Revision Supervision Questions – 3F3

1 (a) A source generates a stream of symbols S_n, $n = 0, 1, \ldots$ and each symbol takes one of two possible values, either A or B. The probability of symbol S_n depends only upon the value of symbol S_{n-1}. Let $p(i_n|i_{n-1})$ denote the probability that $S_n = i_n$ given $S_{n-1} = i_{n-1}$. These probabilities are given in the following table.

$p(i_n\|i_{n-1})$	$S_{n-1} = A$	$S_{n-1} = B$
$S_n = A$	0.7	0.1
$S_n = B$	0.3	0.9

(i) Let $p(i_0)$ denote the probability that $S_0 = i_0$. Explain how $p(i_n)$, the probability that $S_n = i_n$, may be calculated. [10%]

(ii) Show that a possible probability mass function for $p(i_n)$, for all $n \geq 0$, is [10%]

	$p(i_n)$
$i_n = A$	0.25
$i_n = B$	0.75

(iii) Show that the random process $S_0, S_2, S_4, \ldots$, generated by the same source but retaining only source symbols with even time indices, is a Markov chain, and determine its transition probability matrix. [30%]

(b) The characteristic function of a random variable X is defined using the mathematical expectation $\mathbb{E}$ as $\varphi_X(t) = \mathbb{E}[\exp(iXt)]$ where t is a real number.

(i) Let X have probability density function $f_X(x)$. Determine the relationship between $\varphi_X(t)$ and the Fourier transform of $f_X(x)$. [10%]

(ii) Let $f_X(x)$ be the following triangular shaped function

$$f_X(x) = 1/b\left(1 - \frac{|x|}{b}\right) \qquad \text{for } |x| \leq b$$

and $f_X(x) = 0$ for $|x| > b$. Determine $\varphi_X(t)$ (using the Data Book). [10%]

(iii) Express $\varphi_X(t)$ as a power series in t and hence find $\mathbb{E}[X^0]$, $\mathbb{E}[X^2]$ and $\mathbb{E}[X^4]$. [30%]

2 Consider the following autoregressive process

$$X_n + a_1 X_{n-1} + a_2 X_{n-2} = \sigma W_n$$

where $\{W_n\}$ is a zero-mean white noise process with variance 1 and σ a positive constant.

(a) Find the power spectrum of $\{X_n\}$. [10%]

(b) Let $\{V_n\}$ be the moving average process

$$V_n = b_0 E_n + b_1 E_{n-1}$$

where $\{E_n\}$ is a zero-mean white noise process with variance 1. Find the power spectrum of $\{V_n\}$. [10%]

(c) Let Y_n be the noisy measurement of X_n given by

$$Y_n = X_n + V_n.$$

Assume the noise sequences $\{W_n\}$ and $\{E_n\}$ are independent. Find the power spectrum of $\{Y_n\}$. [15%]

(d) Based on measurements of $\{V_n\}$, the power spectrum of $\{V_n\}$ is estimated to be

$$\widehat{S}_V(\omega) = 2 + 2\cos\omega$$

Show that valid estimates of b_0 and b_1 are $b_0 = b_1 = 1$ and $b_0 = b_1 = -1$. [25%]

(e) Show that

$$\begin{bmatrix} R_{XX}[0] & R_{XX}[1] \\ R_{XX}[1] & R_{XX}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = -\begin{bmatrix} R_{XX}[1] \\ R_{XX}[2] \end{bmatrix},$$

$$R_{XX}[0] + a_1 R_{XX}[1] + a_2 R_{XX}[2] = \sigma^2.$$

[30%]

(f) Based on measurements of Y_n as in (c), the following estimates are made for its autocorrelation function:

$$\widehat{R}_{YY}[0] = 4.74, \qquad \widehat{R}_{YY}[1] = 0.54, \qquad \widehat{R}_{YY}[2] = 1.41$$

Use these values to estimate the power spectrum of $\{X_n\}$. [10%]

2 Let X_n follow an equation described by a linear trend in noise,

$$X_n = a + bn + W_n$$

where $\{W_n\}$ is a zero mean wide-sense stationary random process and a, b are unknown constants. Let $x_0, x_1, \ldots, x_{N-1}$ be a batch of N of data points.

(a) Find the least squares estimate $\hat{\theta} = (\hat{a}, \hat{b})^{\mathrm{T}}$ of the unknown parameters $\theta = (a, b)^{\mathrm{T}}$ and show that this least squares estimate is unbiased. (The superscript T denotes transpose.) [30%]

(b) Find the variance $\mathbf{E}(\hat{\theta}\hat{\theta}^{\mathrm{T}}) - \theta\theta^{\mathrm{T}}$. [20%]

(c) Show that the variance tends to 0 as N increases. (You may assume W_n is white noise.) [10%]

(d) The model can be used to describe a data set with a linear trend and a sinusoidal component. Let $W_n = c\sin(\omega_0 n + \phi)$ where ϕ is a Uniform random variable with range $(-\pi, \pi)$,

(i) Show W_n is wide-sense stationary and find its mean and autocorrelation function. [20%]

(ii) Find the power spectral density of W_n. [10%]

(iii) How might an estimate of ω_0 be found from the data? [10%]