CS1231S Discrete Structures

Tutorial 1

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Tutorial Group 3A

Admin Stuff

Safety Measures

- Masks on at all time
- Stay 1.5m apart
- Passing objects require disinfection (e.g. markers)
- Take attendance and picture of seating arrangement

About Me

- Theodore Leebrant
- Year 2 Computer Science & Mathematics + USP
- Living in Cinnamon College
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About This Tutoria

- Safe space
    - Questions, mistakes, comments are welcome.
- Expectations
    - **Come to tutorial prepared for discussion!**
    - (Try to) finish the tutorial questions.
    - If still cannot, at least read the questions.
- Workflow
    - We will go through the tutorial questions
    - I'll stay back for any questions if needed, including discussion questions

Consultations

If you need any consultation,

- PM me on Telegram (preferred) or drop an email
- Either group or 1-to-1 consultations are fine, keep it below 5 people.
- F2F (preferred) or online (through zoom/discord)
- Check for timing, at least 1 day ahead. Most free on Mondays.

Recap

Important Sets

- $\mathbb{R}$: Set of all real numbers
- $\mathbb{Q}$: Set of all rational numbers (can be expressed as fractions)
- $\mathbb{Z}$: Set of all integers
- $\mathbb{N}$: Set of all natural numbers (non-negative integers)

Subscripts and superscripts used: $\mathbb{R}^+$, $\mathbb{R}^-$ $\mathbb{R}_{\leq 10}$

Important notations

- $\in$: element of
- $\forall$: for all
- $\exists$: there exists
- $|$: divisible by
- $\nmid$: not divisible by

Writing Proofs

For CS1231S:

- Introductory module
- Clarity of presentation
- One line for each step
- One law per step

Important notations (Propositional and Conditional Logic)

- $\neg$, or $\sim$: not / negation
- $\wedge$: and
- $\vee$: or
- $\rightarrow$: implies (if)
- $\leftrightarrow$: if and only if (iff)

Order of operations

1. $\neg$ or $\sim$
2. $\wedge$ and $\vee$
3. $\rightarrow$

Use brackets to eliminate ambiguity

Logical equivalences

- Commutativity: $p \vee q \equiv q \vee p$ (also works for $\wedge$)
- Associativity: $(p \vee q) \vee r \equiv p \vee (q \vee r)$ (also works for $\wedge$)
- Distributivity: $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- Identity law: $p \wedge \text{true} \equiv p$; $p \vee \text{false} \equiv p$
- Negation law: $p \vee \neg p \equiv \text{true}$; $p \wedge \neg p \equiv \text{false}$
- Idempotent law: $p \vee p \equiv p$; $p \wedge p \equiv p$
- Universal bound law: $p \vee \text{true} \equiv \text{true}$; $p \wedge \text{false} \equiv \text{false}$
- De Morgan's law: $\neg(p \wedge q) \equiv \neg p \vee \neg q$
- Absorption law: $p \vee (p \wedge q) \equiv p$; $p \wedge (p \vee q) \equiv p$
- Negation of true and false

More logical equivalences

- Implication law: $p \to q \equiv \neg p \vee q$
- Contrapositive: $p \to q \equiv \neg q \to \neg p$
- Biconditional equivalence: $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Converse and Inverse

Consider $p \to q$.

- Converse: $q \to p$
- Inverse: $\neg p \to \neg q$

Rules of Inference

- Modus ponens
- Modus tollense
- Generalisation
- Specialisation
- Conjunction
- Elimination
- Transitivity
- Proof by Division into cases
- Contradiction Rule

Rule of thumb on how to prove a logical statement

- Put the entire statement into consideration (e.g. Tutorial qn D2(a): do not just consider the left of the implication)
- You can try to sketch something, but answer needs to be step-by-step
- If all else fails, truth table (not recommended)

Tutorial Questions (and Photo Taking)

Question 1a

Claim: Assuming $a \in \mathbb{R}$, the negation of $(1 < a < 5)$ is $(1 \geq a \geq 5)$.

$$\begin{gathered} \sim (1 < a < 5) \\ \sim ((1 < a) \wedge (a < 5)) \\ (\sim (1 < a)) \vee (\sim (a < 5)) \\ (1 \geq a) \vee (a \geq 5) \end{gathered}$$

But $(1 \geq a \geq 5)$ means $(1 \geq a) \wedge (a \geq 5) \not\equiv (1 \geq a) \vee (a \geq 5)$.
Hence, (a) is **false**.

Question 1b

Claim: The two statements are logically equivalent:

1. "he's welcome to come along only if he behaves himself"
2. "if he behaves himself then he's welcome to come along"

Let p be "he's welcome to come along" and q be "he behaves himself".
Recall:

- p only if q means $p \to q$
- if p then q means $p \to q$

So Statement 1 $\equiv p \to q$
but Statement 2 $\equiv q \to p$.
Hence, (b) is **false**.

Question 2

a)

$$\sim a \land (\sim a \to (a \land b))$$

$$\begin{aligned}
&\equiv \sim a \land (\sim(\sim a) \lor (a \land b)) && \text{by implication law} \\
&\equiv \sim a \land (a \lor (a \land b)) && \text{by double negative law} \\
&\equiv \sim a \land a && \text{by absorption law} \\
&\equiv a \land \sim a && \text{by commutative law} \\
&\equiv \textbf{false} && \text{by negation law}
\end{aligned}$$

Question 2

b)

$$p \vee \sim q \rightarrow q$$

$$
\begin{aligned}
&\equiv \sim (p \vee \sim q) \vee q && \text{by implication law} \\
&\equiv (\sim p \vee \sim (\sim q)) \vee q && \text{by De Morgan's law} \\
&\equiv (\sim p \vee q) \vee q && \text{by double negative law} \\
&\equiv q \vee (\sim p \vee q) && \text{by commutative law} \\
&\equiv q \vee (q \vee \sim p) && \text{by commutative law} \\
&\equiv q && \text{by absorption law}
\end{aligned}
$$

Question 2

c)

$$\sim (p \vee \sim q) \vee (\sim p \wedge \sim q)$$

$$
\begin{aligned}
&\equiv (\sim p \vee \sim (\sim q)) \vee (\sim p \wedge \sim q) && \text{by De Morgan's Law} \\
&\equiv (\sim p \vee q) \vee (\sim p \wedge \sim q) && \text{by double negative law} \\
&\equiv \sim p \vee (q \wedge \sim q) && \text{by distributive law} \\
&\equiv \sim p \vee \textbf{true} && \text{by distributive law} \\
&\equiv \sim p && \text{by identity law}
\end{aligned}
$$

Question 2

d)

$$(p \to q) \to r$$

$$
\begin{aligned}
&\equiv (\sim p \vee q) \to r && \text{by implication law} \quad (1)\\
&\equiv (\sim (\sim p \vee q)) \vee r && \text{by implication law} \quad (2)\\
&\equiv (\sim (\sim p) \wedge \sim q) \vee r && \text{by De Morgan's law} \quad (3)\\
&\equiv (p \wedge \sim q) \vee r && \text{by the double negative law} \quad (4)
\end{aligned}
$$

Question 3

Answer: $(p \to q) \to r$ and $p \to (q \to r)$ are **not** logically equivalent.
To show this, we just need to get one counterexample from the following truth table:

p	q	r	$p \to q$	$(p \to q) \to r$	$q \to r$	$p \to (q \to r)$
true	true	true	true	true	true	true
true	true	false	true	false	false	false
true	false	true	false	true	true	true
true	false	false	false	true	true	true
false	true	true	true	true	true	true
false	true	false	true	**false**	false	**true**
false	false	true	true	true	true	true
false	false	false	true	**false**	true	**true**

Question 4

"The rule says that to qualify for the draw, SAFRA-DBS credit card holders must 'charge a minimum of S$50 nett to their card during the Qualifying Period', which is 1 July to 30 September 2017."

Let $C =$ "Charge a minimum of S$50 nett"
$P =$ "Charge during the Qualifying Period"
$W =$ "Win 100,000 AirAsia Miles"

Question 4a

Write a conditional statement using C, P and W that describes the rule above.
Note that the qualifying conditions are **necessary but not sufficient** conditions. Thus, C and P are necessary for W, which translates to:

$$\text{if } W \text{ then } (C \land P)$$
$$W \rightarrow (C \land P)$$

Question 4b

Write the converse, inverse, contrapositive and negation forms of the statement in part (a).

Statement:	$W \to (C \wedge P)$
Converse:	$(C \wedge P) \to W$
Inverse:	$\sim W \to \sim (C \wedge P)$
Contrapositive:	$\sim (C \wedge P) \to \sim W$
Negation:	$\sim (W \to (C \wedge P))$

Question 5

Idea: for a valid conditional statement, the transitive rule of inference holds.
If $(p \to q)$ and $(q \to r)$, then $(q \to r)$.
i.e. $((p \to q) \land (q \to r)) \to (p \to r)$ would be always true (tautology)

We have three 'alternative' definition of the conditional statement. To show that they are not correct, we can show that the transitive rule of inference does not hold: in this case, showing a counterexample is sufficient.

Question 5 (Alternative conditional statement a)

Using the first alternative definition of conditional statement,

p	q	r	$p \to q$	$q \to r$	$p \to r$
true	true	true	true	true	true
true	true	false	true	false	false
true	false	true	false	false	true
true	false	false	false	false	false
false	true	true	false	true	false
false	true	false	false	false	false
false	false	true	false	false	false
false	false	false	false	false	false

Question 5 (Alternative conditional statement a)

Therefore,

$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
true	true
false	**false**
false	**false**
false	**false**
false	**false**
false	**false**
false	**false**
false	**false**

Question 5 (alternative conditional statement b)

Using the second alternative definition of conditional statement,

p	q	r	$p \to q$	$q \to r$	$p \to r$
true	true	true	true	true	true
true	true	false	true	false	false
true	false	true	false	true	true
true	false	false	false	false	false
false	true	true	true	true	true
false	true	false	true	false	false
false	false	true	false	true	true
false	false	false	false	false	false

Question 5 (Alternative conditional statement b)

Therefore,

$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
true	true
false	**false**
false	true
false	**false**
true	true
false	**false**
false	true
false	**false**

Question 5 (Alternative conditional statement c)

Using the third alternative definition of conditional statement,

p	q	r	$p \rightarrow q$	$q \rightarrow r$
true	true	true	true	true
true	true	false	true	false
true	false	true	false	false
true	false	false	false	true
false	true	true	false	true
false	true	false	false	false
false	false	true	true	false
false	false	false	true	true

Question 5 (Alternative conditional statement c)

Therefore,

$(p \to q) \wedge (q \to r)$	$((p \to q) \wedge (q \to r)) \to (p \to r)$
true	true
false	true
false	true
false	**false**
false	**false**
false	true
false	true
true	true

Question 6a

Sandra knows Java and Sandra knows C++.
$\therefore$ Sandra knows C++.

Let $p =$ "Sandra knows Java"
Let $q =$ "Sandra knows C++"

$$p \wedge q \qquad \text{(premise)}$$
$$\therefore q \qquad \text{(valid by specialisation)}$$

Question 6b

If at least one of these two numbers is divisible by 6, then the product of these two numbers is divisible by 6.
Neither of these two numbers is divisible by 6.
$\therefore$ The product of these two numbers is not divisible by 6.

Let $p =$ "the first number is divisible by 6"
Let $q =$ "the second number is divisible by 6"
Let $r =$ "the product of these two numbers is divisible by 6"

$$
\begin{aligned}
& p \vee q \to r && \text{(premise)} \\
& \sim p \wedge \sim q && \text{(premise)} \\
& \therefore \sim r && \text{(invalid: inverse error)}
\end{aligned}
$$

Question 6c

If there are as many rational numbers as there are irrational numbers, then the set of all irrational numbers is infinite
The set of all irrational numbers is infinite.
$\therefore$ There are as many rational numbers as there are irrational numbers.

Let $p =$ "there are as many rational numbers as there are irrational numbers"
Let $q =$ "the set of all irrational numbers is infinite"

$$
\begin{array}{lr}
p \to q & \text{(premise)} \\
q & \text{(premise)} \\
\therefore p & \text{(invalid: converse error)}
\end{array}
$$

Question 6d

If I get a Christmas bonus, I'll buy a stereo.
If I sell my motorcycle, I'll buy a stereo.
$\therefore$ If I get a Christmas bonus or I sell my motorcycle, I'll buy a stereo.

Let $p =$ "I get a Christmas bonus"
Let $q =$ "I sell my motorcycle"
Let $r =$ "I'll buy a stereo"

$$
\begin{aligned}
& p \to r && \text{(premise)} \\
& q \to r && \text{(premise)} \\
& (p \to r) \wedge (q \to r) && \text{(by conjunction)} \\
& (\sim p \vee r) \wedge (\sim q \vee r) && \text{(by implication law)} \\
& (\sim p \wedge \sim q) \vee r && \text{(by distributive law)} \\
& (p \vee q) \to r && \text{(by implication law)}
\end{aligned}
$$

Question 7

Prove that $\exists x, y, z \in \mathbb{Z}_{>10}$ such that $x^2 + y^2 = z^2$. What is your proof called? What are these values called?

Proof

1. Let $x = 11, y = 60, z = 61$.
2. Then $x, y, z \in \mathbb{Z}_{>10}$ and
   $x^2 + y^2 = 11^2 + 60^2 = 121 + 3600 = 3721 = 61^2$.
3. Thus $\exists x, y, z \in \mathbb{Z}_{>10}$ such that $x^2 + y^2 = z^2$. $\square$

This is proof by construction. The values are called Pythagorean triples.

Question 8

The island of Wantuutrewan is inhabited by two types of people: knights who always tell the truth and knaves who always lie. You visit the island and have the following encounters with the natives.

Two natives C and D speak to you:

- C says: D is a knave.
- D says: C is a knave.

What are C and D?

Question 8 (cont.)

Proof (by exhaustion)

1. If C is a knight:
    1.1 What C says is true. (by definition of knight)
    1.2 $\therefore D$ is a knave. (what C says)
2. If C is not a knight:
    2.1 Then C is a knave. (one is either a knight or a knave)
    2.2 $\therefore$ what C says is false. (by definition of knave)
    2.3 $\therefore D$ is not a knave. (negation of what C says)
    2.4 $\therefore D$ is a knight (one is either a knight or a knave)
3. In both cases, there is one knight and one knave. $\square$

Question 9

Prove the following statement:

The product of any two odd integers is an odd integer.

Proof (direct)

1. Take any two odd integers m, n.
2. Then $m = 2k + 1$ and $n = 2p + 1$ for $k, p \in \mathbb{Z}$
   (by definition of odd numbers)
3. $mn = (2k + 1)(2p + 1) = 4kp + 2k + 2p + 1 = 2(2kp + k + p) + 1$
   (by basic algebra)
4. Let $q = 2kp + k + p$. Then $q \in \mathbb{Z}$.
   (by closure of integers under $+$ and $\times$)
5. Then $mn = 2q + 1$. So mn is odd.
   (by definition of odd numbers)
6. Therefore, the product of any two odd integers is an odd integer.
   $\square$

Question 10a

The problem with Smart's proof:

In Line 5, Smart claims that $\sqrt{4k^2 + 4k + 4m^2 + 2}$ not an integer, but did not provide a proof nor cite any theorem to support his claim. Thus, Smart's proof is incomplete, which means we cannot be sure if it is correct.

Question 10b

Proof (by contraposition)

1. Take the contraposition: if a, b are both odd, then $a^2 + b^2 \neq c^2$.
2. Suppose a, b are both odd.
3. Then $\exists k, m \in \mathbb{Z} \ni a = 2k+1$ and $b = 2m+1$. (by definition of odd numbers)
4. $a^2 + b^2 = (2k+1)^2 + (2m+1)^2 = 4k^2 + 4k + 4m^2 + 4m + 2$ (by basic algebra)
5. Let $z = k^2 + k + m^2 + m$. Then $a^2 + b^2 = 4z + 2 = 2(2z+1)$. (by basic algebra)
6. Note that z and $(2z+1)$ are both integers. (by closure of integers under addition and multiplication)
7. Hence $a^2 + b^2$ is even. (by definition of even numbers)
8. Moreover, since $a^2 + b^2 = 4z + 2$, it follows that $a^2 + b^2$ has remainder 2 when divided by 4.

Question 10b (cont.)

9. Now, c is either odd or even.
   9.1 Case 1: c is odd.
      9.1.1 Then c^2 is odd. (by Question 9)
      9.1.2 Then $c^2 \neq a^2 + b^2$ since RHS is even (from line 7).
   9.2 Case 2: c is even.
      9.2.1 Then $\exists p \in \mathbb{Z} \ni c = 2p$ (by definition of even numbers)
      9.2.2 Then $c^2 = 4p^2$. (by basic algebra)
      9.2.3 Hence c^2 has a remainder of 0 when divided by 4.
      9.2.4 Therefore $c^2 \neq a^2 + b^2$ since RHS has remainder 2 when divided by 4. (from line 8)
10. In all cases, $c^2 \neq a^2 + b^2$.
11. Therefore, by contraposition, the original statement is true. $\square$

Feedback / Questions

Feedback