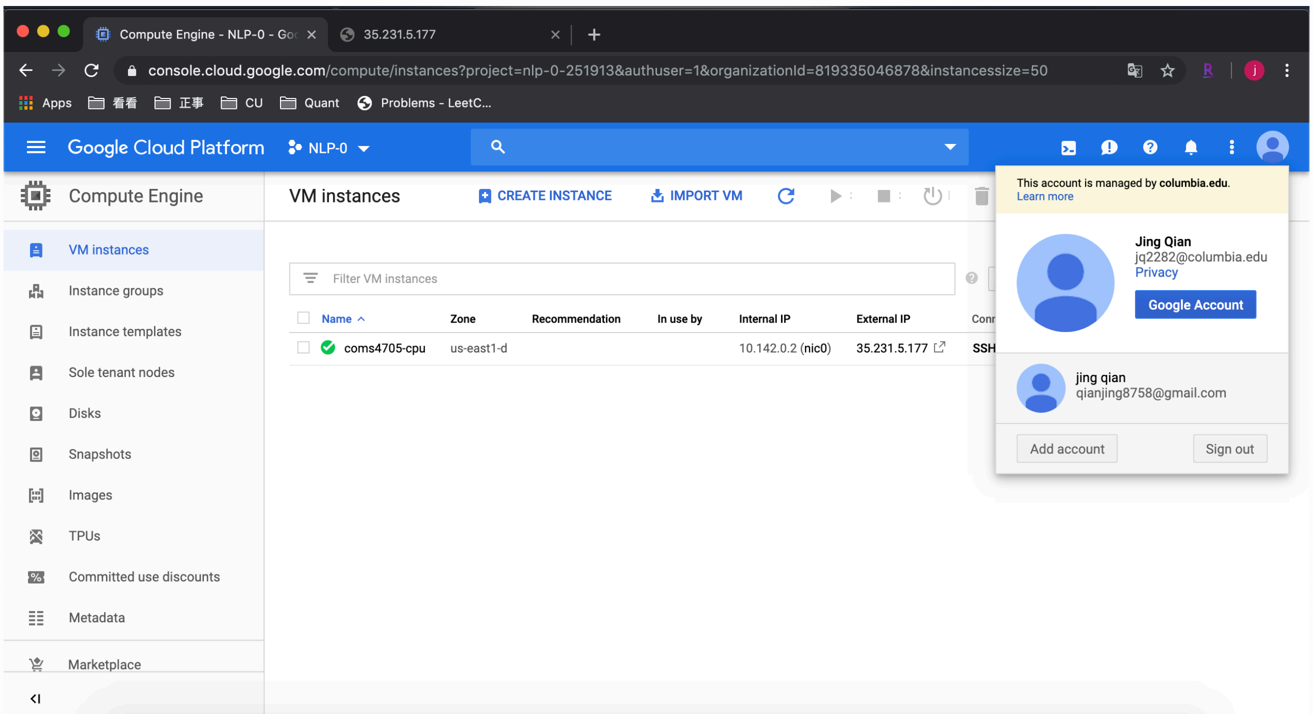
# HOMEWORK 0 (COMS W4705)

Jing Qian (jq2282@columbia.edu)

## 1. Environment Setup and Programming

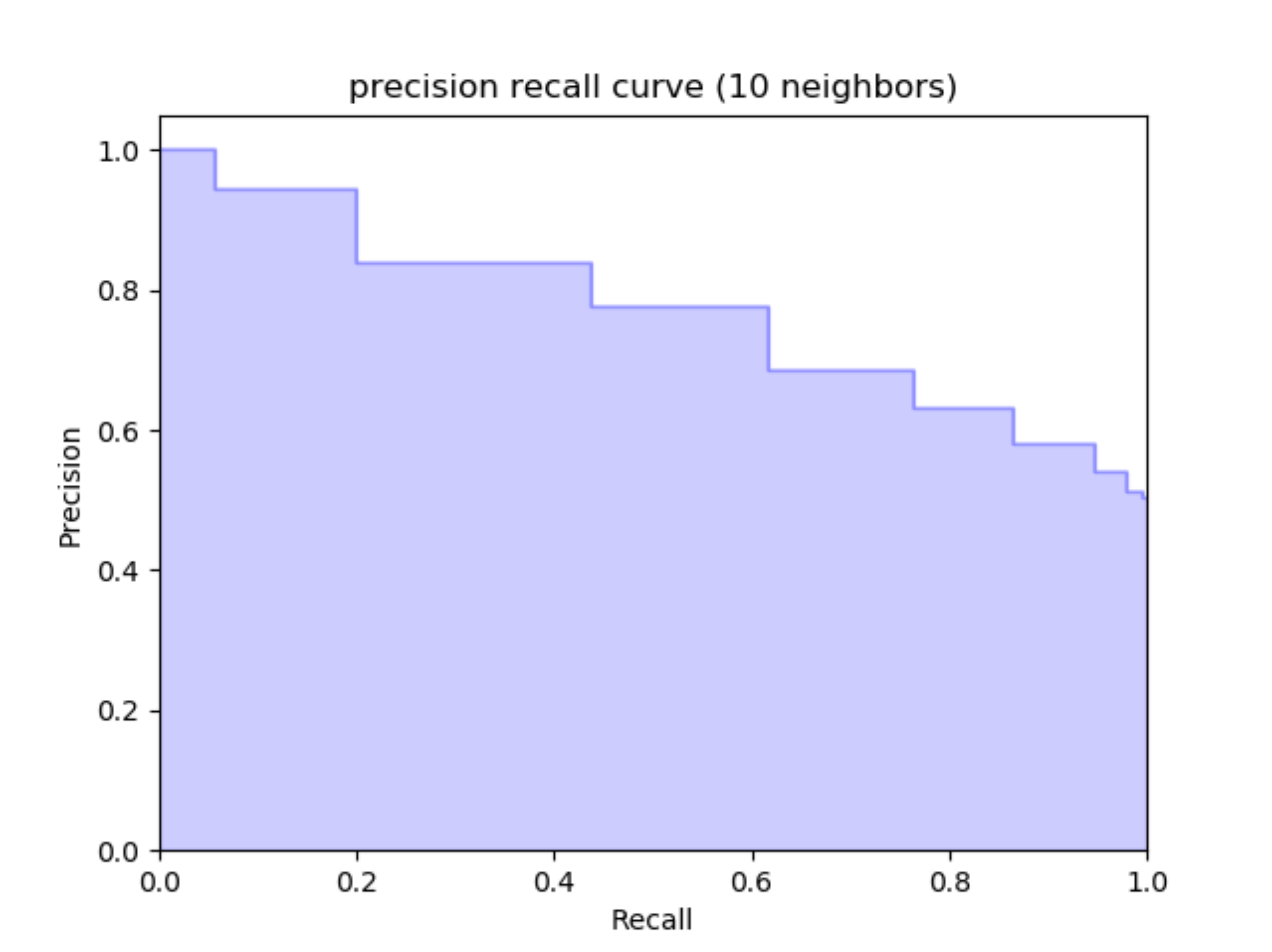
### 1.1 Enviroment Setup

Following is the screenshot of my Google Cloud "VM instances" page showing my virtual machine running.

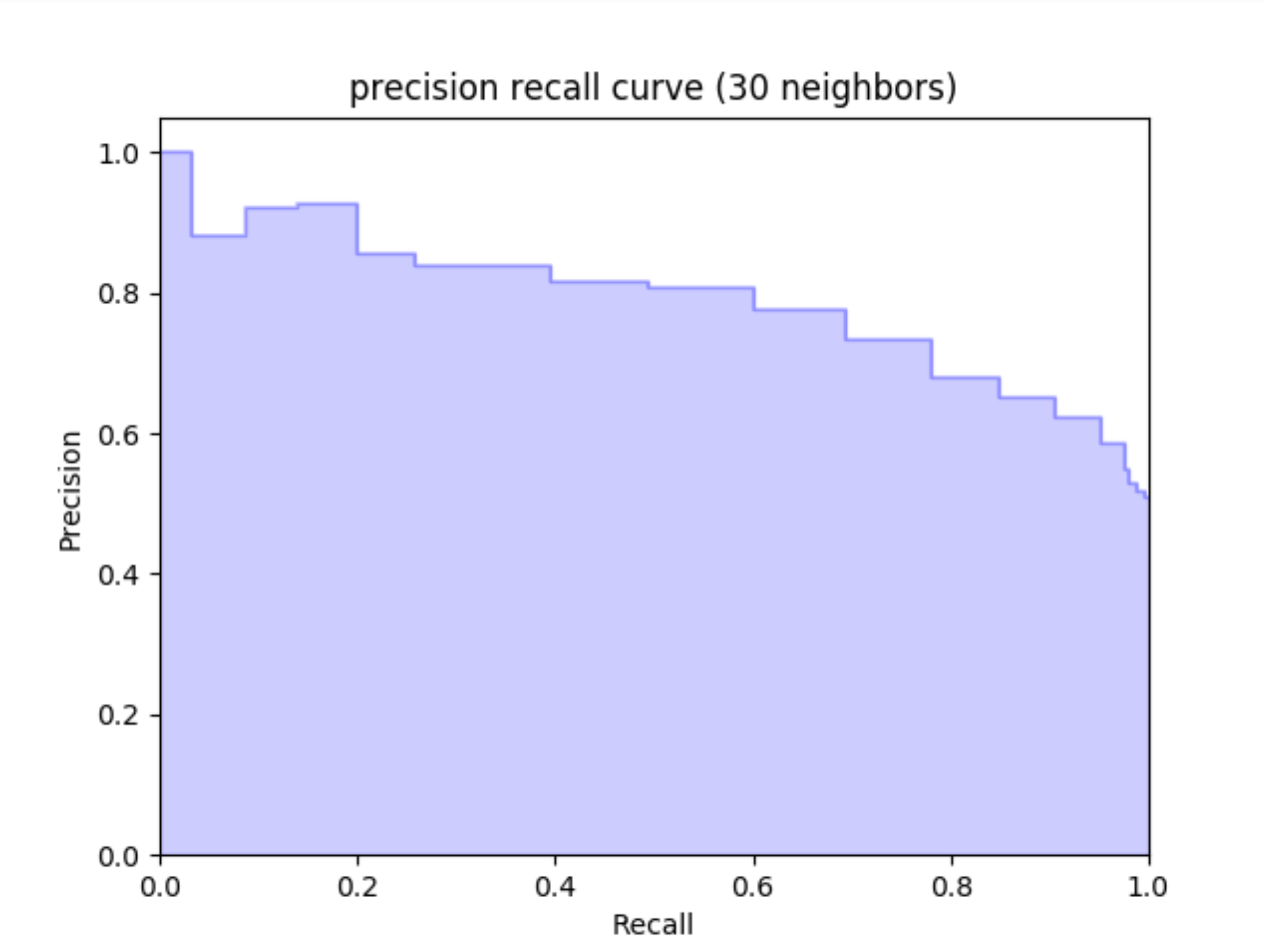



### 1.2 Programming

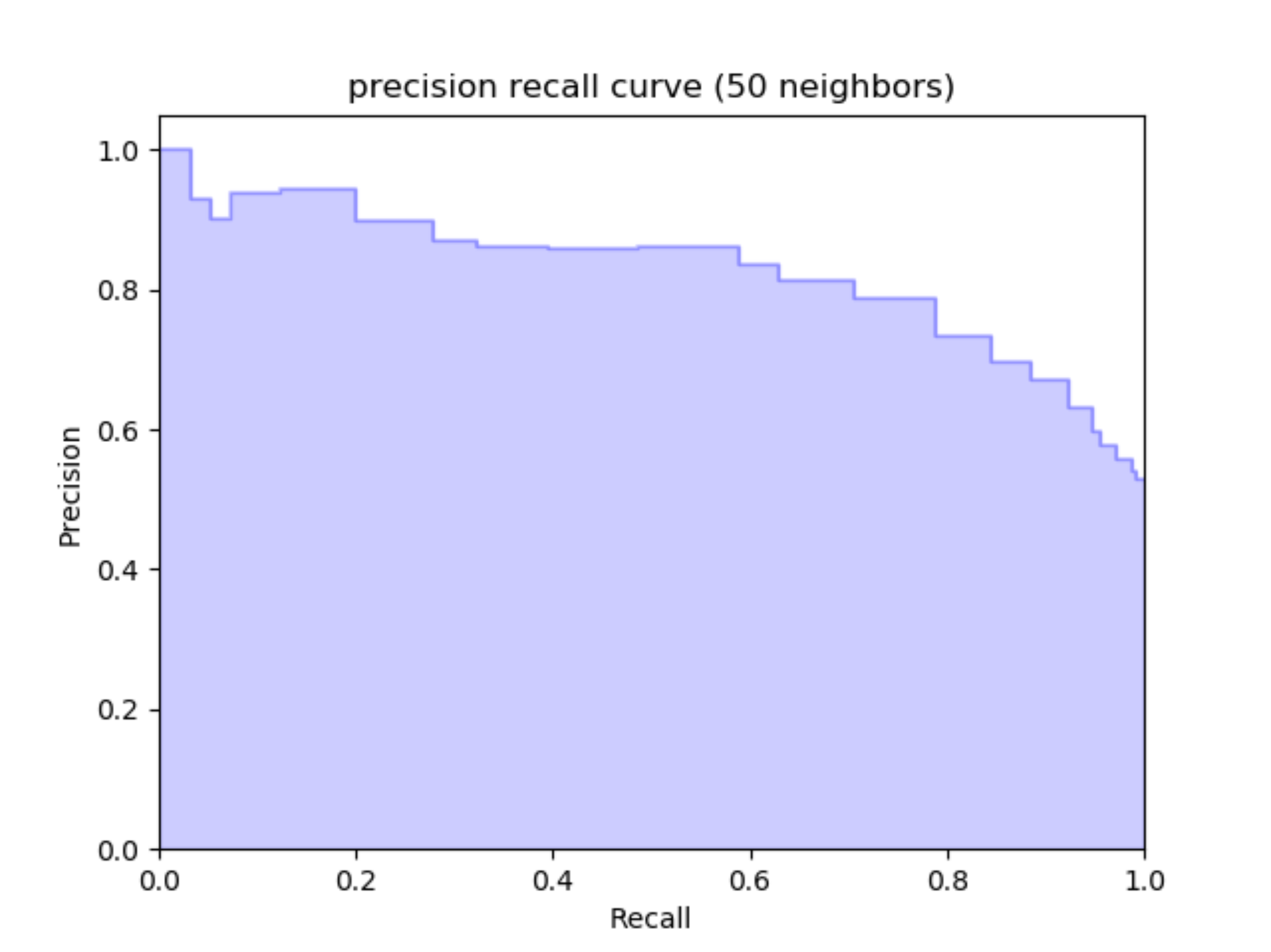
- Plot the precision-recall curve for the existing classifier:




- Plot the precision-recall curve when the number of neighbors is 30 instead.




- Plot the precision-recall curve when the number of neighbors is 50 instead.




# 2. Calculus

## 2.1 Chain rule and multivariate derivatives

**i)** From the functions provided, we have:

$$g(x,y) = x^2y - xh(x^2,y) = x^2y - x(x^2y^2+5) = x^2y - x^3y^2 - 5x$$

So:

$$\frac{\partial f}{\partial x} = g + x\frac{\partial g}{\partial x} = (x^2y - x^3y^2 - 5x) + x(2xy - 3x^2y^2 - 5) = 3x^2y - 4x^3y^2 - 10x$$
$$\frac{\partial f}{\partial y} = x\frac{\partial g}{\partial y} + 2 = x(x^2 - 2x^3y) + 2 = -2x^4y + x^3 + 2$$

**ii)**

$$\frac{\partial f}{\partial x} = 1/y^2 + z\exp(x^2)(2x) = 1/y^2 + 2xz\exp(x^2)$$
$$\frac{\partial f}{\partial y} = x(-2)/y^3 = -2x/y^3$$
$$\frac{\partial f}{\partial z} = \exp(x^2)$$

## 2.2 Maxima and minima

From the expression of $f(x)$, we know that $f(x)$ is symmetrical about $x = 1/2$. Although $\log 0$ is not defined, as $x$ approaches the 0, $\lim_{x->0^+} x\log_2 x = 0$. So at the lower bound 0 or the upper bound 1 of the domain of $x$, the limit of $f(x)$ is 0:

$$\lim_{x->0^+} f(x) = \lim_{x->0^+} x\log_2 x = 0$$
$$\lim_{x->1^-} f(x) = \lim_{x->0^+} x\log_2 x = 0$$

We get the first and second derivatives of $f(x)$ as following:

$$f'(x) = \log_2 x + x\frac{1}{x\ln 2} - \log_2(1-x) + (1-x)\frac{1}{-(1-x)\ln 2} = \log_2\frac{x}{1-x}$$
$$f''(x) = \frac{1}{\frac{x}{1-x}\ln 2}\frac{(1-x)+x}{(1-x)^2} = \frac{1}{x(1-x)\ln 2}$$

For $x \in (0,1)$, the first and second derivatives of $f(x)$ are continuous, which could help us find the maxima and minima.

When $x = 1/2$, $f'(x) = 0$ and $f''(x) > 0$. So the **minima** of $f(x)$ for $x \in [0,1]$ is $f(1/2) = -1$ .Since $f(x)$ is symmetrical about $x = 1/2$, which is also the minima point and $f''(x)$ is positive over $x \in (0,1)$, $f(x)$ increases as $x$ increases from 1/2 to 1. Also, $f(x)$ increases as $x$ decreases from 1/2 to 0. So the **maxima** of $f(x)$ for $x \in [0,1]$ is $f(0) = f(1) = 0$.

# 3. Probability and Statistics

## 3.1 Conditional probability

Here we use $P(b)$ to denote the probability of the first strip is "buffalo", and hence $P(b,b|b)$ denotes the probability that the second strip is also "buffalo" given the first strip is "buffalo". Then the probability that we pull out the following words "buffalo buffalo buffalo" is:

$$P(b,b,b) = P(b)P(b,b|b)P(b,b,b|b,b) = \frac{5}{10} * \frac{5-1}{10-1} * \frac{5-2}{10-2} = \frac{1}{12}$$

## 3.2 Bayes' rule

Here event $X$ is the fact that I get a text from Maria about dogs, event Y is that the sender is Maria B and event Z is that the sender is Maria A. So using the Bayes' rule to calculate the probability this dog-content message is from Maria is:

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)} = \frac{P(X|Y)P(Y)}{P(X|Y)P(Y) + P(X|Z)P(Z)} = \frac{90\% * 50\%}{90\% * 50\% + 10\% * 50\%} = 90\%.$$

# 4 Linear Algebra

## 4.1 Basic matrix operations

**i)** According to the definition of the matrix multiplication, the matrix product $\mathbf{C}_{n\times p} = \mathbf{A}_{n\times m}\mathbf{B}_{m\times p}$ has element calculated as $c_{ij} = \sum_{k=1}^{m} a_{ik}b_{kj}$. So we have:

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 5 & 2 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0*1+2*4 & 0*5+2*3 & 0*2+2*1 \\ 1*1+3*4 & 1*5+3*3 & 1*2+3*1 \\ 2*1+0*4 & 2*5+0*3 & 2*2+0*1 \end{bmatrix} = \begin{bmatrix} 8 & 6 & 2 \\ 13 & 14 & 5 \\ 2 & 10 & 4 \end{bmatrix}$$

**ii)** According to the definition of covariance and standard deviation, we have:

$$\mathrm{corr}(\mathbf{u},\mathbf{v}) = \frac{\mathrm{cov}(\mathbf{u},\mathbf{v})}{\sigma_{\mathbf{u}}\sigma_{\mathbf{v}}} = \frac{\frac{1}{n}\sum_{i=1}^{n} u_i v_i - (\frac{1}{n}\sum_{i=1}^{n} u_i)(\frac{1}{n}\sum_{i=1}^{n} v_i)}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} u_i^2 - (\frac{1}{n}\sum_{i=1}^{n} u_i)^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n} v_i^2 - (\frac{1}{n}\sum_{i=1}^{n} v_i)^2}}$$

Since $\mathbf{u}$ and $\mathbf{v}$ both have zero elementwise mean, then:

$$\mathrm{corr}(\mathbf{u},\mathbf{v}) = \frac{\mathrm{cov}(\mathbf{u},\mathbf{v})}{\sigma_{\mathbf{u}}\sigma_{\mathbf{v}}} = \frac{\frac{1}{n}\sum_{i=1}^{n} u_i v_i}{\sqrt{\frac{1}{n}\sum_{i=1}^{n} u_i^2}\sqrt{\frac{1}{n}\sum_{i=1}^{n} v_i^2}} = \frac{\sum_{i=1}^{n} u_i v_i}{\sqrt{\sum_{i=1}^{n} u_i^2}\sqrt{\sum_{i=1}^{n} v_i^2}} = \frac{|\mathbf{u}||\mathbf{v}|\cos\theta}{|\mathbf{u}||\mathbf{v}|} = \cos\theta.$$

So the correlation between the elements of $\mathbf{u}$ and $\mathbf{v}$ is equal to their cosine similarity.

## 4.2 Singular Value Decomposition

**i)** Since $M = U\Sigma V^T$, $U$ has the same number of rows as that of $M$, which is $m$ and $V^T$ has the same number of columns as that of $M$, which is $n$. So $V$ has $n$ rows. Since $U$ and $V$ are orthogonal matrices, they are both square matrices. So the dimension of $U$ is $m \times m$ and the dimension of $V$ is $n \times n$.

According to the definition of matrix muplitication, the dimension of $\Sigma$ is $m \times n$ to make the multiplication between $U$ and $\Sigma$, $\Sigma$ and $V^T$ possible.

**ii)** Since $U$ and $V$ are orthogonal matrices, the product of each matrix with its transpose are identity matrices, which means, $UU^T = I_m,\ VV^T = I_n$.

If matrix $M$ is invertible, which means $m = n = \mathbf{rank}(M)$, the inverse of $M$ is $M^{-1} = V\Sigma^{-1}U^T$. Here $\Sigma$ is a symmetric diagonal matrics and all its diagonal entries are non-zero, so we could get $\Sigma^{-1}$ by replacing all the diagonal entries of $\Sigma$ with their reciprocal and have $\Sigma\Sigma^{-1} = I_m$. So $MM^{-1} = U\Sigma V^T V\Sigma^{-1}U^T = U\Sigma(V^T V)\Sigma^{-1}U^T = U(\Sigma\Sigma^{-1})U^T = UU^T = I_m$, which suggests that $M^{-1} = V\Sigma^{-1}U^T$ is the inverse matrix of $M$ if $M$ is invertible.

On the other hand, if matrix is not invertible, we could use SVD to get the pseudoinverse in the similar way: $M^+ = V\Sigma^+U^T$. The difference here is that: $m$ may not equal to $n$ and min$(m, n)$ may not equal to the rank of $M$. $\Sigma^+$ is the pseudoinverse of $\Sigma$, which could be calculated by replacing every non-zero diagonal entry in $\Sigma$ by its reciprocal and transposing the resulting matrix. So $\Sigma\Sigma^+$ is a $m \times m$ diagonal matrix and all its non-zero diagonal entries are 1. Then $MM^+ = U\Sigma V^T V\Sigma^+U^T = U(\Sigma\Sigma^+)U^T$, which is also a $m \times m$ diagonal matrix and all its non-zero diagonal entries are 1. Also, $M^+M = V\Sigma^+U^T U\Sigma V^T = V(\Sigma^+\Sigma)V^T$, which is a $n \times n$ diagonal matrix and all its non-zero diagonal entries are 1.