Problem 1 (50pts). Consider the following linear program:

$$\begin{array}{lll} \text{maximize} & 3x_1 + 4x_2 + 3x_3 + 6x_4 & \\ \text{subject to} & 2x_1 + x_2 - x_3 + x_4 & \geq 12 \\ & x_1 + x_2 + x_3 + x_4 & = 8 \\ & -x_2 + 2x_3 + x_4 & \leq 10 \\ & x_1, x_2, x_3, x_4 & \geq 0. \end{array} \tag{1}$$

After transforming the problem into standard form and apply Simplex method, we obtain the final tableau as follow:

B							
	0	2	9	0	3	0	36
1	1	0	−2	0	−1	0	4
4	0	1	3	1	1	0	4
6	0	−2	−1	0	−1	1	6

a) Derive the dual problem of the linear program (1) and calculate a dual solution based on complementarity conditions. Given that the optimal solution to the primal solution is unique, investigate whether the dual solution is unique.

(a) Dual problem: minimize $12y_1 + 8y_2 + 10y_3$

s.t. $2y_1 + y_2 \geq 3$

$y_1 + y_2 - y_3 \geq 4$

$-y_1 + y_2 + 2y_3 \geq 3$

$y_1 + y_2 + y_3 \geq 6$

$y_1 \leq 0$, y_2 free, $y_3 \geq 0$.

Accroding to the complementartity conditions:

$x = [4, 0, 0, 4, 0, 6]$

$$\begin{cases} 2y_1 + y_2 = 3 \\ y_1 + y_2 + y_3 = 6 \\ 12y_1 + 8y_2 + 10y_3 = 36 \end{cases} \Rightarrow y = [-3, 9, 0]$$

Since the optimal solution is unique, accroding to the complementarlty conditions, the dual solution is unique.

b) Do the optimal solution and the objective function value change if we

- decrease the objective function coefficient for x_3 to 0?
- increase the objective function coefficient for x_3 to 9?
- decrease the objective function coefficient for x_4 to 5?
- increase the objective function coefficient for x_1 to 7?

(b). $3 \in N$.

$$\begin{bmatrix} 2 \\ 9 \\ 3 \end{bmatrix} + \lambda \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \geq 0 \Rightarrow \lambda \in [-9, +\infty)$$

① $3 - 0 = 3 > -9$

Hence, decrease the objective function coefficient for x_3 to 0 will not change the optimal solution and objective function value.

② $3-9=-6>-9$

Hence, increase the objective function coefficient for x_3 to 9 will not change the optimal solution and objective function value.

$4 \in B$

$$\begin{bmatrix}2\\9\\3\end{bmatrix} - \lambda \begin{bmatrix}0\\1\\0\end{bmatrix}^T \begin{bmatrix}1 & -1 & 0\\-1 & 2 & 0\\1 & -2 & 1\end{bmatrix} \begin{bmatrix}1 & -1 & 1\\1 & 1 & 0\\-1 & 2 & 0\end{bmatrix} = \begin{bmatrix}2\\9\\3\end{bmatrix} - \lambda \begin{bmatrix}1\\3\\-1\end{bmatrix} \geq 0$$

$\Rightarrow \lambda \in [-3, 2]$

③ $6-5=1 \in [-3,2]$

Hence, decrease the objective function coefficient for x_4 to 5 will not change the optimal solution but objective function value change to 32

$1 \in B$

$$\begin{bmatrix}2\\9\\3\end{bmatrix} - \lambda \begin{bmatrix}1\\0\\0\end{bmatrix}^T \begin{bmatrix}1 & -1 & 0\\-1 & 2 & 0\\1 & -2 & 1\end{bmatrix} \begin{bmatrix}1 & -1 & 1\\1 & 1 & 0\\-1 & 2 & 0\end{bmatrix} = \begin{bmatrix}2\\9\\3\end{bmatrix} - \lambda \begin{bmatrix}0\\-2\\1\end{bmatrix} \geq 0$$

$\lambda \in [-\frac{9}{2}, 3]$

④ $3-7=-4 \in [-\frac{9}{2}, 3]$

Hence, increase the objective function coefficient for x_1 to 7 will not change the optimal solution, but objective function value change to 52.

e) Find the possible range for adjusting the coefficient 8 of the second constraint such that the current basis is kept optimal.

(e)

$$\begin{bmatrix}4\\4\\6\end{bmatrix} + \lambda \begin{bmatrix}2 & 1 & 0\\1 & 1 & 0\\0 & 1 & 1\end{bmatrix}^{-1} \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}4\\4\\6\end{bmatrix} + \lambda \begin{bmatrix}1 & -1 & 0\\-1 & 2 & 0\\1 & -2 & 1\end{bmatrix} \begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}4\\4\\6\end{bmatrix} + \lambda \begin{bmatrix}-1\\2\\-2\end{bmatrix} \geq 0$$

$\Rightarrow \lambda \in [-2, 3]$

Hence, possible range $= [6, 11]$

Problem 2 (50pts). An insurance company is introducing three products: special risk insurance, mortgage insurance, and long-term care insurance. The expected profit is \$500 per unit on special risk insurance, \$250 per unit on mortgage insurance and \$600 per unit on long term care insurance. The work requirements are as follows:

Department	Working hours per unit			Working hours available
	Special risk	Mortgage	Long-term care	
Underwriting	2	1	1	240
Administration	3	1	2	150
Claims	1	2	4	180

The management team wants to establish sales quotas for each product to maximize the total expected profit.

1. Formulate this problem as a linear optimization problem. Specify the decision variables, objective function, and constraints.

1. Decision variable: the sales quotas for special risk insurance, mortgage insurance, long-term care insurance are x_1, x_2, x_3, respectively.

Objective function: $500x_1 + 250x_2 + 600x_3$

constraints: $2x_1 + x_2 + x_3 \le 240$ (working hours available for underwriting)

$3x_1 + x_2 + x_3 \le 150$ (working hours available for administration)

$x_1 + 2x_2 + 4x_3 \le 180$ (working hours available for claims)

$x_1, x_2, x_3 \ge 0$

maximize $500x_1 + 250x_2 + 600x_3$

s.t. $2x_1 + x_2 + x_3 \le 240$

$3x_1 + x_2 + x_3 \le 150$

$x_1 + 2x_2 + 4x_3 \le 180$

$x_1, x_2, x_3 \ge 0$

2. After solving the problem, the final simplex tableau (for the standard form) is given as below (the variables are in the natural order as in the description of the problem):

B	0	50	0	0	140	80	35400
4	0	0.5	0	1	-0.7	0.1	153
1	1	0	0	0	0.4	-0.2	24
3	0	0.5	1	0	-0.1	0.3	39

Show the dual variables corresponding to the services of the three departments. Using complementarity conditions to explain why mortgage insurance is not sold.

Dual: minimize $240y_1 + 150y_2 + 180y_3$

s.t. $2y_1 + 3y_2 + y_3 \ge 500$

$y_1 + y_2 + 2y_3 \ge 250$

$y_1 + y_2 + 4y_3 \ge 600$

$y_1, y_2, y_3 \ge 0$

From the final tableau, we know optimal solution $(0, 140, 80)$

$x_2(y_1 + y_2 + 2y_3 - 250) = 50x_2 = 0$

$\Rightarrow x_2 = 0$

Hence, since $x_2 = 0$, the mortgage insurance is not sold.

3. Find the range of working hours available for underwriting to keep the current basis optimal.

3. $x^* + \lambda A_B^{-1} e_i \geq 0$

$$\begin{bmatrix} 24 \\ 39 \\ 153 \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0.4 & -0.2 \\ 0 & -0.1 & 0.3 \\ 1 & -0.7 & 0.1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ 39 \\ 153 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda \end{bmatrix} \geq 0$$

$\lambda \in [-153, +\infty)$

Hence, the range of working hours available for underwriting $= [87, +\infty)$

4. Find the range of the expected profit on long-term care insurance such that the current basis remains optimal.

4. $3 \in B$

$c_N^T - c_B^T A_B^{-1} A_N - \lambda e_j^T A_B^{-1} A_N \geq 0$

$$\begin{bmatrix} 50 \\ 140 \\ 80 \end{bmatrix} - \lambda \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 & 0.4 & -0.2 \\ 0 & -0.1 & 0.3 \\ 1 & -0.7 & 0.1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 50 \\ 140 \\ 80 \end{bmatrix} - \lambda \begin{bmatrix} 0.5 \\ -0.1 \\ 0.3 \end{bmatrix} \geq 0$$

$\lambda \in [-1400, 100]$

Hence, the range of the expected profit on long-term care insurance $= [-800, 700]$