The Chinese University of Hong Kong, Shenzhen
SDS · School of Data Science
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DDA 3005 — Numerical Methods

Exercise Sheet Nr.: 3

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In the creation of this solution sheet, I worked together with:

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For correction:

Exercise						$\sum$
Grading						

Problem 1 (Computing the LU Factorization): (approx. *20* pts)
Calculate the LU factorization (with pivoting) of the matrix

$$A = \begin{bmatrix} 1 & \frac{5}{3} & 1 & 1 \\ 2 & -1 & -1 & 0 \\ 3 & -1 & 0 & 1 \\ -2 & 1 & -4 & 0 \end{bmatrix}.$$

For each step of the algorithm, clearly mark the current L and U factor and the pivot element.

State the final LU factorization and permutation matrix P.

current L current U

$$P_1 A = \begin{bmatrix} \boxed{3} & -1 & 0 & 1 \\ 2 & -1 & -1 & 0 \\ 1 & \frac{5}{3} & 1 & 1 \\ -2 & 1 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{2}{3} & 1 & 0 & 0 \\ \frac{1}{3} & 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & \boxed{2} & 1 & \frac{2}{3} \\ 0 & \frac{1}{3} & -4 & \frac{2}{3} \end{bmatrix}$$

$$P_2 P_1 A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 1 & \frac{5}{3} & 1 & 1 \\ 2 & -1 & -1 & 0 \\ -2 & 1 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ \frac{2}{3} & -\frac{1}{6} & 1 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & 0 & -\frac{5}{6} & -\frac{5}{9} \\ 0 & 0 & \boxed{-\frac{25}{6}} & \frac{5}{9} \end{bmatrix}$$

$$P_3 P_2 P_1 A = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 1 & \frac{5}{3} & 1 & 1 \\ -2 & 1 & -4 & 0 \\ 2 & -1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 1 & 0 \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{5} & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & 0 & -\frac{25}{6} & \frac{5}{9} \\ 0 & 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

Final $PA = LU$:

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{3} & 1 & 0 & 0 \\ -\frac{2}{3} & \frac{1}{6} & 1 & 0 \\ \frac{2}{3} & -\frac{1}{6} & \frac{1}{5} & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 3 & -1 & 0 & 1 \\ 0 & 2 & 1 & \frac{2}{3} \\ 0 & 0 & -\frac{25}{6} & \frac{5}{9} \\ 0 & 0 & 0 & -\frac{2}{3} \end{bmatrix}$$

Problem 2 (Properties of Triangular Matrices): (approx. *20* pts)
In this exercise, we investigate additional theoretical properties of triangular matrices.

a) Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular upper triangular matrix. Show that A^{-1} is an upper triangular matrix.

Hint: This result can be shown via induction over the dimension n using suitable decompositions of the involved matrices.

① $n=1$

$A = [a_{11}]$, $a_{11} \neq 0$

$\Rightarrow A^{-1} = [\frac{1}{a_{11}}]$ is an upper triangular matrix

② Assum the statement holds for $n=k$, need to show it is also true for $n=k+1$:

$A_{k+1} = \begin{bmatrix} A_k & b \\ 0 & a_{k+1,k+1} \end{bmatrix}$, A_{k+1} is a nonsingular upper triangular matrix, $A_k \in \mathbb{R}^{k\times k}$ is an upper $\triangle$ matrix, $b \in \mathbb{R}^k$, $a_{k+1,k+1} \neq 0$

$$\Rightarrow A_{k+1}^{-1} = \begin{bmatrix} \frac{a_{k+1,k+1}}{a_{k+1,k+1}\cdot A_k} & \frac{-b}{a_{k+1,k+1}\cdot A_k} \\ 0 & \frac{A_k}{a_{k+1,k+1}\cdot A_k} \end{bmatrix}$$

$$= \begin{bmatrix} A_k^{-1} & \frac{-A_k^{-1} b}{a_{k+1,k+1}} \\ 0 & \frac{1}{a_{k+1,k+1}} \end{bmatrix}$$

$\because$ A_k^{-1} is an upper $\triangle$, $\frac{-A_k^{-1} b}{a_{k+1,k+1}}$ does not affect whether A_{k+1}^{-1} is an upper $\triangle$ or not, $\frac{1}{a_{k+1,k+1}} \neq 0$

$\therefore$ $A_{k+1,k+1}^{-1}$ is an upper $\triangle$ matrix.

Hence, the statement holds for $n=k+1$ $\square$

Therefore, the statement is proved.

b) Suppose that $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a unit lower triangular matrix. Prove that the matrix $\boldsymbol{A}^{-1}$ is unit lower triangular as well.

$\because$ $A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 1 \end{bmatrix}$, $AA^{-1} = I$

$\therefore$ assume

$A^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 1 \end{bmatrix} \begin{bmatrix} b_{11} & 0 & 0 & \cdots & 0 \\ b_{21} & b_{22} & 0 & \cdots & 0 \\ b_{31} & b_{32} & b_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

① $a_{11} b_{11} = b_{11} = I_{11} = 1 \Rightarrow \boxed{b_{11} = 1}$

$a_{11} b_{1i} = I_{1i} = 0 \Rightarrow b_{1i} = 0, \ \forall i \in [2, n]$

② $a_{21} b_{11} + a_{22} b_{21} = a_{21} + b_{21} = 0 \Rightarrow b_{21} = -a_{21}$

$a_{21} b_{12} + a_{22} b_{22} = 0 + b_{22} = 1 \Rightarrow \boxed{b_{22} = 1}$

$a_{21} b_{1i} + a_{22} b_{2i} = 0 + b_{2i} = 0 \Rightarrow b_{2i} = 0, \forall i \in [3, n]$

③ $a_{31} b_{11} + a_{32} b_{21} + a_{33} b_{31} = a_{31} - a_{21} a_{32} + b_{31} = 0 \Rightarrow b_{31} = a_{21} a_{32} - a_{31}$

$a_{31} b_{12} + a_{32} b_{22} + a_{33} b_{32} = 0 + a_{32} + b_{32} = 0 \Rightarrow b_{32} = -a_{32}$

$a_{31} b_{13} + a_{32} b_{23} + a_{33} b_{33} = 0 + 0 + b_{33} = 1 \Rightarrow \boxed{b_{33} = 1}$

$a_{31} b_{1i} + a_{32} b_{2i} + a_{33} b_{3i} = b_{3i} = 0 \Rightarrow b_{3i} = 0, \ \forall i \in [4, n]$

④ Assume $b_{kk} = 1$, $b_{ki} = 0$ for $\forall i \in [k+1, n]$, need to show $b_{k+1,k+1} = 1$, $b_{k+1,i} = 0$ for $i \in [k+2, n]$

$a_{k+1,1} b_{1,k+1} + a_{k+1,2} b_{2,k+1} + \cdots + a_{k+1,k+1} b_{k+1,k+1} = 0 + 0 + \cdots + b_{k+1,k+1} = 1 \Rightarrow b_{k+1,k+1} = 1$

$a_{k+1,1} b_{1,i} + a_{k+1,2} b_{2,i} + \cdots + a_{k+1,k+1} b_{k+1,i} = b_{k+1,i} = 0 \Rightarrow b_{k+1,i} = 0. \ \forall i \in [k+2, n]$

Hence, $A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ b_{21} & 1 & 0 & \cdots & 0 \\ b_{31} & b_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & 1 \end{bmatrix}$

Therefore, the statement is proved.

Problem 3 (Solving Linear Least Squares): (approx. *15* pts)
Let $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $m \ge n$, and $\boldsymbol{b} \in \mathbb{R}^m$ be given and suppose that $\boldsymbol{A}$ has full column rank.

Let $\tilde{\boldsymbol{Q}}_r \tilde{\boldsymbol{R}} = [\boldsymbol{A}\ \boldsymbol{b}] \in \mathbb{R}^{m\times(n+1)}$ be the reduced QR factorization of the extended matrix $\tilde{\boldsymbol{A}} = [\boldsymbol{A}\ \boldsymbol{b}]$. Let us further consider the decomposition

$$\tilde{\boldsymbol{R}} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ & \rho \end{bmatrix}$$

where $\boldsymbol{R} \in \mathbb{R}^{n\times n}$ is upper triangular, $\boldsymbol{p} \in \mathbb{R}^n$, and $\rho \in \mathbb{R}$, and let $\boldsymbol{x} \in \mathbb{R}^n$ be the solution of the linear least squares problem

$$\min_{\boldsymbol{x}\in\mathbb{R}^n} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2.$$

Show that $\boldsymbol{R}\boldsymbol{x} = \boldsymbol{p}$ and $|\rho| = \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_2$.

Hint: Decompose $\tilde{\boldsymbol{Q}}_r$ via $\tilde{\boldsymbol{Q}}_r = [\boldsymbol{Q}_r\ \boldsymbol{q}]$.

① $\tilde{Q}_r = [Q_r \ q]$, $Q_r^T Q_r = I$

$\tilde{A} = \tilde{Q}_r \tilde{R}$

$[A \ b] = [Q_r \ q] \begin{bmatrix} R & p \\ & \rho \end{bmatrix}$

$\Rightarrow A = Q_r R$, $b = Q_r p + q\rho$

$\because$ residual $q\rho = 0$

$\therefore \|Ax - b\|_2 = \|q\rho\|_2 = |\rho|$

② $A^T A x = A^T b$

$\because A = Q_r R$

$\therefore R^T Q_r^T Q_r R x = R^T Q_r^T b$

$\because Q_r^T Q_r = I, \quad b = Q_r p$

$\therefore R^T R x = R^T Q_r^T Q_r p = R^T p$

$\therefore Rx = p$

Hence, $Rx = p$, $\|Ax - b\|_2 = |p|$

Problem 4 (LU vs. QR Factorizations): (approx. *45* pts)

For $n \in \mathbb{N}$, the so-called *Wilkinson*-matrix $\boldsymbol{W} \in \mathbb{R}^{n \times n}$ and its inverse $\boldsymbol{W}^{-1}$ are given by

$$\boldsymbol{W} = \begin{bmatrix} 1 & & & & 1 \\ -1 & 1 & & & 1 \\ -1 & -1 & \ddots & & 1 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -1 & -1 & \dots & -1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{W}^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & & & & -\frac{1}{2^{n-1}} \\ & \frac{1}{2} & -\frac{1}{4} & & & -\frac{1}{2^{n-2}} \\ & & \frac{1}{2} & -\frac{1}{4} & & -\frac{1}{2^{n-3}} \\ & & & \ddots & & \vdots \\ & & & & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \dots & \frac{1}{2^{n-1}} & \frac{1}{2^{n-1}} \end{bmatrix}.$$

a) Verify that the matrices

$$\boldsymbol{L} = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ -1 & -1 & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ -1 & -1 & \dots & -1 & 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{U} = \begin{bmatrix} 1 & & & & 1 \\ & 1 & & & 2 \\ & & \ddots & & \vdots \\ & & & 1 & 2^{n-2} \\ & & & & 2^{n-1} \end{bmatrix}$$

define an LU factorization of the Wilkinson-matrix $\boldsymbol{W}$.

$$LU = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 2^{n-1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 1 \\ -1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 2^{n-1} - \sum_{i=0}^{n-2} 2^i \end{bmatrix}$$

$$2^{n-1} - \sum_{i=0}^{n-2} 2^i = 2^{n-1} - (1 + 2 + 4 + \cdots 2^{n-2}) = 2^{n-1} - \frac{1 \times (1 - 2^{n-1})}{1-2} = 2^{n-1} - (2^{n-1} - 1) = 1$$

$$\therefore LU = \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 1 \\ -1 & -1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{bmatrix} = W$$

b) Calculate the condition number of $\boldsymbol{W}$ with respect to the norm $\|\cdot\|_\infty$. (Here, $\|\cdot\|_\infty$ denotes the maximum absolute row sum). Is the matrix $\boldsymbol{W}$ generally well- or ill-conditioned?

$K(W) = \|W\|_\infty \|W^{-1}\|_\infty$

$\|W\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |W_{ij}| = n$

$\|W^{-1}\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |W^{-1}_{ij}| = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^{n-1}} = \frac{\frac{1}{2} \times (1 - \frac{1}{2}^{n-1})}{1 - \frac{1}{2}} + \frac{1}{2^{n-1}} = 1$

$\therefore \text{Cond}(W) = n \cdot 1 = n$

Hence, W generally is ill-conditioned if n is very large

c) Compute the inverse matrix $\boldsymbol{L}^{-1}$ and show that $\|\boldsymbol{L}\|_\infty = n$ and $\|\boldsymbol{L}^{-1}\|_\infty = 2^{n-1}$.

$$① L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & 1 \end{bmatrix}$$

$\because LL^{-1} = I$, L^{-1} is also lower $\triangle$

$\therefore$ assume

$$L^{-1} = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} 1&0&0&\cdots&0\\ -1&1&0&\cdots&0\\ -1&-1&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ -1&-1&-1&\cdots&1\end{bmatrix}\begin{bmatrix} a_{11}\,|\,1 & 0&0&\cdots&0\\ a_{21}\,|\,1 & a_{22}\,|\,1 & 0 &\cdots&0\\ a_{31}\,2 & a_{32}\,|\,1 & a_{33}\,|\,1&\cdots&0\\ \vdots\ 4 & \vdots\ 2 & \vdots\ 1 & \ddots\ 1 & \vdots \\ a_{n1}&a_{n2}&a_{n3}&\cdots&a_{nn}\end{bmatrix} = \begin{bmatrix} 1&0&0&\cdots&0\\ 0&1&0&\cdots&0\\ 0&0&1&\cdots&0\\ \vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&\cdots&1\end{bmatrix}$$

$\Rightarrow\ a_{11}=1$

$-a_{11}+a_{21}=0 \Rightarrow a_{21}=1, \quad a_{22}=1$

$-a_{11}-a_{21}+a_{31}=0 \Rightarrow a_{31}=2$

$-a_{22}+a_{32}=0 \Rightarrow a_{32}=1$

$a_{33}=1$

$-a_{11}-a_{21}-a_{31}+a_{41}=0 \Rightarrow a_{41}=4, \quad a_{42}=2,\ a_{43}=1,\ a_{44}=1$

$$\Rightarrow L^{-1}=\begin{bmatrix} 1&0&0&0&\cdots&0&0\\ 1&1&0&0&\cdots&0&0\\ 2&1&1&0&\cdots&0&0\\ 4&2&1&1&\cdots&0&0\\ \vdots&&&&&&\\ 2^{n-2}&2^{n-3}&2^{n-4}&2^{n-5}&\cdots&1&1\end{bmatrix}$$

② $\|L\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|L_{ij}| = \sum_{j=1}^{n}|L_{nj}| = n$

③ $\|L^{-1}\|_\infty = \max_{1\le i\le n}\sum_{j=1}^{n}|L^{-1}_{ij}| = \sum_{j=1}^{n}|L^{-1}_{nj}| = 1+1+2+4+\cdots+2^{n-2} = 1+\dfrac{1\cdot(1-2^{n-1})}{1-2} = 2^{n-1}$

d) Use Householder transformations to compute a QR factorization of $\boldsymbol{W}$ in the case $n=4$.

Clearly mark and perform the corresponding updates step-by-step. You only need to state the final upper triangular $\boldsymbol{R}$ factor, i.e., computation of $\boldsymbol{Q}$ is not required.

$$W=\begin{bmatrix}1&0&0&1\\-1&1&0&1\\-1&-1&1&1\\-1&-1&-1&1\end{bmatrix}$$

① Find $v_1\in\mathbb{R}^4$ s.t. $H_{v_1}A_{:,1}=\alpha_1(1,0,0,0)^T$

$a=[1\ -1\ -1\ -1]^T, \quad \alpha=\pm\|a\|_2=\pm 2$

$$v_1=a-\alpha e_1=\begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}-(-2)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix}$$

$$H_{v_1}=I-2\frac{vv^T}{\|v\|^2}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-2\cdot\frac{1}{12}\begin{bmatrix}3\\-1\\-1\\-1\end{bmatrix}[3\ -1\ -1\ -1]=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}-\frac{1}{6}\begin{bmatrix}9&-3&-3&-3\\-3&1&1&1\\-3&1&1&1\\-3&1&1&1\end{bmatrix}=\begin{bmatrix}-\frac12&\frac12&\frac12&\frac12\\ \frac12&\frac56&-\frac16&-\frac16\\ \frac12&-\frac16&\frac56&-\frac16\\ \frac12&-\frac16&-\frac16&\frac56\end{bmatrix}$$

$$H_{v_1}W=\begin{bmatrix}-\frac12&\frac12&\frac12&\frac12\\ \frac12&\frac56&-\frac16&-\frac16\\ \frac12&-\frac16&\frac56&-\frac16\\ \frac12&-\frac16&-\frac16&\frac56\end{bmatrix}\begin{bmatrix}1&0&0&1\\-1&1&0&1\\-1&-1&1&1\\-1&-1&-1&1\end{bmatrix}=\begin{bmatrix}-2&-0.5&0&1\\0&1.17&0&1\\0&-0.83&1&1\\0&-0.83&-1&1\end{bmatrix}$$

② Find v_2

$a_2=[1.17\ -0.83\ -0.83]^T, \quad \alpha=\pm 1.66$

$$v_2=a_2-\alpha e_2=\begin{bmatrix}1.17\\-0.83\\-0.83\end{bmatrix}-(-1.66)\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}2.83\\-0.83\\-0.83\end{bmatrix}$$

$$H_{v_2}=I-2\frac{v_2v_2^T}{\|v_2\|^2}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{2}{9.39}\begin{bmatrix}2.83\\-0.83\\-0.83\end{bmatrix}[2.83\ -0.83\ -0.83]=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1.71&-0.5&-0.5\\-0.5&0.15&0.15\\-0.5&0.15&0.15\end{bmatrix}=\begin{bmatrix}-0.71&0.5&0.5\\0.5&0.85&-0.15\\0.5&-0.15&0.85\end{bmatrix}$$

$$H_{v_2}'H_{v_1}W=\begin{bmatrix}1&0&0&0\\0&-0.71&0.5&0.5\\0&0.5&0.85&-0.15\\0&0.5&-0.15&0.85\end{bmatrix}\begin{bmatrix}-2&-0.5&0&1\\0&1.17&0&1\\0&-0.83&1&1\\0&-0.83&-1&1\end{bmatrix}=\begin{bmatrix}-2&-0.5&0&1\\0&-1.67&0&0.29\\0&0&1&1.2\\0&0&-1&1.2\end{bmatrix}$$

③ Find v_3

$a_3=[1\ -1]^T, \quad \alpha=\pm 1.41$

$$v_3=a_3-\alpha e_3=\begin{bmatrix}1\\-1\end{bmatrix}-(-1.41)\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}2.41\\-1\end{bmatrix}$$

$$H_{v_3}=I-2\frac{V_3V_3^T}{\|V_3\|^2}=\begin{bmatrix}-0.71&0.71\\0.71&0.71\end{bmatrix}$$

$$H_{v_3}'H_{v_2}'H_{v_1}W=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&-0.71&0.71\\0&0&0.71&0.71\end{bmatrix}\begin{bmatrix}-2&-0.5&0&1\\0&-1.67&0&0.29\\0&0&1&1.2\\0&0&-1&1.2\end{bmatrix}=\begin{bmatrix}-2&-0.5&0&1\\0&-1.67&0&0.29\\0&0&-1.42&0\\0&0&0&1.7\end{bmatrix}=R$$

$$\Rightarrow R = \begin{bmatrix} -2 & -0.5 & 0 & 1 \\ 0 & -1.67 & 0 & 0.29 \\ 0 & 0 & -1.42 & 0 \\ 0 & 0 & 0 & 1.7 \end{bmatrix}$$

e) Let us consider the linear system of equations $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{b}$ and the associated LU algorithm

$$\boldsymbol{W}\boldsymbol{x} = \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{L}\boldsymbol{U}\boldsymbol{x} = \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{x} = \boldsymbol{U}^{-1}(\boldsymbol{L}^{-1}\boldsymbol{b}) \tag{1}$$

(using forward- and back-substitution to compute $\boldsymbol{y} = \boldsymbol{L}^{-1}\boldsymbol{b}$ and $\boldsymbol{x} = \boldsymbol{U}^{-1}\boldsymbol{y}$, respectively). Is the LU algorithm (1) generally an accurate method to solve the linear system $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{b}$? Explain your answer!

Provide a suitable experiment that confirms your answer numerically. For instance, write a test program (in `MATLAB` or `Python`) and generate $\boldsymbol{W}$ and random vectors $\boldsymbol{b} \sim \mathcal{N}(0,1)^n$ for different n. Using the true inverse $\boldsymbol{W}^{-1}$ and $\boldsymbol{x}^* = \boldsymbol{W}^{-1}\boldsymbol{b}$, you can then report and compare the forward errors $\|\boldsymbol{x}^* - \boldsymbol{x}\|/\|\boldsymbol{x}^*\|$ where $\boldsymbol{x}$ is computed via (1).

Repeat your experiments for the QR-based algorithm

$$\boldsymbol{W}\boldsymbol{x} = \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{Q}\boldsymbol{R}\boldsymbol{x} = \boldsymbol{b} \quad \Longleftrightarrow \quad \boldsymbol{x} = \boldsymbol{R}^{-1}(\boldsymbol{Q}^\top\boldsymbol{b})$$

(with back-substitution to compute $\boldsymbol{x} = \boldsymbol{R}^{-1}(\boldsymbol{Q}^\top\boldsymbol{b})$). You can use `MATLAB` or `Python` in-built code to obtain the QR factorizations of $\boldsymbol{W}$. Does this method perform differently from the LU-based approach? Can you explain your numerical observations?

```
n=  10
LU forward error:  4.583361763156522e-15
QR forward error:  4.021402925899061e-16
n=  50
LU forward error:  0.0019511879993023861
QR forward error:  1.732747904809201e-15
n=  75
LU forward error:  6224.278029030788
QR forward error:  3.3202132744902102e-15
n=  100
LU forward error:  23.26694343030354
QR forward error:  4.978444305475217e-15
n=  250
LU forward error:  1.252909211669882e+56
QR forward error:  1.1057476388704773e-14
n=  500
LU forward error:  2.0366721428489032e+116
QR forward error:  2.125081603584421e-14
n=  1000
LU forward error:  inf
QR forward error:  4.224271138149484e-14
```

① LU algorithm is an accurate method when n is small.
However, LU algorithm performs badly when n is large.
② QR-based algorithm performs well across all range of n.
③ LU and QR perform differently especially when n is large.
QR-based approach is better due to its numerical stability.