} = 0$. You should use the simplifications $\sin\left(\frac{\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{3}\right) = -\cos\left(\frac{2\pi}{3}\right) = \frac{1}{2}$. Note that the constraint matrix A^T is constant since we've specified the pose.

① $(\dot{x} - l\dot{\theta})\sin 0 - \dot{y}\cos 0 = \dot{y} = 0.$

② $(\dot{x} + \frac{1}{2}l\dot{\theta})\sin\frac{2\pi}{3} - (\dot{y} - \frac{\sqrt{3}}{2}l\dot{\theta})\cos\frac{2\pi}{3} = \frac{\sqrt{3}}{2}\dot{x} + \frac{1}{2}\dot{y} = 0.$

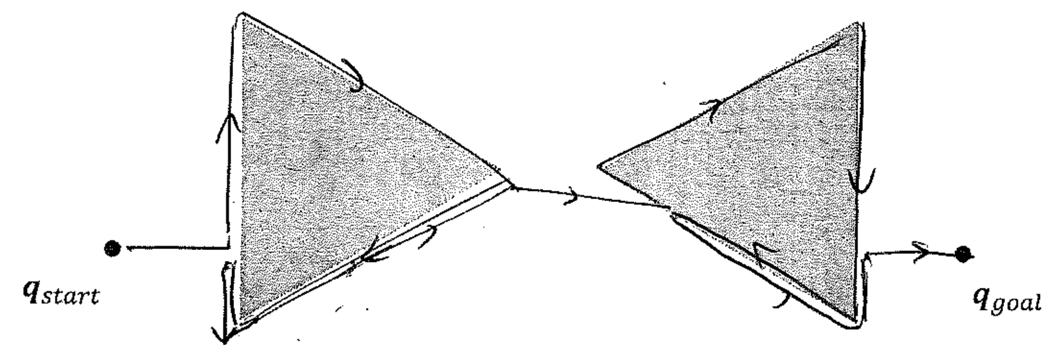
③ $(\dot{x} + \frac{1}{2}l\dot{\theta})\sin\frac{\pi}{3} - (\dot{y} + \frac{\sqrt{3}}{2}l\dot{\theta})\cos\frac{\pi}{3} = \frac{\sqrt{3}}{2}\dot{x} - \frac{1}{2}\dot{y} = 0.$

$$A^T\dot{q} = \begin{pmatrix} 0 & 1 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \end{pmatrix}\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = 0$$

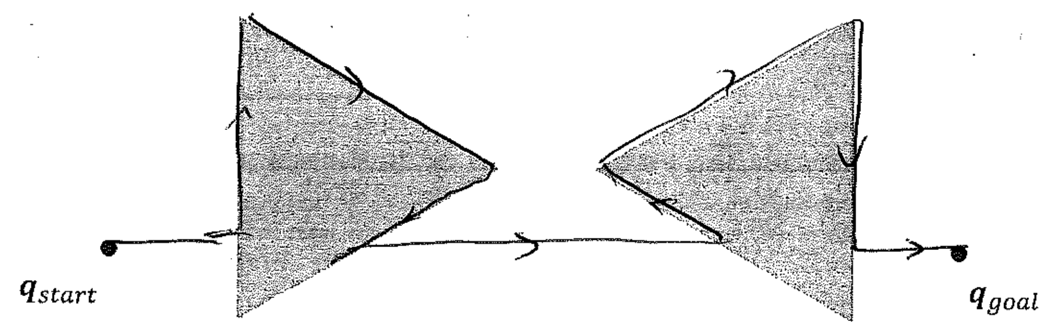
(b) **(4 pt)** How many controllable degrees of freedom does the robot have?

One.

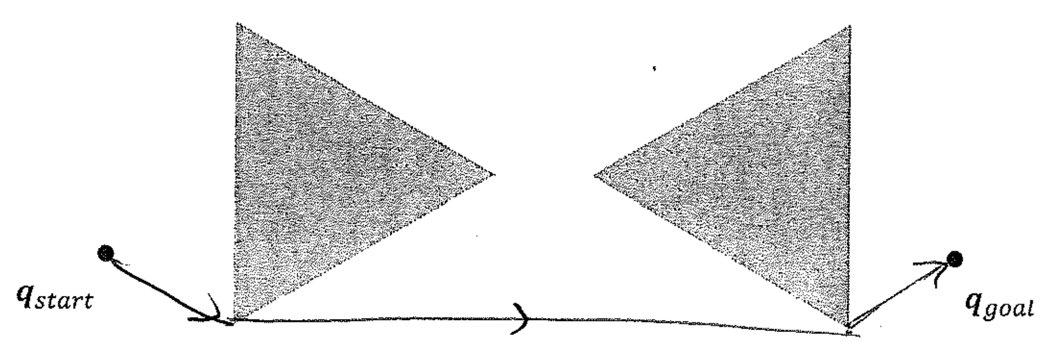
(c) **(4 pt)** In the environment below, draw the path that a Bug 1 algorithm would follow from q_{start} to q_{goal}. Suppose that left turns are always chosen when a random turn is required.



(d) **(4 pt)** In the environment below, draw the path that a Bug 2 algorithm would follow from q_{start} to q_{goal}. Suppose that left turns are always chosen when a random turn is required.

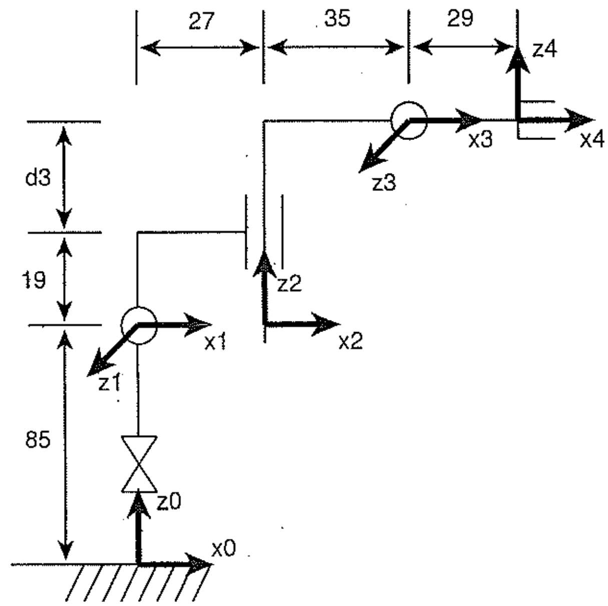


(e) **(4 pt)** In the environment below, draw the path that a Tangent Bug algorithm with infinite sensor range would follow from q_{start} to q_{goal}. Suppose that left turns are always chosen when a random turn is required.



4. **(24 points) Forward Kinematics**

   (a) **(16 pt)** The manipulator below has an assignment of coordinate frames according to the DH convention. The joint variables are θ_1, θ_2, d_3, and θ_4. Fill in the DH parameters that correspond to the frames. Note that z_1 and z_3 point out of the page.



Link	a_i	α_i	d_i	θ_i
1	0	90	85	θ_1
2	27	-90	0	θ_2
3	35	90	d_3+19	~~θ_3~~ 0
4	29	-90	0	θ_4.

   (b) **(8 pt)** Find the homogeneous transformation T_0^2 expressing **frame 0 relative to frame 2**. No need to simplify; you may leave your answer with matrix products or inverses.

$$T_2^0 = \begin{pmatrix} c_1 & 0 & s_1 & 0 \\ s_1 & 0 & -c_1 & 6 \\ 0 & 1 & 0 & 85 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} c_2 & 0 & -s_2 & 27c_2 \\ s_2 & 0 & c_2 & 27s_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underline{T_0^2 = (T_2^0)^{-1}}.$$

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