Fluid Mechanics Homework #1

——杨敬轩

——SZ160310217

1.1 Starting from (2.1) and (2.3), prove (2.7).

证明：已知

$$\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 \tag{1.1}$$

$$\vec{x} = x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3'\vec{e}_3' \tag{1.2}$$

求证

$$x_j = x_i' C_{ji} \tag{1.3}$$

令方向余弦矩阵 $C_{ij} = \vec{e}_i \cdot \vec{e}_j'$，由式(1.1)可知

$$\vec{x} \cdot \vec{e}_j = x_j, \tag{1.4}$$

由式(1.2)可知

$$\vec{x} \cdot \vec{e}_j = x_1'\left(\vec{e}_1' \cdot \vec{e}_j\right) + x_2'\left(\vec{e}_2' \cdot \vec{e}_j\right) + x_3'\left(\vec{e}_3' \cdot \vec{e}_j\right) = x_i' C_{ji}, \tag{1.5}$$

则由式(1.4)和式(1.5)可得

$$x_j = x_i' C_{ji}. \tag{1.6}$$

证毕.

1.2 Using Cartesian coordinates where the position vector is $\vec{x} = \left(x_1, x_2, x_3\right)$ and the fluid velocity is $\vec{u} = \left(u_1, u_2, u_3\right)$, write out the three components of the vector:

$$\left(\vec{u} \cdot \nabla\right)\vec{u} = u_i \frac{\partial u_j}{\partial x_i}$$

解：

$$\begin{aligned}\left(\vec{u} \cdot \nabla\right)\vec{u} &= u_i \frac{\partial u_j}{\partial x_i} = u_1 \frac{\partial u_j}{\partial x_1} + u_2 \frac{\partial u_j}{\partial x_2} + u_3 \frac{\partial u_j}{\partial x_3} \\ &= \begin{bmatrix} u_1 \dfrac{\partial u_1}{\partial x_1} + u_2 \dfrac{\partial u_1}{\partial x_2} + u_3 \dfrac{\partial u_1}{\partial x_3} \\ u_1 \dfrac{\partial u_2}{\partial x_1} + u_2 \dfrac{\partial u_2}{\partial x_2} + u_3 \dfrac{\partial u_2}{\partial x_3} \\ u_1 \dfrac{\partial u_3}{\partial x_1} + u_2 \dfrac{\partial u_3}{\partial x_2} + u_3 \dfrac{\partial u_3}{\partial x_3} \end{bmatrix}\end{aligned} \tag{2.1}$$

1.3 Prove the following relationship: $\delta_{ij}\delta_{ij}=3$.

证明：

$$\begin{aligned}\delta_{ij}\delta_{ij}&=\delta_{1j}\delta_{1j}+\delta_{2j}\delta_{2j}+\delta_{3j}\delta_{3j}\\&=\delta_{11}\delta_{11}+\delta_{12}\delta_{12}+\delta_{13}\delta_{13}+\delta_{21}\delta_{21}+\delta_{22}\delta_{22}+\delta_{23}\delta_{23}+\delta_{31}\delta_{31}+\delta_{32}\delta_{32}+\delta_{33}\delta_{33}\\&=1+0+0+0+1+0+0+0+1\\&=3\end{aligned} \tag{3.1}$$

证毕.

1.4 Show that $\vec{C}\cdot\vec{C}^T=\vec{C}^T\cdot\vec{C}=\delta$, where $\vec{C}$ is the direction cosine matrix and δ is the matrix of the Kronecker delta. Any matrix obeying such a relationship is called an orthogonal matrix because it represents transformation of one set of orthogonal axes into another.

证明：Kronecker delta 写成矩阵形式为单位矩阵

$$\delta=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \tag{4.1}$$

由式(1.2)可知

$$\vec{x}\cdot\vec{e}'_j=x'_j, \tag{4.2}$$

由式(1.1)可知

$$\vec{x}\cdot\vec{e}'_j=x_1\left(\vec{e}_1\cdot\vec{e}'_j\right)+x_2\left(\vec{e}_2\cdot\vec{e}'_j\right)+x_3\left(\vec{e}_3\cdot\vec{e}'_j\right)=x_iC_{ij}, \tag{4.3}$$

则由式(4.2)和式(4.3)可得

$$x'_j=x_iC_{ij}. \tag{4.4}$$

将式(1.6)和式(4.4)分别写成矩阵形式

$$\vec{x}=\vec{C}^T\cdot\vec{x}', \tag{4.5}$$

$$\vec{x}'=\vec{C}\cdot\vec{x}, \tag{4.6}$$

由式(4.5)和式(4.6)可知$\vec{C}$的转置矩阵与逆矩阵相等，

$$\vec{C}^{-1}=\vec{C}^T. \tag{4.7}$$

那么

$$\begin{cases}\vec{C}\cdot\vec{C}^T=\vec{C}\cdot\vec{C}^{-1}=\delta,\\\vec{C}^T\cdot\vec{C}=\vec{C}^{-1}\cdot\vec{C}=\delta,\end{cases} \tag{4.8}$$

即

$$\vec{C}\cdot\vec{C}^T=\vec{C}^T\cdot\vec{C}=\delta. \tag{4.9}$$

证毕.