1 Exercise 13

1.1 Show that $\vec{p}^2 = \|\vec{p}\|^2$

$$\begin{aligned}
&\vec{p}^2 \\
&= \vec{p} \cdot \vec{p} \\
&= \cos 0 \times \|\vec{p}\|\|\vec{p}\| \\
&= \|\vec{p}\|^2
\end{aligned}$$

1.2 Show that $\|\vec{p} + \vec{q}\|^2 = \|\vec{p}\|^2 + 2\vec{p} \cdot \vec{q} + \|\vec{q}\|^2$

$$\begin{aligned}
&\|\vec{p} + \vec{q}\|^2 \\
&= (\vec{p} + \vec{q})^2 \\
&= (\vec{p} + \vec{q}) \cdot (\vec{p} + \vec{q}) \\
&= \vec{p}^2 + 2\vec{p} \cdot \vec{q} + \vec{q}^2 \\
&= \|\vec{p}\|^2 + 2\vec{p} \cdot \vec{q} + \|\vec{q}\|^2
\end{aligned}$$

1.3 Deduce that $\|\vec{p} + \vec{q}\| \leq \|\vec{p}\| + \|\vec{q}\|$

$$\begin{aligned}
&\|\vec{p} + \vec{q}\|^2 - (\vec{p} + \vec{q})^2 \\
&= \|\vec{p}\|^2 + 2\vec{p} \cdot \vec{q} + \|\vec{q}\|^2 - \|\vec{p}\|^2 - 2\|\vec{p}\|\|\vec{q}\| - \|\vec{q}\|^2 \\
&= 2\cos\theta\|\vec{p}\|\|\vec{q}\| - 2\|\vec{p}\|\|\vec{q}\| \\
&= 2\|\vec{p}\|\|\vec{q}\|(\cos\theta - 1) \\
&\cos\theta \leq 1 \\
\Leftrightarrow &\cos\theta - 1 \leq 0 \\
&2\|\vec{p}\|\|\vec{q}\| \geq 0 \\
\therefore &2\|\vec{p}\|\|\vec{q}\|(\cos\theta - 1) \leq 0 \\
\Leftrightarrow &\|\vec{p} + \vec{q}\|^2 \leq (\vec{p} + \vec{q})^2 \\
&\|\vec{p} + \vec{q}\| \geq 0 \\
&\vec{p} + \vec{q} \geq 0 \\
\therefore &\|\vec{p} + \vec{q}\| \leq \|\vec{p}\| + \|\vec{q}\|
\end{aligned}$$