STA4001 HW3

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#1 **Exercise** 4.35 Consider a Markov chain with states 0, 1, 2, 3, 4. Suppose $p_{0,4} = 1$; and suppose that when the chain is in state $i, i > 0$, the next state is equally likely to be any of the states $0, 1, \cdots, i-1$. Find the limiting probabilities of this Markov chain.

Transition matrix P:

$$
\begin{array}{c|ccccc}
 & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 \\
2 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
3 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\
4 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0
\end{array}
$$

set $\pi = (\pi_0, \pi_1, \pi_2, \pi_3, \pi_4)$,

$$
\begin{cases} \pi = \pi P \\ \pi 1 = 1 \end{cases}
\Rightarrow
\begin{cases} \pi_0 = \frac{12}{37} \\ \pi_1 = \frac{6}{37} \\ \pi_2 = \frac{4}{37} \\ \pi_3 = \frac{3}{37} \\ \pi_4 = \frac{12}{37} \end{cases}
$$

#2 **Exercise** 4.50 A Markov chain with states $1, \cdots, 6$ has transition probability matrix

$$
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 \\
\hline
1 & .2 & .4 & 0 & .3 & 0 & .1 \\
2 & .1 & .3 & 0 & .4 & 0 & .2 \\
3 & 0 & 0 & .3 & .7 & 0 & 0 \\
4 & 0 & 0 & .6 & .4 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & .5 & .5 \\
6 & 0 & 0 & 0 & 0 & .2 & .8
\end{array}
$$

(a) Give the classes and tell which are recurrent and which are transient.

(b) Find $\lim_{n\to\infty} P^n_{1,2}$.

(c) Find $\lim_{n\to\infty} P^n_{5,6}$.

(d) Find $\lim_{n\to\infty} P^n_{1,3}$.

(a) 4 classes: $\underbrace{\{1,2\}}_{\text{transient}}, \underbrace{\{3,4\}}_{\text{recurrent}}, \underbrace{\{5,6\}}_{\text{recurrent}}$

(b) since $\{1,2\}$ is transient, $\lim_{n\to\infty} P^n_{1,2} = 0$

(c) $P = \begin{pmatrix} 0.5 & 0.5 \\ 0.2 & 0.8 \end{pmatrix}$

set $\pi = (\pi_5, \pi_6)$:

$$
\begin{cases} \pi = \pi P \\ \pi 1 = 1 \end{cases}
\Rightarrow
\begin{cases} \pi_5 = \frac{2}{7} \\ \pi_6 = \frac{5}{7} \end{cases}
$$

$\Rightarrow \lim_{n\to\infty} P^n_{5,6} = \frac{5}{7}$

(d). $\begin{cases} f_{13} = 0.2 f_{13} + 0.4 f_{23} + 0.3 \\ f_{23} = 0.3 f_{23} + 0.1 f_{13} + 0.4 \end{cases} \Rightarrow f_{13} = \frac{37}{52}$

set $\pi = (\pi_3, \pi_4)$:

$$
\begin{cases} [\pi_3, \pi_4] = [\pi_3, \pi_4] \begin{bmatrix} 0.3 & 0.7 \\ 0.6 & 0.4 \end{bmatrix} \\ \pi_1 + \pi_4 = 1 \end{cases}
\Rightarrow \pi_3 = \frac{6}{13}
$$

$\Rightarrow \lim_{n\to\infty} P^n_{1,3} = f_{1,3} \cdot \pi_3 = \frac{37}{52} \cdot \frac{6}{13} = \frac{111}{338}$

#3 **Exercise** 4.54 Consider the Ehrenfest urn model in which M molecules are distributed between two urns, and at each time point one of the molecules is chosen at random and is then removed from its urn and placed in the other one. Let X_n denote the number of molecules in urn 1 after the nth switch and let $\mu_n = E[X_n]$. Show that

(a) $\mu_{n+1} = 1 + (1 - 2/M)\mu_n$.

(b) Use **(a)** to prove that

$$\mu_n = \frac{M}{2} + \left(\frac{M-2}{M}\right)^n \left(E[X_0] - \frac{M}{2}\right)$$

(a) Assume there are k molecules in urn I at initial state.

with probability $\frac{k}{M}$, $X_{n+1} = X_n - 1$; with probability $\frac{M-k}{M}$, $X_{n+1} = X_n + 1$.

$E[X_{n+1} \mid X_n = k] = \frac{k}{M}(k-1) + \frac{M-k}{M}(k+1) = k + \frac{M-k}{M} - \frac{k}{M} = k(1-\frac{2}{M}) + 1$

since $E[X_n] = \mu_n$, we get:

$\mu_{n+1} = 1 + (1-\frac{2}{M})\mu_n$

(b) $\mu_n = 1 + (1-\frac{2}{M})\mu_{n-1}$

$= 1 + (1-\frac{2}{M})[1 + (1-\frac{2}{M})\mu_{n-2}]$

$= 1 + (1-\frac{2}{M}) + (1-\frac{2}{M})^2 \mu_{n-2}$

$= 1 + (1-\frac{2}{M}) + (1-\frac{2}{M})^2 [1 + (1-\frac{2}{M})\mu_{n-3}]$

$= 1 + (1-\frac{2}{M}) + (1-\frac{2}{M})^2 + (1-\frac{2}{M})^3 \mu_{n-3}$

$= \cdots$

$= 1 + (1-\frac{2}{M}) + \cdots + (1-\frac{2}{M})^{n-1} + (1-\frac{2}{M})^n \mu_0$

$= \dfrac{1-(1-\frac{2}{M})^n}{\frac{2}{M}} + (1-\frac{2}{M})^n \mu_0$

$= \frac{M}{2} - \frac{M}{2}(1-\frac{2}{M})^n + (1-\frac{2}{M})^n \mu_0$

$= \frac{M}{2} + (\frac{M-2}{M})^n (\mu_0 - \frac{M}{2})$

$\Rightarrow \mu_n = \frac{M}{2} + (\frac{M-2}{M})^n (\mu_0 - \frac{M}{2})$

since $\mu_0 = E[X_0]$, we get

$\mu_n = \frac{M}{2} + (\frac{M-2}{M})^n (E[X_0] - \frac{M}{2})$

#4 **Exercise** 4.59 For the gambler's ruin model of Section 4.5.1, let M_i denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of N, given that he starts with $i, i = 0,1,\cdots,N$. Show that M_i satisfies

$$M_0 = M_N = 0;\ M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1,\cdots,N-1$$

Solve these equations to obtain

$$M_i = i(N-i), \qquad if\ p = \frac{1}{2}$$
$$= \frac{i}{q-p} - \frac{N}{q-p}\frac{1-(q/p)^i}{1-(q/p)^N}, if\ p \neq \frac{1}{2}$$

① when $i=0$ or N, no game need to play: $M_0 = M_N = 0$

For $i = 1, \cdots, N-1$:

$M_i = p(1+M_{i+1}) + q(1+M_{i-1}) = (p+q) + pM_{i+1} + qM_{i-1} = 1 + pM_{i+1} + qM_{i-1}$

② $M_i = 1 + pM_{i+1} + qM_{i-1}$

$\Leftrightarrow (p+q)M_i = 1 + pM_{i+1} + qM_{i-1}$

$\Leftrightarrow M_i - M_{i-1} = \frac{q}{p}(M_{i-1} - M_{i-2}) - \frac{1}{p}$

$= \frac{q}{p}[\frac{q}{p}(M_{i-2} - M_{i-3}) - \frac{1}{p}] - \frac{1}{p}$

$= (\frac{q}{p})^2 (M_{i-2} - M_{i-3}) - \frac{q}{p}\cdot\frac{1}{p} - \frac{1}{p}$

$= (\frac{q}{p})^2 [\frac{q}{p}(M_{i-3} - M_{i-4}) - \frac{1}{p}] - \frac{q}{p}\cdot\frac{1}{p} - \frac{1}{p}$

$= (\frac{q}{p})^3 (M_{i-3} - M_{i-4}) - (\frac{q}{p})^2\cdot\frac{1}{p} - \frac{q}{p}\cdot\frac{1}{p} - \frac{1}{p}$

$= \cdots$

$= (\frac{q}{p})^{i-1}(M_1 - M_0) - [\frac{1}{p} + \frac{1}{p}\cdot\frac{q}{p} + \cdots + \frac{1}{p}\cdot(\frac{q}{p})^{i-2}]$

$= (\frac{q}{p})^{i-1} M_1 - \dfrac{1-(\frac{q}{p})^{i-1}}{p-q}$ → if $p=q$, then $(i-1)\frac{1}{p}$

$M_i - M_0 = (M_i - \cancel{M_{i-1}}) + (\cancel{M_{i-1}} - \cancel{M_{i-2}}) + \cdots + (\cancel{M_1} - M_0) = M_1[1 + \frac{q}{p} + \cdots + (\frac{q}{p})^{i-1}] - \frac{i}{p-q} + \dfrac{[1 + \frac{q}{p} + \cdots + (\frac{q}{p})^{i-1}]}{p-q}$

$\Rightarrow M_i = \begin{cases} M_1[\dfrac{p(1-\frac{q}{p}^i)}{p-q}] - \dfrac{i}{p-q} + \dfrac{p(1-\frac{q}{p}^i)}{(p-q)^2}, & p \neq q \\ i[M_1 - (i-1)], & p = q = \frac{1}{2} \end{cases}$

use $M_N = 0$, set $i = N$:

$$M_1 = \begin{cases} \dfrac{N}{p(1-\frac{q}{p}^N)} - \dfrac{1}{p-q}, & p \neq q \\ N-1, & p = q = \frac{1}{2} \end{cases}$$

$$\Rightarrow M_i = \begin{cases} \dfrac{i}{q-p} - \dfrac{N}{q-p}\,\dfrac{1-(\frac{q}{p})^i}{1-(q/p)^N}, & p \neq q \neq \frac{1}{2} \\ i(N-i), & p = q = \frac{1}{2} \end{cases}$$

#5 **Exercise** 4.63 For the Markov chain with states 1, 2, 3, 4 whose transition probability matrix $\boldsymbol{P}$ is as specified below find f_{i3} and s_{i3} for $i = 1,2,3$.

$$\begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \boldsymbol{P} = \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0.4 & 0.2 & 0.1 & 0.3 \\ 0.1 & 0.5 & 0.2 & 0.2 \\ 0.3 & 0.4 & 0.2 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$\left.\begin{aligned} f_{13} &= 0.1 + 0.2 f_{23} + 0.4 f_{13} \\ f_{23} &= 0.2 + 0.5 f_{23} + 0.1 f_{13} \\ f_{33} &= 0.2 + 0.4 f_{23} + 0.3 f_{13} \end{aligned}\right\} \Rightarrow \begin{cases} f_{13} = \frac{9}{28} \\ f_{23} = \frac{13}{28} \\ f_{33} = \frac{27}{56} \end{cases}$$

$$\Rightarrow f_{13} = \frac{9}{28}, \; f_{23} = \frac{13}{28}, \; f_{33} = \frac{27}{56}$$

$$S = (I-P)^{-1} = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.4 & 0.2 \end{pmatrix}\right]^{-1} = \begin{bmatrix} 2.207 & 1.379 & 0.621 \\ 0.966 & 3.103 & 0.897 \\ 1.310 & 2.069 & 1.931 \end{bmatrix}$$

$\Rightarrow S_{13} = 0.621$, $S_{23} = 0.897$, $S_{33} = 1.931$

#6 **Exercise** 4.65 In a branching process having $X_0 = 1$ and $\mu > 1$, prove that π_0 is the *smallest* positive number satisfying Eq. (4.20). $\pi_0 = \sum_{j=0}^{\infty} \pi_0^j P_j$

Hint: Let π be any solution of $\pi = \sum_{j=0}^{\infty} \pi^j P_j$. Show by mathematical induction that $\pi \geq P\{X_n = 0\}$ for all n, and let $n \to \infty$. In using the induction argue that

$$P\{X_n = 0\} = \sum_{j=0}^{\infty} (P\{X_{n-1} = 0\})^j P_j$$

Proof: ① Base Case: $n = 0$

$P(X_0 = 0)$: probability that no individuals in the initial generation.

$\because X_0 = 1$

$\therefore \pi \geq P\{X_0 = 0\}$

② $n > 0$

Assume $\pi \geq P\{X_n = 0\}$ holds for some $n = k$.

we want to show it also holds for $n = k+1$, that $\pi \geq P\{X_{k+1} = 0\}$.

$P\{X_{k+1} = 0\} = \sum_{j=0}^{\infty} P(X_k = j)\, P(X_{k+1} = 0 \mid X_k = j)$

$\because P(X_{k+1} = 0 \mid X_k = j) = \pi^j$

$\therefore P\{X_{k+1} = 0\} = \sum_{j=0}^{\infty} P(X_k = j) \cdot \pi^j$

$\because \pi = \sum_{j=0}^{\infty} \pi^j P_j \geq P\{X_n = 0\}$

$\therefore \pi = \sum_{j=0}^{\infty} \pi^j P_j \geq \sum_{j=0}^{\infty} \pi^j \cdot P(X_k = j) = P\{X_{k+1} = 0\}$

③ $n \to \infty$

when $n \to \infty$, $P(X_n = 0)$ approaches the probability of extinction, π_0.

$\because \pi \geq P\{X_n = 0\}$

$\therefore n \to \infty$, $\pi \geq \pi_0$

Given $\mu > 1$, $P(\text{extinction}) \in (0,1)$.

$\therefore \pi_{min} = \sum_{j=0}^{\infty} \pi_0^j P_j$

$\therefore \pi_0$ is the smallest positive number of $\pi = \sum_{j=0}^{\infty} \pi^j P_j$

#7 **Exercise** 4.66 For a branching process, calculate π_0 when

(a) $P_0 = \frac{1}{4}, P_2 = \frac{3}{4}$.

(b) $P_0 = \frac{1}{4}, P_1 = \frac{1}{2}, P_2 = \frac{1}{4}$.

(c) $P_0 = \frac{1}{6}, P_1 = \frac{1}{2}, P_2 = \frac{1}{3}$.

(a) $\mu = 0 \times \frac{1}{4} + 2 \times \frac{3}{4} = \frac{3}{2} > 1$

$\Rightarrow \pi_0 = \frac{1}{4} + \frac{3}{4}\pi_0^2 \Rightarrow$ root: $\pi_0^1 = \frac{1}{3}$ $\pi_0^2 = 1$

$\Rightarrow \pi_0 = \frac{1}{3}$

(b) $\mu = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4} = 1$

$\Rightarrow \pi_0 = 1$

(c) $\mu = 0 \times \frac{1}{6} + 1 \times \frac{1}{2} + 2 \times \frac{1}{3} = \frac{7}{6} > 1$

$\Rightarrow \pi_0 = \frac{1}{6} + \frac{1}{2}\pi_0 + \frac{1}{3}\pi_0^2 \Rightarrow$ root: $\pi_0^1 = 1, \pi_0^2 = \frac{1}{2}$

$\Rightarrow \pi_0 = \frac{1}{2}$