hw 6

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Problem 1 (25pts). Consider the following function:

$$f(x,y,z) = 3x^2 - 2x - 2xy + 3y^2 - 2y - 2zy + 3z^2 - 2z - 2xz$$

(a). Considering the 1st-order necessary condition, try to find the candidate minimizers of $f(x,y,z)$.

(b). Considering the 2nd-order sufficient condition, whether these candidates are indeed local minimizers?

(c). Is (0,0,0) a local minimizer? Why?

(a) $\nabla f(x) = (6x - 2 - 2y - 2z,\ -2x + 6y - 2 - 2z,\ -2y + 6z - 2 - 2x)$

$= (6x - 2y - 2z - 2,\ 6y - 2z - 2x - 2,\ 6z - 2x - 2y - 2)$

$$\begin{cases} 6x - 2y - 2z - 2 = 0 \\ 6y - 2z - 2x - 2 = 0 \\ 6z - 2x - 2y - 2 = 0 \end{cases} \Rightarrow x = y = z = 1,\ f(x,y,z) = -3$$

(b) $\nabla^2 f(x) = \begin{pmatrix} 6 & -2 & -2 \\ -2 & 6 & -2 \\ -2 & -2 & 6 \end{pmatrix} = A$

$\det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -2 & -2 \\ -2 & 6-\lambda & -2 \\ -2 & -2 & 6-\lambda \end{vmatrix} = (6-\lambda) \times [(6-\lambda)^2 - 4] - (-2) \times [(-2) \times (6-\lambda) - 4] + (-2) \times [4 + 2(6-\lambda)] = -\lambda^3 + 18\lambda^2 - 96\lambda + 128$

$= (\lambda - 2)(-\lambda^2 + 16\lambda - 64) = -(\lambda - 2)(\lambda - 8)^2 = 0$

$\Rightarrow \lambda_1 = 2 \geq 0,\ \lambda_2 = 8 \geq 0,\ \lambda_3 = 8 \geq 0$

Since the eigenvalues are positive, these candidates are indeed local minimizers.

(c) $\nabla f(0,0,0) = (-2, -2, -2) \neq 0$

Hence, (0,0,0) is not a local minimizer.

Problem 2 (25pts). Given a symmetric matrix $A \in R^{n \times n}$, consider the following problem:

$$\begin{array}{ll} \min_{x \in R^n} & \overset{1\times n\ \ n\times n\ \ n\times 1}{x^T A x} \\ \text{subject to} & 2 - \underset{1\times n\ \ n\times 1}{x^T x} = 0 \end{array}$$

(a). Give the KKT conditions of this problem.

(b). If A is positive definite without repeated eigenvalues, how many different KKT points are there at most?

(c). If A is positive definite without repeated eigenvalues, what is the minimum value of this problem, and how many local minimizers? (Hint: According to Rayleigh quotient, $min\{x^T Ax/(x^T x)\} = \lambda_{min}$, where λ_{min} is the minimum eigenvalue of A)

global

(a) $L(x, \lambda) = x^T A x + \lambda(2 - x^T x)$, λ free, x free

① main condition: $2Ax - 2\lambda x = 0$

② dual feasibility: λ free.

③ complementarity: $x_i(2Ax_i - 2\lambda x_i) = 0,\ \forall i$

④ primal feasibility: $2 - x^T x = 0$, x_i free

(b) According to main condition: $Ax = \lambda x$

$\Rightarrow \lambda$ is the eigenvalue of A.

Since A (n×n) is positive definite without repeated eigenvalues.

there are n different eigenvalues.

$\begin{cases} Ax = \lambda x \\ x^T x = 2 \end{cases}$ or $\begin{cases} A(-x) = \lambda(-x) \\ (-x)^T(-x) = 2 \end{cases}$, Hence, there are 2n KKT points at most.

(c) ① $2Ax - 2\lambda x = 0 \Rightarrow Ax - \lambda x = 0$

$\Rightarrow x^T A x - \lambda x^T x = 0$

$\Rightarrow \lambda = \dfrac{x^T A x}{x^T x}$

According to primal feasibility, $x^T x = 2$

$g(x) = \dfrac{x^T A x}{x^T x} = \frac{1}{2}(x^T A x)$, $\min\{x^T Ax / x^T x\} = \min\{\frac{1}{2} x^T A x\} = \lambda_{min}$

$L_{min}(x, \lambda) = x^T A x + \lambda(2 - x^T x) = 2\lambda_{min} + 2\lambda_{min} - 2\lambda_{min} = 2\lambda_{min}$

Hence, the minimum value is $2\lambda_{min}$.

② Since A is positive definite without repeated eigenvalues,

there are 2 global minimizers, x^* and $-x^*$.

Problem 3 (25pts). Construct the KKT conditions for the following linear program:

$$\begin{array}{ll} \text{maximize} & 3x_1 + x_2 + 4x_3 \\ \text{subject to} & x_1 + 3x_2 + x_3 \leq 5 \\ & x_1 + 2x_2 + 2x_3 \leq 8 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

minimize $-(3x_1 + x_2 + 4x_3)$

subject to $x_1 + 3x_2 + x_3 - 5 \leq 0$

$x_1 + 2x_2 + 2x_3 - 8 \leq 0$

$x_1, x_2, x_3 \geq 0$

$L(x_1, x_2, x_3, \lambda, \eta) = -3x_1 - x_2 - 4x_3 + \lambda(x_1 + 3x_2 + x_3 - 5) + \eta(x_1 + 2x_2 + 2x_3 - 8)$

① main condition =

$-3 + \lambda + \eta \geq 0$

$-1 + 3\lambda + 2\eta \geq 0$

$-4 + \lambda + 2\eta \geq 0$

② dual feasibility = $\lambda, \eta \geq 0$

③ complementarity =

$\lambda(x_1 + 3x_2 + x_3 - 5) = 0$

$\eta(x_1 + 2x_2 + 2x_3 - 8) = 0$

$x_1(-3 + \lambda + \eta) = 0$

$x_2(-1 + 3\lambda + 2\eta) = 0$

$x_3(-4 + \lambda + 2\eta) = 0$

④ primal feasiblity =

$x_1 + 3x_2 + x_3 - 5 \leq 0$

$x_1 + 2x_2 + 2x_3 - 8 \leq 0$

$x_1, x_2, x_3 \geq 0$

Problem 4 (25pts). Construct the KKT conditions for the following nonlinear program:

$$\begin{array}{ll} \text{minimize} & x_1 ln(x_1) + (x_2 - 2)^2 + x_3 \\ \text{subject to} & x_1 + x_2 \leq 3 \\ & x_3 - x_2^2 \geq 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$L(x_1, x_2, x_3, \lambda, \eta) = x_1 \ln(x_1) + (x_2 - 2)^2 + x_3 + \lambda(x_1 + x_2 - 3) + \eta(x_3 - x_2^2 - 3)$

① main condition.

$\ln x_1 + 1 + \lambda \geq 0$

$2(x_2 - 2) + \lambda - 2\eta x_2 \geq 0$

$1+\eta \geq 0$

② dual feasibility: $\lambda \geq 0,\ \eta \leq 0$

③ complementarity:

$\lambda(x_1+x_2-3)=0$

$\eta(x_3-x_2^2-3)=0$

$x_1(\ln x_1+1+\lambda)=0$

$x_2(2x_2-4+\lambda-2\eta x_2)=0$

$x_3(1+\eta)=0$

④ primal feasibility:

$x_1+x_2 \leq 3$

$x_3-x_2^2 \geq 3$

$x_1, x_2, x_3 \geq 0$