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HW4

Problem 1 (20pts). Consider the following linear program:

$$\begin{array}{rl} \text{maximize} & 3x_1 + x_2 + 4x_3 \\ \text{subject to} & x_1 + 3x_2 + x_3 \le 5 \\ & x_1 + 2x_2 + 2x_3 \le 8 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

(a). What is the corresponding dual problem?

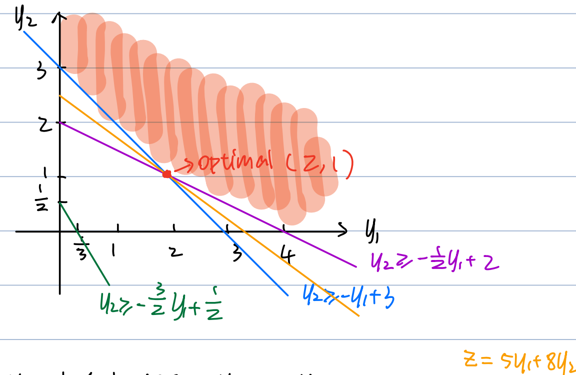
(b). Solve the dual problem graphically.

(c). Use complementarity conditions for the primal-dual pair to solve the primal problem.

(a) dual problem:

$$\begin{array}{rl} \text{minimize} & 5y_1 + 8y_2 \\ \text{subject to} & y_1 + y_2 \geqslant 3 \\ & 3y_1 + 2y_2 \geqslant 1 \\ & y_1 + 2y_2 \geqslant 4 \\ & y_1, y_2 \geqslant 0 \end{array}$$

(b)



optimal solution: $y_1 = 2,\ y_2 = 1$

optimal value: $2 \times 5 + 8 \times 1 = 18$

(c) complementarity conditions: $y_i \cdot (a_i^T x - b_i) = 0$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Hence, optimal solution for the primal problem: $x = [2, 0, 3]$

optimal value: $3 \times 2 + 0 + 4 \times 3 = 18$

Problem 2 (25pts). Consider the following table of food and corresponding nutritional values:

	Protein, g	Carbohydrates, g	Calories	Cost
Bread	4	7	130	3
Milk	6	10	120	4
Fish	20	0	150	8
Potato	1	30	70	2

The ideal intake for an adult is at least 30 grams of protein, 40 grams of carbohydrates, and 400 calories per day. The problem is to find the **least** costly way to achieve those amounts of nutrition by using the four types of food shown in the table.

(a). Formulate this problem as a linear optimization problem (specify the meaning of each decision variable and constraint).

(b). Solve it using MATLAB, find an optimal solution and the optimal value.

(c). Formulate the dual problem. Interpret the dual problem. (Hint: Suppose a pharmaceutical company produces each of the nutrients in pill form and sells them each for a certain price.)

(d). Use MATLAB to solve the dual problem. Find an optimal solution and the optimal value.

(a) Assume the amount of bread, milk, fish and potato are x_1, x_2, x_3, x_4.

minimize $3x_1 + 4x_2 + 8x_3 + 2x_4$

subject to $4x_1 + 6x_2 + 20x_3 + x_4 \geq 30$ # at least 30 grams of protein

$7x_1 + 10x_2 + 30x_4 \geq 40$ # at least 40 grams of carbohydrates

$130x_1 + 120x_2 + 150x_3 + 70x_4 \geq 400$ # at least 400 calories

$x_1, x_2, x_3, x_4 \geq 0$ # the amounts are non-negative.

(b)

```
cvx_begin
    variable x1
    variable x2
    variable x3
    variable x4
    minimize 3*x1+4*x2+8*x3+2*x4
    subject to
        4*x1+6*x2+20*x3+x4>=30;
        7*x1+10*x2+30*x4>=40;
        130*x1+120*x2+150*x3+70*x4>=400;
        x1>=0;
        x2>=0;
        x3>=0;
        x4>=0;
cvx_end

x1
x2
x3
x4
```

```
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +15.2351


x1 =

    1.0732


x2 =

   1.1560e-08


x3 =

    1.2312


x4 =

    1.0829
```

optimal solution: $x = [1.0732, 1.1560e{-}08, 1.2312, 1.0829]$

optimal value: 15.2351

(c) dual problem:

maximize $30y_1 + 40y_2 + 400y_3$

subject to $4y_1 + 7y_2 + 130y_3 \leq 3$

$6y_1+10y_2+120y_3 \le 4$

$20y_1+150y_3 \le 8$

$y_1+30y_2+70y_3 \le 2$

$y_1, y_2, y_3, y_4 \ge 0$

Interpretation: a pharmaceutical company produces each of nutrienes in pill form and sell them each for a certain price. The company wants to maximize the price of pills to earn money, But the price shouldn't be higher than 4 types of food, or the consumers will prefer to buy food directly.

(d)

```
cvx_begin
    variable y1
    variable y2
    variable y3
    variable y4
    maximize 30*y1+40*y2+400*y3
    subject to
        4*y1+7*y2+130*y3<=3;
        6*y1+10*y2+120*y3<=4;
        20*y1+150*y3<=8;
        y1+30*y2+70*y3<=2;
        y1>=0;
        y2>=0;
        y3>=0;
        y4>=0;

cvx_end
y1
y2
y3
y4
```

```
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +15.2351


y1 =

    0.3099


y2 =

    0.0283


y3 =

    0.0120


y4 =

     0
```

optimal solution: $y=[0.3099, 0.0283, 0.0120, 0]$

optimal value: 15.2351

Problem 3 (25pts). Consider the max flow problem on the graph in figure 1 with the orange node being the source node and the green node being the terminal node (the number on each edge is its capacity, see the lecture slide 13). Do the following based on the lecture slides.

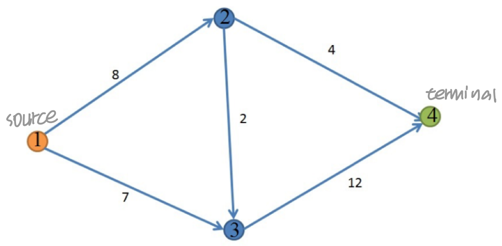


Figure 1: Max flow problem

(a). Formulate it as a linear program and solve it using MATLAB.

(b). Formulate the dual of this problem and solve it using MATLAB.

(c). Find the corresponding maximum flow and minimum cut of the graph.(Please draw the cut on figure 1).

(a) maximize the flow from 1 to 4, denoted by $\triangle$

maximize $\triangle$

subject to $\sum_{j:(i,j)\in E} x_{ji} - \sum_{j:(i,j)\in E} x_{ij} = 0$

$\sum_{j:(j,1)\in E} x_{j1} - \sum_{j:(1,j)\in E} x_{1j} + \triangle = 0$

$\sum_{j:(j,4)\in E} x_{j4} - \sum_{j:(4,j)\in E} x_{4j} - \triangle = 0$

$E=\{(1,2),(1,3),(2,3),(2,4),(3,4)\}$

$$W=\begin{bmatrix} 0 & 8 & 7 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

$x_{ij} \le w_{ij}$

$x_{ij} \ge 0$

```
cvx_begin
    variable x12
    variable x13
    variable x23
    variable x24
    variable x34
    variable y
    maximize y
    subject to
        x12-x23-x24==0;
        x13+x23-x34==0;
        y-x12-x13==0;
        x34+x24-y==0;
        x12<=8;
        x13<=7;
        x23<=2;
        x24<=4;
        x34<=12;
        x12>=0;
        x13>=0;
        x23>=0;
        x24>=0;
        x34>=0;

cvx_end
x12
x13
x23
x24
x34
```

```
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +13


x12 =

    6.0000


x13 =

    7.0000


x23 =

    2.0000


x24 =

    4.0000


x34 =

    9.0000
```

Optimal value (max flow) = 13

(b) dual problem:

minimize $\sum_{(i,j)\in E} w_{ij} z_{ij}$

subject to $z_{ij} \ge y_i - y_j$, $\forall (i,j) \in E$

$y_1 - y_4 = 1$

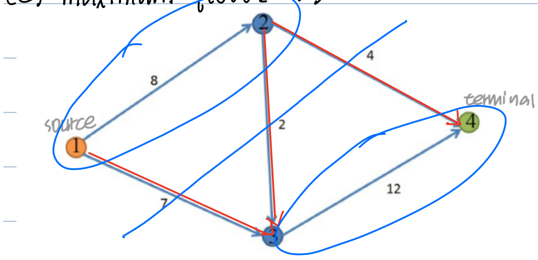
$z_{ij} \ge 0$

```
cvx_begin
    variable z12
    variable z13
    variable z23
    variable z24
    variable z34
    variable y1
    variable y2
    variable y3
    variable y4
    minimize 8*z12+7*z13+2*z23+4*z24+12*z34
    subject to
        z12>=y1-y2;
        z13>=y1-y3;
        z23>=y2-y3;
        z24>=y2-y4;
        z34>=y3-y4;
        y1-y4==1;
        z12>=0;
        z13>=0;
        z23>=0;
        z24>=0;
        z34>=0;
        y1==1;
        y2>=0;
        y3>=0;
        y4==0;

cvx_end
```

```
------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): +13


y1 =

     1


y2 =

    1.0000


y3 =

   2.5106e-09


y4 =

     0
```

optimal value = 13

(c) maximum flow: 13 $\quad z_{13}, z_{23}, z_{24} = 1, \; z_{12} = 0, \; z_{34} = 0$



Problem 4 (15pts). Use linear program duality to show that exactly one of the following systems has a solution

1. $A\boldsymbol{x} \leq \boldsymbol{b}$
2. $\boldsymbol{y}^T A = 0, \boldsymbol{b}^T \boldsymbol{y} < 0, \boldsymbol{y} \geq 0$

Hint: You can first show that they can't both have solutions. Then you show that if the second one is infeasible, the first one must be feasible.

Assume 1 and 2 both have solutions.

1. primal problem: $\qquad$ dual problem:

① maximize 0 $\qquad$ minimize $b^T y$

subject to $Ax \leq b$ $\qquad$ subject to $A^T y = 0$

x free $\qquad$ $y \geq 0$

since 1 has solution, $b^T y \geq 0$, $y \geq 0$, which is contradicted with 2. ($b^T y < 0$, $y \geq 0$).

② minimize 0 $\qquad$ maximize $b^T y$

subject to $Ax \leq b$ $\qquad$ subject to $A^T y = 0$

x free $\qquad$ $y \leq 0$

2. Consider the pair of primal-dual linear program:

primal $\qquad$ dual

min $b^T y$ $\qquad$ max 0

s.t. $y^T A = 0$, $y \geq 0$ $\qquad$ s.t. $Ax \leq b$.

If there exists y s.t.

$$y^T A = 0, \; y \geq 0, \; b^T y < 0$$

By scaling this y, $b^T y$ can be arbitrarily negative, therefore, the primal problem must be unbounded.

If the primal is unbounded, the dual must be infeasible.

Hence, exactly one of the system has a solution.

Problem 5 (15pts).

$\rightarrow M^T = -M$

Suppose M is a square matrix such that $M = -M^T$, for example,

$$M = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -4 \\ -2 & 4 & 0 \end{pmatrix}$$

Consider the following optimization problem:

$$\begin{array}{ll} \text{minimize}_{\boldsymbol{x}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & M\boldsymbol{x} \geq -\boldsymbol{c} \\ & \boldsymbol{x} \geq 0 \end{array}$$

(a). Show that the dual problem of it is equivalent to the primal problem.

(b). Argue that the problem has optimal solution if and only if there is a feasible solution.

$M = -M^T \Rightarrow M^T = (-M^T)^T = -M$

(a) dual problem:

$$\begin{array}{ll} \text{maximize} & -c^T y \\ \text{subject to} & M^T y \leq c \\ & y \geq 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \text{minimize} & c^T y \\ \text{subject to} & -My \leq c \Rightarrow My \geq -c \\ & y \geq 0 \end{array}$$

Hence, the dual problem of it is equivalent to the primal problem.

(b). Since the dual problem is same as the primal problem, the problem can only be feasible or infeasible.

So if there is a feasible solution, the problem has optimal solution.