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Problem 1 (5+5+10=20pts).
Consider the following linear program:

$$
\begin{array}{ll}
\text{maximize} & x_1 + 3x_2 + 3x_3 + 8x_4 \\
\text{subject to} & x_1 - x_2 + x_3 \quad \le 2 \\
& x_3 - x_4 \quad \le 1 \\
& 2x_2 + 3x_3 + 4x_4 \quad \le 10 \\
& x_1, x_2, x_3, x_4 \quad \ge 0.
\end{array}
$$

(a) Transform this problem to standard form and find a trivial BFS.

(b) Compute the reduced costs $\bar{\mathbf{c}}$.

(c) Choose the incoming basis according to **Bland's Rule**, compute the step size θ^* and and jth basic direction, then apply the simplex method to update the current BFS to a neighboring one.

(a) standard form:

minimize $-X_1 - 3X_2 - 3X_3 - 8X_4$

subject to $X_1 - X_2 + X_3 + S_1 = 2$

$X_3 - X_4 + S_2 = 1$

$2X_2 + 3X_3 + 4X_4 + S_3 = 10$

$X_1, X_2, X_3, X_4, S_1, S_2, S_3 \geq 0$

It is easy to see, a trivial BFS is $X = [0,0,0,0,2,1,10]^T$

(b) $C = [-1,-3,-3,-8]^T$ $\quad A = \begin{bmatrix} 1 & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 2 \\ 1 \\ 10 \end{bmatrix}$

Trivial BFS: $X = [0,0,0,0,2,1,10]^T$. $\quad B = \{5,6,7\}$.

$\Rightarrow C_B^T = [0,0,0]$, $\quad C_B^T A_B^{-1} A_j = 0$

$\Rightarrow \bar{C}_1 = C_1 = -1$; $\quad \bar{C}_2 = C_2 = -3$; $\quad \bar{C}_3 = C_3 = -3$; $\quad \bar{C}_4 = C_4 = -8$. $\quad \bar{C} = [-1,-3,-3,-8,0,0,0]$

(c) According to Bland's rule, choose $j=1$.

$-A_B^{-1} A_1 = -\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$ $\quad \Rightarrow d = [-1,0,0,1,0,0,0]^T$

$\theta^* = \min_{i \in B, d_i < 0} \{-\frac{2}{-1}\} = 2$

5 6 7 1 2 3 4

$y = \underset{\substack{\Downarrow \\ [X_B, X_N]^T}}{X} + \theta^* d = [2,1,10,0,0,0,0]^T + 2[-1,0,0,1,0,0,0]^T = [0,1,10,2,0,0,0]^T$

Hence, $\theta^* = 2$, j-th basic direction $= [-1,0,0,1,0,0,0]$, new BFS $= [2,0,0,0,0,1,10]$

Problem 2 (20pts). Consider the same linear program in **Problem 1**, please use simplex tableau to completely solve it. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding objective function value. You can start from the same trivial BFS found in **Problem 1**. Also, compare the BFS obtained in **Problem 1** and the second tableau obtained in this question. They should be consistent.

B	-1	-3	-3	-8	0	0	0	0
5	1	-1	1	0	1	0	0	2
6	0	0	1	-1	0	1	0	1
7	0	2	3	4	0	0	1	10

current basis: $B = \{5, 6, 7\}$

current basic solution: $X = [0, 0, 0, 0, 2, 1, 10]^T$

Objective function value: 0

B	0	-4	-2	-8	1	0	0	2
1	1	-1	1	0	1	0	0	2
6	0	0	1	-1	0	1	0	1
7	0	2	3	4	0	0	1	10

current basis: $B = \{1, 6, 7\}$

current basic solution: $X = [2, 0, 0, 0, 0, 1, 10]^T$

Objective function value: -2

(consistent with Problem 1)

B	0	0	4	0	1	0	2	22
1	1	0	$\frac{5}{2}$	2	1	0	$\frac{1}{2}$	7
6	0	0	1	-1	0	1	0	1
2	0	1	$\frac{3}{2}$	2	0	0	$\frac{1}{2}$	5

current basis: $B = \{1, 2, 6\}$

current basic solution: $X = [7, 5, 0, 0, 0, 1, 0]^T$

Objective function value: -22

Problem 3 (20pts).
Use the two-phase simplex method (implemented by simplex tableau) to completely solve the linear optimization problem. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding (negative) objective function value.

$$
\begin{array}{lll}
\text{minimize} & x_1 + 3x_2 + x_4 - 2x_5 & \\
\text{subject to} & x_1 + 2x_2 + x_4 + x_5 & = 2 \\
 & x_1 + 2x_2 - 6x_4 + x_5 & = 2 \\
 & x_1 + 4x_2 - 3x_3 + x_4 & = -1 \\
 & x_1, x_2, x_3, x_4, x_5 & \geq 0.
\end{array}
$$

Auxiliary problem:

$$
\begin{array}{ll}
\text{minimize} & x_6 + x_7 + x_8 \\
\text{subject to} & x_1 + 2x_2 + x_4 + x_5 + x_6 = 2 \\
 & x_1 + 2x_2 - 6x_4 + x_5 + x_7 = 2 \\
 & -x_1 - 4x_2 + 3x_3 - x_4 + x_8 = 1 \\
 & x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8 \geq 0
\end{array}
$$

Trivial BFS: $[0, 0, 0, 0, 0, 2, 2, 1]$, $B = \{6, 7, 8\}$.

B	-1	0	-3	6	-2	-1	-1	-1	-5
6	1	2	0	1	1	1	0	0	2
7	1	2	0	-6	1	0	1	0	2
8	-1	-4	3	-1	0	0	0	1	1

current basis: $B = \{6, 7, 8\}$

current basic solution: $X = [0, 0, 0, 0, 0, 2, 2, 1]^T$

B	0	2	-3	7	-1	1	0	0	-3
1	1	2	0	1	1	1	0	0	2
7	0	0	0	-7	0	-1	1	0	0
8	0	-2	3	0	1	1	0	1	3

current basis = $B = \{1, 7, 8\}$

current basic solution = $x = [2,0,0,0,0,0,0,0,3]^T$

B	0	0	0	7	0	2	0	1	0
1	1	2	0	1	1	1	0	0	2
7	0	0	0	-7	0	-1	1	0	0
3	0	$-\frac{2}{3}$	1	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	1

→ degenerate

current basis = $B = \{1, 7, 3\}$

current basic solution = $x = [2,0,1,0,0,0,0,0,0]^T$

B	0	0	0	0	0	1	1	1	0
1	1	2	0	0	1	$\frac{6}{7}$	$\frac{1}{7}$	0	2
4	0	0	0	1	0	$\frac{1}{7}$	$-\frac{1}{7}$	0	0
3	0	$-\frac{2}{3}$	1	0	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	1

current basis = $B = \{1, 4, 3\}$

current basic solution = $x = [2,0,1,0,0,0,0,0]^T$

original problem:

B	0	1	0	0	-3	-2
1	1	2	0	0	1	2
4	0	0	0	1	0	0
3	0	$-\frac{2}{3}$	1	0	$\frac{1}{3}$	1

current basis = $B = \{1, 4, 3\}$

current basic solution = $x = [2,0,1,0,0,0,0,0]^T$

B	3	7	0	0	0	4
5	1	2	0	0	1	2
4	0	0	0	1	0	0
3	$-\frac{1}{3}$	$-\frac{4}{3}$	1	0	0	$\frac{1}{3}$

current basis = $B = \{3, 4, 5\}$

current basic solution = $x = [0,0,\frac{1}{3},0,2,0,0,0]^T$

Hence, the corresponding negative objective function value is -4.

Problem 4 (20pts).
Apply the two-phase simplex method (implemented by simplex tableau) to solve the following linear program. For each step, draw the simplex tableau. Clearly mark what is the current basis, the current basic solution, and the corresponding (negative) objective function value.

$$
\begin{array}{lll}
\text{minimize} & x_1 - x_2 + 3x_3 & \\
\text{subject to} & 2x_1 - x_2 + 4x_3 & \le -1 \\
& x_1 - x_2 - x_3 & \le 4 \\
& x_2 - x_4 & = 0 \\
& x_1, x_2, x_3, x_4 & \ge 0.
\end{array}
$$

standard form:

minimize $x_1 - x_2 + 3x_3$

subject to $2x_1 - x_2 + 4x_3 + x_5 = -1$ $(-2x_1 + x_2 - 4x_3 - x_5 = 1)$

$x_1 - x_2 - x_3 + x_6 = 4$

$x_2 - x_4 = 0$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

Auxiliary problem:

minimize $x_7 + x_8 + x_9$

subject to $-2x_1 + x_2 - 4x_3 - x_5 + x_7 = 1$

$x_1 - x_2 - x_3 + x_6 + x_8 = 4$

$x_2 - x_4 + x_9 = 0$

$x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$

Trivial BFS: $x = [0,0,0,0,0,0,1,4,0]^T$, $B = \{7, 8, 9\}$.

B										
B	1	-1	5	1	1	-1	0	0	0	-5
7	-2	1	-4	0	-1	0	1	0	0	1
8	1	-1	-1	0	0	1	0	1	0	4
9	0	1	0	-1	0	0	0	0	1	0

current basis: $B = \{7, 8, 9\}$

current basic solution: $x = [0,0,0,0,0,0,1,4,0]^T$

B	1	0	5	0	1	-1	0	0	1	-5
7	-2	0	-4	1	-1	0	1	0	-1	1
8	1	0	-1	-1	0	1	0	1	1	4
2	0	1	0	-1	0	0	0	0	1	0

current basis: $B = \{2, 7, 8\}$

current basic solution: $x = [0,0,0,0,0,0,1,4,0]^T$

B	2	0	4	-1	1	0	0	1	2	-1
7	-2	0	-4	1	-1	0	1	0	-1	1
6	1	0	-1	-1	0	1	0	1	1	4
2	0	1	0	-1	0	0	0	0	1	0

current basis: $B = \{2, 6, 7\}$

current basic solution: $x = [0,0,0,0,0,4,1,0,0]^T$

B	0	0	0	0	0	0	1	1	1	0
4	-2	0	-4	1	-1	0	1	0	-1	1
6	-1	0	-5	0	-1	1	1	1	0	5
2	-2	1	-4	0	-1	0	1	0	0	1

current basis: $B = \{2, 4, 6\}$

current basic solution: $x = [0,1,0,1,0,5,0,0,0]^T$

Original problem:

B	-1	0	-1	0	-1	0	-1
4	-2	0	-4	1	-1	0	1
6	-1	0	-5	0	-1	1	5
2	-2	1	-4	0	-1	0	1

Since all the elements in the middle part of the tableau is ≤ 0, the optimal value is unbounded

Problem 5 Conditions for a Unique Optimum (10+10=20pts).
Let x^* be a basic feasible solution associated with some basic indices B. Prove the following:

a) If the reduced cost of every non-basic variable is positive, then x^* is the unique optimal solution.

   Hint: Let $y \in \{x \in \mathbb{R}^n : Ax = b,\ x \geq 0\}$ be given with $y \neq x^*$. First, show that there exists $\ell \in N = B^{\mathsf{C}}$ such that $y_\ell > 0$. You can then mimic the proof of the theorem of stopping criterion based on the reduced costs.

b) If x^* is the unique optimal solution and if x^* is nondegenerate, then the reduced cost of every non-basic variable is positive.

   Hint: The construction of the simplex method via basic directions can be helpful.

(a) We assume that $\bar{c} \geq 0$, we let $y \in \{x \in \mathbb{R}^n : Ax = b, x \geq 0\}$ be given with $y \neq x^*$.

We define $d = y - x$.

$Ax = Ay = b \Rightarrow Ad = 0$

$\Rightarrow A_B \cdot d_B + \sum_{i \in N} A_i d_i = 0 \quad (N = B^C)$

$\Rightarrow d_B = -\sum_{i \in N} A_B^{-1} A_i d_i$

$c^T d = c_B^T d_B + \sum_{i \in N} c^T d_i = \sum_{i \in N} (c_i - c_B^T A_B^{-1} A_i) d_i = \sum_{i \in N} \bar{c}_i d_i$

$\forall i \in N$, $x_i = 0$. Since y is feasible, $y_i \geq 0$.

$\Rightarrow d_i \geq 0$, $\bar{c}_i d_i \geq 0$, $\forall i \in N$.

We conclude that $c^T(y - x) = c^T d \geq 0$, and since y is an arbitrary feasible solution, x^* is unique optimal solution.

(b). Suppose x^* is a nondegenerate basic feasible solution and that $\bar{c}_j < 0$ for some j.

Since the reduced cost of a basic variable is always zero, x_j must be a nonbasic variable.

Since x^* is nondegenerate, the j-th basic direction is a feasible direction of cost decrease. By moving in that direction, we obtain feasible solutions whose cost is less than of x^*, and x^* is not optimal.

Hence, if x^* is the unique optimal solution and x^* is nondegenerated, then the reduced cost of every non-basic variable is positive. $\square$