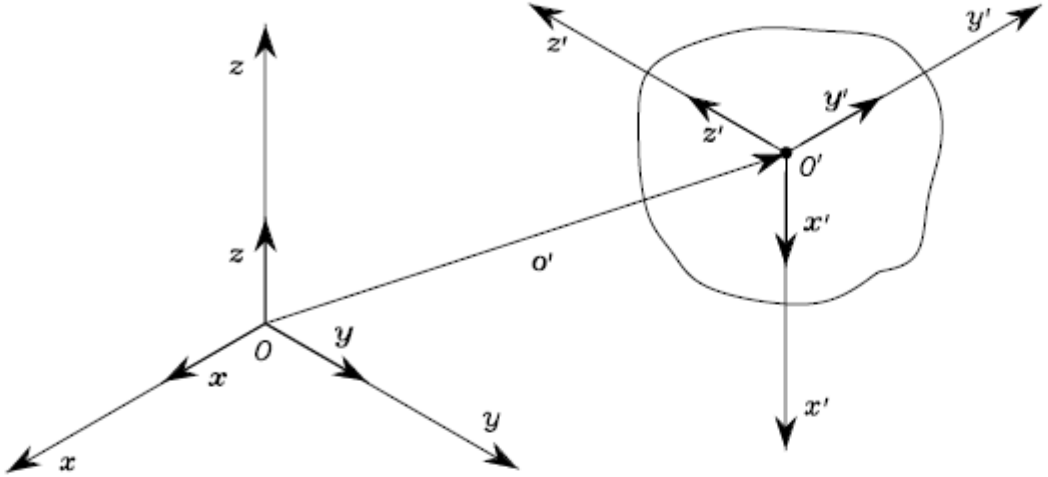
COMS W4733: Computational Aspects of Robotics

Lecture 2: Rigid Body Motions

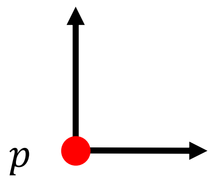


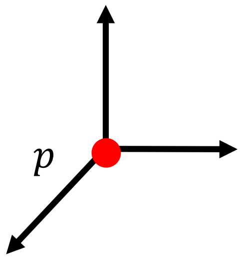
Instructor: Tony Dear

Degrees of Freedom

- Independent parameters defining an object's configuration or state
- Equivalently, independent motions that a mechanical body has

- How many DOFs does a point have in 2D space? In 3D space?



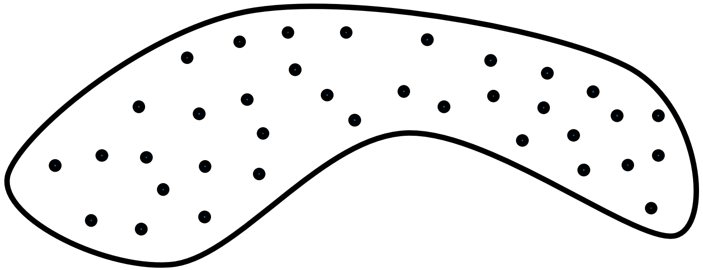


- But real mechanisms are not just points!

Rigid Bodies

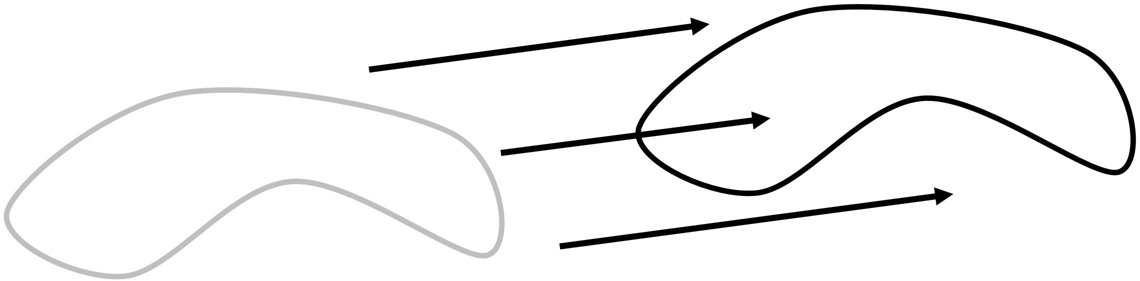
- A **rigid body** is a collection of an infinite number of points such that their relative distances remain constant when the body is subject to a *displacement*.

- A displacement on a rigid body
  - does not change inter-point distances (stretching)
  - does not change handedness of any three points (reflection)

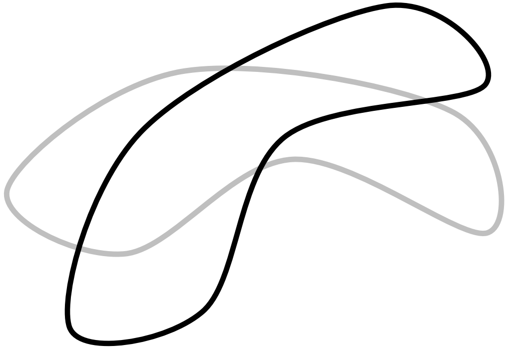


Rigid Body Motions

- Translation DOFs describe **position**
  - 2 DOFs in 2D space
  - 3 DOFs in 3D space

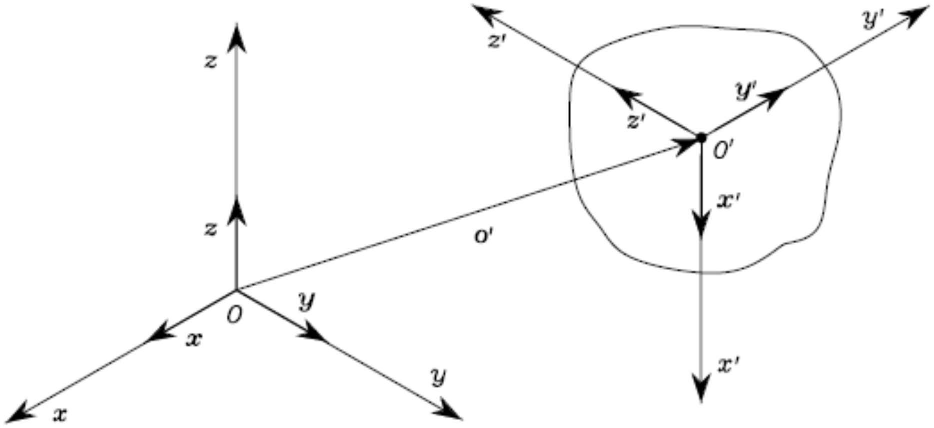


- Rotation DOFs describe **orientation**
  - 1 DOF in 2D space
  - **3 DOFs** in 3D space



Coordinate Frames

- DOFs are described relative to a **coordinate frame**
- Notation: O-xyz, or just O for short
  - (O also refers to the frame's origin)
- Always follow right-hand rule!
  - Point fingers toward x
  - Close palm toward y
  - Thumb points along z



- Points referred to by vectors starting from origin O

$$\boldsymbol{o}' = o'_x\boldsymbol{x} + o'_y\boldsymbol{y} + o'_z\boldsymbol{z} = [o'_x \quad o'_y \quad o'_z]^T$$

Rotating Frames

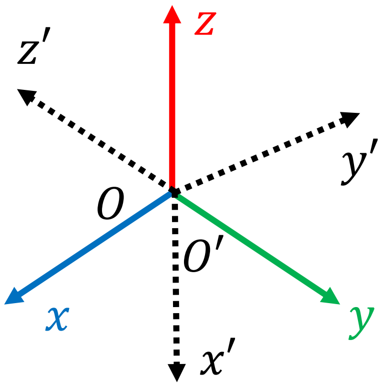
- In addition to specifying points and vectors relative to a frame O, we can also specify *other frames* O'

$$\boldsymbol{x}' = x'_x\boldsymbol{x} + x'_y\boldsymbol{y} + x'_z\boldsymbol{z}$$
$$\boldsymbol{y}' = y'_x\boldsymbol{x} + y'_y\boldsymbol{y} + y'_z\boldsymbol{z}$$
$$\boldsymbol{z}' = z'_x\boldsymbol{x} + z'_y\boldsymbol{y} + z'_z\boldsymbol{z}$$

- Each axis of frame O' written wrt to O:
- Now define the following matrix:

$$\boldsymbol{R}^O_{O'} = [\boldsymbol{x}' \quad \boldsymbol{y}' \quad \boldsymbol{z}'] = \begin{bmatrix} x'_x & y'_x & z'_x \\ x'_y & y'_y & z'_y \\ x'_z & y'_z & z'_z \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{x}' \cdot \boldsymbol{x} & \boldsymbol{y}' \cdot \boldsymbol{x} & \boldsymbol{z}' \cdot \boldsymbol{x} \\ \boldsymbol{x}' \cdot \boldsymbol{y} & \boldsymbol{y}' \cdot \boldsymbol{y} & \boldsymbol{z}' \cdot \boldsymbol{y} \\ \boldsymbol{x}' \cdot \boldsymbol{z} & \boldsymbol{y}' \cdot \boldsymbol{z} & \boldsymbol{z}' \cdot \boldsymbol{z} \end{bmatrix}$$



Rotation Matrices

- $\boldsymbol{R} = [\boldsymbol{x}' \quad \boldsymbol{y}' \quad \boldsymbol{z}']$ is an **rotation** (orthonormal) **matrix**

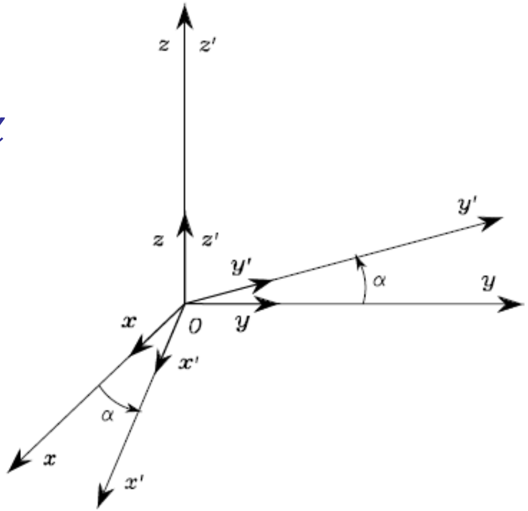
- Columns are orthogonal: $(x')^T y' = (y')^T z' = (z')^T x' = 0$
- Columns are unit-length: $(x')^T x' = (y')^T y' = (z')^T z' = 1$

- Can show that $\boldsymbol{R}^T \boldsymbol{R} = \boldsymbol{I}$, which means that $\boldsymbol{R}^{-1} = \boldsymbol{R}^T$ and $\det(\boldsymbol{R}) = 1$

- $n \times n$ rotation matrices belong to the **special orthogonal group** $\mathrm{SO}(n)$
  - $n = 2$ in 2D, $n = 3$ in 3D

Elementary Rotations

- What do rotation matrices actually look like?
- Let's keep one axis (e.g. $\boldsymbol{z}$) fixed and rotate by an angle α
  - Positive rotations are counterclockwise about fixed axis
  - Right-hand rule: Point thumb along axis and close palm



$$\boldsymbol{R}_{O'}^{O} = \begin{bmatrix} \boldsymbol{x}' \cdot \boldsymbol{x} & \boldsymbol{y}' \cdot \boldsymbol{x} & \boldsymbol{z}' \cdot \boldsymbol{x} \\ \boldsymbol{x}' \cdot \boldsymbol{y} & \boldsymbol{y}' \cdot \boldsymbol{y} & \boldsymbol{z}' \cdot \boldsymbol{y} \\ \boldsymbol{x}' \cdot \boldsymbol{z} & \boldsymbol{y}' \cdot \boldsymbol{z} & \boldsymbol{z}' \cdot \boldsymbol{z} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- All terms involving $\boldsymbol{z}$ or $\boldsymbol{z}'$ are 0, except for $\boldsymbol{z}' \cdot \boldsymbol{z} = 1$
- All other nonzero terms are sinusoidal functions of angle α

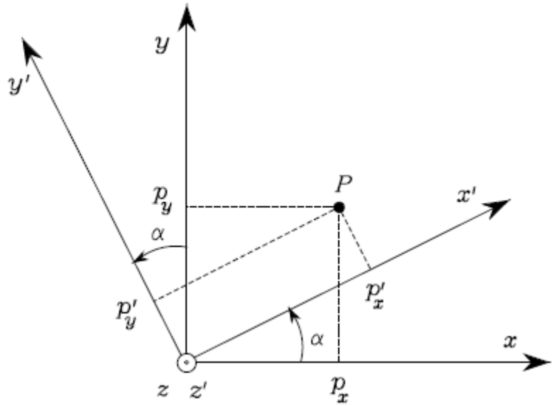
Elementary Rotations

- We can derive elementary rotations about all three axes

$$\boldsymbol{R}_z(\alpha) = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{R}_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad \boldsymbol{R}_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}$$

- What about in 2D? Only one rotation DOF!
- All rotations about an imaginary z axis

$$\boldsymbol{R}_{o'}^{o} = \begin{bmatrix} \boldsymbol{x}' \cdot \boldsymbol{x} & \boldsymbol{y}' \cdot \boldsymbol{x} \\ \boldsymbol{x}' \cdot \boldsymbol{y} & \boldsymbol{y}' \cdot \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$



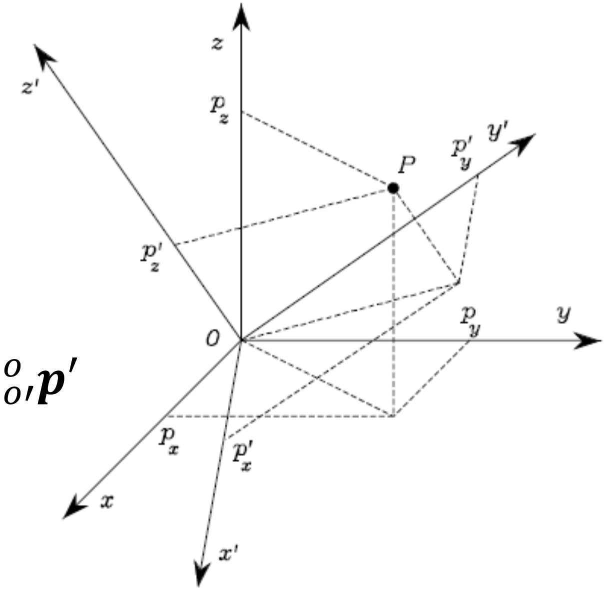
Rotations as Transformations

- A point P has different coordinates depending on frame of reference
- A rotation matrix transforms point coordinates from one frame to another

$$\boldsymbol{p} = p_x\boldsymbol{x} + p_y\boldsymbol{y} + p_z\boldsymbol{z} = [p_x \quad p_y \quad p_z]^T$$

$$\boldsymbol{p}' = p'_x\boldsymbol{x}' + p'_y\boldsymbol{y}' + p'_z\boldsymbol{z}' = [p'_x \quad p'_y \quad p'_z]^T$$

- In frame O: $\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = [\boldsymbol{x}' \quad \boldsymbol{y}' \quad \boldsymbol{z}'] \begin{bmatrix} p'_x \\ p'_y \\ p'_z \end{bmatrix} = \boldsymbol{R}^O_{O'}\boldsymbol{p}'$



Rotations as Transformations

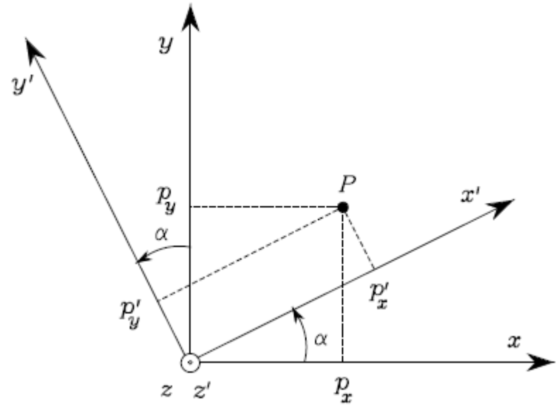
- A point P has different coordinates depending on frame of reference
- A rotation matrix transforms point coordinates from one frame to another

- Ex: transforming 2D point coordinates
- In frame O:

$$\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$\boldsymbol{p} = \boldsymbol{R}_{o'}^{o}\boldsymbol{p}' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} p'_x \\ p'_y \end{bmatrix}$$

$$\boldsymbol{p} = \begin{bmatrix} p'_x \cos\alpha - p'_y \sin\alpha \\ p'_x \sin\alpha + p'_y \cos\alpha \end{bmatrix}$$



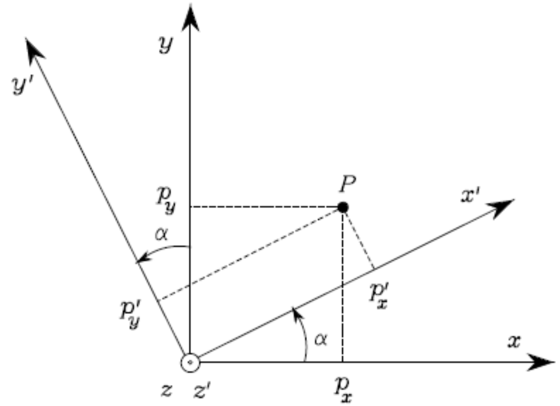
Rotation Inverse

- Frame O' is rotated an angle α relative to frame O
- Equivalently, frame O is rotated an angle $-\alpha$ relative to frame O'

$$\boldsymbol{R}_O^{O'} = \begin{bmatrix} \cos(-\alpha) & -\sin(-\alpha) \\ \sin(-\alpha) & \cos(-\alpha) \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

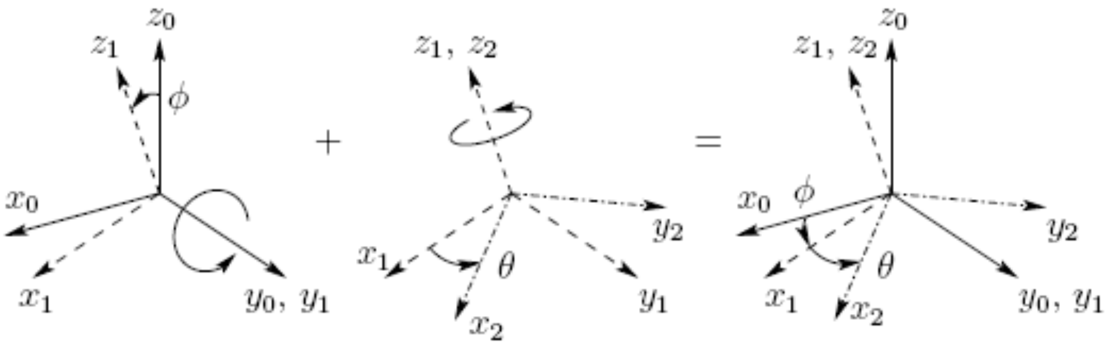
$$\boldsymbol{R}_O^{O'} = \left(\boldsymbol{R}_{O'}^{O}\right)^T = \left(\boldsymbol{R}_{O'}^{O}\right)^{-1}$$

- Ex: How to interpret $\boldsymbol{R}_O^{O'}(\boldsymbol{R}_{O'}^{O}\boldsymbol{p}')$?
  - $\boldsymbol{p} = \boldsymbol{R}_{O'}^{O}\boldsymbol{p}'$ expresses P in frame O
  - $\boldsymbol{R}_O^{O'}\boldsymbol{p}$ brings P back to frame O'
- Rotations cancel each other out! $\boldsymbol{p}' = \boldsymbol{R}_O^{O'}(\boldsymbol{R}_{O'}^{O}\boldsymbol{p}')$



Rotation Composition

- Same composition idea if we have more than two frames



$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0(\phi)\boldsymbol{p}^1 \qquad \boldsymbol{p}^1 = \boldsymbol{R}_2^1(\theta)\boldsymbol{p}^2 \qquad \boldsymbol{p}^0 = \boldsymbol{R}_2^0\boldsymbol{p}^2 = \boldsymbol{R}_1^0\boldsymbol{R}_2^1\boldsymbol{p}^2$$

$$\boxed{\boldsymbol{R}_2^0 = \boldsymbol{R}_1^0\boldsymbol{R}_2^1}$$

- Ex: What is $\boldsymbol{p}^0$ expressed in frame O_2?

Rotation Commutativity

- 2D rotations commute since there is only one axis of rotation
  - $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = \boldsymbol{R}(\theta+\phi)$
  - $\begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{bmatrix} = \boldsymbol{R}(\phi+\theta)$

- 3D rotations do **not** commute when rotating about different axes!
  - $\boldsymbol{R}_y(\phi)\boldsymbol{R}_z(\theta) = \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix}\begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}$
  - $\boldsymbol{R}_z(\theta)\boldsymbol{R}_y(\phi) = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} = \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}$

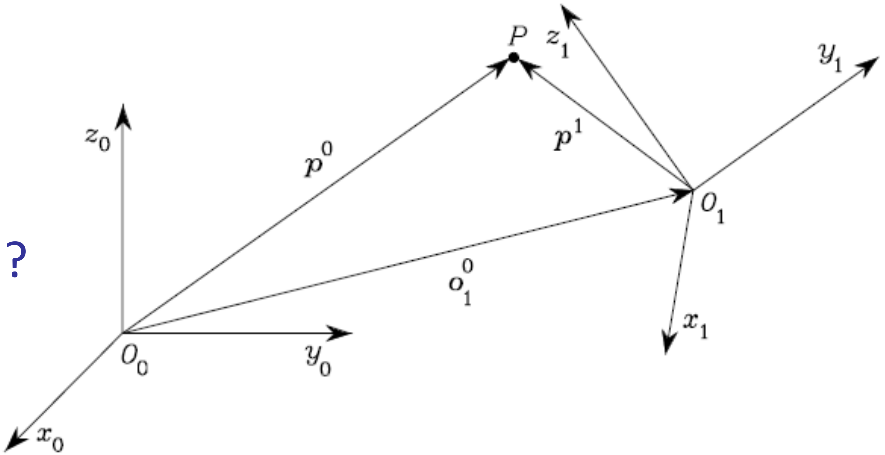
Translations

- Recall that rigid bodies can both **rotate** and **translate**
- How to describe point coordinates when relative frames are not coincident?

- Displacement between frames is o_1^0

$$\boxed{\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1}$$

- What happens when frames are aligned?
- $\boldsymbol{R}_1^0 = \boldsymbol{I}; \boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{p}^1$



Homogeneous Transformation

- We can combine rotation and translation into one linear transformation!

$$\boldsymbol{p}^0 = \boldsymbol{o}_1^0 + \boldsymbol{R}_1^0 \boldsymbol{p}^1 \qquad \begin{bmatrix} \boldsymbol{p}^0 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \underset{\text{Vector of 3 zeroes}}{\boldsymbol{0}^T} & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}^1 \\ 1 \end{bmatrix} \qquad \boxed{\tilde{\boldsymbol{p}}^0 = \boldsymbol{A}_1^0 \tilde{\boldsymbol{p}}^1}$$

- $\tilde{\boldsymbol{p}}$ is the *homogeneous representation* of $\boldsymbol{p}$
- $\boldsymbol{A}_1^0$ is a *homogeneous transformation matrix*

- $\boldsymbol{A}_1^0$ belongs to the **special Euclidean group** $\mathrm{SE}(n) = \mathbb{R}^n \times \mathrm{SO}(n)$
  - $n = 3$: 4×4 matrix; $n = 2$: 3×3 matrix

Inverse Transform

- Inverse rotations were simple due to orthogonality: $R^{-1} = R^T$
- Equivalent to "undoing" the rotation or rotating by negative angle

- Homogeneous transform first rotates then translates: $\boldsymbol{p}^0 = \boldsymbol{R}_1^0\boldsymbol{p}^1 + \boldsymbol{o}_1^0$
- Inverse operation: $\boldsymbol{p}^1 = \boldsymbol{R}_0^1\boldsymbol{p}^0 - \boldsymbol{R}_0^1\boldsymbol{o}_1^0$

$$\begin{bmatrix} \boldsymbol{p}^1 \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_0^1 & -\boldsymbol{R}_0^1\boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{p}^0 \\ 1 \end{bmatrix} = \boldsymbol{A}_0^1\widetilde{\boldsymbol{p}}^0 = (\boldsymbol{A}_1^0)^{-1}\widetilde{\boldsymbol{p}}^0$$

- Note that $(\boldsymbol{A}_1^0)^{-1} \neq (\boldsymbol{A}_1^0)^T$!

Composition of Transforms

- Suppose we have three frames O_0, O_1, O_2

$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 \boldsymbol{p}^1 + \boldsymbol{o}_1^0$$

$$\boldsymbol{p}^1 = \boldsymbol{R}_2^1 \boldsymbol{p}^2 + \boldsymbol{o}_2^1$$

$$\boldsymbol{A}_1^0 = \begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} \quad \boldsymbol{A}_2^1 = \begin{bmatrix} \boldsymbol{R}_2^1 & \boldsymbol{o}_2^1 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

$$\boldsymbol{p}^0 = \boldsymbol{R}_1^0 (\boldsymbol{R}_2^1 \boldsymbol{p}^2 + \boldsymbol{o}_2^1) + \boldsymbol{o}_1^0$$

$$= \underbrace{\boldsymbol{R}_1^0 \boldsymbol{R}_2^1}_{} \boldsymbol{p}^2 + \underbrace{\boldsymbol{R}_1^0 \boldsymbol{o}_2^1 + \boldsymbol{o}_1^0}_{}$$

$$\boldsymbol{p}^0 = \boldsymbol{R}_2^0 \boldsymbol{p}^2 + \boldsymbol{o}_2^0$$

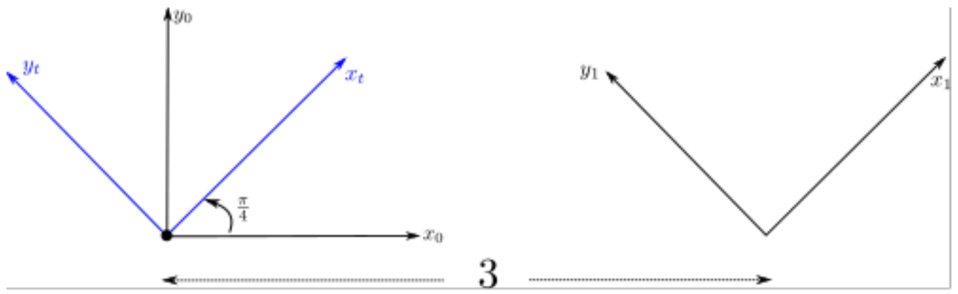
$$\begin{bmatrix} \boldsymbol{R}_1^0 & \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix} \begin{bmatrix} \boldsymbol{R}_2^1 & \boldsymbol{o}_2^1 \\ \boldsymbol{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_1^0 \boldsymbol{R}_2^1 & \boldsymbol{R}_1^0 \boldsymbol{o}_2^1 + \boldsymbol{o}_1^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}$$

$$\boxed{\boldsymbol{A}_2^0 = \boldsymbol{A}_1^0 \boldsymbol{A}_2^1 = \begin{bmatrix} \boldsymbol{R}_2^0 & \boldsymbol{o}_2^0 \\ \boldsymbol{0}^T & 1 \end{bmatrix}}$$

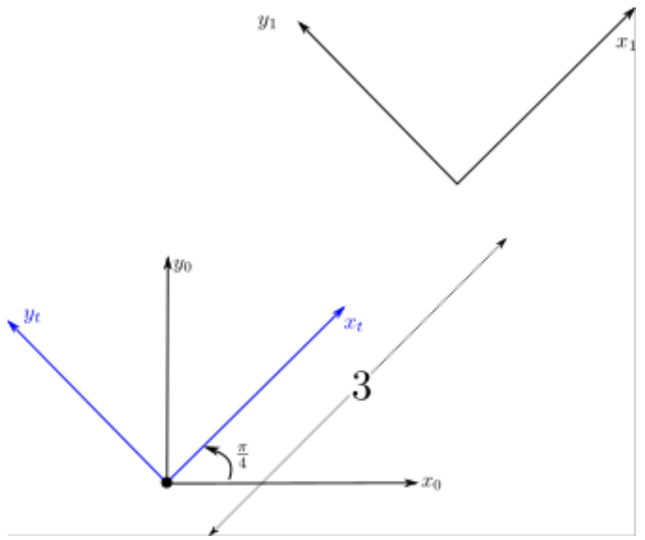
Non-Commutativity

- While 2D rotations were commutative, homogeneous transforms are in general **not** commutative, both in 2D and in 3D

- Ex: Rotation and translation (in 2D)



$$A = Trans_x(3)Rot\left(\frac{\pi}{4}\right)$$



$$A = Rot\left(\frac{\pi}{4}\right)Trans_x(3)$$

Basic Transforms

- 2D:

$$\boldsymbol{Trans}_x(a) = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Trans}_y(b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Rot}(\theta) = \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- 3D:

$$\boldsymbol{Trans}_x(a) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Trans}_y(b) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Trans}_z(c) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{Rot}_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Rot}_y(\beta) = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \boldsymbol{Rot}_z(\gamma) = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Summary

- Rigid bodies are characterized by rotation and translation degrees of freedom
- Rotation matrices express relative orientations between frames; also transform point representations between frames
  - Orthonormal; note properties: inverse, composition, (non-) commutativity
- Homogeneous transformations linearly combine rotation and translation
  - Can also be inverted and composed; generally not commutative
- Homogeneous transformations describe all DOFs of open kinematic chains