STA4001 Homework 5

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#1 **Exercise 1.** Suppose $N_1(t), \dots, N_k(t)$ are independent Poisson processes with corresponding rates $\lambda_1, \dots, \lambda_k$.

(a) Use **Definition 5.2** to prove $N_1(t)+\dots+N_k(t)$ is a Poisson process with rate $\lambda_1+\dots+\lambda_k$.

Definition 5.2 The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if the following axioms hold:

1. $N(0) = 0$.
2. $\{N(t), t \geq 0\}$ has independent increments.
3. $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda(h) + o(h)$.
4. $\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h)$.

Assume $N(t) = N_1(t) + \dots + N_k(t)$.

① $N(0) = N_1(0) + \dots + N_k(0) = 0$

② Since $N_1(t), \dots, N_k(t)$ have independent increments, and $N_1(t), \dots, N_k(t)$ are independent,

Hence $\{N(t), t \geq 0\}$ has independent increments.

③ $\forall\, t, h > 0$,

$P(N(t+h) - N(t) = 1) = P([N_1(t+h) - N_1(t)] + [N_2(t+h) - N_2(t)] + \dots + [N_k(t+h) - N_k(t)] = 1)$

$= [\lambda_1 h + O(h)] + [\lambda_2 h + O(h)] + \dots + [\lambda_k h + O(h)]$

$= (\lambda_1 + \lambda_2 + \dots + \lambda_k) h + O(h)$

Hence, $P(N(t+h) - N(t) = 1) = h \sum_{i=1}^{k} \lambda_i + O(h)$

④ $P(N(t+h) - N(t) \geq 2) = P([N_1(t+h) - N_1(t)] + [N_2(t+h) - N_2(t)] + \dots + [N_k(t+h) - N_k(t)] \geq 2)$

$= \underbrace{O(h) + \dots + O(h)}_{k}$

$= O(h)$

Hence, $P(N(t+h) - N(t) \geq 2) = O(h)$

Since it satisfies 4 axioms, $N_1(t) + \dots + N_k(t)$ is a poisson process with rate $\lambda_1 + \dots + \lambda_k$

(b) Use **Definition 5.2'** to prove $N_1(t)+\dots+N_k(t)$ is a Poisson process with rate $\lambda_1+\dots+\lambda_k$.

Definition 5.2' A Poisson process $N(t)$ with rate λ is a stationary process with independent increments and $N(0) = 0$, $N(t) \sim \text{Poisson}(\lambda t)$.

Again assume $N(t) = N_1(t) + \dots + N_k(t)$

① $N(0) = N_1(0) + \dots + N_k(0) = 0$

② $N_i(t) \sim Poi(\lambda_i t)$, $\forall i \in \{1, \dots, k\}$ ⇒ $N_i(t)$ is a stationary process, $\forall i \in \{1, \dots, k\}$ ⇒ $N(t)$ is a stationary process.

③ $\forall$ finite pairs of time points $s_i < t_i$,

Since $N_1(t), \dots, N_k(t)$ are independent, $N(t_1) - N(s_1), N(t_2) - N(s_2), \dots, N(t_n) - N(s_n)$ are independent of each other.

Therefore, $N(t)$ has independent increments.

⇒ $N(t) = N_1(t) + \dots + N_k(t) \sim Poi(t \sum_{i=1}^{k} \lambda_i)$

Hence, $N_1(t) + \dots + N_k(t)$ is a poisson process with rate $\lambda_1 + \dots + \lambda_k$.

#2 **Exercise 2.** Use **Definition 5.2** and the conclusions from the class to prove that **Definition 5.2** is equivalent to **Definition 5.2'**.

Definition 5.2 The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process with rate $\lambda > 0$ if the following axioms hold:

1. $N(0) = 0$.
2. $\{N(t), t \geq 0\}$ has independent increments.
3. $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda(h) + o(h)$.
4. $\mathbb{P}(N(t+h) - N(t) \geq 2) = o(h)$.

Definition 5.2' A Poisson process $N(t)$ with rate λ is a stationary process with independent increments and $N(0) = 0$, $N(t) \sim \text{Poisson}(\lambda t)$.

① From Definition 5.2 ⟶ Definition 5.2'

1°. From 5.2.2. $\{N(t), t \geq 0\}$ has independent increments, we can get condition $N(t)$ has independent increments in 5.2'.

2°. From 5.2.1 $N(0) = 0$ we get condition $N(0) = 0$ in 5.2'

3°. From 5.2.3 $P(N(t+h) - N(t) = 1) = \lambda(h) + O(h)$, 5.2.4 $P(N(t+h) - N(t) \geq 2) = O(h)$.

We get $P(N(t+h)-N(t)=0) = 1-P(N(t+h)-N(t)=1) - P(N(t+h)-N(t)\geq 2) = 1-\lambda(h)+O(h)$

With the independent increments condition, assume $t=nh$.

While $n\to\infty$, $h\to 0$, $N(t) \sim \text{Binomial}(n, \lambda(h)+O(h))$

$\Rightarrow N(t) \sim \text{Poisson}(\lambda t)$

Hence, we can get Definition 5.2' from Definition 5.2.

② From Definition 5.2' $\longrightarrow$ Definition 5.2.

1°. $N(0)=0$ is given in Definition 5.2'

2°. $\{N(t), t\geq 0\}$ has independent increments is given in Definition 5.2'.

3°. $\forall$ $N(t) \sim Poi(\lambda t)$

$\therefore P(N(t+h)-N(t)=1) = \dfrac{(\lambda h)e^{-\lambda h}}{1!}$

Taylor expansion $\Rightarrow$ $e^{-\lambda h} = 1-\lambda h + O(h^2)$

$\Rightarrow P(N(t+h)-N(t)=1) = (\lambda h)[1-\lambda h+O(h^2)] = \lambda h-(\lambda h)^2+O(h^2)$

As $h\to 0$, $P(N(t+h)-N(t)=1) = \lambda h+O(h)$

4°. $P(N(t+h)-N(t)\geq 2) = 1-P(N(t+h)-N(t)=1) - P(N(t+h)-N(t)=0)$

$= 1-(\lambda h)e^{-\lambda h} - e^{-\lambda h}$

$= 1-(1-\lambda h+O(h^2))\cdot(1+\lambda h)$

$= 1-[1-(\lambda h)^2+\lambda h\, O(h^2)]$

$= (\lambda h)^2+\lambda h\, O(h^2)$

$= O(h)$, as $h\to 0$

Hence, we can get Definition 5.2 from Definition 5.2'

Therefore, Definition 5.2 and Definition 5.2' are equivalent.

#3 **Exercise 3.** Consider two machines that are maintained by a single repairman. Machine i functions for an exponential time with rate μ_i before breaking down, $i=1,2$.. The repair times (for either machine) are exponential with rate μ. Can you analyze this as a birth and death process? If so, what are the parameters? If not, how can we analyze it? (Note that the repair man can only fix one machine at the same time.)

① This is not a birth and death process.

② Assume 5 states $\in \{0,1,2,3,4\}$.

0: both machines are working

1: Machine 1 is working, Machine 2 breaks down

2: Machine 1 breaks down, Machine 2 is working.

3. both break down and Machine 1 is repairing

4. both break down and Machine 2 is repairing

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{\mu_2}{\mu_1+\mu_2} & \frac{\mu_1}{\mu_1+\mu_2} & 0 & 0 \\ \frac{\mu}{\mu+\mu_1} & 0 & 0 & 0 & \frac{\mu_1}{\mu+\mu_1} \\ \frac{\mu}{\mu+\mu_2} & 0 & 0 & \frac{\mu_2}{\mu+\mu_2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

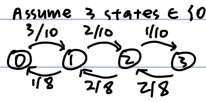
$V_0 = \mu_1+\mu_2$, $V_1 = \mu_1+\mu$, $V_2=\mu_2+\mu$, $V_3=\mu$, $V_4=\mu$.

$$\Rightarrow Q = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} -(\mu_1+\mu_2) & \mu_2 & \mu_1 & 0 & 0 \\ \mu & -(\mu_1+\mu) & 0 & 0 & \mu_1 \\ \mu & 0 & -(\mu_2+\mu) & \mu_2 & 0 \\ 0 & \mu & 0 & -\mu & 0 \\ 0 & 0 & \mu & 0 & -\mu \end{bmatrix} \end{array}$$

#4 **Exercise 4.** A job shop consists of three machines and two repairmen. The amount of time a machine works before breakdown is exponentially distributed with a mean 10. If the amount of time it takes a single repairment to fix a machine is exponentially distributed with a mean 8, then

1. what's the average number of machines not in use?
2. what proportion of time are both repairmen busy?

1. Assume 3 states $\in \{0,1,2,3\}$, each denote # machines are not in use.



$\lambda_{01} = \frac{3}{10}$, $\lambda_{12} = \frac{2}{10}$, $\lambda_{23} = \frac{1}{10}$

$\mu_{10} = \frac{1}{8}$, $\mu_{21} = \frac{2}{8}$, $\mu_{32} = \frac{2}{8}$

$$\begin{cases} \sum_{i=0}^{3} P_i = 1 \\ P_1 = \frac{\lambda_{01}}{\mu_{10}} P_0 \\ P_2 = \frac{\lambda_{12}}{\mu_{21}} P_1 \\ P_3 = \frac{\lambda_{23}}{\mu_{32}} P_2 \end{cases} \Rightarrow P_0 = \frac{250}{1522}$$

$\Rightarrow N = 0 \cdot P_0 + 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 = \frac{1068}{761}$

Hence, the average number of machines not in use is $\frac{1068}{761}$

2. $P_2 + P_3 = \frac{336}{761}$

$\Rightarrow$ Both repairmen are busy for $\frac{336}{761}$ of time.