STA 4001 Homework 2

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#1 **Exercise 4.1** Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state i, $i = 0, 1, 2, 3$ if the first urn contains i white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let X_n denote the state of the system after the n^{th} step. Explain why $\{X_n, n = 0, 1, 2, 3, \dots\}$ is a Markov chain and calculate its transition probability matrix.

① If the distribution of X_{n+1} depends only on X_n for all $n \geq 0$, we say the process X_n is a Markov chain.

In this context, # of white balls in the first urn at the next step depends only on the current distribution of balls and not on how the balls were arranged in previous steps, Hence, $\{X_n, n = 0, 1, 2, 3 \dots\}$ is a Markov chain.

② The state space of the system is finite, consisting of states $\{0, 1, 2, 3\}$.

$$P = \begin{array}{c c} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$P_{01} = 1$: state 0 (all black balls in 1st urn) → state 1 (1 white ball in 1st urn)

$P_{10} = \frac{1}{9}$: state 1 (1st urn = WBB, 2nd urn = BWW) → state 0 (1st urn = BBB, 2nd urn = WWW) = $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$P_{11} = \frac{4}{9}$: state 1 (1st urn = WBB, 2nd urn = BWW): $\frac{1}{3} \times \frac{2}{3} + \frac{2}{3} \times \frac{1}{3} = \frac{4}{9}$

$P_{12} = \frac{4}{9}$: state 1 (1st urn = WBB, 2nd urn = BWW) → state 2 (1st urn = WWB, 2nd urn = BBW) = $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$P_{21} = \frac{4}{9}$: state 2 (1st urn = WWB, 2nd urn = BBW) → state 1 (1st urn = WBB, 2nd urn = BWW) = $\frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$P_{22} = \frac{4}{9}$: state 2 (1st urn = WWB, 2nd urn = BBW): $\frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$

$P_{23} = \frac{1}{9}$: state 2 (1st urn = WWB, 2nd urn = BBW) → state 3 (1st urn = WWW, 2nd urn: BBB): $\frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

$P_{32} = 1$: state 3 (1st urn: WWW, 2nd urn = BBB) → state 2 (1st urn = WWB, 2nd urn = BBW)

Hence, the transition probability matrix is:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} & 0 \\ 0 & \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

#2 **Exercise 4.5** A Markov chain $\{X_n, n \geq 0\}$ with states $\{0, 1, 2\}$ has transition probability matrix:

$$\begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \end{array} \tag{1}$$

If $P(X_0 = 0) = P(X_0 = 1) = \frac{1}{4}$, please find $E(X_3)$.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{5}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{1}{9} & \frac{5}{9} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\ 0 & \frac{1}{3} & \frac{2}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{5}{18} & \frac{7}{18} \\ \frac{1}{3} & \frac{1}{9} & \frac{5}{9} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{13}{36} & \frac{11}{54} & \frac{47}{108} \\ \frac{4}{9} & \frac{4}{27} & \frac{11}{27} \\ \frac{5}{12} & \frac{2}{9} & \frac{13}{36} \end{bmatrix}$$

$$\begin{aligned} E[X_3] &= \sum_{i=0}^{2} E[X_3 \mid X_0 = i] P(X_0 = i) \\ &= \sum_{i=0}^{2} P(X_0 = i) \sum_{j=0}^{2} j P(X_3 = j \mid X_0 = i) \\ &= \sum_{i=0}^{2} P(X_0 = i) \sum_{j=0}^{2} j (P^3)_{i,j} \\ &= \frac{1}{4} \times \left(\frac{11}{54} + 2 \times \frac{47}{108}\right) + \frac{1}{4} \times \left(\frac{4}{27} + 2 \times \frac{11}{27}\right) + \frac{1}{2} \times \left(\frac{2}{9} + 2 \times \frac{13}{36}\right) \\ &= \frac{29}{108} + \frac{13}{54} + \frac{17}{36} \\ &= \frac{53}{54} \end{aligned}$$

Hence, $E[X_3] = \frac{53}{54}$

#3 **Exercise 4.8** An urn initially contains 2 balls, one of which is read and the other blue. At each stage, a ball is randomly selected. If the selected ball is red, then it is replaced with a red ball with a probability of 0.7 or with a blue ball with a probability of 0.3; If the selected ball is blue, then it is equally likely to be replaced by either a red ball or a blue ball.

1. Let X_n equal to 1 if the n^{th} ball selected is red, and let it equal to 0 otherwise. Is $\{X_n, n \geq 1\}$ a Markov chain? If so, give its transition probability matrix.
2. Let Y_n denote the number of red balls in the urn immediately before the n_{th} ball is selected. Is $\{Y_n, n \geq 1\}$ a Markov chain? If so, give the transition probability matrix.
3. Find the probability that the second ball selected is red.
4. Find the probability that the fourth ball selected is red.

1. Not a Markov chain.

The selection of previous ball will affect the component of balls in the current urns,

then further affect the next selection.

2. Yes. $\{Y_n, n \geq 0\}$ is a Markov chain.

Y_n has states: 0, 1, 2

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{20} & \frac{3}{5} & \frac{1}{4} \\ 0 & \frac{3}{10} & \frac{7}{10} \end{bmatrix} \end{array}$$

3. $P(X_1 = \text{red}) = \frac{1}{2}$, $P(X_1 = \text{blue}) = \frac{1}{2}$

$P(\text{second selection is red})$

$= \sum_{i=0}^{2} P(\text{second selection is red} \mid X_2 = i) P(X_2 = i)$

$= 0 \cdot P_{10}^1 + 0.5 \cdot P_{11}^1 + 1 \cdot P_{12}^1$

$= \frac{1}{2} \times \frac{3}{5} + \frac{1}{4}$

$= 0.55$

4. $P(\text{fourth selection is red})$

$= \sum_{i=0}^{2} P(\text{fourth selection is red} \mid X_2 = i) P(X_4 = i)$

$= 0 \cdot P_{10}^3 + 0.5 \cdot P_{11}^3 + 1 \cdot P_{12}^3$

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{20} & \frac{3}{5} & \frac{1}{4} \\ 0 & \frac{3}{10} & \frac{7}{10} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{20} & \frac{3}{5} & \frac{1}{4} \\ 0 & \frac{3}{10} & \frac{7}{10} \end{bmatrix} = \begin{bmatrix} \frac{13}{40} & \frac{11}{20} & \frac{1}{8} \\ \frac{33}{200} & \frac{102}{200} & \frac{13}{40} \\ \frac{9}{200} & \frac{39}{100} & \frac{113}{200} \end{bmatrix}$$

$$P^3 = \begin{bmatrix} \frac{13}{40} & \frac{11}{20} & \frac{1}{8} \\ \frac{33}{200} & \frac{102}{200} & \frac{13}{40} \\ \frac{9}{200} & \frac{39}{100} & \frac{113}{200} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{3}{20} & \frac{3}{5} & \frac{1}{4} \\ 0 & \frac{3}{10} & \frac{7}{10} \end{bmatrix}$$

$P_{11}^3 = \frac{33}{200} \times \frac{1}{2} + \frac{102}{200} \times \frac{3}{5} + \frac{13}{40} \times \frac{3}{10} = \frac{243}{500}$

$P_{12}^3 = \frac{33}{200} \times 0 + \frac{102}{200} \times \frac{1}{4} + \frac{13}{40} \times \frac{7}{10} = \frac{71}{200}$

$\therefore P(\text{fourth selection is red}) = \frac{1}{2} \times \frac{243}{500} + \frac{71}{200} = 0.598$

#4 Let $\{X_n, n \geq 0\}$ be a Markov chain with states 0,1,2,3. Suppose this Markov chain is irreducible and the transition probability is $P_{i,j} > 0$, $i, j = 0, 1, 2, 3$. Let N be the number of transitions, starting from the state 0, until the pattern 1,2,3,1 appears. That is,

$$N = \min\{n \geq 4 : X_{n-3} = 1, X_{n-2} = 2, X_{n-1} = 3, X_n = 1\}.$$

Please define a new Markov chain to model this process and find the transition probability matrix in terms of $P_{i,j}$.

(**Hint:** Please refer to the Example 4.13 in the textbook and the Exercise 3 of Tutorial 3)

There are 4 steps to reach the pattern 1, 2, 3, 1.

Once we are in state 0, the procedure has to restart.

If $X_n = 1$, but $X_{n-1} \neq 3$, the procedure need to restart.

If $X_n = 3$, but $X_{n-1} \neq 2$, the procedure need to restart.

If $X_n = 2$, but $X_{n-1} \neq 1$, the procedure need to restart.

So in total, we need 7 states as follows.

① $Y_n=1$, if $X_n=1$ and $(X_{n-3}, X_{n-2}, X_{n-1}) \neq (1,2,3)$

② $Y_n=2$, if $X_n=2$, $X_{n-1}=1$

③ $Y_n=3$, if $X_{n-2}=1$, $X_{n-1}=2$ and $X_n=3$

④ $Y_n=4$, if pattern 1,2,3,1 has happened by time n.

⑤ $Y_n=5$, if $X_n=2$ and $X_{n-1}\neq 1$

⑥ $Y_n=6$, if $X_n=3$ and $(X_{n-2}, X_{n-1}) \neq (1,2)$

⑦ $Y_n=7$, if $X_n=0$

state 4 is the absorbing state. The transition probability matrix of $\{Y_n\}$ is

$$Q = \begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} & \begin{bmatrix} P_{1,1} & P_{1,2} & 0 & 0 & 0 & P_{1,3} & P_{1,0} \\ P_{2,1} & 0 & P_{2,3} & 0 & P_{2,2} & 0 & P_{2,0} \\ 0 & 0 & 0 & P_{3,1} & P_{3,2} & P_{3,3} & P_{3,0} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ P_{2,1} & 0 & 0 & 0 & P_{2,2} & P_{2,3} & P_{2,0} \\ P_{3,1} & 0 & 0 & 0 & P_{3,2} & P_{3,3} & P_{3,0} \\ P_{0,1} & 0 & 0 & 0 & P_{0,2} & P_{0,3} & P_{0,0} \end{bmatrix} \end{array}$$

#5 **Exercise 4.14** Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

1. $P_1 = \begin{array}{cc} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{array}$

2. $P_2 = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$

3. $P_3 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}$

4. $P_4 = \begin{array}{cc} & \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$

1. The chain has 1 classes: $\overset{\text{recurrent}}{\{0,1,2\}}$
2. The chain has 1 classes: $\overset{\text{recurrent}}{\{0,1,2,3\}}$
3. The chain has 3 classes: $\overset{\text{recurrent}}{\{0,2\}}$, $\overset{\text{transient}}{\{1\}}$, $\overset{\text{recurrent}}{\{3,4\}}$.
4. The chain has 4 classes: $\overset{\text{recurrent}}{\{0,1\}}$, $\overset{\text{recurrent}}{\{2\}}$, $\overset{\text{transient}}{\{3\}}$, $\overset{\text{transient}}{\{4\}}$.