Probability
概率论

Victory won't come to us unless we go to it.

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目　　录

第 1 章 Axioms of Probability

定义 1.1: Sample Space

The sample space Ω of an experiment is the set of all possible outcomes of the experiment. ♣

定义 1.2: Event

An event of an experiment is a subset of the sample space Ω of the experiment. We call Ω the certain event and Φ the impossible event of the experiment. We say that an event A occurs if the outcome of the experiment belongs to A. ♣

定义 1.3: σ-algebra

A σ-algebra $\mathcal{A}$ of subsets of a sample space Ω is a collection of subset of Ω s.t.

(1) $\Omega \in \mathcal{A}$,

(2) $\mathcal{A}$ is closed under complementation, i.e., if $A \in \mathcal{A}$, then $\Omega \backslash A \in \mathcal{A}$,

(3) $\mathcal{A}$ is closed under countable union, i.e., if $A_n \in \mathcal{A}$ for $n = 1, 2, \cdots$, then

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}.$$

♣

定理 1.1: Properties of σ-algebra

Suppose A is a σ-algebra of subsets of a sample space Ω.

(1) $\Phi \in \mathcal{A}$,

(2) A is closed under finite union,

(3) A is closed under countable and finite intersection. ♡

定理 1.2: Intersection of σ-algebra

Suppose Γ is a nonempty collection of σ-algebra of subsets of a sample space Ω. Then the intersection

$$B = \bigcap_{A \in \Gamma} A$$

of the σ-algebra in Γ is also a σ-algebra of subsets of Ω. ♡

推论 1.1: Existence of Smallest σ-algebra

Suppose C is a collection of subsets of a sample space Ω. Then there exists a smallest σ-algebra of subsets of Ω including C. ♡

定义 1.4: Generated σ-algebra

Let C be a collection of subsets of a sample space Ω, we define the σ-algebra of subsets of Ω generated by C as the smallest σ-algebra of subsets of Ω including C and denoted it as $\sigma(C)$. ♣

定义 1.5: Probability Measure

Let A be a σ-algebra of subsets of a sample space Ω, a probability measure $P : \mathcal{A} \to \mathcal{R}$ on A is a real-valued function on A s.t.

(1) Nonnegativity: $P(A) \geqslant 0, \ \forall A \in \mathcal{A}$,

(2) Normalization: $P(\Omega) = 1$,

(3) Countable additivity: If $A_1, A_2, \cdots$ are pairwise disjoint events in A then

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A).$$

For an event $A \in \mathcal{A}$, we call $P(A)$ the probability of the event A. ♣

定义 1.6: Probability Space

A probability space is an ordered triple $(\Omega, \mathcal{A}, P)$ consisting of a sample space Ω , a σ-algebra $\mathcal{A}$ of subsets of Ω, and a probability measure P on $\mathcal{A}$. ♣

定理 1.3: A Kind of Probability Measure

Suppose $\Omega = w_1, w_2, \cdots, \mathcal{A} \in \mathcal{P}(\Omega)$ and

$$P(A) = \sum_{w_i \in \mathcal{A}} P_i, \ \text{for all } A \in \mathcal{P}(\Omega),$$

where $P_i \geqslant 0, \ \forall i = 1, 2, \cdots$ and

$$\sum_{i=1}^{\infty} P_i = 1,$$

then P is a probability measure on $\mathcal{P}(\Omega)$. A similar result holds if $\Omega = w_1, w_2, \cdots, w_N$, where $N \geqslant 1$. ♡

推论 1.2: A Kind of Probability Measure (special)

Suppose $\Omega = w_1, w_2, \cdots, w_N$, $\mathcal{A} \in \mathcal{P}(\Omega)$, and

$$P(A) = \frac{|A|}{N}$$

for all $A \in \mathcal{P}(\Omega)$, then P is a probability measure on $\mathcal{P}(\Omega)$. ♡

定理 1.4: Classical definition of probability

Suppose $\Omega = w_1, w_2, \cdots, w_N$, $\mathcal{A} \in \mathcal{P}(\Omega)$ and P is a probability measure on $\mathcal{P}(\Omega)$ such that $P(w_1) = P(w_2) = \cdots = P(w_N)$, then

$$P(A) = \frac{|A|}{N}$$

for all $A \in \mathcal{P}(\Omega)$. ♡

定理 1.5: Properties of Probability Measure

Suppose $(\Omega, \mathcal{A}, P)$ is a probability space.

(1) $P(\Phi) = 0$,

(2) $P(A) + P(A^c) = 1$. Therefore, $0 \leqslant P(A) \leqslant 1$, for all $A \in \mathcal{A}$.

(3) Finite additivity: If $A_1, A_2, \cdots, A_N$ are pairwise disjoint events in A, then

$$P\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} P(A).$$

♡

定理 1.6: Properties of Probability Measure

Suppose $(\Omega, \mathcal{A}, P)$ is a probability space, and suppose $A, B \in \mathcal{A}$.

(1) If $A_1, A_2, \cdots$ are pairwise disjoint events on A and

$$\bigcup_{n=1}^{\infty} A_n = \Omega,$$

then

$$P(A) = \sum_{i=1}^{\infty} P(A \cap A_n).$$

(2) If $B \subseteq A$, then $P(A) = P(A \cap B) + P(A \cap A^c)$ for all $A, B \in \mathcal{A}$.

(3) $P(A \cap B) \leqslant \min\{P(A), P(B)\} \leqslant \max\{P(A), P(B)\} \leqslant P(A \cup B)$. ♡

推论 1.3: Finite Additivity under Union

Suppose $(\Omega, \mathcal{A}, P)$ is a probability space, $A \in \mathcal{A}$, $A_1, A_2, \cdots$ are pairwise disjoint events in $\mathcal{A}$, and

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = 1,$$

then

$$P(A) = \sum_{n=1}^{\infty} P\left(A \cap A_n\right).$$

♡

定理 1.7: Inclusion-exclusion identity

Suppose $(\Omega, \mathcal{A}, P)$ is a probability space, and suppose $A_1, A_2, \cdots, A_n \in \mathcal{A}$, where $n \geqslant 2$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{k=1}^{n}(-1)^{k+1} \cdot \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} P\left(A_{i_1} \bigcap A_{i_2} \bigcap \cdots \bigcap A_{i_k}\right).$$

♡

引理 1.1: Generated Pairwise Disjoint

Suppose A is a σ-algebra of subsets of a sample space Ω, suppose $A_1, A_2, \cdots \in \mathcal{A}$, $B_1 = A_1$, and

$$B_n = A_n \setminus \bigcup_{i=1}^{n-1} A_i$$

for all $n \geqslant 2$, then $B_1, B_2, \cdots$ are pairwise disjoint events in $\mathcal{A}$,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} B_i$$

for all $n \geqslant 1$, and

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

♡

定理 1.8: Inclusion-exclusion inequality

Suppose $(\Omega, \mathcal{A}, P)$ is a probability space, and suppose $A_1, A_2, \cdots, A_n \in \mathcal{A}$, where $n \geqslant 2$, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) \begin{cases} \leqslant \displaystyle\sum_{k=1}^{m}(-1)^{k+1} \cdot \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} P\left(A_{i_1} \bigcap A_{i_2} \bigcap \cdots \bigcap A_{i_k}\right), & \text{if } m \text{ is odd,} \\ \geqslant \displaystyle\sum_{k=1}^{m}(-1)^{k+1} \cdot \sum_{1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n} P\left(A_{i_1} \bigcap A_{i_2} \bigcap \cdots \bigcap A_{i_k}\right), & \text{if } m \text{ is even,} \end{cases}$$

where $1 \leqslant m \leqslant n$.

In particular,

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leqslant \sum_{i=1}^{n} P(A_i),$$

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geqslant \sum_{i=1}^{n} P(A_i) - \sum_{1\leqslant i<j\leqslant n} P\left(A_i \bigcap A_j\right).$$

♡

定理 1.9: Boole's inequality

Suppose $(\Omega, \mathcal{A}, P)$ is a probability space, and suppose $A_1, A_2, \cdots \in \mathcal{A}$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leqslant \sum_{i=1}^{\infty} P(A_i).$$

♡

定义 1.7: Monotonicity

Let $(\Omega, \mathcal{A}, P)$ be a probability space.

A sequence $\{A_1, A_2, \cdots\}$ of events in A is increasing if $A_1 \subseteq A_2 \subseteq \cdots$

A sequence $\{A_1, A_2, \cdots\}$ of events in A is decreasing if $A_1 \supseteq A_2 \supseteq \cdots$

♣

定义 1.8: Limit of Events

Let $(\Omega, \mathcal{A}, P)$ be a probability space.

(1) The limit $\lim\limits_{n\to\infty} A_n$ of an increasing sequence $\{A_1, A_2, \cdots\}$ of events in A is the event that at least one of the events occurs, i.e.,

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n.$$

(2) The limit $\lim\limits_{n\to\infty} A_n$ of a decreasing sequence $\{A_1, A_2, \cdots\}$ of events in A is the event that all the events occur, i.e.,

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n.$$

♣

定理 1.10: Continuity of probability measure

Let $(\Omega, \mathcal{A}, P)$ be a probability space.

(1) Suppose that $\{A_1, A_2, \cdots\}$ is an increasing sequence of events in A. Then

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

(2) Suppose that $\{A_1, A_2, \cdots\}$ is a decreasing sequence of events in A. Then

$$P\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} P(A_n).$$

♡

备注： If $P(A) = 0$, then it is not necessary that $A = \Phi$, e.g., $\Omega = (0, 1)$ and $A = A_\alpha,\ \alpha \in (0, 1)$. If $P(A) = 1$, then it is not necessary that $A = \Omega$, e.g., $\Omega = (0, 1)$ and $A = A_\alpha^c,\ \alpha \in (0, 1)$.

定义 1.9: Length

The length of the intervals $(a, b),\ [a, b),\ (a, b],\ [a, b]$ are defined to be $(b - a)$. ♣

定义 1.10: Random

A point is said to be randomly selected from an interval (a, b) if any subintervals of (a, b) with the same length are equally likely to contain the randomly selected point. ♣

定理 1.11: Probability of Randomness

The probability that a randomly selected point from (a, b) falls in the subinterval (α, β) of (a, b) is

$$\frac{\beta - \alpha}{b - a}.$$

♡

定义 1.11: Borel Algebra

The σ-algebra of subsets of (a, b) generated by the set of all subintervals of (a, b) is called Borel algebra associated with (a, b) and is denoted $\mathcal{B}_{(a,b)}$. ♣

定理 1.12: Existence of Probability Measure

For any interval (a, b), there exists a unique probability measure P on $\mathcal{B}_{(a,b)}$ s.t.,

$$P\left[(\alpha, \beta)\right] = \frac{\beta - \alpha}{b - a},$$

for all $(\alpha, \beta) \subseteq (a, b)$. ♡

第 2 章 Templates

定义 2.1

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定理 2.1

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推论 2.1

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引理 2.1

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证明：

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备注：