2-D Random Walk

1. Theory Investigation

In a plane, consider a sum of N two-dimensional vectors with random orientations. Use phasor notation, and let the phase of each vector be random. Assume N unit steps are taken in an arbitrary direction (i.e., with the angle theta uniformly distributed in [0,2pi) and not on a lattice), as illustrated above. The position z in the complex plane after N steps is then given by:

$$z = \sum_{j=1}^{N} e^{i\theta_j}$$

which has absolute square:

$$\begin{aligned} |z|^2 &= \sum_{j=1}^{N} e^{i\theta_j} \sum_{k=1}^{N} e^{-i\theta_k} \\ &= \sum_{j=1}^{N} \sum_{k=1}^{N} e^{i(\theta_j - \theta_k)} \\ &= N + \sum_{\substack{j,k=1 \\ k \neq j}}^{N} e^{i(\theta_j - \theta_k)} \end{aligned}$$

Each unit step is equally likely to be in any direction (θ_j and θ_k). The displacements are random variables with identical means of zero, and their difference is also a random variable. Averaging over this distribution, which has equally likely positive and negative values yields an expectation value of 0, so

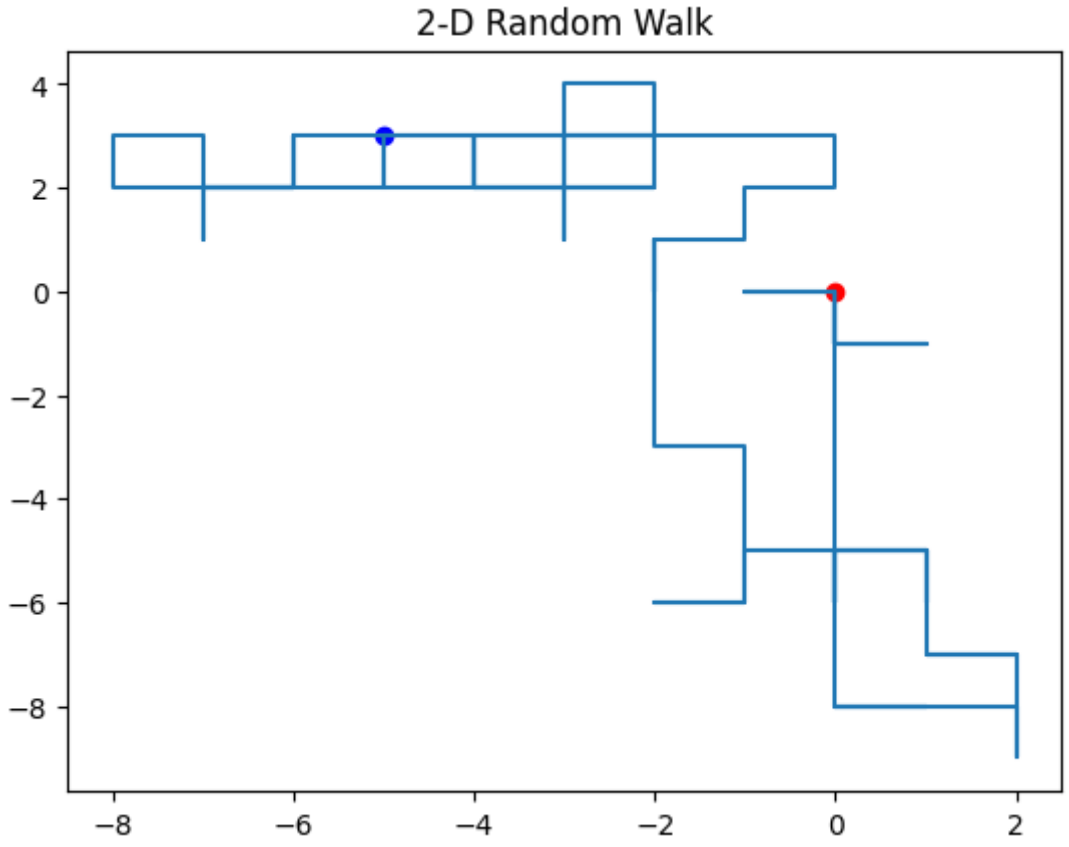
$$\begin{aligned} \left\langle |z|^2 \right\rangle &= N \\ |z|_{\mathrm{rms}} &= \sqrt{N} \end{aligned}$$

2. Simulation on 2-D square lattice

定义需要用的函数

单次游走

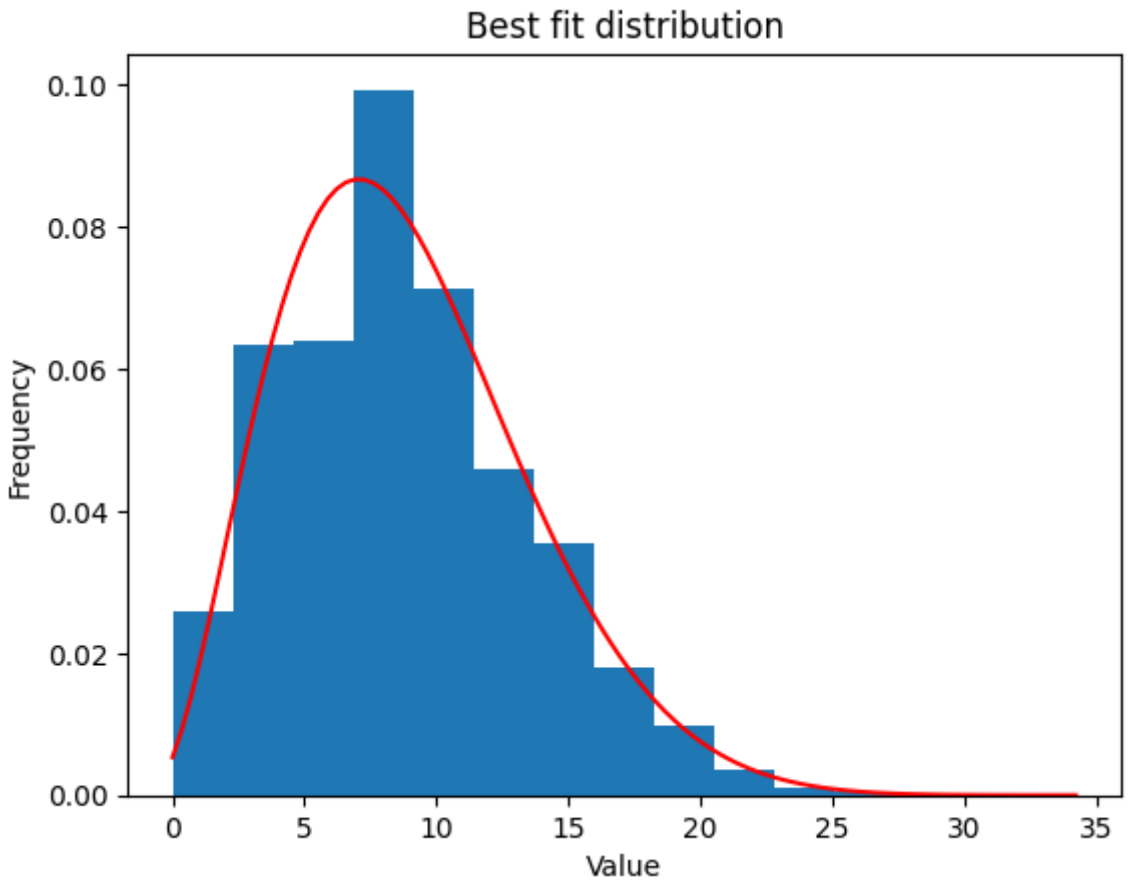
可见这个代码可以正常游走，如下是一段游走轨迹，步长为1，步数为100，最后一个点距离第一个点的距离是5.8




```
5.830951894845301
```

我们进行同步长，同步数的多次游走，统计最后一个点距离初始点的距离,看起来符合某种分布

```
d:\apps\Anaconda3\envs\normal\lib\site-packages\scipy\stats\_continuous_distns.py:639: Runtim
eWarning: invalid value encountered in sqrt
  sk = 2*(b-a)*np.sqrt(a + b + 1) / (a + b + 2) / np.sqrt(a*b)
Best Distribution: beta
```



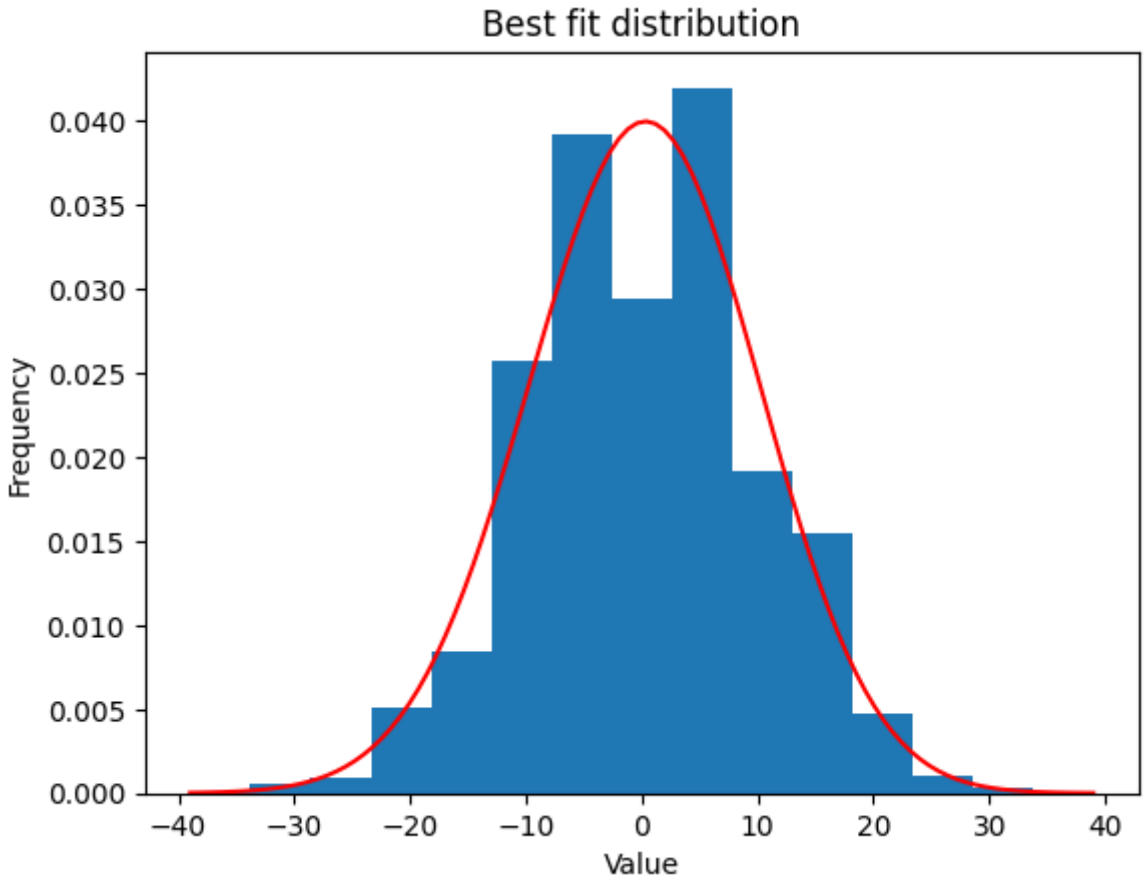
得到的最佳拟合的分布函数是beta分布

函数形式：

$$
\begin{aligned}
f(x;\alpha,\beta) &= \frac{x^{\alpha-1}(1-x)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}du} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1} \\
&= \frac{1}{\mathrm{B}(\alpha,\beta)}x^{\alpha-1}(1-x)^{\beta-1}
\end{aligned}
$$

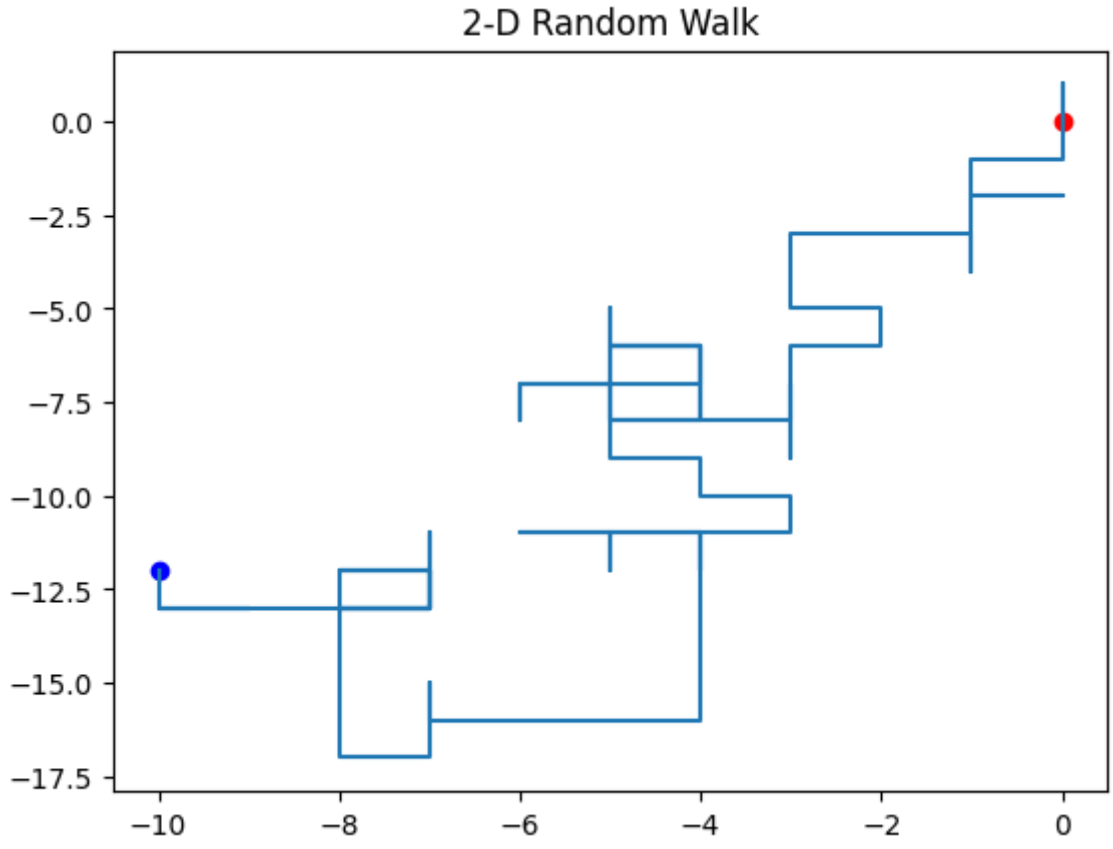
原理上应该是趋于正态分布，故首先怀疑是实验循环采样次数不够，考虑到算力问题，用1D问题示例,步长为1，步数100（降低问题复杂度，辅助收敛）

```
Best Distribution: beta
```

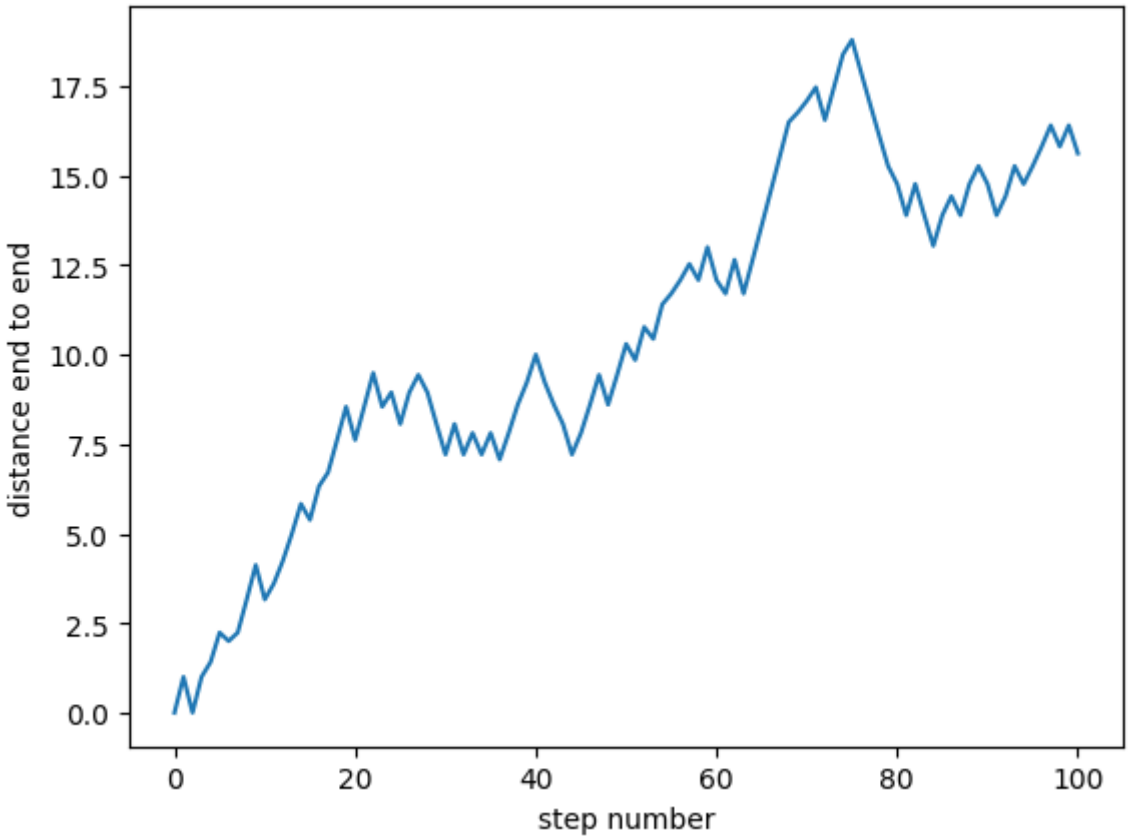


仍旧是BETA分布，但是已经在频率上比较像正态分布了，做出如下分析： 1） 可能确实趋于正态分布，但是由于拟合的时,BETA分布有更多参量，故可能在得分上更有利（主要可能性） 2) 计算机函数的伪随机不是完全的真随机，采样可能不均匀 3） 二维情况下的距离计算（方格模拟和欧氏距离引起的不连续误差)

如果观察首位距离随着游走时间增加的变化呢？ 看起来没有什么规律,忽上忽下(步长为1，步数100)




```
Text(0, 0.5, 'distance end to end')
```

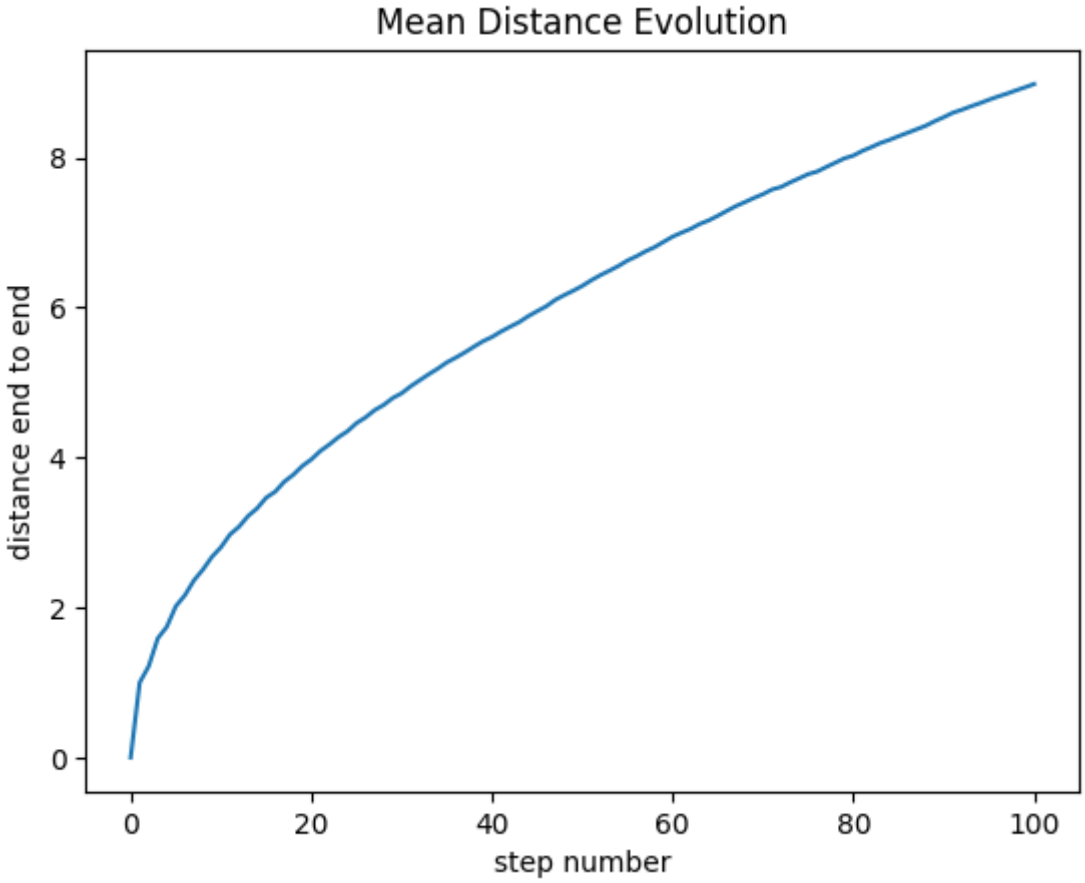


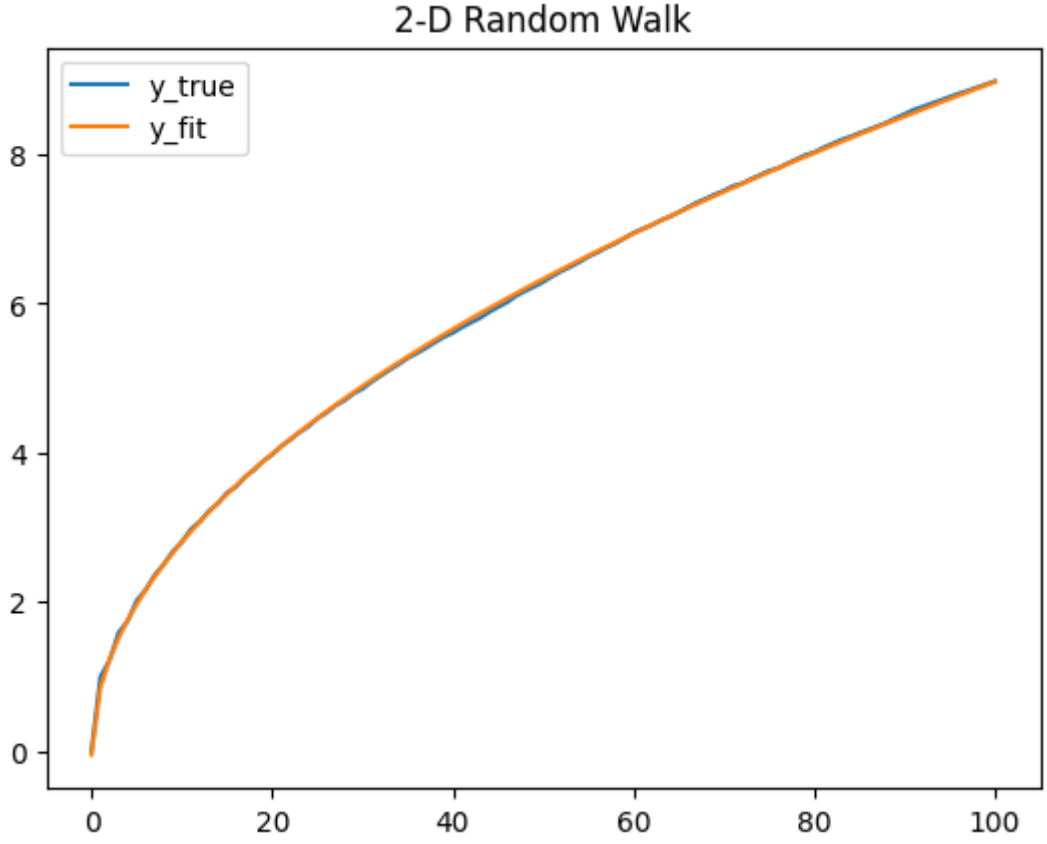
多次游走

如果我们在同一步长和步数的条件下做多次游走，把每次轨迹的end to end 距离trajectory数据平均起来呢？ 我们可以发现，正如之前推导的，函数逐渐收敛至：(步长为1，步数10000)

$$|z|_{\mathrm{rms}} = A\sqrt{N} + B$$

```
Text(0.5, 1.0, 'Mean Distance Evolution')
```






```
y={0.9020}x+{-0.0472}
R^2=  0.9998180662327076
```

3. References

[1] Weisstein, Eric W. "Random Walk--2-Dimensional." From MathWorld--A Wolfram Web Resource. https://mathworld.wolfram.com/RandomWalk2-Dimensional.html

[2] https://zh.wikipedia.org/zh-cn/%CE%92%E5%88%86%E5%B8%83

4. Code Availability

https://github.com/Jingdan-Chen/random_walk