Harbin Institute of Technology, ShenZhen
OPERATIONS RESEARCH
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Homework 2

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2.1. Label each of the following statements about linear programming problems as true or false.

(a) For minimization problems, if the objective function evaluated at a CPF solution is no larger than its value at every adjacent CPF solution then that solution is optimal.

Solution. True

(b) Only CPF solutions can be optimal, so the number of optimal solutions cannot exceed the number of CPF solutions.

Solution. False

(c) If multiple optimal solutions exist, then an optimal CPF solution may have an adjacent CPF solution that also is optimal (the same value of Z).

Solution. True

2.2. Solve the following problem by the simplex method in tabular form.

Maximize

$$Z = 2x_1 - x_2 + x_3$$

subject to

$$\begin{aligned} 3x_1 + x_2 + x_3 &\leqslant 6 \\ x_1 - x_2 + 2x_3 &\leqslant 1 \\ x_1 + x_2 - x_3 &\leqslant 2 \\ x_1,\ x_2,\ x_3 &\geqslant 0 \end{aligned}$$

Solution. Use the following Tab.1 to solve this problem.

Table 1: Simplex tableaux

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
0	Z	(0)	1	-2	1	-1	0	0	0	0
	x_4	(1)	0	3	1	1	1	0	0	6
	x_5	(2)	0	1	-1	2	0	1	0	1
	x_6	(3)	0	1	1	-1	0	0	1	2
1	Z	(0)	1	0	-1	3	0	2	0	2
	x_4	(1)	0	0	4	-5	1	-3	0	3
	x_1	(2)	0	1	-1	2	0	1	0	1
	x_6	(3)	0	0	2	-3	0	-1	1	1
2	Z	(0)	1	0	0	1.5	0	1.5	0.5	2.5
	x_4	(1)	0	0	0	1	1	-1	-2	1
	x_1	(2)	0	1	0	0.5	0	0.5	0.5	1.5
	x_2	(3)	0	0	1	-1.5	0	-0.5	0.5	0.5

At iteration 2, none of the coefficients in row 0 is negative, thus the solution is optimal and the algorithm is finished. Consequently, the optimal solution for this problem is
$x_1 = 1.5,\ x_2 = 0.5,\ x_3 = 0$, with $Z = 2.5$.

2.3. Consider the following problem.

Minimize

$$Z = 2x_1 + 3x_2 + x_3$$

subject to

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &\geqslant 8 \\ 3x_1 + 2x_2 &\geqslant 6 \\ x_1,\ x_2,\ x_3 &\geqslant 0 \end{aligned}$$

Using the Big M method, work through the simplex method step by step to solve the problem.

Solution. First add artificial and surplus variables to reformulate the question as follows:

Maximize

$$-Z = -2x_1 - 3x_2 - x_3 - M\bar{x}_5 - M\bar{x}_7$$

subject to

$$\begin{aligned} x_1 + 4x_2 + 2x_3 - x_4 + \bar{x}_5 &= 8 \\ 3x_1 + 2x_2 - x_6 + \bar{x}_7 &= 6 \\ x_1,\ x_2,\ x_3,\ x_4,\ \bar{x}_5,\ x_6,\ \bar{x}_7 &\geqslant 0 \end{aligned}$$

New row 0: $[-4M+2 \quad -6M+3 \quad -2M+1 \quad M \quad 0 \quad M \quad 0 \quad -14M]$

Use the following Tab.2 to solve this problem.

Table 2: The Big M method

Iter.	B.V.	Eq.	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	x_6	$\bar{x}_7$	R.S.
0	Z	(0)	-1	$-4M+2$	$-6M+3$	$-2M+1$	M	0	M	0	$-14M$
	$\bar{x}_5$	(1)	0	1	4	2	-1	1	0	0	8
	$\bar{x}_7$	(2)	0	3	2	0	0	0	-1	1	6
1	Z	(0)	-1	$\frac{-10M+5}{4}$	0	$\frac{2M-1}{2}$	$\frac{-2M+3}{4}$	$\frac{6M-3}{4}$	M	0	$-2M-6$
	x_2	(1)	0	$\frac{1}{4}$	1	$\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	0	0	2
	$\bar{x}_7$	(2)	0	$\frac{5}{2}$	0	-1	$\frac{1}{2}$	$-\frac{1}{2}$	-1	1	2
2	Z	(0)	-1	0	0	0	$\frac{1}{2}$	$\frac{2M-1}{2}$	$\frac{1}{2}$	$\frac{2M-1}{2}$	-7
	x_2	(1)	0	0	1	$\frac{3}{5}$	$-\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$-\frac{1}{10}$	$\frac{9}{5}$
	x_1	(2)	0	1	0	$-\frac{2}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{2}{5}$	$\frac{4}{5}$

At iteration 2, none of the coefficients in row 0 is negative given that M is big enough, thus the solution is optimal and the algorithm is finished. Consequently, the optimal solution for this problem is $x_1 = \frac{4}{5}$, $x_2 = \frac{9}{5}$, $x_3 = 0$, with $Z = 7$.

2.4. Consider the following problem.

Maximize

$$Z = 2x_1 + 5x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 &\geqslant 20 \\ 2x_1 + 4x_2 + x_3 &= 50 \\ x_1,\ x_2,\ x_3 &\geqslant 0 \end{aligned}$$

Using the two-phase method, work through the simplex method step by step to solve the problem.

Solution. Phase 1 problem:

Maximize

$$Z = -\bar{x}_5 - \bar{x}_6$$

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 + \bar{x}_5 &= 20 \\ 2x_1 + 4x_2 + x_3 + \bar{x}_6 &= 50 \\ x_1,\ x_2,\ x_3,\ x_4,\ \bar{x}_5,\ \bar{x}_6 &\geqslant 0 \end{aligned}$$

New row 0: $[-3 \quad -2 \quad -2 \quad 1 \quad 0 \quad 0 \quad -70]$

Phase 2 problem:

Maximize

$$Z = 2x_1 + 5x_2 + 3x_3$$

subject to

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 20 \\ 2x_1 + 4x_2 + x_3 &= 50 \\ x_1,\ x_2,\ x_3,\ x_4 &\geqslant 0 \end{aligned}$$

Use the following Tab.3 to solve phase 1 of the problem.

Use the following Tab.4 to solve phase 2 of the problem.

At iteration 2, none of the coefficients in row 0 is negative, thus the solution is optimal and the algorithm is finished. Consequently, the optimal solution for this problem is $x_1 = 0,\ x_2 = 0,\ x_3 = 50$, with $Z = 150$.

2.5. Work through the matrix form of the simplex method step by step to solve the following model.

Maximize

$$Z = x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + 3x_2 &\leqslant 8 \\ x_1 + x_2 &\leqslant 4 \\ x_1,\ x_2 &\geqslant 0 \end{aligned}$$

Table 3: Phase 1 of two-phase method

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	$\bar{x}_5$	$\bar{x}_6$	Right Side
0	Z	(0)	1	-3	-2	-2	1	0	0	-70
	$\bar{x}_5$	(1)	0	1	-2	1	-1	1	0	20
	$\bar{x}_6$	(2)	0	2	4	1	0	0	1	50
1	Z	(0)	1	0	-8	1	-2	3	0	-10
	x_1	(1)	0	1	-2	1	-1	1	0	20
	$\bar{x}_6$	(2)	0	0	8	-1	2	-2	1	10
2	Z	(0)	1	0	0	0	0	1	1	0
	x_1	(1)	0	1	0	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{45}{2}$
	x_2	(2)	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{5}{4}$

Solution. In this case,

$$\mathbf{c} = [1, 2], \quad [\mathbf{A}, \mathbf{I}] = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{x}_s = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

Initialization

The initial basic variables are slack variables, so

$$\mathbf{x}_B = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}, \quad \mathbf{c}_B = [0, 0], \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{B}^{-1}$$

Optimality test

The coefficients of the nonbasic variables are

$$\mathbf{c}_B \mathbf{B}^{-1} \mathbf{A} - \mathbf{c} = [0, 0] - [1, 2] = [-1, -2]$$

so these negative coefficients indicate that initial BF solution is not optimal.

Iteration 1

Since -2 ia larger in absolute value than -1, then entering basic variable is x_2. Performing only the relevant portion of a matrix multiplication, the coefficients of x_2 in every equation except Eq.(0) are

$$\mathbf{B}^{-1} \mathbf{A} = \begin{bmatrix} - & 3 \\ - & 1 \end{bmatrix}$$

Table 4: Phase 2 of two-phase method

Iteration	Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	Right Side
0	Z	(0)	1	0	0	$\frac{17}{8}$	$\frac{1}{4}$	$\frac{205}{4}$
	x_1	(1)	0	1	0	$\frac{3}{4}$	$-\frac{7}{2}$	$\frac{45}{2}$
	x_2	(2)	0	0	1	$-\frac{1}{8}$	$\frac{1}{4}$	$\frac{5}{4}$
1	Z	(0)	1	$\frac{17}{6}$	0	0	$-\frac{7}{6}$	115
	x_3	(1)	0	$\frac{4}{3}$	0	1	$-\frac{2}{3}$	30
	x_2	(2)	0	$\frac{1}{6}$	1	0	$\frac{1}{6}$	5
2	Z	(0)	1	4	7	0	0	150
	x_3	(1)	0	2	2	1	0	50
	x_4	(2)	0	1	6	0	1	30

and the right-hand side of these equations are given by the value of $\mathbf{x}_B$ shown in the initial step. Therefore, the minimum ratio test indicates that the leaving basic variable is x_3 since $8/3 < 4/1$. Update the matrices:

$$\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} \frac{8}{3} \\ \frac{4}{3} \end{bmatrix}, \quad \mathbf{c}_B = [2, 0]$$

Optimality test

The nonbasic variables now are x_1 and x_3, and their coefficients in Eq.(0) are

$$\text{For } x_1: \quad [x_1, x_2] = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [2, 0] \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} - [1, 2] = \left[-\frac{1}{3}, -\right]$$

$$\text{For } x_3: \quad [x_3, x_4] = \mathbf{c}_B\mathbf{B}^{-1} = [2, 0] \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} = \left[\frac{2}{3}, -\right]$$

Since x_1 has negative coefficient, the current BF is not optimal, so we go on to the next iteration.

Iteration 2

Since x_1 is the one nonbasic variable with a negative coefficient in Eq.(0), it now becomes the entering basic variable. Its coefficients in the other equations are

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & - \\ \frac{2}{3} & - \end{bmatrix}$$

Also using $\mathbf{x}_B$ obtained at the end of the preceding iteration, the minimum ratio test indicates that x_4 is the leaving basic variable since $2 < 8$. Update the matrices:

$$\mathbf{B} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix}, \quad \mathbf{x}_B = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix} = \mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad \mathbf{c}_B = [2, 1]$$

so x_1 has replaced x_4 in $\mathbf{x}_B$, in providing an element of $\mathbf{c}_B$ from [1,2,0,0], and in providing a column from $[\mathbf{A}, \mathbf{I}]$ in $\mathbf{B}$.

Optimality test

The nonbasic variables are now x_3 and x_4. Their coefficients in Eq.(0) are

$$[x_3, x_4] = \mathbf{c}_B\mathbf{B}^{-1} = [2, 1] \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2} \end{bmatrix} = \left[\frac{1}{2}, \frac{1}{2}\right]$$

Since neither these coefficients are negative, the current BF solution is optimal, i.e., $x_1 = 2,\ x_2 = 2$, with $Z = 6$.

2.6. Consider the following problem.

Maximize

$$Z = 4x_1 + 3x_2 + x_3 + 2x_4$$

subject to

$$\begin{aligned} 4x_1 + 2x_2 + x_3 + x_4 &\leqslant 5 \\ 3x_1 + x_2 + 2x_3 + x_4 &\leqslant 4 \\ x_1,\ x_2,\ x_3,\ x_4 &\geqslant 0 \end{aligned}$$

Let x_5 and x_6 denote the slack variables for the respective constraints. After applying the simplex method, a portion of the final simplex tableau is as follows:

Table 5: A portion of the final simplex tableau

Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1					1	1	
x_2	(1)	0					1	-1	
x_4	(2)	0					-1	2	

(a) Use the fundamental insight presented in Topic 3 to identify the missing numbers in the final simplex tableau. Show your calculations.

Solution. From the problem we can immediately know that

$$\mathbf{A} = \begin{bmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \quad \mathbf{c}_B\mathbf{B}^{-1} = [1, 1], \quad \mathbf{c} = [4, 3, 1, 2]$$

Therefore

$$\mathbf{B}^{-1}\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 3 & 1 \end{bmatrix}$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{A} - \mathbf{c} = [1, 1] \begin{bmatrix} 4 & 2 & 1 & 1 \\ 3 & 1 & 2 & 1 \end{bmatrix} - [4, 3, 1, 2] = [3, 0, 2, 0]$$

$$\mathbf{B}^{-1}\mathbf{b} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\mathbf{c}_B\mathbf{B}^{-1}\mathbf{b} = [1, 1]\begin{bmatrix}5\\4\end{bmatrix} = 9$$

The portion of the final simplex tableau filled is shown in Tab.6.

Table 6: A portion of the final simplex tableau (filled)

Basic Variable	Eq.	Z	x_1	x_2	x_3	x_4	x_5	x_6	Right Side
Z	(0)	1	3	0	2	0	1	1	9
x_2	(1)	0	1	1	-1	0	1	-1	1
x_4	(2)	0	2	0	3	1	-1	2	3

(b) Identify the defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau.

Solution. The defining equations of the CPF solution corresponding to the optimal BF solution in the final simplex tableau are

$$\begin{cases} x_1 = 0, \\ x_3 = 0, \\ 2x_2 + x_4 = 5, \\ x_2 + x_4 = 4. \end{cases}$$