西安交通大學

数理统计作业

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目　录

第一章　数理统计的基本概念

1.1 设总体 $X \sim N(\mu, \sigma^2)$，X_1，...，X_n 为总体 X 的一个样本，$\bar{X}$ 为样本均值，如要

$$P\{|\bar{X} - \mu| < 1\} = 0.95$$

则样本容量 n 应取多大？

解: 由**定理 1.2.4** 得 $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$，所以 $\frac{\sqrt{n}\cdot(\bar{X}-\mu)}{\sigma} \sim N(0,1)$，又由条件 $P\{|\bar{X}-\mu| < 1\} = 0.95$ 可得，$P\{|\frac{\sqrt{n}\cdot(\bar{X}-\mu)}{\sigma}| < \frac{\sqrt{n}}{\sigma}\} = 0.95$，有

$$\begin{aligned}
\alpha &= (1 - 0.95) \div 2 \\
&= 0.025 \\
\mu_{\alpha=0.025} &= 1.96 \\
1.96 &< \frac{\sqrt{n}}{\sigma} \\
n &= \lceil (1.96 \cdot \sigma)^2 \rceil
\end{aligned}$$

1.2 设电子元件的寿命（小时）$X \sim Exp(0.0015)$，独立测试 6 个元件并记下它们的失效时间，试求：

(1) 至 800 小时，没有一个元件失效的概率；

(2) 至 3000 小时，所有元件都失效的概率.

解: (1) 由题意知 $F(x) = e^{-0.0015\cdot x}$，则

$$\begin{aligned}
P\{X_1, X_2, X_3, X_4, X_5, X_6; X \geq 800\} &= (F(800))^6 \\
&= (e^{-0.0015\times 800})^6 \\
&= e^{-7.2}
\end{aligned}$$

(2)

$$\begin{aligned}
P\{X_1, X_2, X_3, X_4, X_5, X_6; X < 3000\} &= (1 - F(3000))^6 \\
&= (1 - e^{-0.0015\times 3000})^6 \\
&= (1 - e^{-4.5})^6
\end{aligned}$$

1.4 设总体 X 服从对数正态分布，即 X 具有概率密度

$$f(x) = \begin{cases} \frac{1}{x\sqrt{2\pi}\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}, & 0 < x < \infty \\ 0, & \text{其他} \end{cases}$$

$X_1, ..., X_n$ 为总体 X 的一个样本，试写出 $X_1, ..., X_n$ 的联合概率密度.

解:

$$\begin{aligned}
x > 0, \quad \prod_{i=1}^{n} f(x_i) &= \prod_{i=1}^{n} \frac{1}{x_i\sqrt{2\pi}\sigma} \cdot e^{-\frac{(\ln x_i-\mu)^2}{2\sigma^2}} \\
&= \frac{e^{-\frac{1}{2\sigma^2}\cdot\sum\limits_{i=1}^{n}(\ln x_i-\mu)^2}}{(2\pi\sigma^2)^{\frac{n}{2}} \cdot \prod_{i=1}^{n} x_i}
\end{aligned}$$

则：

$$f(x_1,...,x_n)=\begin{cases}\frac{e^{-\frac{1}{2\sigma^2}\cdot\sum\limits_{i=1}^{n}(\ln x_i-\mu)^2}}{(2\pi\sigma^2)^{\frac{n}{2}}\cdot\prod_{i=1}^{n}x_i}, & 0<x<\infty\\ 0, & \text{其他}\end{cases}$$

1.6 证明下列等式

(1) $\sum\limits_{i=1}^{n}(X_i-\mu)^2=\sum\limits_{i=1}^{n}(X_i-\bar{X})^2+n(\bar{X}-\mu)^2$；

(2) $\sum\limits_{i=1}^{n}(X_i-\bar{X})^2=\sum\limits_{i=1}^{n}X_i^2-n\bar{X}^2$.

解：(1)

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\mu)^2&=\sum_{i=1}^{n}[(X_i-\bar{X})+(\bar{X}-\mu)]^2\\&=\sum_{i=1}^{n}[(X_i-\bar{X})^2+2(X_i-\bar{X})(\bar{X}-\mu)+(\bar{X}-\mu)^2]\\&=\sum_{i=1}^{n}(X_i-\bar{X})^2+2(\bar{X}-\mu)\cdot\sum_{i=1}^{n}(X_i-\bar{X}+\sum_{i=1}^{n}(\bar{X}-\mu)^2\\&=\sum_{i=1}^{n}(X_i-\bar{X})^2+n(\bar{X}-\mu)^2\end{aligned}$$

(2)

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\bar{X})^2&=\sum_{i=1}^{n}(X_i-\bar{X})(X_i-\bar{X})\\&=\sum_{i=1}^{n}[(X_i-\bar{X})X_i-(X_i-\bar{X})\bar{X}]\\&=\sum_{i=1}^{n}[X_i^2-\bar{X}\cdot X_i-(X_i-\bar{X})\bar{X}]\\&=\sum_{i=1}^{n}X_i^2-\bar{X}\sum_{i=1}^{n}X_i-\bar{X}\sum_{i=1}^{n}(X_i-\bar{X})\\&=\sum_{i=1}^{n}X_i^2-n\bar{X}^2\end{aligned}$$

1.7 设 $\bar{X}_n$ 和 S_n^2 分别为样本 $X_1,X_2,\ldots,X_n$ 的样本均值和样本方差，再抽出一个样品 X_{n+1}，记样本 $X_1,X_2,\ldots,X_{n+1}$ 的样本均值和样本方差为分别 $\bar{X}_{n+1}$ 和 S_{n+1}^2，求证：

$$\bar{X}_{n+1}=\bar{X}_n+\frac{1}{n+1}(X_{n+1}-\bar{X}_n)$$
$$S_{n+1}^2=\frac{n}{n+1}[S_n^2+\frac{1}{n+1}(X_{n+1}-\bar{X}_n)^2]$$

解：(1)

$$
\begin{aligned}
\bar{X}_n + \frac{1}{n+1}(X_{n+1} - \bar{X}_n) &= \frac{1}{n}\sum_{i=1}^{n} X_i + \frac{1}{n+1}(X_{n+1} - \frac{1}{n}\sum_{i=1}^{n} X_i) \\
&= \frac{(n+1)\sum_{i=1}^{n} X_i + nX_{n+1} - \sum_{i=1}^{n} X_i}{n(n+1)} \\
&= \frac{n\sum_{i=1}^{n} X_i + nX_{n+1}}{n(n+1)} \\
&= \frac{\sum_{i=1}^{n+1} X_i}{n(n+1)} \\
&= \frac{1}{n+1}\sum_{i=1}^{n+1} X_i \\
&= \bar{X}_{n+1}
\end{aligned}
$$

(2)

$$
\begin{aligned}
S_n^2 &= \frac{1}{n}(\sum_{i=1}^{n}(X_i - \bar{X}_n))^2 \\
&= \frac{1}{n}(\sum_{i=1}^{n} X_i^2 - n\bar{X}_n^2)
\end{aligned}
$$

$$
S_{n+1}^2 = \frac{1}{n+1}(\sum_{i=1}^{n+1} X_i^2 - (n+1)\bar{X}_{n+1}^2)
$$

$$
\begin{aligned}
\text{左边-右边} &= S_{n+1}^2 - \frac{n}{n+1}[S_n^2 + \frac{1}{n+1}(X_{n+1} - \bar{X}_n)^2] \\
&= \frac{1}{n+1}[\sum_{i=1}^{n+1} X_i^2 - (n+1)\bar{X}_{n+1}^2] - \frac{n}{n+1}[\frac{1}{n}(\sum_{i=1}^{n} X_i^2 - n\bar{X}_n^2) + \frac{1}{n+1}(X_{n+1} - \bar{X}_n)^2] \\
&= \frac{1}{n+1}[\sum_{i=1}^{n+1} X_i^2 - (n+1)\bar{X}_{n+1}^2 - \sum_{i=1}^{n} X_i^2 + n\bar{X}_n^2 - \frac{n}{n+1}(X_{n+1}^2 - 2X_{n+1}\bar{X}_n + \bar{X}_n^2)] \\
&= \frac{1}{n+1}[\frac{1}{n+1}X_{n+1}^2 - (n+1)\bar{X}_{n+1}^2 + \frac{n^2}{n+1}\bar{X}_n^2 + \frac{2n}{n+1}X_{n+1}\bar{X}_n] \\
&= \frac{1}{n+1}[\frac{1}{n+1}(X_{n+1}^2 + 2nX_{n+1}\bar{X}_n + n\bar{X}_n^2) - (n+1)\bar{X}_{n+1}^2] \\
&= \frac{1}{n+1}[\frac{1}{n+1}(X_{n+1} + n\bar{X}_n)^2 - (n+1)\bar{X}_{n+1}^2] \\
&= \frac{1}{n+1}[(n+1)\bar{X}_{n+1}^2 - (n+1)\bar{X}_{n+1}^2] \\
&= 0
\end{aligned}
$$