3F3 Revision question.

1. (i). $P(i_n=A) = P(i_{n-1}=A)\,P(i_n=A \mid i_{n-1}=A) + P(i_{n-1}=B)\,P(i_n=A \mid i_{n-1}=B)$

$P(i_n=B) = P(i_{n-1}=A)\,P(i_n=B \mid i_{n-1}=A) + P(i_{n-1}=B)\,P(i_n=B \mid i_{n-1}=B)$

(ii) Try $P(i_n) = P(i_{n-1})$

And let $P(i_n=A) = a = P(i_{n-1}=A)$

$P(i_n=B) = 1-a = P(i_{n-1}=B)$

$$a = a \cdot 0.7 + (1-a)\cdot 0.1$$

$$1-a = a\cdot 0.3 + (1-a)\cdot 0.9$$

$\Rightarrow a = 0.25, \quad 1-a = 0.75$

$\Rightarrow$ ~~$P(i_n=A)$~~ $P(S_n=A) = 0.25$

$P(S_n=B) = 0.75$

(iii) $P(i_n=A) = P(i_{n-1}=A)\cdot 0.7 + P(i_{n-1}=B)\cdot 0.1$

$= [P(i_{n-2}=A)\cdot 0.7 + P(i_{n-2}=B)\cdot 0.1]\cdot 0.7$
$+ [P(i_{n-2}=A)\cdot 0.3 + P(i_{n-2}=B)\cdot 0.9]\cdot 0.1$

$= P(i_{n-2}=A)\cdot(0.49+0.03) + P(i_{n-2}=B)\cdot(0.07+0.09)$

$P(i_n=B) = P(i_{n-1}=A)\cdot 0.3 + P(i_{n-1}=B)\cdot 0.9$

$= [P(i_{n-2}=A)\cdot 0.7 + P(i_{n-2}=B)\cdot 0.1]\cdot 0.3$
$+ [P(i_{n-2}=A)\cdot 0.3 + P(i_{n-2}=B)\cdot 0.9]\cdot 0.9$

$= P(i_{n-2}=A)\cdot(0.21+0.27) + P(i_{n-2}=B)\,(0.81+0.03)$

Therefore, for $S_0, S_2, S_4$, the value of $S_{n+2}$ only requires $S_n$, do not require further previous values.

Transition matrix $Q$: =
$$\begin{bmatrix} P(S_{n+2}=A \mid S_n=A) & P(S_{n+2}=B \mid S_n=A) \\ P(S_{n+2}=A \mid S_n=B) & P(S_{n+2}=B \mid S_n=B) \end{bmatrix}$$

$\cancel{P(S_{n+2} \mid S_n) : \quad A \mid A \Rightarrow P(S_{n+2}=A) = 0.52 \cdot P(S_n=A) + 0.16(1-P(S_n=A))}$

$\cancel{P(S_{n+2}=A) = 0.16 + 0.36\,P(S_n=A)}$

~~By Bayes theorem:~~ $\cancel{P(A \mid B)}$

$\cancel{P(S_{n+2}=A) = P(S_{n+1}=A)\,P(S_{n+2}=A \mid S_{n+1}=A) + P(S_{n+1}=B)\,P(S_{n+2}=A \mid S_{n+1}=A)}$

$\cancel{= [P(S_{n+1}=A \mid S_n=A)\,P(S_n=A) + P(S_{n+1}=A \mid S_n=B)\,P(S_n=B)] \cdot P(S_{n+2}=A \mid S_{n+1}=A)}$

$$P(S_{n+2}=A \mid S_n=A) = 0.49 + 0.03 = 0.52$$

$$P(S_{n+2}=A \mid S_n=B) = 0.07 + 0.09 = 0.16$$

$$P(S_{n+2}=B \mid S_n=A) = 0.21 + 0.27 = 0.48$$

$$P(S_{n+2}=B \mid S_n=B) = 0.81 + 0.03 = 0.84$$

$$\therefore Q = \begin{bmatrix} 0.52 & 0.48 \\ 0.16 & 0.84 \end{bmatrix}$$

(b) (i) F.T. : $\int_{-\infty}^{\infty} y(t)\, e^{-i\omega t}\, dt$

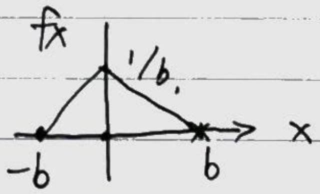
$$\int_{-\infty}^{\infty} f_X(x)\, e^{-i\omega x}\, dx$$

$$E[e^{(ixt)}] = \int_{-\infty}^{\infty} f_X(x)\, e^{itx}\, dx$$

The characteristic function is equivalent to Fourier transform if setting the frequency in F.T. to $-t$ in $\varphi_X(t)$.

(ii) $\varphi_X(t)$ Equivalent to $F(w)$ by setting $w=-t$.

$$F(w) = F.T.\left(\frac{1}{b}\left(1-\frac{|x|}{b}\right)\right)$$




$$F(w) = \frac{1}{b}\cdot b \,\mathrm{sinc}^2\left(\frac{wb}{2}\right)$$

$$= \mathrm{sinc}^2\left(\frac{wb}{2}\right)$$

$$\varphi_X(t) = F(-t) = \mathrm{sinc}^2\left(\frac{-bt}{2}\right)$$

$$= \frac{4\sin^2\left(\frac{-bt}{2}\right)}{b^2t^2}$$

(iii)

$$\varphi_X(t) = \frac{2\cdot 2\sin^2(-bt/2)}{b^2t^2}$$

$$= \frac{2\left(1-\cos(-bt)\right)}{b^2t^2}$$

$$= \frac{2\left(1-\cos(bt)\right)}{b^2t^2}$$

$$\cancel{= \frac{2}{b^2t^2} - \left[\frac{1-\frac{b^2t^2}{2}+\frac{b^4t^4}{4!}-\frac{b^6t^6}{6!}}{b^2t^2}\right]}$$

$$\cancel{= \frac{2}{b^2t^2} - \frac{1}{b^2t^2} + \frac{1}{2} - \frac{b^2t^2}{4!} + \frac{b^4t^4}{6!}}$$

$$\cancel{= \frac{2}{b^2t^2} - \left[\frac{\frac{b^2t^2}{2}-\frac{b^4t^4}{4!}+\frac{b^6t^6}{6!}}{b^2t^2}\right]}$$

$$\cancel{= \frac{2}{b^2t^2} - \frac{1}{2} + \frac{b^2t^2}{4!}}$$

$$= (b_0 + b_1 e^{jwt})(b_0 + b_1 e^{-jwt})$$

$$= \frac{2}{b^2t^2}\left(\frac{b^2t^2}{2} - \frac{b^4t^4}{4!} + \frac{b^6t^6}{6!} - \dots\right)$$

$$= 1 - \frac{2b^2t^2}{4!} + \frac{2b^4t^4}{6!}$$

$E[X^0]: = \varphi_X(0) = 1$

$E[X^2]:$ $\frac{d^2\varphi(t)}{dt^2} = E\{i^2x^2\exp(itx)\}$

$$\left.\frac{d^2\varphi(t)}{dt^2}\right|_{t=0} = E\{-x^2\} = -\frac{4b^2}{4!}$$

$$= -\frac{b^2}{6}$$

$$E\{x^2\} = \frac{b^2}{6}$$

$E[X^4]:$ $\left.\frac{d^4\varphi(t)}{dt^2}\right|_{t=0} = E\{x^4\} = \frac{2\cdot 4\cdot 3\cdot 2 b^4}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$

$$= \frac{b^4}{15}$$

2. (a) ~~$R_{xx}(k) = E\{X_{n+k} X_n\} = a_1 E\{$~~

$X_n = -a_1 X_{n-1} - a_2 X_{n-2} + \sigma W_n$.

~~$R_{xx}(1) = E\{X_{n+1} X_n\}$~~ $= E\{X_n X_{n+1}\}$.

$= E\{X_{n-1} \cdot (-a_1 X_{n-1} - a_2 X_{n-2} + \sigma W_n)\}$

~~$= -a_1 E\{X_{n-1}^2\} - a_2 E\{X_{n-1} X_{n-2}\}$~~ $+ 0$.

Take z transform to map Noise to process X.

$X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) = \sigma W(z)$

$$X(z) = \frac{\sigma \cdot W(z)}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

so $S_X(e^{j\omega T}) = S_W(e^{j\omega T}) \, |H(e^{j\omega T})|^2$

$$= 1 \cdot \left| \frac{\sigma}{1 + a_1 z^{-1} + a_2 z^{-2}} \right|^2_{(z = e^{j\omega T})}$$

$$= \frac{\sigma^2}{(1 + a_1 e^{j\omega T} + a_2 e^{j\omega T})(1 + a_1 e^{-j\omega T} + a_2^{-j\omega T})}$$

(b). $V(z) = b_0 E(z) + b_1 z^{-1} E(z)$.

$V(z) = (b_0 + b_1 z^{-1}) E(z)$

$S_V(e^{j\omega T}) = S_E(e^{j\omega T}) \cdot |H(e^{j\omega T})|^2$

$= 1 \cdot |b_0 + b_1 e^{-j\omega T}|^2$

$= (b_0 + b_1 e^{j\omega T})(b_0 + b_1 e^{-j\omega T})$

(c). PSD of $Y_n = \sum_{k=0}^{\infty} R_{YY}(k) e^{-j\omega k}$

$R_{YY}(k) = E[Y_n Y_{n+k}]$

$= E[(X_n + V_n)(X_{n+k} + V_{n+k})]$

$= E[X_n X_{n+k}] + E[V_n V_{n+k}] + E[X_n V_{n+k}] + E[V_n X_{n+k}]$

Since the noise sequences are independent
$E[X_n V_{n+k}] = E[V_n X_{n+k}] = 0.$

So PSD of $Y_n = S_Y = S_X + S_V$

$$= \frac{\sigma^2}{(1+a_1 e^{j\omega} + a_2 e^{j\omega})(1+a_1 e^{-j\omega} + a_2 e^{-j\omega})} + (b_0 + b_1 e^{j\omega}) \cdot (b_0 + b_1 e^{-j\omega})$$

(d). $S_V(\omega) = b_0^2 + b_1^2 + b_0 b_1 e^{j\omega} + b_0 b_1 e^{-j\omega}$
$= (b_0^2 + b_1^2) + b_0 b_1 (\cos\omega + j\sin\omega + \cos\omega - j\sin\omega)$
$= b_0^2 + b_1^2 + 2b_0 b_1 \cos\omega$

$$\begin{cases} 2 = b_0^2 + b_1^2 \\ 2 = 2b_0 b_1 \end{cases} \Rightarrow \cancel{b_0^2+b_1^2+2b_0b_1} \quad \cancel{(b_0+b_1)}$$

$b_0 = \frac{1}{b_1}$

$b_1^2 + \left(\frac{1}{b_1}\right)^2 = 2$

$b_1^4 + 1 = 2b_1^2$

$b_1^4 - 2b_1^2 + 1 = 0$

$(b_1^2 - 1) = 0$

$b_1 = \pm 1$

$b_0 = \pm 1$ $\Rightarrow \begin{cases} b_0 = 1 \\ b_1 = 1 \end{cases}$ or $\begin{cases} b_0 = -1 \\ b_1 = -1 \end{cases}$

$$x_n = -a_1 x_{n-1} - a_2 x_{n-2} + 6 W_n$$

(e) $R_{xx}[1] = E\{x_{n-1} x_n\}$

$$= E\{x_{n-1}(-a_1 x_{n-1} - a_2 x_{n-2} + 6W_n)\}$$

$$R_{xx}[1] = -a_1 R_{xx}[0] - a_2 E\{x_{n-1} x_{n-2}\}$$

$$R_{xx}[1] = -a_1 R_{xx}[0] - a_2 R_{xx}[-1]$$

$$R_{xx}[1] = -a_1 R_{xx}[0] - a_2 R_{xx}[1] \quad ①$$

$$R_{xx}[2] = E\{x_{n-2} x_n\}$$

$$= E\{x_{n-2} \cdot (-a_1 x_{n-1} - a_2 x_{n-2} + 6W_n)\}$$

$$R_{xx}[2] = -a_1 R_{xx}[1] - a_2 R_{xx}[0] \quad ②$$

① + ② Gives:
$$\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \\ R_{xx}[1] & R_{xx}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} R_{xx}[1] \\ R_{xx}[2] \end{bmatrix}$$

(f) Find $a_1, a_2$ and $6$ to find PSD of $\{x_n\}$.
Since: Psd $\rightarrow$ $S_V(w) = \sum_{k=-\infty}^{\infty} e^{-jwk} \cdot r_V(k)$

$R_W[0] = 2.$

$R_W[1] = R_W[-1] = a$

$[e^{-jw} + e^{jw}]a = a 2\cos w \quad = 2\cos w$

$a = 1$

$R_W[1] = 1$

$R_W[2] = 0.$

So. $R_{xx}[0] = R_{YY}[0] - R_{VV}[0] = 2.74$

$R_{xx}[1] = -0.46.$

$R_{xx}[2] = 1.41$

$$x_n = -a_1 x_{n-1} - a_2 x_{n-2} + \sigma W_n$$

(e) $R_{xx}[1] = E\{x_{n-1} x_n\}$

$$= E\{x_{n-1}(-a_1 x_{n-1} - a_2 x_{n-2} + \sigma W_n)\}$$

$$R_{xx}[1] = -a_1 R_{xx}[0] - a_2 E\{x_{n-1} x_{n-2}\}$$

$$R_{xx}[1] = -a_1 R_{xx}[0] - a_2 R_{xx}[-1]$$

$$R_{xx}[1] = -a_1 R_{xx}[0] - a_2 R_{xx}[1] \quad ①$$

$$R_{xx}[2] = E\{x_{n-2} x_n\}$$

$$= E\{x_{n-2} \cdot (-a_1 x_{n-1} - a_2 x_{n-2} + \sigma W_n)\}.$$

$$R_{xx}[2] = -a_1 R_{xx}[1] - a_2 R_{xx}[0]. \quad ②$$

① + ② Gives: $\begin{bmatrix} R_{xx}[0] & R_{xx}[1] \\ R_{xx}[1] & R_{xx}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} R_{xx}[1] \\ R_{xx}[2] \end{bmatrix}$

(f) Find $a_1, a_2$ and $\sigma$ to find PSD of $\{x_n\}$.

Since: Psd $\rightarrow S_V(\omega) = \sum_{k=-\infty}^{\infty} e^{-j\omega k} \cdot r_{vv}(k)$

$R_V[0] = 2.$

$R_V[1] = R_V[-1] = a$

$[e^{-j\omega} + e^{j\omega}]a = a 2\cos\omega = 2\cos\omega$

$a = 1$

$R_{VV}[1] = 1$

$R_V[2] = 0.$

So. $R_{xx}[0] = R_{YY}[0] - R_{VV}[0] = 2.74$

$R_{xx}[1] = -0.46.$

$R_{xx}[2] = 1.41$

$$\begin{bmatrix} 2.74 & -0.46 \\ -0.46 & 2.74 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} -0.46 \\ 1.41 \end{bmatrix}$$

$$a_1 = 0.0839 \qquad a_2 = -0.5$$

$$\sigma^2 = 2.74 - 0.46 \cdot 0.0839 + -0.5 \cdot 1.41$$
$$= 1.996.$$

3\. (a). For the linear model! $x_n = a + bn + w_n$

rewrite into $x = G\theta + e$.

The best fit minimise the error $e$

$$J = \sum_{n=0}^{N-1} e_n^2 = e^T e.$$

$$\begin{aligned} J = e^T e &= (x - G\theta)^T (x - G\theta) \\ &= x^T x + \theta^T G^T G \theta - x^T G\theta - (G\theta)^T x \\ &= x^T x + \theta^T G^T G \theta - (G\theta)^T x - (G\theta)^T x \\ &= x^T x + \theta^T G^T G \theta - 2\theta^T G^T x \end{aligned}$$

$$\frac{dJ}{d\theta} = 2G^T G\theta - 2G^T x = 0.$$

$$G^T G \theta = G^T x$$

$$\theta^{OLS} = (G^T G)^{-1} G^T x$$

$$\begin{aligned} E[\theta^{OLS}] &= (G^T G)^{-1} G^T E[x]. \\ &= (G^T G)^{-1} G^T (E[G\theta + e]) \\ &= (G^T G)^{-1} G^T G \theta + 0. \\ &= \theta \qquad \Rightarrow \text{unbiased}. \end{aligned}$$

$$E(\theta^{OLS}\theta^{OLS\,T}) - \theta^{OLS}\theta^{OLS\,T}$$

$$= E\left[(G^TG)^{-1}G^Tx\,((G^TG)^{-1}G^Tx)^T\right].$$

$$= E\left[(G^TG)^{-1}G^Tx\,x^TG\,[(G^TG)^{-1}]^T\right]$$

$$= (G^TG)^{-1}G^T\,E[xx^T]\;G\,[(G^TG)^{-1}]^T$$

$$E[xx^T] = \cancel{E\,E\,G\theta\,\theta E G} = E[(G\theta+e)\,(G\theta+e)^T]$$

$$= E[\,G\theta\,(G\theta)^T + ee^T + G\theta e^T + e(G\theta)^T\,].$$

$$= E[\,G\theta\theta^TG^T + ee^T\,] + E(G\theta e^T + e(G\theta)^T]$$

$$\underset{0.}{\shortparallel}$$

(d)(i) $E[W_n] = \int_{-\pi}^{\pi} c\sin(w_0n+\phi)\,d\phi = 0.$

$$E[W_nW_m] = R_{WW}[m-n] = E[\,c\sin(w_0n+\phi)\cdot c\sin(w_0m+\phi)\,]$$

$$= c^2E\left(\tfrac{1}{2}[\cos(w_0(n-m)) - \cos(w_0(n+m)+2\phi)\,]\right).$$

$\uparrow \int \cos(2\phi + \text{constant})\,d\phi = 0$

$$= \frac{c^2}{2}\cdot\cos(w_0(n-m))$$

(ii). $$S_W(e^{jw}) = \sum_{m=-\infty}^{\infty} R_{WW}[m]\,e^{-jmw}$$

$$= \sum_{m=-\infty}^{\infty} \frac{c^2}{2}\cos(w_0m)e^{-jmw}$$

$$= \sum_{m=-\infty}^{\infty} \frac{c^2}{2\cdot2}\left(e^{-jmw_0} + e^{+jmw_0}\right)e^{-jmw}$$

$$= \frac{c^2}{4}\sum_{m=-\infty}^{\infty} \delta(w - w_0 - 2m\pi) + \delta(w + w_0 - 2m\pi)$$