EGT2
ENGINEERING TRIPOS PART IIA

Friday 22 April 2016   9.30 to 11

Module 3F3

SIGNAL AND PATTERN PROCESSING

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

1 (a) A complex-coefficient digital filter has a transfer function of the following form:

$$H(z) = \frac{z^{-1} - r\exp(-j\phi)}{1 - z^{-1}r\exp(j\phi)}$$

where $0 < r < 1$ and $0 \leq \phi \leq \pi$.

(i) Sketch the pole-zero diagram for such a filter when $r = 0.7$ and $\phi = \pi/4$. [10%]

(ii) Determine the frequency response of such a filter for any (r, ϕ) and show by geometrical arguments from the pole-zero diagram, or otherwise, that all frequencies are passed with equal gain, i.e. the filter is *all-pass*. [25%]

(iii) Sketch the phase response of the above filter when $r = 0.95$ and $\phi = \pi/4$, for a range of normalised frequencies from 0 to 2π. Your justification should be geometric and based on the pole-zero diagram; you may utilise the fact that the pole is close to the zero and to the unit circle when $r = 0.95$. [20%]

(iv) Explain how a real-coefficient filter can be generated from a combination of two such structures, retaining the all-pass property but with modified phase response. [10%]

(b) A second digital filter has the following transfer function:

$$H(z) = \frac{1 - z^{-P}}{1 - rz^{-P}}$$

where $0 < r < 1$ and $P > 1$ is an integer.

(i) Sketch the pole-zero diagram for this filter when $P = 8$ and $r = 0.8$. [10%]

(ii) Sketch also the frequency magnitude response of this filter when r is close to 1, and hence suggest an application for such a filter. What trade-off would be made by choice of the value of r? [25%]

2 The discrete Fourier transform (DFT) of a sequence x_n, $n = 0, 1, ..., N-1$ is given by:

$$X_p = \sum_{n=0}^{N-1} x_n e^{-\frac{jnp2\pi}{N}}$$

(a) Show that the DFT spectrum values X_p are related to the true Discrete-time Fourier Transform (DTFT) spectrum $X(e^{j\theta})$ of x_n, $n = -\infty, ..., -1, 0, 1, 2..., \infty$, by the following convolution formula:

$$X_p = \frac{1}{2\pi} \int_0^{2\pi} W(e^{j\theta}) X(e^{j(2p\pi/N - \theta)}) \, d\theta$$

where $W(e^{j\theta})$ is the DTFT of the appropriate rectangular window function. What frequency, in rad/s, would correspond to $\theta = 0.2$, if the digital sampling frequency is 44.1kHz? [30%]

(b) Explain with the aid of diagrams how the DFT modifies the DTFT spectrum of a complex exponential signal $\exp(j\omega_0 t)$ where ω_0 is a fixed frequency. Describe the effects of spectral smearing and spectral leakage, how the use of window functions might aid the analysis, and why this is important for analysis of multiple frequency components. [20%]

(c) Show that when N is even, the DFT of a data sequence x_n may be expressed in the form:

$$X_p = A_p + W^p B_p \text{ and } X_{p+N/2} = A_p - W^p B_p$$

where A_p and B_p are DFTs of length $N/2$ sub-sequences of x_n, and W is a suitable complex exponential. Your solution should include a derivation for the terms A_p, B_p and W. [30%]

(d) Explain briefly how the representation in (c) above allows a very efficient implementation of the DFT when N is a power of 2. Show that the computational load is approximately $(N/2)\log_2(N)$ complex multiplications and additions. How does this compare with direct evaluation of the DFT? [20%]

3 (a) For a discrete-time random process, explain the concept of *stationarity*, and define the terms autocorrelation function, wide-sense stationarity and power spectrum. Give an expression for the average power of a real-valued process between two normalised frequencies ω_1 and ω_2, with $0 \leq \omega_1 < \omega_2 < \pi$. [30%]

(b) A random process $\{y_n\}$ is passed through a causal linear system having impulse response $h_m = \alpha^m$:

$$z_n = \sum_{m=0}^{+\infty} h_m y_{n-m}.$$

(i) Show how the linear system may be implemented as a first-order infinite impulse response (IIR) digital filter. [10%]

(ii) If $\{y_n\}$ has autocorrelation function $E[Y_k Y_l] = R_{YY}[k,l]$, find an expression for the cross-correlation function between $\{y_n\}$ and $\{z_n\}$, and also the autocorrelation function of the output process $\{z_n\}$. [20%]

(iii) If $\{y_n\}$ is wide-sense stationary, show that $\{z_n\}$ is also wide-sense stationary, provided the condition $|\alpha < 1|$ applies. [20%]

(iv) If $\{y_n\}$ is zero-mean white noise with variance 1, and $|\alpha| < 1$, determine the power spectrum of $\{y_n\}$, the autocorrelation function for $\{z_n\}$ and the power spectrum of $\{z_n\}$. Sketch the power spectrum $S_Z(\exp(j\theta))$ for $\alpha = 0.8$ over the range of normalised frequencies $\theta = 0$ to 2π. [20%]

4 Consider the k-means clustering algorithm which seeks to minimise the cost function

$$C = \sum_{n=1}^{N} \sum_{k=1}^{K} s_{nk} \|x_n - m_k\|^2$$

where m_k is the mean (centre) of cluster k, x_n is data point n, $s_{nk} = 1$ signifies that data point n is assigned to cluster k, and there are N data points and K clusters.

(a) Assume that the cluster assignments s_{nk} have all been determined, under the constraint that each data point must be assigned to one cluster, that is, $\sum_k s_{nk} = 1$ for all n, and $s_{nk} \in \{0,1\}$ for all n and k. Now derive the value of the means $\{m_k\}$ which minimise the cost C above, and give an interpretation in terms of the k-means algorithm. [30%]

(b) Give an interpretation of the k-means algorithm in terms of a probabilistic model. Describe three generalisations based on this probabilistic model. [40%]

(c) You are applying the k-means algorithm to a large collection of images, where most of the images are not labelled, but you have labels for a few of the images (e.g. “cat”, “dog”, “person”, “car”). You would like to modify your k-means algorithm so that images with the same label are always in the same cluster, and images with different labels are never in the same cluster. Describe a modified version of the algorithm that would achieve this. [30%]

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