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1. Reform NLP to LP:

minimize $2x_2 + y_1$

s.t. $y_1 \geq x_1 - x_3$

$y_1 \geq x_3 - x_1$

$y_2 + y_3 \leq 5$

$y_2 \geq x_1 + 2$

$y_2 \geq -x_1 - 2$

$y_3 \geq x_2$

$y_3 \geq -x_2$

$x_3 \leq 1$

$x_3 \geq -1$

Transform NLP to standard LP.

Define $x_1 = x_1^+ - x_1^-$, $x_2 = x_2^+ - x_2^-$, with $x_1^+, x_1^-, x_2^+, x_2^- \geq 0$

minimize $2(x_2^+ - x_2^-) + y_1$

s.t. $y_2 + y_3 + s_1 = 5$

$x_3 + s_2 = 1$

$x_3 - s_3 = -1$

$y_1 - (x_1^+ - x_1^-) + x_3 - s_4 = 0$

$y_1 + (x_1^+ - x_1^-) - x_3 - s_5 = 0$

$y_2 - (x_1^+ - x_1^-) - s_6 = 2$

$y_2 + (x_1^+ - x_1^-) - s_7 = -2$

$y_3 - (x_2^+ - x_2^-) - s_8 = 0$

$y_3 + (x_2^+ - x_2^-) - s_9 = 0$

$x_1^+, x_1^-, x_2^+, x_2^-, y_1, y_2, y_3 \geq 0$

$s_i \geq 0$, $i \in [1, 9]$

2.

(a) standard form:

minimize $-x_1 - 2x_2 - 4x_3$

s.t. $x_1 + x_3 + s_1 = 8$

$x_2 + 2x_3 + s_2 = 15$

$x_1, x_2, x_3, s_1, s_2 \geq 0$

(b) The feasible set is $\{x : Ax = b, x \geq 0\}$, where

$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 8 \\ 15 \end{bmatrix}$

$\therefore A \in \mathbb{R}^{2\times5}$ $b \in \mathbb{R}^{2\times1}$

choose 2 independent columns of A:

$A_B x_B = b$

$\Rightarrow x_B = A_B^{-1} b$

$\because A_B^{-1} \in \mathbb{R}^{2\times2}$, $b \in \mathbb{R}^{2\times1}$

$\therefore x_B \in \mathbb{R}^{2\times1}$

Hence, this LP must exist an optimal solution with no more than 2 positive variables.

(c) The feasible set is $\{x: Ax=b, x\geq 0\}$, where

$$A=\begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 1 \end{bmatrix} \qquad b=\begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

Choose 2 independent columns of A:

① $B=\{1,2\}$

$$x_B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

② $B=\{1,3\}$

$$x_B=\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ \frac{15}{2} \end{bmatrix}$$

③ $B=\{1,5\}$

$$x_B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

④ $B=\{2,3\}$

$$x_B=\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} -1 \\ 8 \end{bmatrix}$$

⑤ $B=\{2,4\}$

$$x_B=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 15 \\ 8 \end{bmatrix}$$

⑥ $B=\{3,4\}$

$$x_B=\begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} \frac{15}{2} \\ \frac{1}{2} \end{bmatrix}$$

⑦ $B=\{3,5\}$

$$x_B=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 8 \\ -1 \end{bmatrix}$$

⑧ $B=\{4,5\}$

$$x_B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 15 \end{bmatrix}=\begin{bmatrix} 8 \\ 15 \end{bmatrix}$$

Hence, basic solutions: $\{8,15,0,0,0\}$, $\{\frac{1}{2},0,\frac{15}{2},0,0\}$, $\{8,0,0,0,15\}$, $\{0,-1,8,0,0\}$, $\{0,15,0,8,0\}$, $\{0,0,\frac{15}{2},\frac{1}{2},0\}$, $\{0,0,8,0,-1\}$, $\{0,0,0,8,15\}$

basic feasible solutions: $\{8,15,0,0,0\}$, $\{\frac{1}{2},0,\frac{15}{2},0,0\}$, $\{8,0,0,0,15\}$, $\{0,15,0,8,0\}$, $\{0,0,\frac{15}{2},\frac{1}{2},0\}$, $\{0,0,0,8,15\}$.

(d) objective function: To minimize $(-x_1-2x_2-4x_3)$.

① $\{8,15,0,0,0\}$: $-8-2\times15-4\times0=-38$

② $\{\frac{1}{2},0,\frac{15}{2},0,0\}$: $-\frac{1}{2}-2\times0-4\times\frac{15}{2}=-\frac{61}{2}$

③ $\{8,0,0,0,15\}$: $-8-2\times0-4\times0=-8$

④ $\{0,15,0,8,0\}$: $-0-2\times15-4\times0=-30$

⑤ $\{0,0,\frac{15}{2},\frac{1}{2},0\}$: $-0-0-4\times\frac{15}{2}=-30$

⑥ $\{0,0,0,8,15\}$: $-0-2\times0-4\times0=0$

Hence, the optimal value is 38, the optimal solution is $\{8,15,0,0,0\}$.

3.

(a) Transform into LP:

$$\min_{x \in \mathbb{R}^n} c^T x$$

subject to $A_i x - b_i \le \delta$, $\forall i \in [n]$

$b_i - A_i x \le \delta$, $\forall i \in [n]$

$x \ge 0$

(b) ① formulate a LP.

assume produce x_1 number of salad A, x_2 number of salad B.

maximize $10x_1 + 20x_2$

subject to $\frac{1}{4}x_1 + \frac{1}{2}x_2 = 25$

$\frac{1}{8}x_1 + \frac{1}{4}x_2 = 10$

$3x_1 + x_2 = 120$

$x_1, x_2 \ge 0$

② standard form:

minimize $-10x_1 - 20x_2$

subject to $\frac{1}{4}x_1 + \frac{1}{2}x_2 = 25$

$\frac{1}{8}x_1 + \frac{1}{4}x = 10$

$3x_1 + x_2 = 120$

$x_1, x_2 \ge 0$

③ this LP is NOT solvable.

The feasible set is $\{x : Ax = b, x \ge 0\}$, where

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ \frac{1}{8} & \frac{1}{4} \\ 3 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 25 \\ 10 \\ 120 \end{bmatrix}$$

$\text{rank}(A) = 3 > 2$, hence this LP is not solvable.

(c) minimize $-10x_1 - 20x_2$

subject to $\frac{1}{4}x_1 + \frac{1}{2}x_2 - 25 \le 5$

$\frac{1}{8}x_1 + \frac{1}{4}x_2 - 10 \le 5$

$3x_1 + x_2 - 120 \le 5$

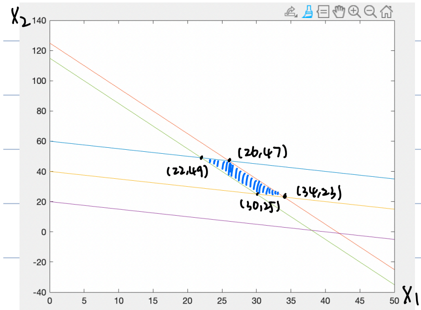
$-\frac{1}{4}x_1 - \frac{1}{2}x_2 + 25 \le 5$

$-\frac{1}{8}x_1 - \frac{1}{4}x_2 + 10 \le 5$

$-3x_1 - x_2 + 120 \le 5$

$x_1, x_2 \ge 0$

① feasible set:



② object function: $-10x_1-20x_2$

$-10\times22-20\times49 = 220+980 = -1200$

$-10\times26-20\times47 = -1200$

$-10\times30-20\times25 = -800$

$-10\times34-20\times23 = -800$

Hence, optimal value $= -1200$, optimal solution set $\{120-2x_2, x_2\}$, $x_1\in[22,26]$, $x_2\in[47,49]$

③ active constraints:
$$\begin{cases} \frac{1}{4}x_1+\frac{1}{2}x_2-25\le 5 \\ -\frac{1}{4}x_1-\frac{1}{2}x_2+25\le 5 \\ 3x_1+x_2-120\le 5 \\ -3x_1-x_2+120\le 5 \end{cases}$$

④ integer solution $=\{x_1=22, x_2=49\}$