LEHIGH UNIVERSITY

ISE403: Research Methods

Set-valued Optimization

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Contents

Abstract

Set-value optimization, a growing area within the optimization domain, distinguishes itself by having objective functions whose value can be sets rather than numbers. It has recently gained attention due to its ability to offer a flexible framework for addressing inherent uncertainty within problem settings. Moreover, numerous underdeveloped areas persist, including the intersection between set-value optimization and machine learning. This paper presents an overview and mathematical foundation of set-valued optimization, accompanied by a literature review encompassing three recent developments in the field. Each of these papers adopts a distinctive method to deal with set-valued optimization: one adopts algorithms of a sorting type, another employs a scalarization approach, and the third develops a first-order descent method.

1 Introduction

Set-valued optimization (SVO) is an exciting extension of traditional optimization theory. In classical optimization, the goal is to find the solution that optimizes a scalar objective function. However, the objective function may not have a single value in SVO. Instead, the value can be a set. Generally speaking, SVO aims to find a solution that optimizes specific criteria while satisfying the set-valued constraints.

Set-valued optimization problems have become crucial because SVO finds applications in various fields, including engineering, finance, economics, and decision-making under uncertainty. It provides a more flexible framework for modeling and solving problems compared to traditional optimization, particularly in scenarios where inherent uncertainty or ambiguity is present in the problem formulation. For example, SVO can be used in robust optimization [8], where the goal is to find solutions that perform well under various uncertain or perturbed conditions. Also, SVO can solve multi-objective optimization (MOO) problems [3] by leveraging the fact that a Pareto front is a set.

However, solving an SVO is difficult because there is no unique ordering on a real vector space or a power set. In order to compare sets, several preorders have been introduced [7]. With these preorders, it is then possible to formulate set-valued optimization problems and define their minimal solutions.

There are various advancing methods and algorithms to solve the SVO problems. For example, to solve SVO problems with finite feasible sets, Kobis developed algorithms of a sorting type [1]. The main idea of this kind of algorithm is comparisons between images of set-valued mappings.

Jahn used derivative-free optimization (DFO) [4] to solve the SVO problems. The researchers designed these DFO algorithms to deal with unconstrained problems, and they assume no particular structure of the set-valued objective mapping. Some algorithms have been developed based on a scalarization method. This method is suitable for problems where the set-valued objective mapping has a particular structure from the so-called robust counterpart of a vector optimization problem under uncertainty, see [3].

There are also some undeveloped open questions about SVO. For example, the application of set-valued optimization in the context of data science and machine learning is an emerging and appealing research direction. Research in this area includes exploring how set-valued optimization can be used to model uncertainty in machine learning models and improve the robustness of the solutions.

We organize the remaining sections of the paper as follows. Section 2 introduces the notation and relevant mathematical concepts needed to understand our paper. In section 3, we discuss three recent papers related to SVO. These three papers adopted three different solution methods. One involved so-called sorting-type algorithms, one used the scalarization approach, and the most recent one developed a first-order descent method to tackle some specific settings of the SVO problems.

2 Mathematical background

In this section, we introduce the notation that can help the readers understand the content about set-valued optimization better. We use $\mathbb{R}$ to denote the set of real numbers, $\mathbb{R}^n$ to denote the set of n-dimensional real vectors, and $\mathbb{R}^{m\times n}$ to denote the set of m-by-n-dimensional real matrices.

If a nonempty subset C of $\mathbb{R}^n$ satisfies that for any positive scalar α, every $x \in C$ implies $\alpha x \in C$, we say C is a cone. If a nonempty subset C of $\mathbb{R}^n$ satisfies that for any positive scalar α, we have $\alpha C = C$ and $C + C = C$, we say C is a convex cone. A convex cone C is said to be pointed if $\mathbf{0}$ is in C. We use $\text{int}(C)$ to denote the interior of the cone C. Moreover, if $\text{int}(C) \neq \emptyset$, we say C is solid.

Due to the characteristic that the value of an SVO objective function can be a set rather than a single value, and given the absence of a unique ordering on a real vector space or a power set, the establishment of order relations becomes crucial in formulating SVO problems. Therefore, we introduce the definitions of partial orders and strict partial orders, which will help define the SVO problems.

Definition 1. (Partial orders [6]) A reflexive, weak, or non-strict partial order, commonly referred to simply as a partial order, is a homogeneous relation $\leq$ on a set P that is reflexive, antisymmetric, and transitive. That is, for all $(a, b, c) \in P^3$, it must satisfy:

1. Reflexivity: $a \leq a$, $i.e.$, every element is related to itself.

2. Antisymmetry: if $a \leq b$ and $b \leq a$, then $a = b$, $i.e.$, no two distinct elements precede each other.

3. Transitivity: if $a \leq b$ and $b \leq c$, then $a \leq c$.

Definition 2. (Strict partial orders [6]) An irreflexive, strong, or strict partial order is a homogeneous relation $<$ on a set P that is irreflexive, asymmetric, and transitive; that is, it satisfies the following conditions for all $(a,b,c) \in P^3$:

1. Irreflexivity: $a \not< a$, *i.e.*, no element is related to itself (also called anti-reflexive).
2. Asymmetry: if $a < b$, then $b \not< a$.
3. Transitivity: if $a < b$ and $b < c$, then $a < c$.

To aid in grasping the problem context of SVO, it might be noteworthy that a non-strict partial order is also known as an antisymmetric preorder. Similarly, a strict partial order is recognized as an asymmetric strict preorder.

In the following article, we let Y be a linear space ordered by a proper closed convex and pointed cone C. The ordering relation on Y is described by $y \leq_C y'$ if and only if $y' - y \in C$ for all $y \in Y$ and $y' \in Y$. Let X also be a linear space. We say $F : X \rightrightarrows Y$ is a set-valued map if each $x \in X$ is mapped to a set $F(x) \subseteq Y$. We denote the graph of the set-valued map F by $\text{graph}F := \{(x,y) \in X \times Y | y \in F(x)\} \subseteq X \times Y$. We denote the domain of the set-valued map F by $\text{dom}F := \{x \in X : F(x) \neq \varnothing\}$. Furthermore, given a subset $\mathscr{X}$ of X, we denote the image of $\mathscr{X}$ by $F(\mathscr{X}) := \cup_{x\in\mathscr{X}} F(x)$. Next, we introduce the definitions of three partial order relations between sets with respect to the cone C.

Definition 3. (Set less order relation [3]) Let $C \subset Y$ be a proper closed convex and pointed cone. Let $A \subset Y$ and $B \subset Y$ be arbitrarily chosen sets. Then the set less order relation $\preceq_C^s$ is defined by

$$A \preceq_C^s B \Leftrightarrow A \subseteq B - C \text{ and } B \subseteq A + C.$$

Remark 1. For $A \subseteq B - C$ and $B \subseteq A + C$, we have

$$A \subseteq B - C \Leftrightarrow \text{for all } a \in A, \text{ there exists some } b \in B \text{ such that } a \leq_C b, \text{ and}$$
$$B \subseteq A + C \Leftrightarrow \text{for all } b \in B, \text{ there exists some } a \in A \text{ such that } a \leq_C b.$$

Definition 4. (Upper-type set relation [7]) Let $A \subset Y$ and $B \subset Y$ be arbitrarily chosen sets and $C \subset Y$ a proper closed convex and pointed cone. Then the u-type set relation $\preceq_C^u$ is defined by

$$A \preceq_C^u B \Leftrightarrow A \subseteq B - C \Leftrightarrow \text{for all } a \in A, \text{ there exists some } b \in B \text{ such that } a \leq_C b.$$

Definition 5. (Lower-type set relation [7]) Let $A \subset Y$ and $B \subset Y$ be arbitrarily chosen sets and $C \subset Y$ a proper closed convex and pointed cone. Then the l-type set relation $\preceq_C^l$ is defined by

$$A \preceq_C^l B \Leftrightarrow B \subseteq A + C \Leftrightarrow \text{for all } b \in B, \text{ there exists some } a \in A \text{ such that } a \leq_C b.$$

Remark 2. We have the relationship between $\preceq_C^u$, $\preceq_C^l$ and $\preceq_C^s$ as follows:

$$A \preceq_C^s B \Leftrightarrow A \preceq_C^u B \text{ and } A \preceq_C^l B.$$

Now, we will introduce the concept related to the set minimal points. Let $A \subset Y$ and $a_0 \in A$. We say a_0 is a minimal point of A with respect to cone C if $A \cap (a_0 - C) = a_0$. The set of all minimal points of A is denoted by $\min(A, C)$. Similarly, we say a_0 is a weak minimal point of A with respect to cone C if $A \cap (a_0 - \text{int}(C)) = \varnothing$. The set of all weak minimal points of A is denoted by $\text{Wmin}(A, C)$.

After we introduce set order relations and set minimal points, we can formally formulate the set-valued optimization problem. Let $F : X \rightrightarrows Y$ be a set-valued map, $\mathscr{X}$ be a subset of X, and $\preceq$ be a preorder on the power set (*i.e.*, the set of all subsets, empty set, and the original set itself) of Y. The set-valued optimization problem ($\mathbf{SP}-\preceq$) is given by

$$\preceq - \min_{x \in \mathscr{X}} F(x).$$

Then the minimal solutions of the set-valued optimization problem (**SP**$- \preceq$) are defined in the following.

Definition 6. (Minimal solutions of (**SP**$- \preceq$) w.r.t. the preorder $\preceq$ [3]) Given a set-valued optimization problem (**SP**$- \preceq$), an element $\overline{x} \in \mathscr{X}$ is called a minimal solution to (**SP**$- \preceq$) if for any $x \in \mathscr{X}$ such that $F(x) \preceq F(\overline{x})$, we have $F(\overline{x}) \preceq F(x)$.

Furthermore, the following definition of a minimizer of a set-valued optimization problem is often used in the theory of set optimization and is given below.

Definition 7. (Minimizer of a set-valued optimization problem [3]) Let $\overline{x} \in \mathscr{X}$ and $(\overline{x}, \overline{y}) \in \operatorname{graph} F$. The pair $(\overline{x}, \overline{y}) \in \operatorname{graph} F$ is called a minimizer of $F : X \rightrightarrows Y$ over $\mathscr{X}$ with respect to C if $\overline{y} \in \min(F(\mathscr{X}), C)$.

3 Literature review

This section will explore recent advancements in addressing the SVO problem through different types of solution methods under different problem settings.

One method in set-valued optimization involves sorting-type algorithms designed to tackle set optimization problems with finite feasible sets. These algorithms rely on comparisons between the images of set-valued objective mapping. Recent work by Gunther, Kobis, and Popovici in [1] introduced two novel algorithms for computing all strictly minimal elements within a nonempty finite family of sets in a real linear space with respect to a preorder relation defined on the power set of that space. The first algorithm resembles Jahn-Graef-Younes type methods [5] and comprises a forward reduction procedure, a backward reduction procedure, and a final comparison procedure. The second algorithm involves an initial reduction step that eliminates some non-strictly minimal elements in Phase 1, followed by applying the first algorithm to the reduced set in Phase 2. Because the sorting procedures within these two algorithms eliminate unnecessary comparison between the images of the set-valued objective mapping, these methods perform more efficiently than a naive implementation in which every pair of sets must be compared once.

Another class of methods adopts a scalarization approach, which is particularly suitable for problems where the set-valued objective mapping exhibits a specific structure. This structure often arises from the so-called robust counterpart of a vector optimization problem under uncertainty. Recently, in [8], Xiong, Jiang, and Cao introduced the augmented weighted Tschebyscheff scalarization method for obtaining robust and efficient solutions for uncertain multiobjective problems (UMOPs). Furthermore, they proposed two generalized robustness concepts based on set-less order relations for UMOPs. These concepts help filter and retain meaningful solutions, excluding relatively bad choices. The authors explored the mutual relationship between these two robustness concepts. Importantly, they also established a connection between these concepts and set-valued optimization problems, demonstrating that, under certain conditions, computing robust solutions for UMOPs can be equivalent to solving set-valued optimization problems.

In a recent development, Quintana et al. introduced a first-order descent method in [2] specifically tailored for unconstrained set-valued optimization problems. A set-valued objective mapping characterizes these problems, and this set-valued mapping is identified by a finite number of continuously differentiable selections. The authors derived optimality conditions for these problems based on the first-order information of the selections in the decomposition process. They also developed an algorithm tailored for problems with such structural characteristics. One noteworthy feature of this method is its guarantee of convergence toward points satisfying the derived optimality conditions.

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