

# 西安交通大学

## 数理统计作业

姓名： 唐靖凯

班级： 5102 班

学号： 3115370067

邮箱： jingkaitang@qq.com

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## 第一章 数理统计的基本概念

1.1 设总体  $X \sim N(\mu, \sigma^2)$ ,  $X_1, \dots, X_n$  为总体  $X$  的一个样本,  $\bar{X}$  为样本均值, 如要

$$P\{|\bar{X} - \mu| < 1\} = 0.95$$

则样本容量  $n$  应取多大?

**解:** 由定理 1.2.4 得  $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ , 所以  $\frac{\sqrt{n} \cdot (\bar{X} - \mu)}{\sigma} \sim N(0, 1)$ , 又由条件  $P\{|\bar{X} - \mu| < 1\} = 0.95$  可得,  $P\{|\frac{\sqrt{n} \cdot (\bar{X} - \mu)}{\sigma}| < \frac{\sqrt{n}}{\sigma}\} = 0.95$ , 有

$$\alpha = (1 - 0.95) \div 2$$

$$= 0.025$$

$$\mu_{\alpha=0.025} = 1.96$$

$$1.96 < \frac{\sqrt{n}}{\sigma}$$

$$n = \lceil (1.96 \cdot \sigma)^2 \rceil$$

1.2 设电子元件的寿命 (小时)  $X \sim \text{Exp}(0.0015)$ , 独立测试 6 个元件并记下它们的失效时间, 试求:

(1) 至 800 小时, 没有一个元件失效的概率;

(2) 至 3000 小时, 所有元件都失效的概率.

**解:** (1) 由题意知  $F(x) = e^{-0.0015 \cdot x}$ , 则

$$\begin{aligned} P\{X_1, X_2, X_3, X_4, X_5, X_6; X \geq 800\} &= (F(800))^6 \\ &= (e^{-0.0015 \times 800})^6 \\ &= e^{-7.2} \end{aligned}$$

(2)

$$\begin{aligned} P\{X_1, X_2, X_3, X_4, X_5, X_6; X < 3000\} &= (1 - F(3000))^6 \\ &= (1 - e^{-0.0015 \times 3000})^6 \\ &= (1 - e^{-4.5})^6 \end{aligned}$$

1.4 设总体  $X$  服从对数正态分布, 即  $X$  具有概率密度

$$f(x) = \begin{cases} \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, & 0 < x < \infty \\ 0, & \text{其他} \end{cases}$$

$X_1, \dots, X_n$  为总体  $X$  的一个样本, 试写出  $X_1, \dots, X_n$  的联合概率密度.

**解:**

$$\begin{aligned} x > 0, \quad \prod_{i=1}^n f(x_i) &= \prod_{i=1}^n \frac{1}{x_i \sqrt{2\pi}\sigma} \cdot e^{-\frac{(\ln x_i - \mu)^2}{2\sigma^2}} \\ &= \frac{e^{-\frac{1}{2\sigma^2} \cdot \sum_{i=1}^n (\ln x_i - \mu)^2}}{(2\pi\sigma^2)^{\frac{n}{2}} \cdot \prod_{i=1}^n x_i} \end{aligned}$$

则:

$$f(x_1, \dots, x_n) = \begin{cases} \frac{e^{-\frac{1}{2\sigma^2} \cdot \sum_{i=1}^n (\ln x_i - \mu)^2}}{(2\pi\sigma^2)^{\frac{n}{2}} \cdot \prod_{i=1}^n x_i}, & 0 < x < \infty \\ 0, & \text{其他} \end{cases}$$

**1.6** 证明下列等式

$$(1) \sum_{i=1}^n (X_i - \mu)^2 = \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2;$$

$$(2) \sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - n\bar{X}^2.$$

**解:** (1)

$$\begin{aligned} \sum_{i=1}^n (X_i - \mu)^2 &= \sum_{i=1}^n [(X_i - \bar{X}) + (\bar{X} - \mu)]^2 \\ &= \sum_{i=1}^n [(X_i - \bar{X})^2 + 2(X_i - \bar{X})(\bar{X} - \mu) + (\bar{X} - \mu)^2] \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + 2(\bar{X} - \mu) \cdot \sum_{i=1}^n (X_i - \bar{X}) + \sum_{i=1}^n (\bar{X} - \mu)^2 \\ &= \sum_{i=1}^n (X_i - \bar{X})^2 + n(\bar{X} - \mu)^2 \end{aligned}$$

(2)

$$\begin{aligned} \sum_{i=1}^n (X_i - \bar{X})^2 &= \sum_{i=1}^n (X_i - \bar{X})(X_i - \bar{X}) \\ &= \sum_{i=1}^n [(X_i - \bar{X})X_i - (X_i - \bar{X})\bar{X}] \\ &= \sum_{i=1}^n [X_i^2 - \bar{X} \cdot X_i - (X_i - \bar{X})\bar{X}] \\ &= \sum_{i=1}^n X_i^2 - \bar{X} \sum_{i=1}^n X_i - \bar{X} \sum_{i=1}^n (X_i - \bar{X}) \\ &= \sum_{i=1}^n X_i^2 - n\bar{X}^2 \end{aligned}$$

**1.7** 设  $\bar{X}_n$  和  $S_n^2$  分别为样本  $X_1, X_2, \dots, X_n$  的样本均值和样本方差, 再抽出一个样品  $X_{n+1}$ , 记样本  $X_1, X_2, \dots, X_{n+1}$  的样本均值和样本方差为分别  $\bar{X}_{n+1}$  和  $S_{n+1}^2$ , 求证:

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + \frac{1}{n+1}(X_{n+1} - \bar{X}_n) \\ S_{n+1}^2 &= \frac{n}{n+1}[S_n^2 + \frac{1}{n+1}(X_{n+1} - \bar{X}_n)^2] \end{aligned}$$

解: (1)

$$\begin{aligned}
\bar{X}_n + \frac{1}{n+1}(X_{n+1} - \bar{X}_n) &= \frac{1}{n} \sum_{i=1}^n X_i + \frac{1}{n+1}(X_{n+1} - \frac{1}{n} \sum_{i=1}^n X_i) \\
&= \frac{(n+1) \sum_{i=1}^n X_i + nX_{n+1} - \sum_{i=1}^n X_i}{n(n+1)} \\
&= \frac{n \sum_{i=1}^n X_i + nX_{n+1}}{n(n+1)} \\
&= \frac{\sum_{i=1}^{n+1} X_i}{n(n+1)} \\
&= \frac{1}{n+1} \sum_{i=1}^{n+1} X_i \\
&= \bar{X}_{n+1}
\end{aligned}$$

(2)

$$\begin{aligned}
S_n^2 &= \frac{1}{n} \left( \sum_{i=1}^n (X_i - \bar{X}_n)^2 \right) \\
&= \frac{1}{n} \left( \sum_{i=1}^n X_i^2 - n\bar{X}_n^2 \right)
\end{aligned}$$

$$S_{n+1}^2 = \frac{1}{n+1} \left( \sum_{i=1}^{n+1} X_i^2 - (n+1)\bar{X}_{n+1}^2 \right)$$

$$\begin{aligned}
\text{左边-右边} &= S_{n+1}^2 - \frac{n}{n+1} \left[ S_n^2 + \frac{1}{n+1} (X_{n+1} - \bar{X}_n)^2 \right] \\
&= \frac{1}{n+1} \left[ \sum_{i=1}^{n+1} X_i^2 - (n+1)\bar{X}_{n+1}^2 \right] - \frac{n}{n+1} \left[ \frac{1}{n} \left( \sum_{i=1}^n X_i^2 - n\bar{X}_n^2 \right) + \frac{1}{n+1} (X_{n+1} - \bar{X}_n)^2 \right] \\
&= \frac{1}{n+1} \left[ \sum_{i=1}^{n+1} X_i^2 - (n+1)\bar{X}_{n+1}^2 - \sum_{i=1}^n X_i^2 + n\bar{X}_n^2 - \frac{n}{n+1} (X_{n+1}^2 - 2X_{n+1}\bar{X}_n + \bar{X}_n^2) \right] \\
&= \frac{1}{n+1} \left[ \frac{1}{n+1} X_{n+1}^2 - (n+1)\bar{X}_{n+1}^2 + \frac{n^2}{n+1} \bar{X}_n^2 + \frac{2n}{n+1} X_{n+1}\bar{X}_n \right] \\
&= \frac{1}{n+1} \left[ \frac{1}{n+1} (X_{n+1}^2 + 2nX_{n+1}\bar{X}_n + n\bar{X}_n^2) - (n+1)\bar{X}_{n+1}^2 \right] \\
&= \frac{1}{n+1} \left[ \frac{1}{n+1} (X_{n+1} + n\bar{X}_n)^2 - (n+1)\bar{X}_{n+1}^2 \right] \\
&= \frac{1}{n+1} [(n+1)\bar{X}_{n+1}^2 - (n+1)\bar{X}_{n+1}^2] \\
&= 0
\end{aligned}$$

**1.14** 设总体  $X \sim \Gamma(\alpha, \lambda)$ ,  $X_1, X_2, \dots, X_n$  为总体的一个样本, 试求样本均值  $\bar{X}$  的概率密度。

解:

$$\begin{aligned}
 X_1, X_2, \dots, X_n &\sim \Gamma(n\alpha, \lambda) \\
 f(x) &= \frac{\lambda^{n\alpha}}{\Gamma(n\alpha)} x^{n\alpha-1} e^{-\lambda x}, \quad x > 0 \\
 \text{令 } t &= \frac{x}{n}, \\
 f_{\bar{x}}(t) &= \frac{\lambda^{n\alpha}}{\Gamma(n\alpha)} (nt)^{n\alpha-1} e^{-\lambda nt} |(nt)'| \\
 &= \frac{(\lambda n)^{n\alpha}}{\Gamma(n\alpha)} t^{n\alpha-1} e^{-(\lambda n)t}, \quad t > 0
 \end{aligned}$$

则:

$$f_X(x) = \begin{cases} \frac{(\lambda n)^{n\alpha}}{\Gamma(n\alpha)} x^{n\alpha-1} e^{-(\lambda n)x}, & x > 0 \\ 0, & \text{其他} \end{cases}$$

**1.17** 证明: 若  $X \sim \Gamma(\alpha, \lambda)$ , 则  $Y = \frac{X}{k} \sim \Gamma(\alpha, k\lambda)$ , 其中  $k > 0$ 。

解:

$$\begin{aligned}
 f(x) &= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad x > 0 \\
 \text{令 } t &= \frac{x}{k}, \\
 f_{\frac{X}{k}}(t) &= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} (kt)^{\alpha-1} e^{-\lambda kt} |(kt)'| \\
 &= \frac{(k\lambda)^{\alpha}}{\Gamma(\alpha)} t^{\alpha-1} e^{-k\lambda t} \\
 \frac{X}{k} &\sim \Gamma(\alpha, k\lambda)
 \end{aligned}$$

**1.18** 证明: 若  $X \sim \beta(a, b)$ , 则  $E(X) = \frac{a}{a+b}$ ,  $D(X) = \frac{ab}{(a+b)^2(a+b+1)}$ 。

解:

$$\begin{aligned}
 f(x) &= \frac{x^{a-1}(1-x)^{b-1}}{B(a, b)} \\
 E(X) &= \int_0^1 x \frac{x^{a-1}(1-x)^{b-1}}{B(a, b)} dx \\
 &= \frac{\int_0^1 x^a (1-x)^{b-1} dx}{B(a, b)} \\
 &= \frac{\frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+b+1)}}{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}} \\
 &= \frac{\frac{a\Gamma(a)\Gamma(b)}{(a+b)\Gamma(a+b)}}{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}} \\
 &= \frac{a}{a+b}
 \end{aligned}$$

$$\begin{aligned}
E(X^2) &= \int_0^1 x^2 \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} dx \\
&= \frac{B(a+2,b)}{B(a,b)} \\
&= \frac{\frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)}}{\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}} \\
&= \frac{a(a+1)}{(a+b+1)(a+b)}
\end{aligned}$$

$$\begin{aligned}
D(X) &= E(X^2) - [E(X)]^2 \\
&= \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2 \\
&= \frac{a(a+1)(a+b) - a^2(a+b+1)}{(a+b)^2(a+b+1)} \\
&= \frac{ab}{(a+b)^2(a+b+1)}
\end{aligned}$$

**1.19** 证明: 若  $X \sim F(n, m)$ , 则  $Y = \frac{\frac{n}{m}X}{1 + \frac{n}{m}X} \sim \beta(\frac{n}{2}, \frac{m}{2})$ 。

**解:**

$$X \sim F(n, m), f(x) = \frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \frac{n}{m} \left(\frac{n}{m}x\right)^{\frac{n}{2}-1} \left(1 + \frac{n}{m}x\right)^{-\frac{n+m}{2}}, \quad x > 0$$

$$\begin{aligned}
Y &= \frac{\frac{n}{m}X}{1 + \frac{n}{m}X} \\
f_Y(y) &= \frac{(\frac{n}{m})^{\frac{n}{2}} \Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} \left(\frac{m}{n} \frac{y}{1-y}\right)^{\frac{n}{2}-1} \left(1 + \frac{y}{1-y}\right)^{-\frac{n+m}{2}} \frac{m}{n} \frac{1}{(1-y)^2} \\
&= \frac{\Gamma(\frac{n+m}{2})}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2})} y^{\frac{n}{2}-1} (1-y)^{\frac{m}{2}-1} \\
&= \frac{y^{\frac{n}{2}-1} (1-y)^{\frac{m}{2}-1}}{B(\frac{n}{2}, \frac{m}{2})}
\end{aligned}$$

$$Y = \frac{\frac{n}{m}X}{1 + \frac{n}{m}X} \sim \beta\left(\frac{n}{2}, \frac{m}{2}\right)$$

**1.20** 已知  $X \sim t(n)$ , 求证:  $X^2 \sim F(1, n)$ 。

**解:**

$$X \sim t(n), f(x) = \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

$$\begin{aligned}
\text{令 } t &= x^2, \\
f_{X^2}(t) &= \frac{\Gamma(\frac{n+1}{2})}{\sqrt{n\pi}\Gamma(\frac{n}{2})} t^{-\frac{1}{2}} \left(1 + \frac{t}{n}\right)^{-\frac{n+1}{2}} \\
&= \frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{n}{2})} \frac{1}{n} \left(\frac{x}{n}\right)^{-\frac{1}{2}} \left(1 + \frac{x}{n}\right)^{-\frac{n+1}{2}}
\end{aligned}$$

$$X^2 \sim F(1, n)$$