hw 7.

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Problem 1 (25pts). Suppose that $f : \mathbf{R} \to \mathbf{R}$ is convex, and $a, b \in \operatorname{dom} f$ with $a < b$, where dom denotes the domain of the function. More specifically, $f : \mathbf{R}^p \to \mathbf{R}^q$ means that f is an $\mathbf{R}^p$-valued function on some *subset* of $\mathbf{R}^p$, and this *subset* of $\mathbf{R}^p$ is the domain of the function f. Show that

(a)

$$f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b), \text{ for all } x \in (a,b)$$

Hint: (Jensen's Inequality) If $p_1, \ldots, p_n$ are positive numbers which sum to 1 and f is a real continuous function that is convex, then

$$f\left(\sum_{i=1}^n p_i x_i\right) \le \sum_{i=1}^n p_i f(x_i)$$

(a) since f is a convex function, (a,b) is a convex set.

Assume $x = \lambda a + (1-\lambda) b$, $\forall \in [0,1]$

$f(x) = f(\lambda a + (1-\lambda) b) \le \lambda f(a) + (1-\lambda) f(b)$

$\because x \in (a,b)$

$\therefore 0 < b-x < b-a$, $0 < x-a < b-a$

$\therefore 0 < \frac{b-x}{b-a} < 1$, $0 < \frac{x-a}{b-a} < 1$

$\because \frac{b-x}{b-a} + \frac{x-a}{b-a} = \frac{b-a}{b-a} = 1$

let $\lambda = \frac{b-x}{b-a}$, $1-\lambda = \frac{x-a}{b-a}$

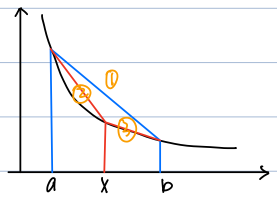
$\therefore f(x) \le \frac{b-x}{b-a} f(a) + \frac{x-a}{b-a} f(b)$

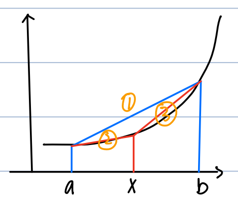
(b)

$$\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$$

for all $x \in (a,b)$. Draw a sketch that illustrates this inequality.

(b)





$\frac{f(x)-f(a)}{x-a}$ is the slop of line 2

$\frac{f(b)-f(a)}{b-a}$ is the slop of line 1

$\frac{f(b)-f(x)}{b-x}$ is the slop of line 3

Hence, $\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$

(c) Suppose f is differentiable. Use the result in (b) to show that:

$$f'(a) \le \frac{f(b)-f(a)}{b-a} \le f'(b)$$

Note that these inequalities also follow form:

$$f(b) \ge f(a) + f'(a)(b-a), \quad f(a) \ge f(b) + f'(b)(a-b)$$

(c) From (b), we get $\frac{f(x)-f(a)}{x-a} \le \frac{f(b)-f(a)}{b-a} \le \frac{f(b)-f(x)}{b-x}$

$f'(a) = \lim\limits_{\Delta h \to 0} \frac{f(a+\Delta h)-f(a)}{\Delta h}$

Assume $\Delta h = x-a$, $x = \Delta h + a$. $\frac{f(x)-f(a)}{x-a} = \frac{f(\Delta h+a)-f(a)}{\Delta h}$

From graph in (b), we get $f'(a) \le \frac{f(x)-f(a)}{x-a}$.

In the same way, we get $f'(b) \geq \frac{f(b)-f(x)}{b-x}$

Hence, $f'(a) \leq \frac{f(x)-f(a)}{x-a} \leq \frac{f(b)-f(a)}{b-a} \leq \frac{f(b)-f(x)}{b-x} \leq f(b)$

$\Rightarrow f'(a) \leq \frac{f(b)-f(a)}{b-a} \leq f'(b)$

(d) Suppose f is twice differentiable. Use the result in (c) to show that $f''(a) \geq 0$ and $f''(b) \geq 0$.

(d). $f(b) \geq f(a) + f'(a)(b-a) \geq f(b) + f'(b)(a-b) + f'(a)(b-a)$

$\Rightarrow [f'(b) - f'(a)](b-a) \geq 0$

$\because b > a$

$\therefore f'(b) - f'(a) \geq 0$

$f''(a) = \lim_{\Delta h \to 0} \frac{f'(a+\Delta h) - f'(a)}{\Delta h}$, $f''(b) = \lim_{\Delta h \to 0} \frac{f'(b+\Delta h) - f'(b)}{\Delta h}$

$\because f'(b) - f'(a) \geq 0$

$\therefore f'(x) - f'(a) \geq 0$, $x \in (a,b)$

$\therefore f'(a+\Delta h) - f'(a) \geq 0$, $f'(b+\Delta h) - f'(b) \geq 0$

$\therefore f''(a) \geq 0$, $f''(b) \geq 0$

Problem 2 (30pts) Show that the following functions are convex:

(a) $f(x) = -\log\left(-\log\left(\sum_{i=1}^{m} e^{a_i^T x + b_i}\right)\right)$ on $\operatorname{dom} f = \left\{x \mid \sum_{i=1}^{m} e^{a_i^T x + b_i} < 1\right\}$. You can use the fact that $\log\left(\sum_{i=1}^{n} e^{y_i}\right)$ is convex.

(a) let $y_i = a_i^T x_i + b_i$

since $a_i^T x_i + b_i$ is linear, y_i is convex.

$\operatorname{dom} f = \{x \mid \sum_{i=1}^{m} e^{a_i^T x + b_i} < 1\} = \{y \mid \sum_{i=1}^{m} e^{y_i} < 1\}$ is a convex set.

$f(x) = -\log(-\log(\sum_{i=1}^{m} e^{a_i^T x + b_i})) = -\log(-\log(\sum_{i=1}^{m} e^{y_i}))$

$\because \log(\sum_{i=1}^{m} e^{y_i})$ is convex

$\therefore -\log(\sum_{i=1}^{m} e^{y_i})$ is concave.

$g(x) = -\log(x)$ is concave.

$\therefore f(x) = -\log(-\log(\sum_{i=1}^{m} e^{a_i^T x + b_i}))$ is convex.

(b)

$$f(x,u,v) = -\log\left(uv - x^T x\right) \text{ on } \operatorname{dom} f = \left\{(x,u,v) \mid uv > x^T x, u, v > 0\right\}$$

(b) let $g(x) = uv - x^T x$, $u, v > 0$.

$g'(x) = -2x$, $g''(x) = -2 < 0$

$\Rightarrow g(x) = uv - x^T x$ is a concave function.

$f(x,u,v) = -\log(uv - x^T x) = -\log(g(x))$

$\because$ $f(x) = -\log(x)$ is a concave function

$\therefore$ $f(x,u,v) = -\log(uv - x^T x)$ is a convex function.

(c) Let $T(x, \omega)$ denote the trigonometric polynomial

$$T(x, \omega) = x_1 + x_2 \cos\omega + x_3 \cos 2\omega + \cdots + x_n \cos(n-1)\omega$$

Show that the function

$$f(x) = \ominus \boxed{\int_0^{2\pi} \log T(x, \omega) d\omega}$$ concave

is convex on $\{x \in \mathbf{R}^n \mid T(x, \omega) > 0, 0 \le \omega \le 2\pi\}$.
Hint: Nonnegative weighted sum of convex functions is still convex. Let this property extend to infinite sums and integrals. Assume that $f(x, y)$ is convex in x for each $y \in \mathcal{A}$ and $w(y) \ge 0$ for each $y \in \mathcal{A}$ and integral exists. Then the function g defined as

$$g(x) = \int_{\mathcal{A}} w(y) f(x, y) dy$$

is convex in x.

(c) Assume w is fixed, $T(x,w) = x_1 + x_2 \cos w + x_3 \cos 2w + \cdots + x_n \cos(n-1)w$ is linear.

$\Rightarrow$ $T(x,w)$ is convex in x for each $w \in [0, 2\pi]$.

$\because$ $\log(x)$ is a concave function, $T(x,w) > 0$

$\therefore$ $-\log T(x,w)$ is convex in x for each w in $[0, 2\pi]$.

Since nonnegative weighted sum of convex functions is still convex,

$f(x) = \int_0^{2\pi} -\log T(x,w) dw = -\int_0^{2\pi} \log T(x,w) dw$ is convex.

Problem 3 (20pts). Consider the following function:

$$\begin{array}{ll} \text{minimize} & -x_1 - x_2 + \max\{x_3, x_4\} \\ \text{s.t.} & (x_1 - x_2)^2 + (x_3 + 2x_4)^4 \le 5 \\ & x_1 + 2x_2 + x_3 + 2x_4 \le 6 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{array}$$

(a) Verify this is a convex optimization problem.

(b) Use CVX to solve the problem.

(a) ① objective function: $f(x) = -x_1 - x_2 + \max\{x_3, x_4\}$.

x_3, x_4 are linear, $\max\{x_3, x_4\}$ is a convex function.

$\Rightarrow$ $f(x) = -x_1 - x_2 + \max\{x_3, x_4\}$ is a convex function.

② constraints:

$x_1 + 2x_2 + x_3 + 2x_4 \le 6$ is linear (convex set).

$g(x_1, x_2, x_3, x_4) = (x_1 - x_2)^2 + (x_3 + 2x_4)^4$

$\nabla g(x_1, x_2, x_3, x_4) = [2(x_1 - x_2), -2(x_1 - x_2), 4(x_3 + 2x_4)^3, 8(x_3 + 2x_4)^3]$

$$\nabla^2 g(x_1, x_2, x_3, x_4) = \begin{bmatrix} 2 & -2 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & 0 & 12(x_3+2x_4)^2 & 24(x_3+2x_4)^2 \\ 0 & 0 & 24(x_3+2x_4)^2 & 48(x_3+2x_4)^2 \end{bmatrix} = A$$

$\det(A) = 2 \times 2 \times [12(x_3+2x_4)^2 \times 48(x_3+2x_4)^2 - 24(x_3+2x_4)^2 \times 24(x_3+2x_4)^2] + 2 \times (-2) \times [12(x_3+2x_4)^2 \times 48(x_3+2x_4)^2 - [24(x_3+2x_4)]^2]$

$= 0$

$tr(A) = 4 + 60(x_3 + 2x_4)^2 \geq 0$

$\Rightarrow$ Hessian matrix is PSD.

$\Rightarrow g(x_1, x_2, x_3, x_4)$ is convex.

$\therefore (x_1 - x_2)^2 + (x_3 + x_4)^4 \leq 5$ is a convex set.

Hence, this is a convex optimization problem.

(b)

编辑器 - /Users/yangjinglan/Desktop/hw7_q3.m

hw7_q3.m

```
cvx_begin
    variables x(5)
    minimize -x(1)-x(2)+x(5)
    subject to
        (x(1)-x(2))^(2)+(x(3)+2*x(4))^(4)<=5;
        x(1)+2*x(2)+x(3)+2*x(4)<=6;
        x(5)>=x(3);
        x(5)>=x(4);
        x>=0;
cvx_end
x
```

命令行窗口

```
 DIMACS: 8.2e-16  0.0e+00  1.8e-12  0.0e+00  4.0e-09  4.0e-09
------------------------------------------------------------------

------------------------------------------------------------
Status: Solved
Optimal value (cvx_optval): -4.74536

x =

    3.4907
    1.2546
    0.0000
    0.0000
    0.0000
```

optimal value $= -4.74536$

optimal solution: $x = [3.4907, 1.2546, 0, 0]$

Problem 4 (25pts). To model the influence of price on customer purchase probability, the following logit model is often used:

$$\lambda(p) = \frac{e^{-p}}{1 + e^{-p}}$$

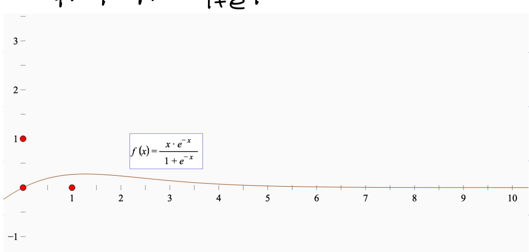
where p is the price, $\lambda(p)$ is the purchase probability.

Assume the variable cost of the product is 0 (e.g., iPhone Apps). As the seller, you want to maximize the expected revenue by choosing the optimal price. That is, you want to solve:

$$\text{maximize}_p \quad p\lambda(p)$$

(a) Draw a picture of $r(p) = p\lambda(p)$ (for p from 0 to 10) and use the picture to show that $r(p)$ is not concave (thus maximize $r(p)$ is not a convex optimization problem)

(a) $r(p) = p\lambda(p) = \dfrac{pe^{-p}}{1 + e^{-p}}$



From the graph, we get that $r(p)$ is not concave.

(b) Write down p as a function of λ (the inverse function of $\lambda(p)$). Show that you can write the objective function as a function of $\lambda : \tilde{r}(\lambda)$, where $\tilde{r}(\lambda)$ is concave in λ.

(b) $\lambda(p) = \dfrac{e^{-p}}{1 + e^{-p}}$

$\Rightarrow e^{-p} = \dfrac{\lambda}{1 - \lambda}$

$\Rightarrow -p = \ln\left(\dfrac{\lambda}{1-\lambda}\right)$

$\Rightarrow p = \ln\left(\dfrac{1-\lambda}{\lambda}\right)$

$\Rightarrow p = \ln(1-\lambda) - \ln(\lambda)$

Objective function: $p\lambda(p) = \lambda[\ln(1-\lambda) - \ln(\lambda)]$

$\Rightarrow \tilde{r}(\lambda) = \lambda[\ln(1-\lambda) - \ln(\lambda)]$

$$\tilde{r}'(\lambda) = \ln(1-\lambda) - \ln(\lambda) + \frac{-\lambda}{1-\lambda} - 1 = \ln(1-\lambda) - \ln(\lambda) + \frac{1}{\lambda-1}$$

$$\tilde{r}''(\lambda) = \frac{1}{\lambda-1} - \frac{1}{\lambda} - \frac{1}{(\lambda-1)^2} = \frac{-1}{\lambda(\lambda-1)^2}$$

Assume $\lambda > 0$, $\tilde{r}''(\lambda) < 0$

Hence, $\tilde{r}(\lambda) = \lambda[\ln(1-\lambda) - \ln(\lambda)]$ is concave in λ.

(c) From part 2, write the KKT condition for the optimal λ. Then transform it back to an optimal condition in p.

(c) $\text{minimize}_\lambda \; -\lambda \ln \frac{1-\lambda}{\lambda}$ $\left(\lambda \ln \frac{\lambda}{1-\lambda}\right)$

subject to $\lambda - \frac{1}{2} \leq 0$

$\lambda \geq 0$

$$L(\lambda, v) = \lambda \ln \frac{\lambda}{1-\lambda} + v\left(\lambda - \frac{1}{2}\right)$$

KKT condition for optimal λ =

① main condition =

$$-\ln(1-\lambda) + \ln(\lambda) + \frac{1}{1-\lambda} + v \geq 0$$

② dual feasibility = $v \geq 0$

③ complementarity =

$$v\left(\lambda - \frac{1}{2}\right) = 0$$

$$\lambda\left(-\ln(1-\lambda) + \ln(\lambda) + \frac{1}{1-\lambda} + v\right) = 0$$

④ primal feasibility =

$\lambda - \frac{1}{2} \leq 0$, $\lambda \geq 0$

KKT condition for optimal p =

① main condition =

$$-p + 1 + e^{-p} + v \geq 0$$

② dual feasibility = $v \geq 0$

③ complementarity =

$$v\left(\frac{e^{-p}}{1+e^{-p}} - \frac{1}{2}\right) = 0$$

$$e^{-p} + \frac{e^{-p}}{1+e^{-p}}(v - p) = 0$$

④ primal feasibility =

$$0 \leq \frac{e^{-p}}{1+e^{-p}} \leq \frac{1}{2}$$