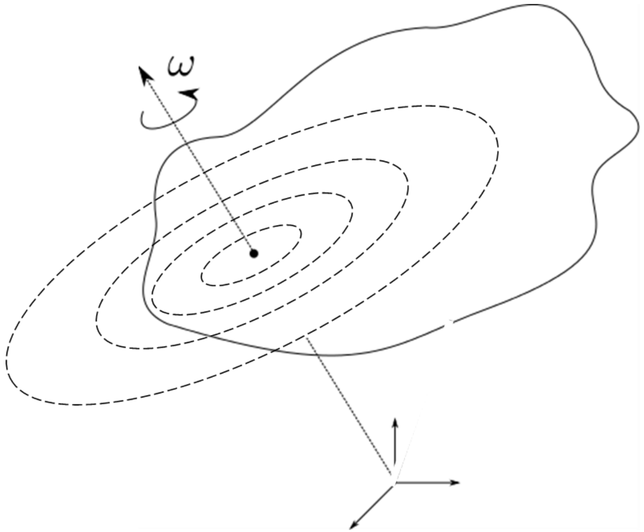
COMS W4733: Computational Aspects of Robotics

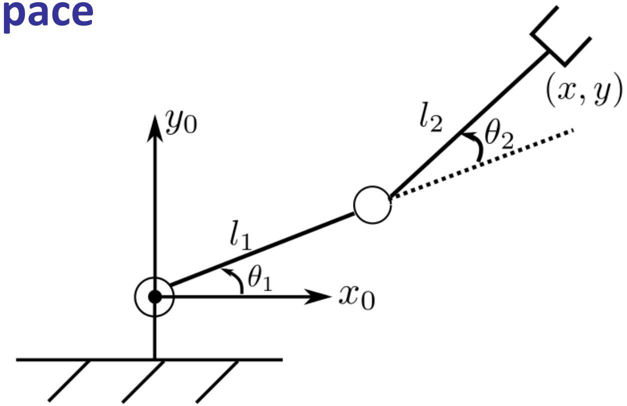
Lecture 5: Differential Kinematics



Instructor: Tony Dear

Review: Kinematics So Far

- Joint variables $\boldsymbol{q} = (q_1, \ldots, q_n)^T$ ∈ **joint/configuration space**
  - Ex: $\boldsymbol{q} = (\theta_1, \theta_2)^T$ for the RR arm
- End effector pose $\boldsymbol{x}_e$ ∈ **operational space**
  - $\boldsymbol{x}_e = (x, y, \phi)^T$ in 2D space
  - $\boldsymbol{x}_e = \{(x, y, z)^T, \boldsymbol{R}_{3\times3}\}$ in 3D space



- Forward kinematics: $\boldsymbol{x}_e = \boldsymbol{k}(\boldsymbol{q})$
  - I move the joints; what is the pose of the end effector?
- Inverse kinematics: $\boldsymbol{q} = \boldsymbol{k}^{-1}(\boldsymbol{x}_e)$
  - How do I control the joints to reach a desired pose?

Frame Velocities

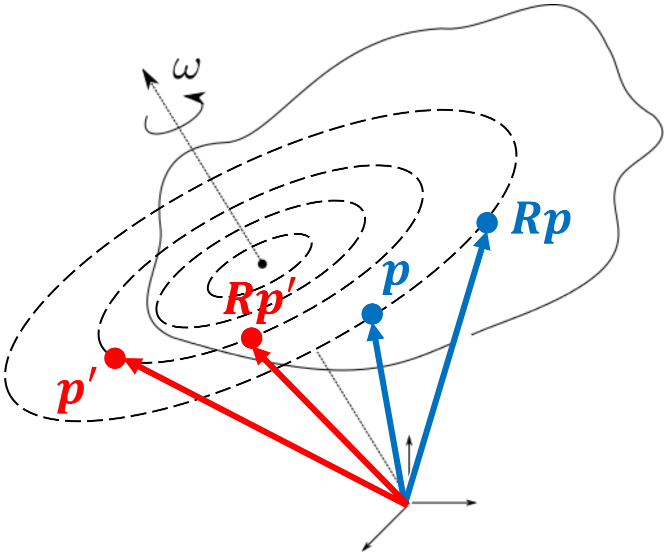
- Our manipulators are not staying still—they will generally be moving around
- How are joint velocities $\dot{\boldsymbol{q}}$ and operational velocities $\dot{\boldsymbol{x}}_e$ related?
- We'll show that $\dot{\boldsymbol{x}}_e$ is a sum of velocity contributions from individual joints
- From forward kinematics: $\boldsymbol{T}_e(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}_e(\boldsymbol{q}) & \boldsymbol{p}_e(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{bmatrix}$
- If we take the time derivative of $\boldsymbol{p}_e(\boldsymbol{q})$, we obtain end effector's *linear velocity*
- What about $\boldsymbol{R}_e(\boldsymbol{q})$?

Derivative of a Rotation Matrix

- $\boldsymbol{R}(t)$ is orthogonal: $$\boldsymbol{R}(t)\boldsymbol{R}^T(t) = \boldsymbol{I}$$
- Differentiate both sides wrt time: $$\dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t) + \boldsymbol{R}(t)\dot{\boldsymbol{R}}^T(t) = \boldsymbol{O}$$
- Define $\boldsymbol{S}(t) = \dot{\boldsymbol{R}}(t)\boldsymbol{R}^T(t)$: $$\boldsymbol{S}(t) + \boldsymbol{S}^T(t) = \boldsymbol{O}$$
- $\boldsymbol{S}(t)$ is skew-symmetric: $$\boldsymbol{S}(t) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$$
- Post-multiply $\boldsymbol{S}(t)$ by $\boldsymbol{R}(t)$: $$\boldsymbol{S}(t)\boldsymbol{R}(t) = \dot{\boldsymbol{R}}(t)$$
- Cross product property: $$\dot{\boldsymbol{R}}(t)\boldsymbol{p} = \boldsymbol{\omega}(t) \times \boldsymbol{R}(t)\boldsymbol{p} \qquad \boldsymbol{\omega}(t) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$
  - (arbitrary vector $\boldsymbol{p}$)

Angular Velocity

- $\dot{\boldsymbol{R}}(t)\boldsymbol{p} = \boldsymbol{S}(t)\boldsymbol{R}(t)\boldsymbol{p} = \boldsymbol{\omega}(t) \times \boldsymbol{R}(t)\boldsymbol{p}$
  - $\boldsymbol{S}(t) = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}, \boldsymbol{\omega}(t) = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$



- $\dot{\boldsymbol{p}} = \frac{d}{dt}(\boldsymbol{R}(t)\boldsymbol{p}) = \dot{\boldsymbol{R}}(t)\boldsymbol{p}$ is the linear velocity of $\boldsymbol{R}(t)\boldsymbol{p}$
- From mechanics, $\boldsymbol{\omega}(t)$ is the *angular velocity* of $\boldsymbol{R}(t)\boldsymbol{p}$

- Since $\boldsymbol{p}$ is arbitrary, we can associate $\boldsymbol{\omega}$ with the rotation specified by $\boldsymbol{R}$
- Compute $\dot{\boldsymbol{R}}(t) = \boldsymbol{S}(\boldsymbol{\omega}(t))\boldsymbol{R}(t)$ and extract the elements of $\boldsymbol{S}$

Angular Velocity Jacobian

- Going back to general form of forward kinematics: $\boldsymbol{T}_e(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{R}_e(\boldsymbol{q}) & \boldsymbol{p}_e(\boldsymbol{q}) \\ \boldsymbol{0}^T & 1 \end{bmatrix}$
- Can show that $\dot{\boldsymbol{R}}_e(\boldsymbol{q}) = \boldsymbol{S}(\boldsymbol{\omega}_e)\boldsymbol{R}_e(\boldsymbol{q})$ leads to

$$\boldsymbol{\omega}_e = \sum_{i=1}^{n} \boldsymbol{\omega}_{e,i}^0 = \sum_{i=1}^{n} \boldsymbol{\rho}_i \dot{q}_i \boldsymbol{z}_{i-1}^0$$

$\rho_i = \boldsymbol{I}$ if joint i is revolute

$\rho_i = \boldsymbol{O}$ if joint i is prismatic (no angular velocity from these)

Angular velocity contribution of ith joint in frame 0

- $\boldsymbol{z}_{i-1}^0$ is the unit vector describing z axis of frame $i-1$ relative to frame 0
- We can factor the above and write the *angular velocity Jacobian* $\boldsymbol{J}_O(\boldsymbol{q})$:

$$\boldsymbol{\omega}_e = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} \boldsymbol{\rho}_1 \boldsymbol{z}_0^0 & \cdots & \boldsymbol{\rho}_n \boldsymbol{z}_{n-1}^0 \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \boldsymbol{J}_O(\boldsymbol{q})\dot{\boldsymbol{q}}$$

$3 \times n$ matrix relating joint velocities $\dot{\boldsymbol{q}}$ to end effector angular velocities $\boldsymbol{\omega}_e$

Linear Velocity Jacobian

- Linear velocity from position forward kinematics: $\dot{\boldsymbol{p}}_e(\boldsymbol{q}) = (\dot{x}, \dot{y}, \dot{z})^T$

$$\dot{\boldsymbol{p}}_e(\boldsymbol{q}) = \frac{d}{dt}\begin{pmatrix} p_{e,x}(\boldsymbol{q}) \\ p_{e,y}(\boldsymbol{q}) \\ p_{e,z}(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} \frac{\partial p_{e,x}}{\partial q_1}\dot{q}_1 + \frac{\partial p_{e,x}}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial p_{e,x}}{\partial q_n}\dot{q}_n \\ \frac{\partial p_{e,y}}{\partial q_1}\dot{q}_1 + \frac{\partial p_{e,y}}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial p_{e,y}}{\partial q_n}\dot{q}_n \\ \frac{\partial p_{e,z}}{\partial q_1}\dot{q}_1 + \frac{\partial p_{e,z}}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial p_{e,z}}{\partial q_n}\dot{q}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial p_{e,x}}{\partial q_1} & \cdots & \frac{\partial p_{e,x}}{\partial q_n} \\ \frac{\partial p_{e,y}}{\partial q_1} & \cdots & \frac{\partial p_{e,y}}{\partial q_n} \\ \frac{\partial p_{e,z}}{\partial q_1} & \cdots & \frac{\partial p_{e,z}}{\partial q_n} \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}$$

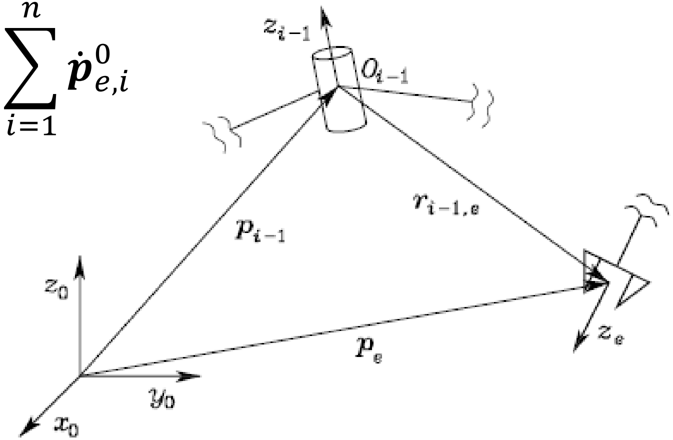
- $\boldsymbol{J}_P(\boldsymbol{q})$: *linear velocity Jacobian* relating joint velocities $\dot{\boldsymbol{q}}$ to end effector velocities $\dot{\boldsymbol{p}}_e$
  - $3 \times n$ matrix in 3D ($2 \times n$ in planar case)

- *Linear* mapping, dependent on joint configuration $\boldsymbol{q}$

Linear Velocity Jacobian

- Linear velocity from position forward kinematics: $\dot{\boldsymbol{p}}_e(\boldsymbol{q}) = (\dot{x}, \dot{y}, \dot{z})^T$
- As with $\boldsymbol{J}_O$, we can also break down $\boldsymbol{J}_P$ as sum of joint contributions

$$\dot{\boldsymbol{p}}_e = \sum_{i=1}^{n} \dot{\boldsymbol{p}}_{e,i}^0$$



- Linear velocity contribution from prismatic joint:

$$\dot{\boldsymbol{p}}_{e,i}^0 = \dot{d}_i \boldsymbol{z}_{i-1}^0 \qquad [J_{Pi}] = [\boldsymbol{z}_{i-1}^0]$$

- Linear velocity contribution from revolute joint:
  - Recall that $\dot{\boldsymbol{p}} = \boldsymbol{\omega} \times \boldsymbol{r}$

$$\dot{\boldsymbol{p}}_{e,i}^0 = \dot{\theta}_i \boldsymbol{z}_{i-1}^0 \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1}) \qquad [J_{Pi}] = [\boldsymbol{z}_{i-1}^0 \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})]$$

Vector from revolute joint to end effector

The Full Jacobian

- Together, the linear and angular velocity Jacobians fully describe the mapping between joint velocities $\dot{q}$ and end effector velocities $\boldsymbol{v}_e = (\dot{\boldsymbol{p}}_e, \boldsymbol{\omega}_e)^T$:

$$\boldsymbol{v}_e = \begin{pmatrix} \dot{\boldsymbol{p}}_e \\ \boldsymbol{\omega}_e \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_O(\boldsymbol{q}) \end{pmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

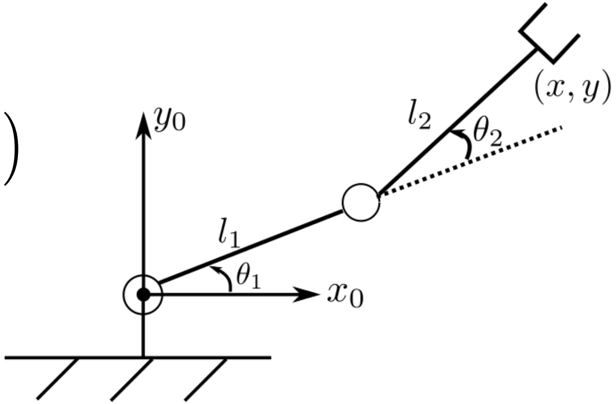
- For general 3D manipulators, J_P is $3 \times n$, J_O is $3 \times n$, and J is $6 \times n$
- For planar 2D manipulators, J_P is $2 \times n$, J_O is $1 \times n$, and J is $3 \times n$

- Unlike position kinematics, this mapping is linear(!!!)
- $\boldsymbol{J}$ depends on $\boldsymbol{q}$; velocity relationship changes depending on configuration

Example: RR Arm

- Planar robot, 2 joints—2×2 linear velocity Jacobian $\boldsymbol{J_P}$
- Partial derivatives of end effector position wrt $\boldsymbol{q}$:

$$\boldsymbol{J}_P = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \end{pmatrix} = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$



- Angular velocity Jacobian: $\boldsymbol{J}_O = \begin{pmatrix} z_0^0 & z_1^0 \end{pmatrix}$
- Both joints have constant, parallel z axes $(0,0,1)^T$
- Planar robot, so only need ω_z component:

$$\boldsymbol{J}_O = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

Also equal to $\begin{pmatrix} \frac{\partial \phi}{\partial q_1} & \frac{\partial \phi}{\partial q_2} \end{pmatrix}$!

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$
$$\phi = \theta_1 + \theta_2$$

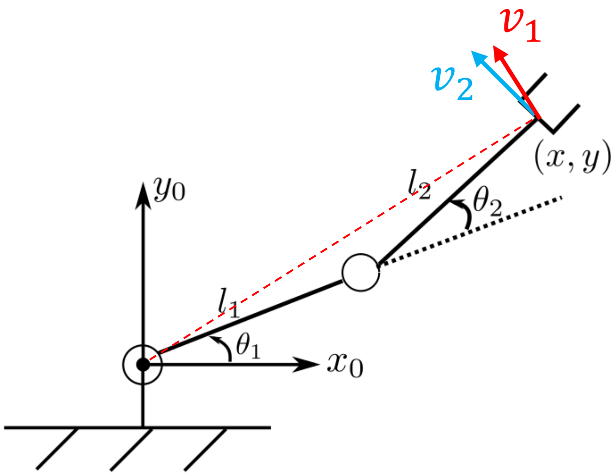
Example: RR Arm

- Putting the entire Jacobian together:

$$\begin{array}{c}
\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \\
\qquad\qquad\qquad {\color{red} v_1} \qquad\qquad\qquad\qquad\qquad {\color{cyan} v_2}
\end{array}$$

- Again, this is a *linear, configuration-dependent* mapping

- Each *row* shows the contribution of each joint to a component of the end effector velocity
- Each *column* shows the end effector's velocity due to unit velocity of a single joint



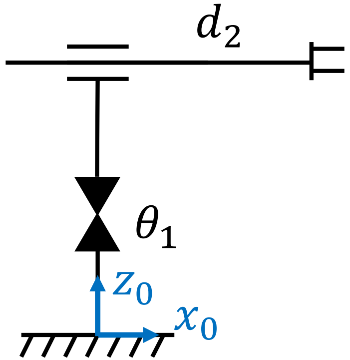
$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$
$$\phi = \theta_1 + \theta_2$$

Example: RP Arm

- Non-planar robot, 2 joints—3 × 2 linear velocity Jacobian

$$J_P = \begin{pmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} \end{pmatrix} = \begin{pmatrix} -d_2 \sin\theta_1 & \cos\theta_1 \\ d_2 \cos\theta_1 & \sin\theta_1 \\ 0 & 0 \end{pmatrix}$$

No velocity can be achieved in z direction!



$$x = d_2 \cos\theta_1$$
$$y = d_2 \sin\theta_1$$
$$z = l_1$$

- 3 × 2 angular velocity Jacobian
- Joint 1 is revolute, z axis given by $(0,0,1)^T$ in frame 0
- Joint 2 is prismatic, no angular velocity contribution

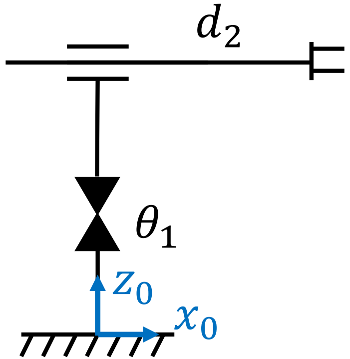
$$J_O = (\mathbf{z}_0^0 \quad \mathbf{0}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}$$

No angular velocity can be achieved in ω_x, ω_y!

Example: RP Arm

- Full Jacobian is 6×2:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = \begin{pmatrix} -d_2 \sin\theta_1 & \cos\theta_1 \\ d_2 \cos\theta_1 & \sin\theta_1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{pmatrix}$$



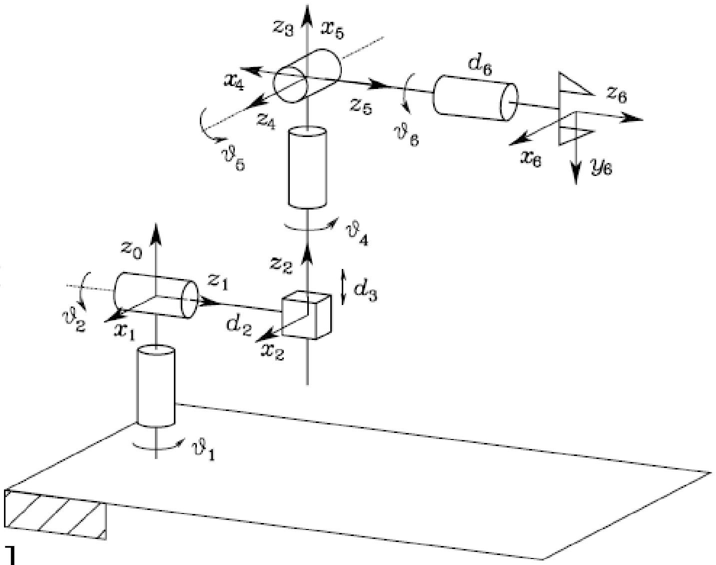
$$x = d_2 \cos\theta_1$$
$$y = d_2 \sin\theta_1$$
$$z = l_1$$

- This manipulator is effectively planar, since the end effector is limited to sweeping out a planar workspace

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \omega_z \end{pmatrix} = \begin{pmatrix} -d_2 \sin\theta_1 & \cos\theta_1 \\ d_2 \cos\theta_1 & \sin\theta_1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{d}_2 \end{pmatrix}$$

Example: Stanford Manipulator

- Recall: Spherical arm plus spherical wrist
- 2 revolute joints, 1 prismatic, 3 revolute joints



- DH gives us the intermediate transformations:

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Stanford Manipulator

- Building up consecutive homogeneous transforms between frames 0 and i:

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^0 = T_1^0 A_2 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & -d_2 s_1 \\ s_1 c_2 & c_1 & s_1 s_2 & d_2 c_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_3^0 = T_2^0 A_3 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 & c_1 & s_1 s_2 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 & 0 & c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_4^0 = T_3^0 A_4 = \begin{bmatrix} c_1 c_2 c_4 - s_1 s_4 & -c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 c_4 + c_1 s_4 & -s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 & -c_2 & s_2 s_4 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_5^0 = T_4^0 A_5 = \begin{bmatrix} (c_1 c_2 c_4 - s_1 s_4) c_5 - c_1 s_2 s_5 & c_1 c_2 s_4 + s_1 c_4 & (c_1 c_2 c_4 - s_1 s_4) s_5 + c_1 s_2 c_5 & d_3 c_1 s_2 - d_2 s_1 \\ (s_1 c_2 c_4 + c_1 s_4) c_5 - s_1 s_2 s_5 & s_1 c_2 s_4 - c_1 c_4 & (s_1 c_2 c_4 + c_1 s_4) s_5 + s_1 s_2 c_5 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 c_5 - c_2 s_5 & -s_2 s_4 & -s_2 c_4 s_5 + c_2 c_5 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_6^0 = T_5^0 A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & c_1 s_2 d_3 - s_1 d_2 + d_6 (c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ r_{21} & r_{22} & r_{23} & s_1 s_2 d_3 - c_1 d_2 + d_6 (c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ r_{31} & r_{32} & r_{33} & c_2 d_3 + d_6 (c_2 c_5 - c_4 s_2 s_5) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad p_e$$

Linear velocity Jacobian $\boldsymbol{J}_P$ can be found via partial derivatives here

Example: Stanford Manipulator

- Building up consecutive homogeneous transforms between frames 0 and i:

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & \boxed{\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}} \\ s_1 & 0 & c_1 & \\ 0 & -1 & 0 & \\ 0 & 0 & 0 & 1 \end{bmatrix} p_1 \quad T_2^0 = T_1^0 A_2 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & -d_2s_1 \\ s_1c_2 & c_1 & s_1s_2 & d_2c_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^0 = T_2^0 A_3 = \begin{bmatrix} c_1c_2 & -s_1 & c_1s_2 & \boxed{\begin{matrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{matrix}} \\ s_1c_2 & c_1 & s_1s_2 & \\ -s_2 & 0 & c_2 & \\ 0 & 0 & 0 & 1 \end{bmatrix} p_3$$

$$T_4^0 = T_3^0 A_4 = \begin{bmatrix} c_1c_2c_4 - s_1s_4 & -c_1s_2 & -c_1c_2s_4 - s_1c_4 & \boxed{\begin{matrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{matrix}} \\ s_1c_2c_4 + c_1s_4 & -s_1s_2 & -s_1c_2s_4 + c_1c_4 & \\ -s_2c_4 & -c_2 & s_2s_4 & \\ 0 & 0 & 0 & 1 \end{bmatrix} p_4$$

$$T_5^0 = T_4^0 A_5 = \begin{bmatrix} (c_1c_2c_4 - s_1s_4)c_5 - c_1s_2s_5 & c_1c_2s_4 + s_1c_4 & (c_1c_2c_4 - s_1s_4)s_5 + c_1s_2c_5 & \boxed{\begin{matrix} d_3c_1s_2 - d_2s_1 \\ d_3s_1s_2 + d_2c_1 \\ d_3c_2 \end{matrix}} \\ (s_1c_2c_4 + c_1s_4)c_5 - s_1s_2s_5 & s_1c_2s_4 - c_1c_4 & (s_1c_2c_4 + c_1s_4)s_5 + s_1s_2c_5 & \\ -s_2c_4c_5 - c_2s_5 & -s_2s_4 & -s_2c_4s_5 + c_2c_5 & \\ 0 & 0 & 0 & 1 \end{bmatrix} p_5$$

$$T_6^0 = T_5^0 A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & \boxed{\begin{matrix} c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) \\ s_1s_2d_3 - c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) \\ c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) \end{matrix}} \\ r_{21} & r_{22} & r_{23} & \\ r_{31} & r_{32} & r_{33} & \\ 0 & 0 & 0 & 1 \end{bmatrix} p_e$$

$\boldsymbol{J}_P$ can also be found column by column using $[\boldsymbol{z}_{i-1}^0 \times (\boldsymbol{p}_e - \boldsymbol{p}_{i-1})]$ for revolute joints and $[\boldsymbol{z}_{i-1}^0]$ for prismatic

Example: Stanford Manipulator

- Building up consecutive homogeneous transforms between frames 0 and i:

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_1^0$$

$$T_2^0 = T_1^0 A_2 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & -d_2 s_1 \\ s_1 c_2 & c_1 & s_1 s_2 & d_2 c_1 \\ -s_2 & 0 & c_2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_2^0$$

$$T_3^0 = T_2^0 A_3 = \begin{bmatrix} c_1 c_2 & -s_1 & c_1 s_2 & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 & c_1 & s_1 s_2 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 & 0 & c_2 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_3^0$$

$$T_4^0 = T_3^0 A_4 = \begin{bmatrix} c_1 c_2 c_4 - s_1 s_4 & -c_1 s_2 & -c_1 c_2 s_4 - s_1 c_4 & d_3 c_1 s_2 - d_2 s_1 \\ s_1 c_2 c_4 + c_1 s_4 & -s_1 s_2 & -s_1 c_2 s_4 + c_1 c_4 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 & -c_2 & s_2 s_4 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_4^0$$

$$T_5^0 = T_4^0 A_5 = \begin{bmatrix} (c_1 c_2 c_4 - s_1 s_4) c_5 - c_1 s_2 s_5 & c_1 c_2 s_4 + s_1 c_4 & (c_1 c_2 c_4 - s_1 s_4) s_5 + c_1 s_2 c_5 & d_3 c_1 s_2 - d_2 s_1 \\ (s_1 c_2 c_4 + c_1 s_4) c_5 - s_1 s_2 s_5 & s_1 c_2 s_4 - c_1 c_4 & (s_1 c_2 c_4 + c_1 s_4) s_5 + s_1 s_2 c_5 & d_3 s_1 s_2 + d_2 c_1 \\ -s_2 c_4 c_5 - c_2 s_5 & -s_2 s_4 & -s_2 c_4 s_5 + c_2 c_5 & d_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_5^0$$

$$T_6^0 = T_5^0 A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & c_1 s_2 d_3 - s_1 d_2 + d_6 (c_1 c_2 c_4 s_5 + c_1 c_5 s_2 - s_1 s_4 s_5) \\ r_{21} & r_{22} & r_{23} & s_1 s_2 d_3 - c_1 d_2 + d_6 (c_1 s_4 s_5 + c_2 c_4 s_1 s_5 + c_5 s_1 s_2) \\ r_{31} & r_{32} & r_{33} & c_2 d_3 + d_6 (c_2 c_5 - c_4 s_2 s_5) \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad z_6^0$$

Angular velocity Jacobian $\boldsymbol{J_O}$ can be built as $\left[\boldsymbol{\rho}_i \boldsymbol{z}_{i-1}^0\right]$ by extracting third column of each transform's rotation

Kinematic Singularities

- Since $\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ is a linear mapping, we may be tempted to invert it
- Even if $\boldsymbol{J}(\boldsymbol{q})$ is square, certain configurations may not be "invertible"

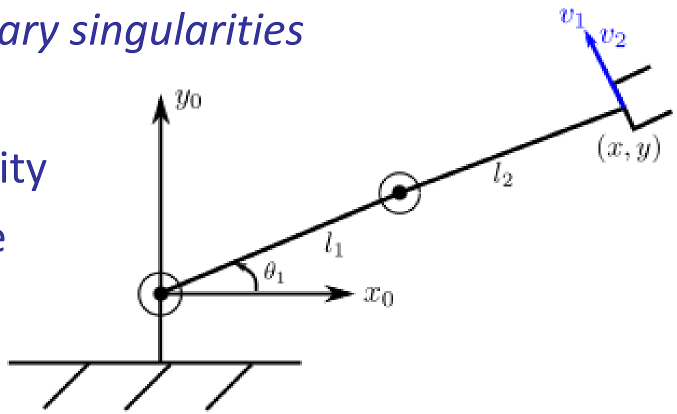
- **Kinematic singularity**: Configuration $\boldsymbol{q}_s$ for which $\boldsymbol{J}(\boldsymbol{q})$ is not invertible
  - Algebraically, $\boldsymbol{J}(\boldsymbol{q})$ loses rank and $\det \boldsymbol{J}(\boldsymbol{q}) = 0$; rows and columns become linearly dependent and $\boldsymbol{J}(\boldsymbol{q})$ gains eigenvalues equal to 0

- Physically, mobility of the robot is reduced; it may be very difficult or require large velocities in the joints to produce end effector movement

Example: RR Arm

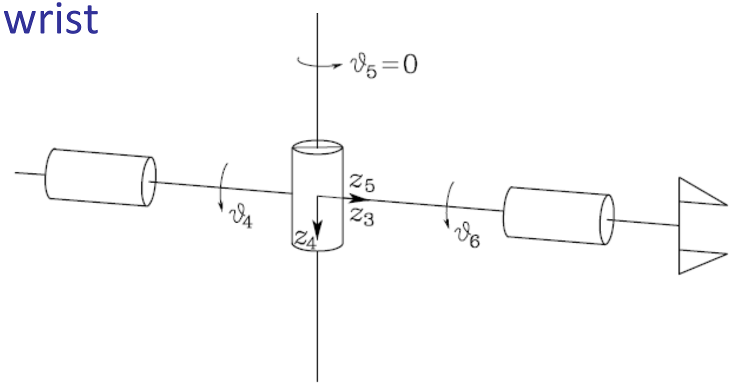
$$J_P = \begin{pmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{pmatrix}$$

- Consider the linear velocity Jacobian of the RR arm, since $\boldsymbol{J}_O$ is constant
- $\boldsymbol{J}_P$ loses rank when $0 = \det \boldsymbol{J}_P = -l_2 c_{12}(l_1 s_1 + l_2 s_{12}) + l_2 s_{12}(l_1 c_1 + l_2 c_{12}) = l_1 l_2 s_2$
- In other words, $\theta_2 = 0$ or $\theta_2 = \pi$: examples of *boundary singularities*
- Moving either joint produces same end effector velocity
- Allowable velocity directions no longer span the plane



Wrist Singularities

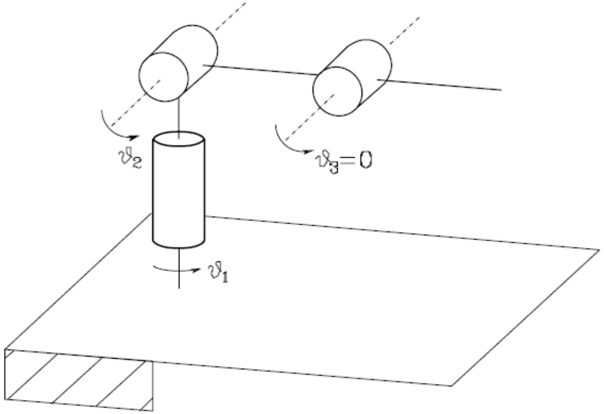
- Consider angular velocity Jacobian $\boldsymbol{J}_O$ of spherical wrist

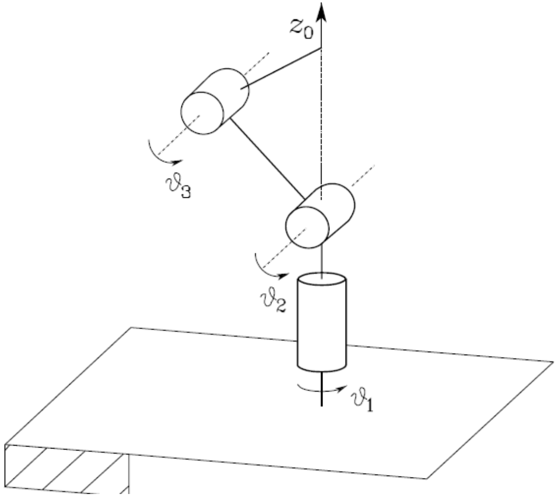


- Singularities occur when $\theta_5 = 0$, $\theta_5 = \pi$
- z_3 and z_5 rotation axes are aligned

- Loss of mobility since rotations in θ_4 and θ_6 produce same end effector movements
- Attempting to solve inverse problem will not yield unique solutions

Arm Singularities





- *Elbow singularity*: $\theta_3 = 0$, $\theta_3 = \pi$
- Outstretched arm at workspace boundary

- *Shoulder singularity*: $a_2 c_2 + a_3 c_{23} = 0$
- Arm end point intersects z_0 rotation axis
- θ_1 rotations produce no end effector movement

Summary

- Differential kinematics provides a linear, configuration-dependent mapping between joint velocities and operational velocities
- Jacobian derivation can be done separately for linear and angular velocities
- Jacobian shows individual joint contributions to overall end effector velocity
- Manipulators singularities can occur at workspace boundaries or in configurations where the arm loses operational degrees of freedom