Fluid Mechanics Homework #3

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1、写出直角坐标系中，广义牛顿公式的九个分量.

解：广义牛顿公式的张量形式为

$$\tau_{ij} = -p\delta_{ij} + 2\mu\left(s_{ij} - \frac{1}{3}s_{kk}\delta_{ij}\right) \tag{1.1}$$

式中

$$s_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) & \frac{\partial w}{\partial z} \end{bmatrix} \tag{1.2}$$

$$s_{kk} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{1.3}$$

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \tag{1.4}$$

所以，直角坐标系中广义牛顿公式的九个分量为

$$\begin{cases} \tau_{xx} = -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial u}{\partial x} \\ \tau_{yy} = -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial v}{\partial y} \\ \tau_{zz} = -p - \frac{2}{3}\mu\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\mu\frac{\partial w}{\partial z} \\ \tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \end{cases} \tag{1.5}$$

2、写出柱坐标系中，连续性方程的形式。

解：

方法一：

不依赖坐标系选择的连续性方程的形式为

$$\frac{\partial\rho}{\partial t}+\nabla\cdot\left(\rho\vec{V}\right)=0 \tag{2.1}$$

柱坐标系与直角坐标系之间的转化关系为

$$x=r\cos\theta, y=r\sin\theta, z=z \tag{2.2}$$

所以在柱坐标系中拉梅系数（Lame coefficients）为

$$\begin{cases} H_1=\sqrt{\left(\frac{\partial x}{\partial r}\right)^2+\left(\frac{\partial y}{\partial r}\right)^2+\left(\frac{\partial z}{\partial r}\right)^2}=\sqrt{\cos^2\theta+\sin^2\theta+0^2}=1 \\ H_2=\sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2+\left(\frac{\partial y}{\partial \theta}\right)^2+\left(\frac{\partial z}{\partial \theta}\right)^2}=\sqrt{\left(-r\sin\theta\right)^2+\left(r\cos\theta\right)^2+0^2}=r \\ H_3=\sqrt{\left(\frac{\partial x}{\partial z}\right)^2+\left(\frac{\partial y}{\partial z}\right)^2+\left(\frac{\partial z}{\partial z}\right)^2}=\sqrt{0^2+0^2+1^2}=1 \end{cases} \tag{2.3}$$

对于矢量$\vec{a}=\left(a_1,a_2,a_3\right)$，散度的定义为

$$\nabla\cdot\vec{a}=\lim_{V\to 0}\frac{\oint_S\vec{a}\cdot\vec{n}\mathrm{d}S}{V} \tag{2.4}$$

矢量$\vec{a}$的通量为

$$\begin{aligned}\oint_S\vec{a}\cdot\vec{n}\mathrm{d}S&=\frac{\partial\left(a_1H_2H_3\right)}{\partial r}\mathrm{d}r\cdot\mathrm{d}\theta\mathrm{d}z+\frac{\partial\left(a_2H_3H_1\right)}{\partial\theta}\mathrm{d}\theta\cdot\mathrm{d}z\mathrm{d}r+\frac{\partial\left(a_3H_1H_2\right)}{\partial z}\mathrm{d}z\cdot\mathrm{d}r\mathrm{d}\theta\\&=\left[\frac{\partial\left(a_1H_2H_3\right)}{\partial r}+\frac{\partial\left(a_2H_3H_1\right)}{\partial\theta}+\frac{\partial\left(a_3H_1H_2\right)}{\partial z}\right]\mathrm{d}r\mathrm{d}\theta\mathrm{d}z\end{aligned} \tag{2.5}$$

且体积微元为

$$\mathrm{d}V=H_1H_2H_3\mathrm{d}r\mathrm{d}\theta\mathrm{d}z \tag{2.6}$$

式(2.5)除以式(2.6)并取极限，得$\vec{a}$的散度为

$$\begin{aligned}\nabla\cdot\vec{a}&=\frac{1}{H_1H_2H_3}\left[\frac{\partial\left(a_1H_2H_3\right)}{\partial r}+\frac{\partial\left(a_2H_3H_1\right)}{\partial\theta}+\frac{\partial\left(a_3H_1H_2\right)}{\partial z}\right]\\&=\frac{1}{r}\left[\frac{\partial\left(ra_1\right)}{\partial r}+\frac{\partial a_2}{\partial\theta}+\frac{\partial\left(ra_3\right)}{\partial z}\right]\\&=\frac{1}{r}\frac{\partial\left(ra_1\right)}{\partial r}+\frac{1}{r}\frac{\partial a_2}{\partial\theta}+\frac{\partial a_3}{\partial z}\end{aligned} \tag{2.7}$$

令速度矢量$\vec{V}=\left(v_r,v_\theta,v_z\right)$，则

$$\nabla\cdot\left(\rho\vec{V}\right)=\frac{1}{r}\frac{\partial\left(\rho r v_r\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\rho v_\theta\right)}{\partial\theta}+\frac{\partial\left(\rho v_z\right)}{\partial z} \tag{2.8}$$

将式(2.8)代入式(2.1)中，得柱坐标系中连续性方程的形式为

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial\left(\rho r v_r\right)}{\partial r}+\frac{1}{r}\frac{\partial\left(\rho v_\theta\right)}{\partial\theta}+\frac{\partial\left(\rho v_z\right)}{\partial z}=0 \tag{2.9}$$

方法二：

直角坐标系中连续性方程为

$$\frac{\partial\rho}{\partial t}+\frac{\partial\left(\rho u\right)}{\partial x}+\frac{\partial\left(\rho v\right)}{\partial y}+\frac{\partial\left(\rho w\right)}{\partial z}=0 \tag{2.10}$$

柱坐标与直角坐标之间的关系为

$$\begin{cases} x=r\cos\theta,\ y=r\sin\theta,\ z=z, \\ x_r'=\cos\theta,\ x_\theta'=-r\sin\theta,\ x_z'=0, \\ y_r'=\sin\theta,\ y_\theta'=r\cos\theta,\ y_z'=0, \\ z_r'=z_\theta'=0,\ z_z'=1, \\ r=\sqrt{x^2+y^2},\ \theta=\arctan\dfrac{y}{x},\ z=z, \\ r_x'=\dfrac{x}{\sqrt{x^2+y^2}}=\dfrac{x}{r}=\cos\theta,\ r_y'=\dfrac{y}{\sqrt{x^2+y^2}}=\dfrac{y}{r}=\sin\theta,\ r_z'=0, \\ \theta_x'=\dfrac{1}{1+\left(\dfrac{y}{x}\right)^2}\left(-\dfrac{y}{x^2}\right)=-\dfrac{y}{x^2+y^2}=-\dfrac{\sin\theta}{r},\ \theta_y'=\dfrac{1}{1+\left(\dfrac{y}{x}\right)^2}\left(\dfrac{1}{x}\right)=\dfrac{x}{x^2+y^2}=\dfrac{\cos\theta}{r},\ \theta_z'=0, \\ z_x'=z_y'=0,\ z_z'=1 \end{cases}$$

直角坐标系三个速度分量与柱坐标系三个速度分量之间的关系为

$$\begin{cases} u=\dfrac{\mathrm{d}x}{\mathrm{d}t}=\dfrac{\partial x}{\partial r}\dfrac{\partial r}{\partial t}+\dfrac{\partial x}{\partial\theta}\dfrac{\partial\theta}{\partial t}+\dfrac{\partial x}{\partial z}\dfrac{\partial z}{\partial t}=\cos\theta\cdot v_r-\sin\theta\cdot v_\theta \\ v=\dfrac{\mathrm{d}y}{\mathrm{d}t}=\dfrac{\partial y}{\partial r}\dfrac{\partial r}{\partial t}+\dfrac{\partial y}{\partial\theta}\dfrac{\partial\theta}{\partial t}+\dfrac{\partial y}{\partial z}\dfrac{\partial z}{\partial t}=\sin\theta\cdot v_r+\cos\theta\cdot v_\theta \\ w=\dfrac{\mathrm{d}z}{\mathrm{d}t}=v_z \end{cases} \tag{2.11}$$

上式中用到了以下事实：$v_r=\dfrac{\mathrm{d}r}{\mathrm{d}t},\ v_\theta=r\dfrac{\mathrm{d}\theta}{\mathrm{d}t},\ v_z=\dfrac{\mathrm{d}z}{\mathrm{d}t}$. 将直角坐标系中散度的三个量分别转换到柱坐标系中，由链导法则可知第一个量为

$$
\begin{aligned}
\frac{\partial(\rho u)}{\partial x} &= \frac{\partial(\rho u)}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial(\rho u)}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial(\rho u)}{\partial z}\frac{\partial z}{\partial x} \\
&= \cos\theta \frac{\partial\left[\rho\left(\cos\theta v_r - \sin\theta v_\theta\right)\right]}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\left[\rho\left(\cos\theta v_r - \sin\theta v_\theta\right)\right]}{\partial \theta} \\
&= \cos^2\theta \frac{\partial(\rho v_r)}{\partial r} - \sin\theta\cos\theta \frac{\partial(\rho v_\theta)}{\partial r} \\
&\quad + \frac{\sin^2\theta}{r}\rho v_r - \frac{\sin\theta\cos\theta}{r}\frac{\partial(\rho v_r)}{\partial\theta} + \frac{\sin\theta\cos\theta}{r}\rho v_\theta + \frac{\sin^2\theta}{r}\frac{\partial(\rho v_\theta)}{\partial\theta}
\end{aligned}
\tag{2.12}
$$

注意到 $\cos\theta$ 并非 r 的函数，而 $\cos\theta$, $\sin\theta$ 为 θ 的函数可以得到上式第三个等号. 第二个量为

$$
\begin{aligned}
\frac{\partial(\rho v)}{\partial y} &= \frac{\partial(\rho v)}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial(\rho v)}{\partial \theta}\frac{\partial \theta}{\partial y} + \frac{\partial(\rho v)}{\partial z}\frac{\partial z}{\partial y} \\
&= \sin\theta \frac{\partial\left[\rho\left(\sin\theta v_r + \cos\theta v_\theta\right)\right]}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\left[\rho\left(\sin\theta v_r + \cos\theta v_\theta\right)\right]}{\partial \theta} \\
&= \sin^2\theta \frac{\partial(\rho v_r)}{\partial r} + \sin\theta\cos\theta \frac{\partial(\rho v_\theta)}{\partial r} \\
&\quad + \frac{\cos^2\theta}{r}\rho v_r + \frac{\sin\theta\cos\theta}{r}\frac{\partial(\rho v_r)}{\partial\theta} - \frac{\sin\theta\cos\theta}{r}\rho v_\theta + \frac{\cos^2\theta}{r}\frac{\partial(\rho v_\theta)}{\partial\theta}
\end{aligned}
\tag{2.13}
$$

第三个量为

$$
\begin{aligned}
\frac{\partial(\rho w)}{\partial z} &= \frac{\partial(\rho w)}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial(\rho w)}{\partial \theta}\frac{\partial \theta}{\partial z} + \frac{\partial(\rho w)}{\partial z}\frac{\partial z}{\partial z} \\
&= \frac{\partial(\rho v_z)}{\partial z}
\end{aligned}
\tag{2.14}
$$

所以散度在柱坐标系中转化为

$$
\begin{aligned}
\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} &= \frac{\partial(\rho v_r)}{\partial r} + \frac{\rho v_r}{r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z} \\
&= \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z}
\end{aligned}
\tag{2.15}
$$

上式中第一个等号后面的前两项由乘积求导法则的反向运算合并为一项.

所以，柱坐标系中连续性方程的形式为

$$
\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial(\rho r v_r)}{\partial r} + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial\theta} + \frac{\partial(\rho v_z)}{\partial z} = 0 \tag{2.16}
$$

3、在不可压缩流体时，写出直角坐标系中牛顿流体的运动方程.

解：直角坐标系中流体的运动方程为

$$\begin{cases}\dfrac{\partial(\rho u)}{\partial t}+\dfrac{\partial(\rho uu)}{\partial x}+\dfrac{\partial(\rho uv)}{\partial y}+\dfrac{\partial(\rho uw)}{\partial z}=\rho g_x+\dfrac{\partial\tau_{xx}}{\partial x}+\dfrac{\partial\tau_{xy}}{\partial y}+\dfrac{\partial\tau_{xz}}{\partial z}\\ \dfrac{\partial(\rho v)}{\partial t}+\dfrac{\partial(\rho vu)}{\partial x}+\dfrac{\partial(\rho vv)}{\partial y}+\dfrac{\partial(\rho vw)}{\partial z}=\rho g_y+\dfrac{\partial\tau_{yx}}{\partial x}+\dfrac{\partial\tau_{yy}}{\partial y}+\dfrac{\partial\tau_{yz}}{\partial z}\\ \dfrac{\partial(\rho w)}{\partial t}+\dfrac{\partial(\rho wu)}{\partial x}+\dfrac{\partial(\rho wv)}{\partial y}+\dfrac{\partial(\rho ww)}{\partial z}=\rho g_z+\dfrac{\partial\tau_{zx}}{\partial x}+\dfrac{\partial\tau_{zy}}{\partial y}+\dfrac{\partial\tau_{zz}}{\partial z}\end{cases}\tag{3.1}$$

在不可压缩流体时，

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\tag{3.2}$$

式(3.2)分别对 x，y，z 求偏导得

$$\begin{cases}\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 v}{\partial x\partial y}+\dfrac{\partial^2 w}{\partial x\partial z}=0\\ \dfrac{\partial^2 u}{\partial y\partial x}+\dfrac{\partial^2 v}{\partial y^2}+\dfrac{\partial^2 w}{\partial y\partial z}=0\\ \dfrac{\partial^2 u}{\partial z\partial x}+\dfrac{\partial^2 v}{\partial z\partial y}+\dfrac{\partial^2 w}{\partial z^2}=0\end{cases}\tag{3.3}$$

牛顿流体满足广义牛顿公式，由式(1.5)和式(3.3)可知

$$\begin{aligned}\frac{\partial\tau_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}&=-\frac{\partial p}{\partial x}+\frac{4}{3}\mu\frac{\partial^2 u}{\partial x^2}-\frac{2}{3}\mu\frac{\partial^2 v}{\partial x\partial y}-\frac{2}{3}\mu\frac{\partial^2 w}{\partial x\partial z}\\ &\quad+\mu\frac{\partial^2 u}{\partial y^2}+\mu\frac{\partial^2 v}{\partial x\partial y}+\mu\frac{\partial^2 u}{\partial z^2}+\mu\frac{\partial^2 w}{\partial x\partial z}\\ &=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\frac{1}{3}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\partial y}+\frac{\partial^2 w}{\partial x\partial z}\right)\\ &=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\end{aligned}\tag{3.4}$$

同理可得

$$\begin{cases}\dfrac{\partial\tau_{yx}}{\partial x}+\dfrac{\partial\tau_{yy}}{\partial y}+\dfrac{\partial\tau_{yz}}{\partial z}=-\dfrac{\partial p}{\partial y}+\mu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}+\dfrac{\partial^2 v}{\partial z^2}\right)\\ \dfrac{\partial\tau_{zx}}{\partial x}+\dfrac{\partial\tau_{zy}}{\partial y}+\dfrac{\partial\tau_{zz}}{\partial z}=-\dfrac{\partial p}{\partial z}+\mu\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}\right)\end{cases}\tag{3.5}$$

将式(3.4)和式(3.5)代入式(3.1)，得不可压缩时直角坐标系中牛顿流体的运动方程为

$$\begin{cases} \dfrac{\partial(\rho u)}{\partial t}+\dfrac{\partial(\rho uu)}{\partial x}+\dfrac{\partial(\rho uv)}{\partial y}+\dfrac{\partial(\rho uw)}{\partial z}=\rho g_x-\dfrac{\partial p}{\partial x}+\mu\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}\right) \\ \dfrac{\partial(\rho v)}{\partial t}+\dfrac{\partial(\rho vu)}{\partial x}+\dfrac{\partial(\rho vv)}{\partial y}+\dfrac{\partial(\rho vw)}{\partial z}=\rho g_y-\dfrac{\partial p}{\partial y}+\mu\left(\dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}+\dfrac{\partial^2 v}{\partial z^2}\right) \\ \dfrac{\partial(\rho w)}{\partial t}+\dfrac{\partial(\rho wu)}{\partial x}+\dfrac{\partial(\rho wv)}{\partial y}+\dfrac{\partial(\rho ww)}{\partial z}=\rho g_z-\dfrac{\partial p}{\partial z}+\mu\left(\dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}\right) \end{cases} \tag{3.6}$$