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**Chapter.1 Axioms of Probability**

**Definition 1.1 Sample Space**

The **sample space** Ω of an experiment is the set of **all possible outcomes** of the experiment.

**Definition 1.2 Event**

An **event** of an experiment is a **subset** of the sample space Ω of the experiment.

We call **Ω** the **certain** event and **Φ** the **impossible** event of the experiment.

We say that an event **occurs** if the outcome of the experiment **belongs** to .

**Definition 1.3 σ-algebra**

A **σ-algebra** Aof subsets of a sample space Ω is **a collection of subset** of Ω s.t.

(1)

(2) is **closed under complementation**, i.e., if , then

(3) is **closed under countable union**, i.e., if for ,

then .

**Theorem 1.1 Properties of σ-algebra**

Suppose A is a σ-algebra of subsets of a sample space Ω.

(1)

(2) A is closed under finite union

(3) A is closed under countable and finite intersection.

**Theorem 1.2 Intersection of σ-algebra**

Suppose *Γ* is a nonempty collection of σ-algebra of subsets of a sample space Ω.

Then the **intersection** of the σ-algebra in *Γ* is **also** a σ-algebra of subsets of Ω.

**Corollary 1.1 Existence of Smallest σ-algebra**

Suppose C is a **collection of subsets** of a sample space Ω.

Then there exists a **smallest** **σ-algebra** of subsets of Ω including C.

**Definition 1.4** **Generated σ-algebra**

Let C be a collection of subsets of a sample space Ω,

we define the σ-algebra of subsets of Ω **generated** by C as the **smallest σ-algebra** of subsets of Ω including C and denoted it as **σ(C)**.

**Definition 1.5** **Probability Measure**

Let A be a σ-algebra of subsets of a sample space Ω, a **probability measure**

on A is a real-valued function on A s.t.

(1) **Nonnegativity**: 0

(2) **Normalization**:

(3) **Countable additivity**: If are pairwise disjoint events in A

then .

For an event , we call the probability of the event .

**Definition 1.6 Probability Space**

A probability space is an **ordered triple** consisting of a sample space ,

a σ-algebra of subsets of , and a probability measure on .

**Theorem 1.3** **A Kind of Probability Measure**

Suppose , and for all ,

where and , then is a **probability measure** on . A similar result holds if , where 1.

**Corollary 1.2** **A Kind of Probability Measure (*special*)**

Suppose , , and for all ,

then is a **probability measure** on P(Ω).

**Theorem 1.4**  **Classical definition of probability**

Suppose , andis a **probability measure** on

such that , then for all .

**Theorem 1.5** **Properties of Probability Measure**

Suppose is a probability space.

(1)

(2) . Therefore, , for all .

(3) **Finite additivity**: If are **pairwise disjoint** events in A,

**Theorem 1.6** **Properties of Probability Measure**

Suppose is a probability space, and suppose .

(1) If are **pairwise disjoint** events on A and ,

then .

(2) If , then for all .

(3) .

**Corollary 1.3** **Finite Additivity under Union**

Suppose is a probability space, , are **pairwise disjoint** events in A, and , then

**Theorem 1.7 Inclusion-exclusion identity**

Suppose is a probability space, and suppose ,

where , then

**Lemma 1.1**  **Generated Pairwise Disjoint**

Suppose A is a σ-algebra of subsets of a sample space Ω, suppose ,

, and for all ,

then are **pairwise disjoint** events in A, for all ,

and .

**Theorem 1.8** **Inclusion-exclusion inequality**

Suppose is a probability space, and suppose ,

where , then

where.

**In particular**,

**Theorem 1.9 Boole’s inequality**

Suppose is a probability space, and suppose ,

then .

**Definition 1.7** **Monotonicity**

Let be a probability space.

A sequence {} of events in A is **increasing** if

A sequence {} of events in A is **decreasing** if

**Definition 1.8** **Limit of Events**

Let be a probability space.

(1) The **limit** of an **increasing** sequence {} of events in A is the event that **at least one** of the events occurs, i.e., .

(2) The **limit** of a **decreasing** sequence {} of events in A is the event that **all** the events occur, i.e., .

**Theorem 1.10**  **Continuity of probability measure**

Suppose is a probability space.

(1) Suppose that is an **increasing** sequence of events in A.

Then .

(2) Suppose that is a **decreasing** sequence of events in A.

Then .

**Remark 1.1 Not Certain or Impossible**

If , then it is **not necessary** that ,

e.g., and .

If , then it is **not necessary** that ,

e.g., and .

**Definition 1.9 Length**

The **length** of the intervals are defined to be.

**Definition 1.10** **Random**

A point is said to be **randomly** selected from an interval if **any** subintervals of with the same length are **equally likely** to contain the randomly selected point.

**Theorem 1.11** **Probability of Randomness**

The **probability** that a randomly selected point from falls in the subinterval of is

**Definition 1.11** **Borel Algebra**

The σ-algebra of subsets of generated by the set of all subintervals of is called **Borel algebra** associated with and is denoted .

**Theorem 1.12** **Existence of Probability Measure**

For **any** interval , there **exists** a **unique** probability measure on s.t.,

for all .

**Chapter.2 Combinational Methods**

**Theorem 2.1 Counting Principle**

There are different ways in which we can first choose an element from a set of elements, then an element from a set of elements,, and finally an element from a set of elements.

**Definition 2.1** **Permutation**

An **ordered** arrangement of *r* objects from a set containing *n* objects is called an

*r*-arrangement permutation of , where .

An *n*-element permutation of is called a permutation of .

The **number** of different *r*-permutation **permutations** of is given by

**Theorem 2.2** **Permutation with Types**

The **number** of different (w.r.t. types) **permutations** of *n* objects of ***k* different types** is

where are alike, are alike,, are alike, and .

**Definition 2.2 Combination**

An **unordered** arrangement of *r* objects from a set containing objects is called an -element **combination** of .

The **number** of different *r*-element **combinations** of is given by

**Theorem 2.3** **Property of Combination**

**Theorem 2.4 Multinomial Expansion**

**Corollary 2.1 Binomial Expansion**

**Theorem 2.5 Stirling’s Formula**

Therefore,

**Chapter.3 Conditional Probability and Independence**

**Definition 3.1** **Conditional Probability**

Let be a probability space, and . The **conditional probability** of given , denoted , is given by

**Remark 3.1** **Property of** **Conditional Probability**

**Theorem 3.1 Conditional Probability Space**

Suppose is a probability space, and suppose , for some .

Then the **conditional** probability function is a **probability measure** on A, and hence is a **probability space**.

**Theorem 3.2 Reduction of Probability Space**

Suppose is a probability space, and suppose , for some .

Let and for all .

Then is a **σ-algebra** of subsets of B and is a **probability measure** on , and hence is a **probability space**.

**Remark 3.2**  **Conversion of Reduced and Conditional Probability Space**

Note that

And .

**Theorem 3.3 Law of Multiplication**

Suppose is a probability space, and suppose .

Then.

**Theorem 3.4 Law of Total Probability (*infinite*)**

Suppose is a probability space, and suppose are pairwise disjoint and .

Then, (1)

(2)

**Corollary 3.1**  **Law of Total Probability (*finite*)**

Suppose is a probability space, and suppose

are pairwise disjoint and .

Then, (1)

(2)

**Theorem 3.5 Bayes’ Theorem (*infinite*)**

Suppose is a probability space, and suppose are pairwise disjoint and .

Then

**Corollary 3.2 Bayes’ Theorem (*finite*)**

Suppose is a probability space, and suppose are pairwise disjoint and .

Then

**Theorem 3.6 Properties of** **Conditional Probability**

Suppose is a probability space, and suppose .

(1)

(2)

(3)

**Definition 3.2 Independence**

Let be a probability space, and .

If, thenandare said to be **independent**, denoted .

If and are not independent, they are said to be **dependent**.

Furthermore, if , then and are said to be **positively** correlated,

and if , then and are said to be **negatively** correlated.

**Theorem 3.7 Properties of Independence**

Suppose is a probability space, and suppose.

(1) If or , then .

(2) If and , then either or .

(3) If and are disjoint and , , then they are dependent.

**Theorem 3.8 Independence of Two Events**

Suppose is a probability space, and suppose , and .

Then , i.e.,

**Corollary 3.3 Conditional Probability with Independence**

Suppose is a probability space, and suppose , and .

If

If

**Remark 3.3 Conditional Probability with Independence**

If and , then knowledge about the occurrence of **does not** change the probability of the occurrence of .

If and , then knowledge about the occurrence of**does not** change the probability of the occurrence of .

**Definition 3.3 Independent Set**

Let is a probability space, and , where .

If

Thenare said to be independent; otherwise, they are said to be dependent.

**Remark 3.4 Sub Independent Set**

If are independent,

then are independent,

**Theorem 3.9 Equivalent Statements of Independence**

Suppose is a probability space, , where.

The following statements are **equivalent**:

(1) are independent.

(2)

(3) .

**Definition 3.4 Independent Set**

Let be a probability space, and , where is an index set,

then is said to be **independent** if **any finite subset** of is independent; otherwise, it is said to be **dependent**.

**Corollary 3.4 Independence under Finite Union**

Suppose is a probability space, and suppose

are independent. Then

**In particular,**

if

Then

**Chapter.4 Distribution Functions and Discrete Random Variables**

**4.1 Random Variables**

**Definition 4. 1.1 Measurable Space**

A measurable space is an ordered pair consisting of a sample space Ω and a σ-algebra A of subsets of Ω.

**Definition 4. 1.2 Measurable Function**

Let be measurable spaces.

A function from to is called a measurable function from to if , where is the

pre-image of under *f*.

**Lemma 4. 1.1 σ-algebra under Function**

Suppose is a function from to .

(1) Ifis a **σ-algebra** of subsets of *,* then is a

**σ-algebra** of subsets of.

(2) If is a **σ-algebra** of subsets of, then is a

**σ-algebra** of subsets of.

**Theorem 4. 1.1 σ-algebra Including Subset**

Suppose is a measurable space and is a function from to .

If, then .

**Corollary 4.1.1 A Kind of Measurable Function**

Suppose are measurable spaces, and is a function from to . Suppose and .

Thenis a **measurable function** from to .

**Theorem 4.1.2 Composite Measurable Function**

Suppose are measurable spaces,

is a **measurable function** from to,

and *g* is a **measurable function** from to.

Then is a **measurable function** from to.

**Definition 4.1.3 Open Set**

A set in is called an **open set** in if for all , where .

**Definition 4.1.4 Borel σ-algebra**

The σ-algebra generated by the set of all open sets in is called the **Borel**

**σ-algebra** of subsets of and is denoted by .

We call a set in a **Borel set** in .

**Theorem 4.1.3 Measurable Function from Continuity**

Supposeis a **continuous** function from to .

Thenis a **measurable** function from to .

**Definition 4.1.5 Cell**

A **cell** in is a finite interval of the form for some

.

A **cell** in , where , is a Cartesian product of cells ,

i.e., .

**Definition 4.1.6 Open Cube**

Let

The **open cube** in with center *x* and side length is defined as the **open cell** .

**Theorem 4.1.4 Set from Cells**

Every **open set** in is a **countable union** of **open cells** in .

**Theorem 4.1.5 Measurable Function on Open Cells**

Suppose is a measurable space and is a function from Ω to .

Suppose that for **all open cells** in .

Then is a **measurable function** from to .

**Theorem 4.1.6 Components of Measurable Function**

Supposeis a measurable space, is a function from Ω to.

Then is a **measurable function** from to

are **measurable functions** from to .

**Theorem 4.1.7 Elementary Operation of Measurable Function**

Suppose and are measurable functions from to , and .

Then are **measurable functions** from to .

**Theorem 4.1.8 Limit of Measurable Functions**

Suppose that are measurable functions from to

and , where is a function from Ω to .

Then is **also** a measurable function from to .

**Theorem 4.1.9 Equivalence of Nine Types of Set**

Suppose is a measurable space andis a function from Ω to .

Let be the set of all open sets in ,

, ,

, ,

, ,

, .

Thenis a measurable function from to

if for **any**

**Theorem 4.1.10 Induced Probability Space under Function**

Suppose is a measurable function from to .

Suppose is a probability measure on .

Then the function on given by  
is a probability measure.

We call the probability space **induced** from under .

**Remark 4.1.1 Conventional Denotation**

(1) The set is **conventionally** denoted as “”.

Therefore .

(2) If ,then , and hence

**Definition 4.1.7 Random Variable**

Let be a probability space.

A measurable function from to is called a **random variable** (r.v.) of the probability space .

A measurable function from to is called a **random vector** (r.vect.) of the probability space .

**Remark 4.1.2 Conventional Denotation of Random Variable**

If is a r.v. of the probability space ,

then .

**Theorem 4.1.11 Additivity of Countable Points**

Suppose is a r.vect. of a probability space , and is a “**countable**” subset of , then , and

**4.2 Distribution Functions**

**Definition 4.2.1 Cumulative Distribution Function**

Let be a r.v. of a probability space .

The **cumulative distribution function** (c.d.f) of the r.v. is a function from to , given by

**Theorem 4.2.1 Properties of C.D.F**

Suppose is a r.v. of a probability space .

(1) is **increasing**.

(2) .

(3) .

(4) . is **right continuous**.

(5) .

**Corollary 4.2.1 Properties of C.D.F**

Suppose is a r.v. of a probability space .

(1)

(2)

(3) .

(4) ,

.

**Theorem 4.2.2 Existence of C.D.F**

Suppose is a function s.t. *F* is increasing and right continuous,

Then there **exists** a r.v. of some probability space (Ω, A, P),

s.t. the c.d.f. of is equal to *F*.

We call such function a **c.d.f.**

**4.3 Discrete Random Variables**

**Definition 4.3.1 Discrete R.V.**

A r.v. of a probability space is called a **discrete r.v.**

if is **countable**.

**Definition 4.3.2 Probability Mass Function**

Let be a discrete r.v. of a probability space s.t.

The **probability mass function** (p.m.f) of is a function from to given by .

**Theorem 4.3.1 Properties of P.M.F**

Suppose is a discrete r.v. of a probability space . Then,

(1) .

(2) .

(3) .

Therefore if , then,

(1) .

(2) .

(3) .

**Theorem 4.3.2 Existence of P.M.F**

Suppose is a function s.t.

(1)

(2) .

(3) .

for some distinct .

Then there **exists** a discrete r.v. of some probability space s.t.

the p.m.f. of is equal to .

We call such a function a p.m.f.

**Theorem 4.3.3 Step Distribution Function for Discrete R.V.**

Suppose is a discrete r.v. of a probability space s.t. , where .

Then the distribution function of is a **step function** given by

where

**4.4 Expectations of Discrete Random Variables**

**Definition 4.4.1 Expectation**

Let be a discrete r.v. of a probability space .

The **expectation (or expected value, or mean)** of is given by

if the sum **converges absolutely**.

And if the sum diverges to , .

**Remark 4.4.1 Explanations of Expectation**

(1) The expectation is the **weighted average** of

with weights .

(2) The expectation is the **center of gravity** of

.

**Theorem 4.4.1 Expectation of Constant**

Suppose is a **discrete r.v.** of a probability space s.t. is a **constant** with probability **1**, i.e., for some .

Then , and .

In particular, if X is a **constant r.v.** of , i.e., , for some

, then .

**Theorem 4.4.2 Composition of Function and R.V.**

Suppose is a **discrete r.v**. of a probability spaceand be a **measurable function** from to .

Then is a discrete r.v. of and

**Corollary 4.4.1 Linearity of Expectation**

Suppose is a discrete r.v. of a probability space ,

are measurable functions from to ,

and ,

Then  
is a discrete r.v. of and

**4.5 Variances and Moments of Discrete Random Variables**

**Definition 4.5.1 Variance and Standard Deviation**

Let be a discrete r.v. of a probability space and suppose exists.

The **variance** of is given by ,

and the **standard deviation** of is given by .

**Remark 4.5.1 Explanation about Variance**

The variance of a discrete r.v. measures the **dispersion (or spread)** of its probability masses about its expectation (center of gravity of its probability masses).

**Theorem 4.5.1 Calculating Variance**

Suppose is a discrete r.v. of a probability spaceand suppose exists.

Then .

**Theorem 4.5.2 Minimum Distance with Expectation**

Suppose is a discrete r.v. of a probability spaceand suppose exists.

If , then .

**Theorem 4.5.3 With Probability**

Suppose is a discrete r.v. of a probability space .

(1) , “” holds **with probability** , i.e., .

(2) If exists, then , “” holds **with probability** ,

i.e., .

**Theorem 4.5.4 Calculating Linear Combination**

Suppose is a discrete r.v. of a probability spaceand suppose exists.

Then and .

**Definition 4.5.2 Moment and Absolute Moment**

Let be a discrete r.v. of a probability space , and .

If the respective sum **converges absolutely**.

**Theorem 4.5.6 Existence of Lower Order Moment**

Suppose is a discrete r.v. of a probability space and suppose .

If exists, then exists.

That is, the existence of a higher order moment of **guarantees the existence** of a lower order moment of .

**4.6 Standardized Random Variables**

**Definition 4.6.1 Standardized R.V.**

Let be a discrete r.v. of a probability space .

If exists and , then the **standardized r.v.** of is given by

i.e., is the number of **standard deviation units** by which differs from .

**Theorem 4.6.1 Expectation and Variance of Standardized R.V.**

Suppose is a discrete r.v. of a probability space and exists,

.

Then and

**Theorem 4.6.2 Independence of Units**

Suppose is a discrete r.v. of a probability space and exists,

.

Then the standardized r.v. of is **independent of the units** in which is measured.

**Remark 4.6.1 Standardization for Comparison**

Standardization can be useful when **comparing** r.v.’s with different distributions.

**Charper.5 Special Discrete Distributions**

**5.1 Bernoulli R.V.’s and Binomial R.V.’s**

**Definition 5.1.1 Bernoulli Trial**

A **Bernoulli trial** is an experiment that has **only two** outcomes, say success and failure, so that its sample space is given by .

* Let be the number of success in a Bernoulli trial.

whereis the probability of success.

**Definition 5.1.2**

A discrete r.v. of a probability space is called a **Bernoulli r.v.** with parameter where , denoted , if its p.m.f is given by

Such a p.m.f is called a Bernoulli p.m.f with parameter .

**Theorem 5.1.1 Expectation and Variance of Bernoulli R.V.**

Suppose , where .

Then and .

◎ Consider an experiment in which independent Bernoulli trials with the same

probability of success, say, are performed.

The sample space of the experiment is

and, where *.*

Let be the number of successes in the Bernoulli trials.

**Definition 5.1.3**

A discrete r.v. of a probability space is called a **binomial r.v.** with parameter and whereand , denoted , if its p.m.f is given by

Such a p.m.f is called a binomial p.m.f with parameter and .

**Remark 5.1.1 Bernoulli R.V. from Binomial R.V.**

(1) A Bernoulli r.v. with parameter **is** a binomial r.v. with parameter and .

(2)  
 is a p.m.f.

**Theorem 5.1.2 Expectation and Variance of Binomial R.V.**

Suppose , whereand .

Then and .

**Theorem 5.1.3 Maximum Point of Binomial Probability**

Suppose , whereand .

Then

**5.2 Poisson R.V.’s**

◎ If is difficult to calculate if is

large.

◎ A recursive relation: *,*

◎ An approximation for large,small,and moderate, sayfor some

constant *.*

**Definition 5.2.1 Poisson R.V.**

A discrete r.v. of a probability space is called a **Poisson r.v.** with parameter where , denoted , if its p.m.f is given by

Such a p.m.f is called a Poisson p.m.f with parameter .

**Remark 5.2.1 Poisson R.V. from Binomial R.V.**

(1) A Poisson r.v. with parameter is an approximation of a binomial p.m.f. with

parameters and such that is large and is small, and

(2)  
 is a p.m.f.

**Theorem 5.2.1 Expectation and Variance of Poisson R.V.**

Suppose , where . Then and .

**Theorem 5.2.2 Maximum Point of Poisson Probability**

Suppose , where . Then

**5.3 Geometric R.V.’s, Negative Binomial R.V.’s and Hypergeometric R.V.’s**

* Consider an experiment in which independent Bernoulli trials with the same

probability of success, say , are performed until the first success occurs.

The sample space of the experiment is

Let be the number of Bernoulli trials until the first success occurs

**Definition 5.3.1 Geometric R.V.**

A discrete r.v. of a probability space is called a **geometric r.v.** with parameter where , denoted , if its p.m.f is given by

Such a p.m.f is called a geometric p.m.f with parameter .

**Remark 5.3.1 Justification of P.M.F.**

is a p.m.f.

**Theorem 5.3.1 Expectation and Variance of Geometric R.V.**

Suppose , where .

Then  
and

**Theorem 5.3.2 Memoryless Property**

Suppose is a discrete r.v. of a probability space with .

Then is a geometric r.v.

* Consider an experiment in which independent Bernoulli trials with the same

probability of success, say , are performed until the success occurs,

where .

Let be the number of Bernoulli trials until the success occurs.

**Definition 5.3.2 Negative Binomial R.V.**

A discrete r.v. of a probability space is called a **negative binomial r.v.** with parameters and where and , denoted , if its p.m.f is given by

Such a p.m.f is called a negative binomial p.m.f with parameters and .

**Remark 5.3.2 Geometric R.V. from Negative Binomial R.V.**

(1) A geometric r.v. with parameter is a negative binomial r.v. with parameters

and .

(2)

is a p.m.f.

**Theorem 5.3.3 Expectation and Variance of Negative Geometric R.V.**

Suppose , where and .

Then  
and

**Theorem 5.3.4 Maximum Point of Negative Geometric Probability**

Suppose , where and . Then

* A box contains red balls and blue balls.

Suppose that balls are randomly drawn from the box, one by one and without replacement.

Let be the number of “red” balls drawn

**Definition 5.3.3 Hypergeometric R.V.**

A discrete r.v. of a probability space is called a **hypergeometric r.v.** with parameter ,and where and,

denoted , if its p.m.f is given by

Such a p.m.f is called a hypergeometric r.v. with parameter ,and .

**Remark 5.3.3 Justification of P.M.F.**

(1) If

(2)

the coefficient of is

where .

is a p.m.f.

**Theorem 5.3.5 Expectation and Variance of Hypergeometric R.V.**

Suppose ,

where and.

Then  
and

**Remark 5.3.4 Binomial Approximation for Hypergeometric**

balls are drawn with replacement

Therefore, if , then drawing with replacement is a good approximation of drawing without replacement.

**Theorem 5.3.6 Maximum Point of Hypergeometric Probability**

Suppose ,

where and.

Then

**Remark 5.3.5 Binomial and Poisson Approximation for Hypergeometric**

(1) If , then

(2) If , then

where .

**Chapter.6 Continuous Random Variables**

**6.1 Probability Density Function**

**Definition 6.1.1 Probability Density Function**

Let be a r.v. of a probability space .

is called an absolutely continuous (or a continuous) r.v. if there exists a nonnegative real-valued function s.t.

The function is called the **probability density function** (p.d.f.) of .

**Remark 6.1.1 Approximation of Probability**

for some .

If is **continuous** at

So , if is continuous at and is very small.

**Theorem 6.1.1 C.D.F and Probability from P.D.F.**

Suppose is a continuous r.v. of a probability space .

(1)

Therefore, is a **continuous** function.

(2)

(3) If is continuous at , then .

Therefore, if is a continuous function, then .

(4) . Therefore,

**Theorem 6.1.2 Existence of P.D.F.**

Suppose is a **nonnegative** real-valued function s.t.

Then there exists a continuous r.v. of some probability space (Ω, A, P) s.t. the p.d.f. is equal to .

**Definition 6.1.2 Sufficient Conditions of P.D.F.**

A **nonnegative** real-valued function s.t.

is called a p.d.f.

The c.d.f. associated with is given by

**Remark 6.1.2 Neither Discrete Nor Continuous R.V.**

There are r.v.’s that are neither discrete nor continuous,

e.g., , where .

**6.2 The Probability Density Function of A Function of A R.V.**

**Theorem 6.2.1 Method of Distribution Functions**

Suppose is a continuous r.v. of a probability space .

If , then

**Theorem 6.2.2 Method of Transformations**

Supposeis a continuous r.v. of a probability space

such that its p.d.f. is continuous.

Suppose , where is a measurable function from to .

(1) If is a **discrete** r.v., then

(2) If is a **continuous** r.v., exists,

and , where .

Then,

**6.3 Expectations and Variances**

**Definition 6.3.1 Expectation**

Let be a continuous r.v. of a probability space s.t. its p.d.f. is continuous.

The **expectation** (or mean) of is given by

if is **absolutely integrable**, i.e.,

and is given by, if the integration diverges to .

**Remark 6.3.1 Necessary and Sufficient Condition of Absolutely Integrable**

**Theorem 6.3.1 Calculation of Expectation**

Suppose is a continuous r.v. of a probability space .

Then

**Corollary 6.3.1 Calculation of**  **Moment**

Suppose is a **nonnegative** continuous r.v. of a probability space ,

and

Then

In particular,

**Theorem 6.3.2 Approximation of Expectation**

Suppose is a continuous r.v. of a probability space .

Then

Therefore,

**Theorem 6.3.3 Infinite Zero**

Suppose is a continuous r.v. of a probability space .

Then,

**Theorem 6.3.4 Expectation of Measurable Function**

Suppose is a continuous r.v. of a probability space ,

and suppose is a **measurable function** from to .

Then

**Corollary 6.3.2 Expectation of Linear Combination of Measurable Functions**

Suppose is a continuous r.v. of a probability space .

are **measurable functions** from to ,

and .

Then

**Definition 6.3.2 Variance and Standard Deviation**

Let be a continuous r.v. of a probability space and suppose exists.

The **variance** of is given by .

And the **standard deviation** of is given by .

**Theorem 6.3.5 Minimum Distance with Expectation**

Suppose is a continuous r.v. of a probability space ,

and suppose exists.

If , then .

**Theorem 6.3.6 Calculation of Linear Combination**

Suppose is a continuous r.v. of a probability space ,

and suppose exists.

Then

(1)

(2)  
and

**Definition 6.3.3 Moment and Absolute Moment**

Let be a continuous r.v. of a probability space , and .

If the respective sum **converges absolutely**.

**Theorem 6.3.7 Existence of Lower Order Moment**

Supposeis a continuous r.v. of a probability spaceand suppose .

If exists, then exists.

That is, the existence of a higher order moment of **guarantees the existence** of a lower order moment of .

**Theorem 6.3.8 Positive Variance**

Suppose is a continuous r.v. of a probability space .

Then

Therefore

**Chapter.7 Special Continuous Distributions**

**7.1 Uniform R.V.’s**

**Definition 7.1.1 Uniform R.V.**

A continuous r.v. of a probability space is called a **uniform r.v.** over ,

where and , denoted , if its p.d.f. is given by

**Remark 7.1.2 P.D.F. and C.D.F.**

(1) , and

is a p.d.f.

(2)

**Theorem 7.1.1 Expectation and Variance of Uniform R.V.**

Suppose , where and .

Then

Therefore

and

**Remark 7.1.2 Expectation and Variance of Discrete “Uniform R.V.”**

Suppose , where .

Then

and

**Theorem 7.1.2 Linear Generated R.V.**

Suppose , where and .

Suppose , where and .

Then

**7.2 Normal (Gaussian) R.V.’s**

**Definition 7.2.1 Normal (Gaussian) R.V.**

A continuous r.v. of a probability space is called a **normal (Gaussian)** r.v. with parameters and , where , denoted , if its p.d.f. is given by

**Remark 7.2.1 P.D.F. and C.D.F.**

(1) , and let .

is a p.d.f.

(2) If , , then is called a **standard** normal (Gaussian) r.v.

(3)

where

**Theorem 7.2.1 Symmetric about**

Suppose .

(1) is **symmetric** about , with maximum at , and **inflection** points

at .

(2) and

Therefore, and

**Theorem 7.2.2 Linear Generated R.V.**

Suppose , where .

Suppose , where and .

Then,

In particular, if  
then

**Definition 7.2.2 Gamma Function**

The function given by

is called the **gamma function**.

**Theorem 7.2.3 Properties of Gamma Function**

(1)

(2)

(3)

**Theorem 7.2.4 Calculation of Moment and Absolute Moment**

Suppose , where .

(1)

(2)

(3)

**Theorem 7.2.5 De Moivre-Laplace Theorem**

Suppose , whereand .

Then

**Theorem 7.2.6 Approximation of**

**Theorem 7.2.7 Expectation of Exponential Function**

Suppose , where , and .

Then

**7.3 Gamma R.V.’s, Erlang R.V.’s and Exponential R.V.’s**

**Definition 7.3.1 Gamma R.V., Erlang R.V. and Exponential R.V.**

A continuous r.v. of a probability space is called a **gamma** r.v. with parameters and, where , denoted , if its p.d.f. is given by

If , then is called an **Erlang** r.v. with parameters and ,

denoted .

If , then is called an **exponential** r.v. with parameters , denoted .

**Remark 7.3.1 Properties of P.D.F.**

(1)

is a p.d.f.

(2)

**Theorem 7.3.1 Calculation of C.D.F.**

Suppose , where .

Then

where

is the **incomplete** gamma function.

If , then

where .

**Theorem 7.3.2 Expectation and Variance of Gamma R.V.**

Suppose , where .

Then

where

Therefore,

**Theorem 7.3.3 Linear Generated Gamma R.V.**

Suppose , where , and , where Then

**Theorem 7.3.4 Gamma R.V. from Normal R.V.**

Suppose , where and . Then

**Lemma 7.3.1 Plus to Multiply Property of Exponential Function**

Suppose is **right continuous** on

and .

Then there either or s.t. .

**Theorem 7.3.5 Memoryless Property**

Suppose is a **nonnegative** continuous r.v. of a probability space .

Then , for some .

**Remark 7.3.2 Analog of Geometric R.V.**

Exponential r.v.’s are the **continuous analog** of geometric r.v.’s.

**Theorem 7.3.6 Geometric R.V. from Exponential R.V.**

Suppose where and . Then .

**Definition 7.3.2 Independent Set**

A set of r.v.’s of a probability space is called **independent**,

if for **any finite subset** of the events

are independent for all .

Otherwise, is called dependent.

**Definition 7.3.3 Continuous R.Vect.**

A r.vect. of a probability space is called an absolute continuous **r.vect.** (or continuous r.vect.) if there exists a nonnegative real-valued function s.t.

for all .

Then the function is called the **p.d.f.** of the r.vect. ,

or the joint p.d.f. of the r.v.’s .

**Theorem 7.3.7 P.D.F. and C.D.F. of Continuous R.Vect.**

Suppose is a continuous r.vect. and

Then

Furthermore, if are independent, then

**Theorem 7.3.8 Convolution Theorem**

If is a continuous r.vect. and . Then

Furthermore, if , then

**Definition 7.3.4 Beta Function**

The function is given by

is called **beta function**.

**Lemma 7.3.2 Calculation of Beta Function**

**Theorem 7.3.9 Independent Additivity of Gamma R.V.**

Suppose where are **independent**,

and . Then

**Theorem 7.3.10 Independent Minimum of Exponential R.V.**

Suppose where , and are **independent**.

(1) If , then

(2)

**Definition 7.3.5 Stochastic Process**

A **stochastic process** (s.p.) is a collection of r.v.’s of a probability space (Ω, A, P).

If or , then we call a **discrete-time** S.P.

If or , then we call a **continuous-time** S.P.

**Definition 7.3.6 Counting Process and Poisson Process**

Let be a **discrete-time** S.P. s.t. , is the time of occurrence of the event, and .

Let , where be the interoccurrence time between the and the events,

and be the number of events occurring in ,

so that is called the **counting process** of the S.P. .

Then we call a **Poisson process** with rate ,

if are **independent and identically distributed** (i.i.d.)

and .

**Theorem 7.3.11 Necessary and Sufficient Condition of Poisson Process**

Suppose is a S.P. s.t. and its interoccurrence times

are i.i.d., where .

Then is a Poisson process with rate

**Remark 7.3.3 Negative BinomialGeometric vs Gamma Exponential**

(1) A negative binomial r.v. is the

number of i.i.d. Bernoulli trials with the same probability of success until the

success occurs, where is the number of Bernoulli trials

between the and the successes, and are independent.

(2) A gamma r.v. is the time of occurrence of the

event of a Poisson process with rate , where is the interoccurrence

time between the and the events, and are independent.

**Theorem 7.3.12 Merging and Splitting of Poisson Process**

(1) Suppose that independent Poisson processes with rates are merged

into a S.P. .

Then is a **Poisson process** with rate .

(2) Suppose that in a Poisson process with rate , an event is a type- event with

probability .

Then the S.P. of the times of the occurrences of the type-events is a

**Poisson process** with rate .

**7.4 Beta R.V.’s**

**Definition 7.4.1 Beta R.V.**

A continuous r.v. of a probability space is called a beta r.v. with parameter and , where , denoted , if its p.d.f. is given by

where

**Remark 7.4.1 P.D.F. and C.D.F.**

(1) is a p.d.f.

(2) Beta r.v.’s are good **approximations** of r.v.’s that vary between **two limits**.

(3) If are i.i.d. and is the **smallest** r.v. of

so that , then

(4)

**Theorem 7.4.1 Expectation and Variance of Beta R.V.**

Suppose , then

Therefore,

and

**Theorem 7.4.2 Beta R.V. vs Binomial R.V.**

Suppose , and , where .

Then

and

**Chapter.8** **Bivariate and Multivariate Distributions**

**8.1 Joint Distributions of Two or More R.v.’s**

**Definition 8.1.1 Joint P.M.F. of Multiple R.v.’s**

Let be discrete r.v.’s of a probability space .

The nonnegative function given by

is called the **joint p.m.f.** of .

**Remark 8.1.1 Properties of Joint P.M.F.**

(1)

(2)

(3)

(4)

**Theorem 8.1.1 Joint Marginal P.M.F.**

Suppose are discrete r.v.’s of a probability space .

Then

We call the **joint p.m.f. marginalized** over . If , we call the **marginal p.m.f.** of .

**Theorem 8.1.2 Expectation of Measurable Function**

Suppose are **discrete r.v.’s** of a probability space ,

and is a **measurable function** from to .

Then

**Corollary 8.1.1 Expectation of Linear Combined Measurable Function**

Suppose are discrete r.v.’s of a probability space ,

and are **measurable functions** from to .

and ,

Then is a discrete r.v. of (Ω, A, P) and

**Definition 8.1.2 Joint P.D.F.**

Let be r.v.’s of a probability space .

We say that are **jointly continuous** r.v.’s if there exists a nonnegative function s.t.

The function is called the **joint p.d.f.** of .

**Theorem 8.1.3 Joint Marginal P.D.F.**

Suppose are **jointly continuous** r.v.’s of a probability space .

Then are also jointly continuous r.v.’s of a probability space

with joint p.d.f.

and the integral has terms.

We callthe **joint p.d.f.** marginalized over.

If , we call the **marginal p.d.f.** of .

**Theorem 8.1.4 Expectation of Measurable Function**

Suppose are **jointly continuous** r.v.’s of a probability space ,

and is a measurable function from to . Then

**Remark 8.1.2 Properties of Joint P.D.F.**

(1)

(2)

(3)

*.*

(4)

(5)

for some if is continuous.

and .

**Corollary 8.1.2 Expectation of Linear Combined Measurable Function**

Suppose are **jointly continuous** r.v.’s of a probability space ,

and are **measurable functions** from to .

and ,

Then  
is a continuous r.v. of (Ω, A, P) and

**Definition 8.1.3 Joint C.D.F.**

Let be r.v.’s of a probability space .

The **joint c.d.f.** of is given by

**Theorem 8.1.5 Joint Marginal C.D.F.**

Suppose are r.v.’s of a probability space .

Then

We callthe **joint c.d.f.** marginalized over .

If , we call the **marginal c.d.f.** of .

**Theorem 8.1.6 Properties of Joint C.D.F.**

Suppose are r.v.’s of a probability space .

(1) is **increasing** and **right continuous** in each argument

(2) if there exists at least one such that .

(3) .

(4) If are **jointly continuous** r.v.’s, then

**8.2 Independent R.V.’s**

**Definition 8.2.1 Independent Set**

Let be r.v.’s of a probability space .

We say that the r.v.’s are **independent**

if for any finite subset of ,

the events are independent

Otherwise, the r.v.’s are dependent.

**Theorem 8.2.1 Equivalent Statements of Independence**

Suppose are r.v.’s of a probability space .

The following three statements are **equivalent**:

(1) are independent.

(2)

(3)

**Theorem 8.2.2 Necessary and Sufficient Condition of Independence**

Suppose are r.v.’s of a probability space .

(1) If are **discrete** r.v.’s,

then are independent

(2) If are **jointly continuous** r.v.’s,

then are independent

**Definition 8.2.2 Indicator Function**

Let be a probability space, and .

The **indicator function** of the event is given by

**Theorem 8.2.3 Indicator Function is a Discrete Measurable Function**

Suppose is a probability space.

is a **discrete r.v.** of for all .

**Theorem 8.2.4 Indicator R.V.’s Indicates Independence**

Suppose is a probability space, and .

The events are **independent**

the **indicator r.v.’s** are **independent**.

**Theorem 8.2.5 Expectation of Measurable Functions of Independent R.V.**

Suppose are independent r.v.’s of a probability space ,

and are measurable functions from to .

Then are independent and

**Remark 8.2.1 Independent Expectations Can’t Imply Independence of R.V.’s**

The converse is **not true**, i.e.,

**8.3 Conditional Distributions**

**Recall 8.3.1 Properties of Conditional Probability**

Suppose be a probability space,

and .

(1) If , then regarded as a function on is a **probability**

**measure.**

(2) **Multiplication theorem:**

.

(3) **Total probability theorem:**

If is a partition of Ω, then

(4) **Bayes’ theorem:**

If and is a partition of Ω, then

◎ and are discrete r.v.’s

**Definition 8.3.1 P.M.F. and C.D.F. of D-D**

Let and be discrete r.v.’s of a probability space and .

The conditional p.m.f. of given that is given by

The conditional c.d.f. of given that is given by

**Remark 8.3.1 Joint P.M.F.**

(1) .

(2) A similar definition can be made for discrete **random** **vectors**.

**Theorem 8.3.1 Properties of D-D Conditional Probability**

Suppose are discrete r.v.’s of a probability space .

(1) If and , then is a p.m.f.

(2)

(3)

(4) If and , then

◎ and are jointly continuous r.v.’s

**Definition 8.3.2 C.D.F. and P.D.F. of C-C**

Let and be jointly continuous r.v.’s of a probability spaceand .

The conditional c.d.f. of given that is given by

The conditional p.d.f. of given that is given by

**Remark 8.3.2 Joint P.D.F.**

(1)

(2) A similar definition can be made for jointly continuous **random vectors**.

**Theorem 8.3.2 Properties of C-C Conditional Probability**

Suppose are jointly continuous r.v.’s of a probability space

.

(1) If and , then is a p.d.f.

(2)

(3)

(4) If and , then

◎ is a continuous r.v. and is a discrete r.v.

**Definition 8.3.3 C.D.F., P.D.F. and P.M.F. of C-D and D-C**

Let be a continuous r.v. and be a discrete r.v. of a probability space .

The conditional **c.d.f.** of given that is given by

The conditional **p.d.f.** of given that is given by

The conditional **p.m.f.** of given that is given by

The conditional **c.d.f.** of given that is given by

**Remark 8.3.3 Calculation of C-D P.D.F. and D-C P.M.F.**

(1)

(2) If and , then

If and , then

**Theorem 8.3.3 Properties of C-D and D-C Conditional Probability**

Supposeis a continuous r.v. andis a discrete r.v. of a probability space .

(1) If and , then is a p.d.f.

If and , then is a p.m.f.

(2)

(3) If and , then

If and , then

**Definition 8.3.4 Expectation of Conditional R.V.**

Let and be r.v.’s of a probability space and .

The conditional expectation of given that is given by

**Theorem 8.3.4 Expectation of Conditional Measurable Function**

Suppose and are r.v.’s of a probability space ,

and is a measurable function from to . Then

**8.4 Transformations of Two R.V.’s**

**Theorem 8.4.1 Transformations of Two R.V.’s**

Suppose and are r.v.’s of a probability space,and are measurable functions from to , and and .

(1) If and are discrete r.v.’s, then and are discrete r.v.’s and

(2) If and are jointly continuous r.v.’s, and are discrete r.v.’s, then

(3) If and are jointly continuous r.v.’s, and are jointly continuous r.v.’s, and

,

where is the Jacobian determinant, , then

**Theorem 8.4.2 Convolution Theorem**

Suppose and are two independent r.v.’s of a probability space

and .

(1) If and are discrete r.v.’s, then

(2) If and are jointly continuous r.v.’s, then

**8.5 Order Statistics**

**Definition 8.5.1 Order Statistic**

Let be i.i.d. r.v.’s of a probability space .

The order statistic of is defined as the **smallest** value in so that ,

namely, the smallest value infor all .

In particular, and .

**Remark 8.5.1 Without Equal & Not I.I.D.**

(1) If are jointly continuous r.v.’s, then

(2) is a function of

are **neither independent** **nor identically distributed**

in general.

**Definition 8.5.2 Random Sample**

A **random sample** of size of a probability space is a sequence of i.i.d. r.v.’s of .

**Definition 8.5.3 Range, Midrange, Median and Mean of Random Sample**

Let be a random sample of size of a probability space .

The **sample range** is given by .

The **sample midrange** is given by .

The **sample median** is given by

**Remark 8.5.1 Forced Decline**

If , then

and .

**Theorem 8.5.1 C.D.F. and P.D.F. of Jointly Order R.V.’s**

Suppose are i.i.d. jointly continuous r.v.’s of a probability space

with common c.d.f. and common p.d.f. .

If , then

**Corollary 8.5.1 Beta R.V. vs Binomial R.V.**

Suppose are i.i.d. r.v.’s , then

*Proof*:

**Corollary 8.5.1 Cases One, Two and Order R.V.’s**

Suppose are jointly continuous r.v.’s of a probability space with continuous c.d.f. and continuous p.d.f. .

In particular, ,

And

(2)

(3)

**8.6 Multinomial Distributions**

◎ Consider an experiment with possible outcomes .

Let be the event that the outcome is

and let

Suppose that such an experiment is independently and successively performed

times.

Let be the number of times that event occurs.

Then

**Definition 8.6.1 Multinomial Joint R.V.’s**

Let be discrete r.v.’s of a probability space .

We call multinomial joint r.v.’s with parameters ,

where , if the joint p.m.f. is given by

Such a joint p.m.f. is called a **multinomial** joint p.m.f. with parameters .

**Remark 8.6.1 Verification of P.M.F.**

and

is a p.m.f.

**Theorem 8.6.1 Splitting of Multinomial Joint R.V.’s**

Suppose are multinomial r.v.’s of a probability space , with parameters, where.

Then are multinomial joint r.v.’s

with parameters .

**Chapter.9 More Expectations and Variance**

**9.1 Expected Values of Sums of R.V.’s**

**Theorem 9.1.1 Expectations of Sum of Finite R.V.’s**

Suppose are r.v.’s of a probability space , then

**Theorem 9.1.2 Expectations of Sum of Infinite R.V.’s**

Suppose are r.v.’s of a probability space .

If or if is nonnegative for all then

**Remark 9.1.1 General Expectations of Sum of Infinite R.V.’s**

In general,

**Corollary 9.1.1 Expectation of Integer-Valued R.V.**

Suppose is an integer-valued r.v. of a probability space , then

**9.2 Covariance and Correlation Coefficients**

**Theorem 9.2.1 Cauchy-Schwarz Inequality**

Suppose and are r.v.’s of a probability space ,

and suppose and exists. Then

with probability 1 or with probability 1

or with probability 1, where

**Remark 9.2.1 Cauchy-Schwarz Equalities**

Suppose that and , then

with probability 1, where

with probability 1, where

**Corollary 9.2.1 Variance Larger Than or Equal to Zero**

Suppose is a r.v. of a probability space and suppose exists, then

**Definition 9.2.1 Covariance**

Let and be r.v.’s of a probability space with means and , resp.

The covariance (or ) of and is given by

We say that and are positively correlated, negatively correlated and uncorrelated if and , resp.

**Remark 9.2.2 Covariance of Linear Combination of Two R.V.’s**

(1) is a measure of the spread or dispersion of .

is a measure of the spread or dispersion of .

is a measure of the joint spread or

dispersion of and .

(2)

is a measure of the spread or dispersion along the -direction.

**Theorem 9.2.2 Calculating Covariance**

Suppose and are r.v.’s of a probability space .

(1) .

(2)

(3)

with probability 1 or with probability 1

or with probability 1, where

If and , then

with probability 1, where

with probability 1, where

**Theorem 9.2.3 Covariance of Two Linear Combined R.V.’s**

Suppose are r.v.’s of a probability space .

In particular, if are pairwise uncorrelated, then

**Theorem 9.2.4 Independence Implies Uncorrelated**

Suppose and are r.v.’s of a probability space .

If , then and are uncorrelated, i.e.,

**Remark 9.2.3 Uncorrelated Can’t Imply Independence**

The inverse is not true, i.e.,

**Definition 9.2.2 Correlation Coefficient**

Letandbe r.v.’s of a probability spacewith .

The correlation coefficient between and is given by

**Remark 9.2.4 Properties of Correlation Coefficient**

is independent of the units in which and is measured.

(2)

with probability 1, where

with probability 1, where

**9.3 Conditioning on R.V.’s**

**Definition 9.3.1 Conditional Expectation on R.V.’s**

Letandbe r.v.’s of a probability space.

Let .

We denote as the r.v. . Note that is a function of .

**Theorem 9.3.1 Marginal Expectation**

Suppose and are r.v.’s of a probability space .

Then

**Theorem 9.3.2 Marginal Expectation of Measurable Function**

Suppose and are r.v.’s of a probability space .

Then

**Theorem 9.3.3 Wald’s Equations**

Suppose are i.i.d. r.v.’s and is a positive integer-valued r.v. of a probability space , and .

(1) If and , then

(2) If and , then

**Theorem 9.3.4 Law of Total Probability**

Suppose is an event and is a r.v. of a probability space , then

**Theorem 9.3.5 Conditional Variance on R.V.’s**

Suppose and are r.v.’s of a probability space , then

**9.4 Bivariate Normal (Gaussian) Distribution**

**Definition 9.4.1 Bivariate Normal (Gaussian) R.V.’s**

Let andbe r.v.’s of a probability space.

We call and jointly normal (Gaussian) r.v.’s with parameters

denoted , if their joint p.d.f. is given by

where .

Such a joint p.d.f. is called a bivariate normal p.d.f. with parameters and .

**Theorem 9.4.1 Explicitly Normal (Gaussian) R.V.**

Suppose and are r.v.’s of a probability space,

and suppose .

(1) and . Therefore

(2)

and

(3) .

Therefore

and

**Remark 9.4.1 Mean Vector and Covariance Matrix**

**Lemma 9.4.1 Linear Conditional Expectation and Constant Variance**

Suppose and are jointly continuous r.v.’s of a probability space

with .

(1) If is a linear function in , then

(2) If is a linear function in ,

and is a constant, then

**Theorem 9.4.2 Derivation of Jointly Normal R.V.’s**

Suppose and are r.v.’s of a probability space.

Suppose

(1) is a normal r.v.

(2) is a normal r.v. for all .

(3) is a linear function in , and

is a constant.

Then and are **jointly normal** r.v.’s.

**Theorem 9.4.3 Independence mutually Implies Uncorrelated**

Suppose and are jointly normal r.v.’s of a probability space.

Then and are independent and are uncorrelated.

**Theorem 9.4.4 Linearly Generated Normal R.V.**

Suppose and , where is **nonsingular**, i.e., .

Then

**Chapter.10 Sums of Independent R.V.’s and Limit Theorems**

**10.1 Moment Generating Functions**

**Definition 10.1.1 Moment Generating Function**

The moment generating function (m.g.f.) of a r.v. is given by

, if is defined for all .

**Theorem 10.1.1 Moment Generation**

(1)

(2) Maclaurin’s series for :

**Remark 10.1.1 Sufficient Condition for**  **Moment to Converge**

If for some , then for all .

But the converse is not true.

**Theorem 10.1.2 Same M.G.F. Implies Same C.D.F.**

If for all for some ,

then the c.d.f. of and are the same.

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| Bernoulli(): |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
| : |  |  |  |
|  |  |  |  |
|  | 0 |  |  |

**10.2 Sums of Independent R.V.’s**

**Theorem 10.2.1 M.G.F. of Sums of Independent R.V.’s**

Suppose are **independent** r.v.’s with m.g.f.’s

respectively.

Then the m.g.f. of their **sum** is

**Theorem 10.2.2 M.G.F. of Sums of Normal R.V.’s**

Suppose are **independent** r.v.’s and

and suppose .

**Corollary 10.2.1 M.G.F. of Sums of I.I.D. Normal R.V.’s**

Suppose are **i.i.d.** , then

**10.3 Markov and Chebyshev Inequalities**

**Theorem 10.3.1 Markov’s Inequality**

Suppose is a nonnegative r.v., then

**Theorem 10.3.2 Chebyshev’s Inequality**

In particular,

**Remark 10.3.1 Not Tight Bounds**

The bounds obtained by Markov and Chebyshev inequalities usually **not very tight**.

**Theorem 10.3.3 Zero Absolute Moment**

**Corollary 10.3.1 Zero Variance**

**Theorem 10.3.4 Chebyshev’s Inequality for I.I.D R.V.’s**

Suppose are **i.i.d.** r.v.’s with mean and variance .

**Theorem 10.3.5 Chebyshev’s Inequality for I.I.D. Bernoulli R.V.’s**

Suppose are i.i.d. Bernoulli.

**10.4 Laws of Large Numbers (LLN’s)**

**Definition 10.4.1 Converge in Probability**

Let be r.v.’s of a probability space .

We say that converges to **in probability**, denoted

**Theorem 10.4.1 Weak Law of Large Numbers (WLLN)**

Suppose are i.i.d. r.v.’s with mean and variance .

**Remark 10.4.1 Relative Frequency Converges to Probability in Probability**

Let an experiment be repeated independently and let be the number of times an event occurs in the first repetitions of the experiment.

**Definition 10.4.2 Converge Almost Surely**

Let be r.v.’s of a probability space .

We say that converges to **almost surely** (a.s.), denoted

if

**Theorem 10.4.2 Strong Law of Large Numbers (SLLN)**

Suppose are i.i.d. r.v.’s with mean .

**Remark 10.4.2 Relative Frequency Converges Almost Surely**

**Theorem 10.4.3 Converge Almost Surely Implies Convergence in Probability**

**10.5 Central Limit Theorem (CLT)**

**Theorem 10.5.1 Levy Continuity Theorem**

Suppose are r.v.’s of a probability space .

**Theorem 10.5.2 Central Limit Theorem (CLT)**

Suppose are i.i.d. r.v.’s with mean and variance .

Then

i.e.,

Equivalently,