Fluid Mechanics Homework #5

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1、写出不可压缩牛顿流体的连续性方程和运动方程在直角坐标系中无量纲形式的方程组和边界条件。

解：设外力为重力，重力方向沿z轴负方向，则不可压缩粘性流体运动的方程组及初边条件具有下列形式：

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right) \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\nu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}+\dfrac{\partial^2 v}{\partial z^2}\right) \\ \dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}=-g-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\nu\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}\right) \end{cases} \tag{1.1}$$

边界条件：

$$\begin{cases} (1)\ \text{在固壁面上}\ \boldsymbol{v}=0 \\ (2)\ \text{在自由面上}\ p_{nn}=-p_0,\ p_{n\tau}=0 \\ (3)\ \text{在无穷远处}\ \boldsymbol{v}=\boldsymbol{v}_\infty \end{cases} \tag{1.2}$$

初始条件：在$t=t_0$时

$$\boldsymbol{v}=\boldsymbol{v}(x,y,z),\ p=p(x,y,z) \tag{1.3}$$

引进特征时间T，特征长度L，特征速度V_∞，特征压力P，有量纲量和无量纲量之间的关系如下：

$$\begin{cases} t=Tt',\ x=Lx',\ y=Ly',\ z=Lz' \\ u=V_\infty u',\ v=V_\infty v',\ w=V_\infty w',\ p=Pp' \end{cases} \tag{1.4}$$

定义$\mathrm{St}=\dfrac{L}{V_\infty T}$为 Strouhal 数，$E=\dfrac{P}{\rho V_\infty^2}$为 Euler 数，$\mathrm{Re}=\dfrac{V_\infty L}{\nu}$为 Reynolds 数，$F=\dfrac{V_\infty^2}{gL}$为 Froude 数，将式(1.4)代入式(1.1)、式(1.2)和式(1.3)中得到无量纲形式的方程组及边界条件：

$$
\begin{cases}
\dfrac{\partial u'}{\partial x'}+\dfrac{\partial v'}{\partial y'}+\dfrac{\partial w'}{\partial z'}=0 \\
\mathrm{St}\dfrac{\partial u'}{\partial t'}+u'\dfrac{\partial u'}{\partial x'}+v'\dfrac{\partial u'}{\partial y'}+w'\dfrac{\partial u'}{\partial z'} \\
\qquad =-E\dfrac{\partial p'}{\partial x'}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 u'}{\partial x'^2}+\dfrac{\partial^2 u'}{\partial y'^2}+\dfrac{\partial^2 u'}{\partial z'^2}\right) \\
\mathrm{St}\dfrac{\partial v'}{\partial t'}+u'\dfrac{\partial v'}{\partial x'}+v'\dfrac{\partial v'}{\partial y'}+w'\dfrac{\partial v'}{\partial z'} \\
\qquad =-E\dfrac{\partial p'}{\partial y'}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 v'}{\partial x'^2}+\dfrac{\partial^2 v'}{\partial y'^2}+\dfrac{\partial^2 v'}{\partial z'^2}\right) \\
\mathrm{St}\dfrac{\partial w'}{\partial t'}+u'\dfrac{\partial w'}{\partial x'}+v'\dfrac{\partial w'}{\partial y'}+w'\dfrac{\partial w'}{\partial z'} \\
\qquad =-\dfrac{1}{F}-E\dfrac{\partial p'}{\partial z'}+\dfrac{1}{\mathrm{Re}}\left(\dfrac{\partial^2 w'}{\partial x'^2}+\dfrac{\partial^2 w'}{\partial y'^2}+\dfrac{\partial^2 w'}{\partial z'^2}\right)
\end{cases}
\tag{1.5}
$$

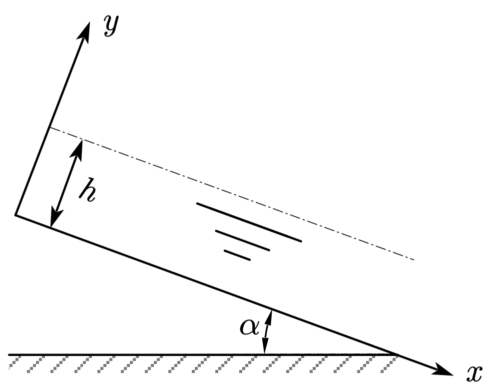
边界条件：

$$
\begin{cases}
\text{(1) 在无量纲化后的固壁面上 } v'=0 \\
\text{(2) 在无量纲化后的自由面上 } p'_{nn}=-\dfrac{p_0}{P},\ p'_{n\tau}=0 \\
\text{(3) 在无穷远处 } v'=\dfrac{v_\infty}{V_\infty}
\end{cases}
\tag{1.6}
$$

初始条件：在$t'=t'_0$时

$$
v'=v'(x,y,z),\ p'=p'(x,y,z) \tag{1.7}
$$

2、带有自由面的不可压缩粘性流体在倾斜板上由于重力作用发生流动。设斜板为无限平面，它与水平面的倾角为$\boldsymbol{\alpha}$。设流动是定常的平行直线运动，流体深为$\boldsymbol{h}$。求流体速度分布、流量、平均速度、最大速度及作用在板上的摩擦力。



题 2 图

解：建立如题 2 图所示的直角坐标系，依题意可知，$v=w=0$，$\frac{\partial}{\partial x}=\frac{\partial}{\partial z}=0$，$u=u(y)$，由粘性不可压缩流体的连续性方程和运动方程可得：

$$
\begin{cases}
\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \\
\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=0 \\
\qquad =g_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \\
\qquad =g\sin\alpha+\nu\frac{\partial^2 u}{\partial y^2} \\
\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=0 \\
\qquad =g_y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \\
\qquad =-g\cos\alpha-\frac{1}{\rho}\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=0 \\
\qquad =g_z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \\
\qquad =0
\end{cases}
$$

即：

$$
\begin{cases}
\frac{\partial^2 u}{\partial y^2}=-\frac{g\sin\alpha}{\nu} \\
\frac{\partial p}{\partial y}=-\rho g\cos\alpha
\end{cases}
\tag{2.1}
$$

解式(2.1)得：

$$
\begin{cases}
u=-\frac{g\sin\alpha}{2\nu}y^2+C_1y+C_2 \\
p=-\rho gy\cos\alpha
\end{cases}
\tag{2.2}
$$

边界条件为：

$$
\begin{cases}
y=0, \quad u=0 \\
y=h, \quad \frac{\partial u}{\partial y}=0
\end{cases}
\tag{2.3}
$$

将边界条件式(2.3)代入式(2.2)可知：

$$C_2=0,\ C_1=\frac{gh\sin\alpha}{v} \tag{2.4}$$

所以，流体的速度分布为：

$$u=\frac{g\sin\alpha}{2v}(2hy-y^2) \tag{2.5}$$

流体速度u为二次函数，易知最大速度在$y=h$处取得，最大速度为：

$$u_{\max}=\frac{gh^2\sin\alpha}{2v} \tag{2.6}$$

在$y=0$到$y=h$上的平均速度为：

$$\begin{aligned}\overline{u}&=\frac{1}{h}\int_0^h u(y)\,\mathrm{d}y\\&=\frac{1}{h}\int_0^h\frac{gh\sin\alpha}{2v}(2hy-y^2)\,\mathrm{d}y\\&=\frac{1}{h}\frac{gh\sin\alpha}{2v}\left(hy^2-\frac{1}{3}y^3\right)\bigg|_0^h\\&=\frac{gh^2\sin\alpha}{3v}\end{aligned} \tag{2.7}$$

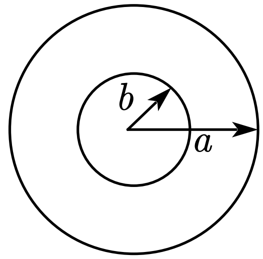
流量为平均速度与流体深度的乘积：

$$Q=\overline{u}h=\frac{gh^3\sin\alpha}{3v} \tag{2.8}$$

作用在板上的摩擦切应力为：

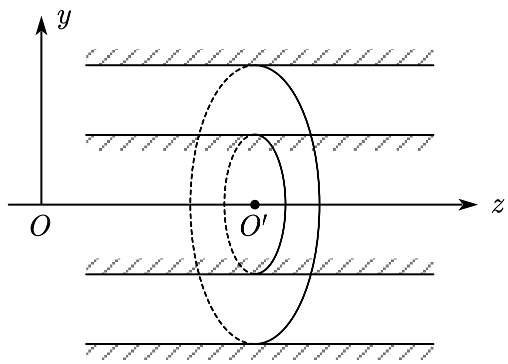
$$\begin{aligned}\tau_{\text{板}}&=\mu\frac{\partial u}{\partial y}\bigg|_{y=0}\\&=\mu\cdot\frac{gh\sin\alpha}{v}\\&=\rho gh\sin\alpha\end{aligned} \tag{2.9}$$

3、考虑两个同轴圆柱面间的粘性不可压缩流体由于压力梯度而产生的运动。设两圆柱的半径分别为a和b，长度为无限。试求该流动的速度分布和管壁上所受的粘性摩擦力。设运动定常，不计外力，沿管轴方向的压力梯度为常数。



题 3 图

解：如下图所示，以轴线为z建立坐标系：



题 3 解图

依题意，$v_\theta = v_r = 0$，$\dfrac{\partial}{\partial\theta} = \dfrac{\partial}{\partial z} = 0$，$v_z = v_z(r)$，柱坐标下粘性不可压缩流体的连续性方程和运动方程为：

$$\begin{cases} \nabla\cdot\mathbf{v} = \dfrac{1}{r}\dfrac{\partial(rv_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial\theta} + \dfrac{\partial v_z}{\partial z} = 0 \\ \dfrac{\partial v_r}{\partial t} + \mathbf{v}\cdot\nabla v_r - \dfrac{v_\theta^2}{r} = F_r - \dfrac{1}{\rho}\dfrac{\partial p}{\partial r} + \nu\left(\Delta v_r - \dfrac{2}{r^2}\dfrac{\partial v_\theta}{\partial\theta} - \dfrac{v_r}{r^2}\right) \\ \dfrac{\partial v_\theta}{\partial t} + \mathbf{v}\cdot\nabla v_\theta + \dfrac{v_r v_\theta}{r} = F_\theta - \dfrac{1}{\rho r}\dfrac{\partial p}{\partial\theta} + \nu\left(\Delta v_\theta + \dfrac{2}{r^2}\dfrac{\partial v_r}{\partial\theta} - \dfrac{v_\theta}{r^2}\right) \\ \dfrac{\partial v_z}{\partial t} + \mathbf{v}\cdot\nabla v_z = F_z - \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} + \nu\Delta v_z \end{cases} \tag{3.1}$$

其中：

$$\mathbf{v}\cdot\nabla = v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z} \tag{3.2}$$

$$\Delta=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2} \tag{3.3}$$

所以，

$$\begin{cases}\dfrac{\partial p}{\partial r}=0\\ \dfrac{\partial p}{\partial\theta}=0\\ \dfrac{\partial p}{\partial z}=\mu\cdot\dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial v_z}{\partial r}\right)\end{cases} \tag{3.4}$$

解偏微分方程得：

$$\begin{aligned}&\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)=r\cdot\frac{1}{\mu}\frac{\partial p}{\partial z}\\ &\xRightarrow{\text{两边对}r\text{积分}} r\frac{\partial v_z}{\partial r}=\frac{1}{2\mu}\frac{\partial p}{\partial z}r^2+C_1\\ &\xRightarrow{\text{两边除以}r}\frac{\partial v_z}{\partial r}=\frac{1}{2\mu}\frac{\partial p}{\partial z}r+\frac{C_1}{r}\\ &\xRightarrow{\text{两边再对}r\text{积分}} v_z=\frac{1}{4\mu}\frac{\partial p}{\partial z}r^2+C_1\ln r+C_2\end{aligned} \tag{3.5}$$

边界条件：

$$\begin{cases}r=a,\ v_z=0\\ r=b,\ v_z=0\end{cases} \tag{3.6}$$

将式(3.6)代入式(3.5)可得：

$$\begin{cases}0=\dfrac{1}{4\mu}\dfrac{\partial p}{\partial z}a^2+C_1\ln a+C_2 & ①\\ 0=\dfrac{1}{4\mu}\dfrac{\partial p}{\partial z}b^2+C_1\ln b+C_2 & ②\end{cases} \tag{3.7}$$

$$\begin{aligned}&\xRightarrow{①-②}\frac{1}{4\mu}\frac{\partial p}{\partial z}a^2+C_1\ln a=\frac{1}{4\mu}\frac{\partial p}{\partial z}b^2+C_1\ln b\\ &\Longrightarrow C_1(\ln a-\ln b)=\frac{1}{4\mu}\frac{\partial p}{\partial z}(b^2-a^2)\\ &\Longrightarrow C_1=\frac{\partial p}{\partial z}\frac{b^2-a^2}{4\mu(\ln a-\ln b)}\end{aligned} \tag{3.8}$$

$$\xRightarrow{①\times\ln b-②\times\ln a}\frac{1}{4\mu}\frac{\partial p}{\partial z}a^2\ln b+C_2\ln b=\frac{1}{4\mu}\frac{\partial p}{\partial z}b^2\ln a+C_2\ln a$$

$$\Longrightarrow C_2(\ln a-\ln b)=\frac{1}{4\mu}\frac{\partial p}{\partial z}(a^2\ln b-b^2\ln a) \tag{3.9}$$

$$\Longrightarrow C_2=\frac{\partial p}{\partial z}\frac{a^2\ln b-b^2\ln a}{4\mu(\ln a-\ln b)}$$

所以，速度分布为：

$$\begin{aligned}
v_z&=\frac{1}{4\mu}\frac{\partial p}{\partial z}r^2+C_1\ln r+C_2\\
&=\frac{1}{4\mu}\frac{\partial p}{\partial z}r^2+\frac{\partial p}{\partial z}\frac{b^2-a^2}{4\mu(\ln a-\ln b)}\ln r+\frac{\partial p}{\partial z}\frac{a^2\ln b-b^2\ln a}{4\mu(\ln a-\ln b)}\\
&=\frac{1}{4\mu}\frac{\partial p}{\partial z}\left[r^2+\frac{b^2-a^2}{\ln a-\ln b}\ln r+\frac{a^2\ln b-b^2\ln a}{\ln a-\ln b}\right]\qquad(*)\\
&=\frac{1}{4\mu}\frac{\partial p}{\partial z}\left[r^2+\frac{(b^2-a^2)\ln r}{\ln a-\ln b}+\frac{a^2\ln b-a^2\ln a+a^2\ln a-b^2\ln a}{\ln a-\ln b}\right]\\
&=\frac{1}{4\mu}\frac{\partial p}{\partial z}\left[r^2-a^2+\frac{(b^2-a^2)(\ln r-\ln a)}{\ln a-\ln b}\right]
\end{aligned} \tag{3.10}$$

速度沿径向的梯度为：

$$\xRightarrow{(*)\text{式对}r\text{求导}}\frac{\partial v_z}{\partial r}=\frac{1}{4\mu}\frac{\partial p}{\partial z}\left(2r+\frac{b^2-a^2}{\ln a-\ln b}\frac{1}{r}\right) \tag{3.11}$$

所以，管壁受到的摩擦切应力为：

$$\begin{aligned}
\tau|_{r=a}=\mu\left.\frac{\partial v_z}{\partial r}\right|_{r=a}&=\frac{1}{4}\frac{\partial p}{\partial z}\left(2a+\frac{b^2-a^2}{\ln a-\ln b}\frac{1}{a}\right)\\
&=\frac{a}{2}\frac{\partial p}{\partial z}+\frac{1}{4a}\frac{\partial p}{\partial z}\frac{b^2-a^2}{\ln a-\ln b}\\
\tau|_{r=b}=\mu\left.\frac{\partial v_z}{\partial r}\right|_{r=b}&=\frac{1}{4}\frac{\partial p}{\partial z}\left(2b+\frac{b^2-a^2}{\ln a-\ln b}\frac{1}{b}\right)\\
&=\frac{b}{2}\frac{\partial p}{\partial z}+\frac{1}{4b}\frac{\partial p}{\partial z}\frac{b^2-a^2}{\ln a-\ln b}
\end{aligned} \tag{3.12}$$

4、从N-S方程出发，利用平均化运算法则推导平均物理量满足的方程组。

解：考虑不可压缩流体情形，假设体积力可以忽略，此时 N-S 方程为：

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 \\ \dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\Delta u \\ \dfrac{\partial v}{\partial t}+u\dfrac{\partial v}{\partial x}+v\dfrac{\partial v}{\partial y}+w\dfrac{\partial v}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\nu\Delta v \\ \dfrac{\partial w}{\partial t}+u\dfrac{\partial w}{\partial x}+v\dfrac{\partial w}{\partial y}+w\dfrac{\partial w}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\nu\Delta w \end{cases} \tag{4.1}$$

将连续性方程加到运动方程中，利用乘积求导的反向运算可得：

$$\begin{cases} \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 \\ \dfrac{\partial u}{\partial t}+\dfrac{\partial (uu)}{\partial x}+\dfrac{\partial (uv)}{\partial y}+\dfrac{\partial (uw)}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}+\nu\Delta u \\ \dfrac{\partial v}{\partial t}+\dfrac{\partial (vu)}{\partial x}+\dfrac{\partial (vv)}{\partial y}+\dfrac{\partial (vw)}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial y}+\nu\Delta v \\ \dfrac{\partial w}{\partial t}+\dfrac{\partial (wu)}{\partial x}+\dfrac{\partial (wv)}{\partial y}+\dfrac{\partial (ww)}{\partial z}=-\dfrac{1}{\rho}\dfrac{\partial p}{\partial z}+\nu\Delta w \end{cases} \tag{4.2}$$

对方程组(4.2)两边进行平均化运算可知

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial x}+\dfrac{\partial \overline{v}}{\partial y}+\dfrac{\partial \overline{w}}{\partial z}=0 \\ \dfrac{\partial \overline{u}}{\partial t}+\dfrac{\partial (\overline{u}\,\overline{u})}{\partial x}+\dfrac{\partial (\overline{u}\,\overline{v})}{\partial y}+\dfrac{\partial (\overline{u}\,\overline{w})}{\partial z}+\dfrac{\partial \left(\overline{u'u'}\right)}{\partial x}+\dfrac{\partial \left(\overline{u'v'}\right)}{\partial y}+\dfrac{\partial \left(\overline{u'w'}\right)}{\partial z} \\ \qquad =-\dfrac{1}{\rho}\dfrac{\partial \overline{p}}{\partial x}+\nu\Delta\overline{u} \\ \dfrac{\partial \overline{v}}{\partial t}+\dfrac{\partial (\overline{v}\,\overline{u})}{\partial x}+\dfrac{\partial (\overline{v}\,\overline{v})}{\partial y}+\dfrac{\partial (\overline{v}\,\overline{w})}{\partial z}+\dfrac{\partial \left(\overline{v'u'}\right)}{\partial x}+\dfrac{\partial \left(\overline{v'v'}\right)}{\partial y}+\dfrac{\partial \left(\overline{v'w'}\right)}{\partial z} \\ \qquad =-\dfrac{1}{\rho}\dfrac{\partial \overline{p}}{\partial y}+\nu\Delta\overline{v} \\ \dfrac{\partial \overline{w}}{\partial t}+\dfrac{\partial (\overline{w}\,\overline{u})}{\partial x}+\dfrac{\partial (\overline{w}\,\overline{v})}{\partial y}+\dfrac{\partial (\overline{w}\,\overline{w})}{\partial z}+\dfrac{\partial \left(\overline{w'u'}\right)}{\partial x}+\dfrac{\partial \left(\overline{w'v'}\right)}{\partial y}+\dfrac{\partial \left(\overline{w'w'}\right)}{\partial z} \\ \qquad =-\dfrac{1}{\rho}\dfrac{\partial \overline{p}}{\partial z}+\nu\Delta\overline{w} \end{cases} \tag{4.3}$$

利用平均化运算后的连续性方程化简运动方程，可得平均物理量满足的方程组为：

$$\begin{cases}
\dfrac{\partial \overline{u}}{\partial x}+\dfrac{\partial \overline{v}}{\partial y}+\dfrac{\partial \overline{w}}{\partial z}=0 \\
\rho\left[\dfrac{\partial \overline{u}}{\partial t}+\dfrac{\partial(\overline{u}\,\overline{u})}{\partial x}+\dfrac{\partial(\overline{u}\,\overline{v})}{\partial y}+\dfrac{\partial(\overline{u}\,\overline{w})}{\partial z}\right] \\
\qquad =-\dfrac{\partial \overline{p}}{\partial x}+\mu\Delta\overline{u}+\dfrac{\partial\left(-\rho\overline{u'u'}\right)}{\partial x}+\dfrac{\partial\left(-\rho\overline{u'v'}\right)}{\partial y}+\dfrac{\partial\left(-\rho\overline{u'w'}\right)}{\partial z} \\
\rho\left[\dfrac{\partial \overline{v}}{\partial t}+\dfrac{\partial(\overline{v}\,\overline{u})}{\partial x}+\dfrac{\partial(\overline{v}\,\overline{v})}{\partial y}+\dfrac{\partial(\overline{v}\,\overline{w})}{\partial z}\right] \\
\qquad =-\dfrac{\partial \overline{p}}{\partial y}+\mu\Delta\overline{v}+\dfrac{\partial\left(-\rho\overline{v'u'}\right)}{\partial x}+\dfrac{\partial\left(-\rho\overline{v'v'}\right)}{\partial y}+\dfrac{\partial\left(-\rho\overline{v'w'}\right)}{\partial z} \\
\rho\left[\dfrac{\partial \overline{w}}{\partial t}+\dfrac{\partial(\overline{w}\,\overline{u})}{\partial x}+\dfrac{\partial(\overline{w}\,\overline{v})}{\partial y}+\dfrac{\partial(\overline{w}\,\overline{w})}{\partial z}\right] \\
\qquad =-\dfrac{\partial \overline{p}}{\partial z}+\mu\Delta\overline{w}+\dfrac{\partial\left(-\rho\overline{w'u'}\right)}{\partial x}+\dfrac{\partial\left(-\rho\overline{w'v'}\right)}{\partial y}+\dfrac{\partial\left(-\rho\overline{w'w'}\right)}{\partial z}
\end{cases} \tag{4.4}$$

5、定义 $\overline{f}$ 函数为物理量 f 对时间的平均值，即

$$\overline{f}(x,y,z,t)=\frac{1}{T}\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} f(x,y,z,t)\,\mathrm{d}t,$$

其中，T 为平均周期，是常数；且 $f=\overline{f}+f'$，f' 为物理量相对平均值的脉动量。求证：

$$\overline{f\cdot g}=\overline{f}\cdot\overline{g}+\overline{f'\cdot g'}.$$

证明：

$$\begin{aligned}
\overline{f\cdot g} &= \overline{\left(\overline{f}+f'\right)\cdot\left(\overline{g}+g'\right)} \\
&= \overline{\overline{f}\cdot\overline{g}+\overline{f}\cdot g'+f'\cdot\overline{g}+f'\cdot g'} \\
&= \overline{\overline{f}\cdot\overline{g}}+\overline{\overline{f}\cdot g'}+\overline{f'\cdot\overline{g}}+\overline{f'\cdot g'} \\
&= \overline{f}\cdot\overline{g}+\overline{f}\cdot\overline{g'}+\overline{f'}\cdot\overline{g}+\overline{f'\cdot g'} \\
&= \overline{f}\cdot\overline{g}+\overline{f'\cdot g'}
\end{aligned}$$

□