Exercises and Solutions for **An Introduction to Stochastic Differential Equations**

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Abstract

Problem 1 Show, using the formal manipulations for Ito's chain rule, that

$$Y(t) := e^{W(t)-\frac{t}{2}}$$

solves the stochastic differential equation

$$\begin{cases} dY & = Y dW \\ Y(0) & = 1. \end{cases}$$

(Hint: If $X(t) := W(t) - \frac{t}{2}$, then $dX = -\frac{dt}{2} + dW$.)

Solution of Problem 1 According to the Ito's chain rule: since function $X(t) := W(t) - \frac{t}{2}$ satisfies the equation $X(t) = -\frac{dt}{2} + dW$, for $Y(t) = \exp(X(t))$, it holds that:

$$dY = \left(e^X \cdot \left(-\frac{1}{2}\right) + \frac{1}{2} \cdot e^X\right) dt + e^X dW = 0 + Y dW = Y dW.$$

Also, it's easy to verify that $Y(0) = 1$. Therefore, the solution

$$Y(t) := e^{W(t)-\frac{t}{2}}$$

actually solves the SDE we want.

Problem 2 Show that,

$$S(t) = s_0 e^{\sigma W(t) + \left(\mu - \frac{\sigma^2}{2}\right) t}$$

solves

$$\begin{cases} dS & = \mu S dt + \sigma S dW \\ S(0) & = s_0. \end{cases}$$

Solution of Problem 2 Again we use the Ito's chain rule: for function $X(t) := W(t) + \frac{\mu - \sigma^2/2}{\sigma} t$, we have:

$$dX = dW + \frac{\mu - \sigma^2/2}{\sigma} dt.$$

Therefore, for function $S(t) = s_0 e^{\sigma \cdot X(t)}$, it holds that:

$$\begin{aligned} dS &= \left(\sigma s_0 e^{\sigma X(t)} \cdot \frac{\mu - \sigma^2/2}{\sigma} + \frac{1}{2} \cdot \sigma^2 s_0 e^{\sigma X(t)}\right) dt + \sigma s_0 e^{\sigma X(t)} dW \\ &= \mu s_0 e^{\sigma X(t)} dt + \sigma s_0 e^{\sigma X(t)} dW = \mu S dt + \sigma S dW. \end{aligned}$$

Also, it's easy to verify that $S(0) = s_0$, which comes to our conclusion.

Problem 3 (1) Let (Ω, μ, P) be a probability space and let $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n \subseteq \ldots$ be events. Show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{m\to\infty} P(A_m).$$

(Hint: Look at the disjoint events $B_n := A_{n+1} - A_n$.)

(2) Likewise, show that if $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supset \ldots$, then

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{m\to\infty} P(A_m).$$

Solution of Problem 3

(1) Consider the disjoint events $B_n = A_{n+1} - A_n$, then we know that these events are disjoint since for $\forall m < n$, it holds that:

$$B_m = A_{m+1} - A_m \subseteq A_{m+1} \subseteq A_n,\ A_{n+1} - A_n = B_n \cap A_n = \varnothing,$$

so we can conclude that $B_m \cap B_n = \varnothing$, which means $\{B_n\}$ are disjoint events. Also, we have $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=0}^{\infty} B_n$. It is because for $\forall x \in \bigcup_{n=1}^{\infty} A_n$, there exists a smallest k such that $x \in A_k \Rightarrow x \in A_k - A_{k-1} = B_k \subseteq \bigcup_{n=0}^{\infty} B_n$, which leads to

$$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=0}^{\infty} B_n.$$

On the other hand, for $\forall x \in \bigcup_{n=0}^{\infty} B_n$, there exists k such that $x \in B_k \subseteq A_{k+1} \subseteq \bigcup_{n=1}^{\infty} A_n$, which leads to

$$\bigcup_{n=1}^{\infty} A_n \supseteq \bigcup_{n=0}^{\infty} B_n.$$

Therefore, we conclude that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=0}^{\infty} B_n$. Also, notice that $A_m = B_0 \cup B_1 \cup \ldots B_{m-1}$, which leads to

$$P(A_m) = \sum_{n=0}^{m-1} P(B_m).$$

Here, we use the countable additivity of $P(\cdot)$ under disjoint events $\{B_n\}$. Finally, it holds that:

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=0}^{\infty} B_n\right) = \sum_{n=0}^{\infty} P(B_n) = \lim_{m\to\infty} \sum_{n=0}^{m-1} P(B_n) = \lim_{m\to\infty} P(A_m)$$

In the final step, we use the fact that the sequence $p_m := \sum_{n=0}^{m-1} P(B_n) = P(A_m)$ is non-decreasing and upper bounded by 1, and therefore it converges.

(2) Similarly, we have $A_1^c \subseteq A_2^c \subseteq \ldots \subseteq A_n^c \subseteq \ldots$. By using the conclusion of (1), we have:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1 - P\left(\bigcup_{n=1}^{\infty} A_n^c\right) = 1 - \lim_{m\to\infty} P(A_m^c) = \lim_{m\to\infty} P(A_m),$$

which comes to our conclusion.

Problem 4 Let Ω be any set and $\mathcal{A}$ any collections of subsets of Ω. Show that there exists a unique smallest σ-algebra μ of subsets of Ω containing $\mathcal{A}$. We call μ the σ-algebra generated by $\mathcal{A}$. (Hint: Take the intersection of all the σ-algebras containing $\mathcal{A}$.)

Solution of Problem 4 Obviously, there exists a σ-algebra containing $\mathcal{A}$ (for example, 2^{Ω} itself). Now, we take the intersection of all the σ-algebras containing $\mathcal{A}$, denoted by μ. In order to prove that there exists a unique smallest σ-algebra μ of subsets of Ω containing $\mathcal{A}$, we only need to prove that the intersection of all σ-algebras is also a σ-algebra of Ω containing $\mathcal{A}$. On one hand, since all the σ-algebras considered by us contain $\mathcal{A}$, so their intersection μ also contains $\mathcal{A}$. On the other hand,

- For all σ-algebra S that contains $\mathcal{A}$, we have $\varnothing \in S$ by the definition of σ-algebra. Therefore, the intersection holds $\varnothing \in \mu$ obviously.
- For $\forall A \in \mu$, we have $A \in S$ for any σ-algebra S that contains $\mathcal{A}$, so it holds that $A^c \in S$. Therefore, we have $A^c \in \mu$.
- For $A_1, A_2, \ldots, \in \mu$, we know that for any σ-algebra S that contains $\mathcal{A}$, $\bigcup_{i=1}^{+\infty} A_i \in S$. Therefore:
$$\bigcup_{i=1}^{+\infty} A_i \in \mu.$$

After combining all the items above, we know that μ is also a σ-algebra of Ω containing $\mathcal{A}$, which comes to our conclusion.

Problem 5 Show that if $A_1, A_2, \ldots, A_n$ are events, then

$$\begin{aligned} P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - &\sum_{1\leqslant i<j\leqslant n} P(A_i \cap A_j) \\ + &\sum_{1\leqslant i<j<k\leqslant n} P(A_i \cap A_j \cap A_k) \\ - &\ldots + (-1)^n P(A_1 \cap A_2 \cap \ldots \cap A_n). \end{aligned}$$

(Hint: Do the case $n = 2$ first and then the general case by induction.)

Solution of Problem 5 We use the method of induction. When $n = 2$, we know that:

$$P(A \cup B) = P(A) + P((A \cup B) - A) = P(A) + P(B - (A \cap B)) = P(A) + P(B) - P(A \cap B).$$

Suppose the condition holds for n, then for $n + 1$, we have:

$$\begin{aligned} &P(A_1 \cup A_2 \cup \ldots \cup A_{n+1}) = P((A_1 \cup A_2 \cup \ldots \cup A_n) \cup A_{n+1}) \\ &= P(A_1 \cup A_2 \cup \ldots \cup A_n) + P(A_{n+1}) - P((A_1 \cup A_2 \cup \ldots \cup A_n) \cap A_{n+1}) \\ &= P(A_{n+1}) + \sum_{i=1}^{n} P(A_i) - \sum_{1\leqslant i<j\leqslant n} P(A_i \cap A_j) + \ldots + (-1)^n P(A_1 \cap A_2 \cap \ldots \cap A_n) \end{aligned}$$

$$
\begin{aligned}
&- P\left((A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \ldots \cup (A_n \cap A_{n+1})\right) \\
&= P(A_{n+1}) + \sum_{i=1}^{n} P(A_i) - \sum_{1 \leqslant i < j \leqslant n} P(A_i \cap A_j) + \ldots + (-1)^n P(A_1 \cap A_2 \cap \ldots \cap A_n) \\
&- \sum_{i=1}^{n} P(A_i \cap A_{n+1}) + \sum_{1 \leqslant i < j \leqslant n} P((A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})) - \ldots \\
&- (-1)^n P((A_1 \cap A_{n+1}) \cap (A_2 \cap A_{n+1}) \cap \ldots \cap (A_n \cap A_{n+1})) \\
&= \sum_{i=1}^{n+1} P(A_i) - \sum_{1 \leqslant i < j \leqslant n+1} P(A_i \cap A_j) + \ldots + (-1)^{n+1} P(A_1 \cap A_2 \cap \ldots \cap A_{n+1}),
\end{aligned}
$$

which completes the induction and comes to our conclusion.

Problem 6 Let $X = \sum_{i=1}^{k} a_i \chi_{A_i}$ be a simple random variable, where the real numbers a_i are distinct, the events A_i are pairwise disjoint, and $\Omega = \bigcup_{i=1}^{k} A_i$. Let $\mu(X)$ be the σ-algebra generated by X.
(1) Describe precisely which sets are in $\mu(X)$.
(2) Suppose the random variable Y is $\mu(X)$-measurable. Show that Y is constant on each set A_i.
(3) Show that therefore Y can be written as a function of X.

Solution of Problem 6

Problem 8 Suppose A and B are independent events in some probability space. Show that A^c and B are independent. Likewise, show that A^c and B^c are independent.

Solution of Problem 8 Since A and B are independent events, we have:

$$P(A \cap B) = P(A)P(B).$$

Therefore, we have the following two equations:

$$P(A^c \cap B) = P(B) - P(A \cap B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(A^c)P(B).$$

$$
\begin{aligned}
P(A^c \cap B^c) &= 1 - P(A \cup B) = 1 - P(A) - P(B) + P(A \cap B) = 1 - P(A) - P(B) + P(A)P(B) \\
&= (1 - P(A))(1 - P(B)) = P(A^c) \cdot P(B^c).
\end{aligned}
$$

These two equations conclude that A^c and B are independent. Also, A^c and B^c are independent.

Problem 10 Suppose that $A_1, A_2, \ldots, A_m$ are disjoint events, each of positive probability, such that $\Omega = \bigcup_{j=1}^{m} A_j$. Prove Bayes' formula:

$$P(A_k|B) = \frac{P(B|A_k)P(A_k)}{\sum_{j=1}^{m} P(B|A_j)P(A_j)}$$

for $k = 1, 2, \ldots, m$ provided $P(B) > 0$.

Solution of Problem 10 Since $A_1, A_2, \ldots, A_m$ all have positive probabilities, so:

$$P(B|A_j)P(A_j) = P(B \cap A_j).$$

Also, they are disjoint and $\bigcup_{j=1}^m A_j = \Omega$, which leads to $\{B \cap A_j\}_{j\in[m]}$ are also disjoint and $\bigcup_{j=1}^m (B \cap A_j) = B \cap \Omega = B$. Therefore:

$$\frac{P(B|A_k)P(A_k)}{\sum_{j=1}^m P(B|A_j)P(A_j)} = \frac{P(B \cap A_k)}{\sum_{j=1}^m P(B \cap A_j)} = \frac{P(B \cap A_k)}{P(B)} = P(A_k|B),$$

which comes to our conclusion. Here, we used the additivity of probability measures.