STA4001 Homework 4

杨景岚 Jinglan Yang 121090699

#1 **(Chapter 5, Exercise 3)** Let X be an exponential random variable. Without any computations, tell which one of the following is correct. Explain your answer.

(a) $E[X^2|X>1] = E[(X+1)^2]$

(b) $E[X^2|X>1] = E[X^2]+1$

(c) $E[X^2|X>1] = (1+E[X])^2$

Assume $X \sim Exp(\lambda)$, $f(x) = \lambda\exp(-\lambda x)$, $x>0$

$P(X>x) = 1-F(X<x) = 1-\int_0^x \lambda\exp(-\lambda x)\,dx = \exp(-\lambda x)$

From the memoryless property of exponential distribution:

$\forall a>0,\ P(X>a+x \mid X>a) = \exp(-\lambda x)$

$\Rightarrow X-a \mid X>a \sim Exp(\lambda)$

$\Rightarrow X-a \mid X>a$ and X have the same distribution

$\Rightarrow X \mid X>a$ and $X+a$ have the same distribution

$\Rightarrow X^2 \mid X>1$ and $(X+1)^2$ have the same distribution

Therefore, (a) $E[X^2|X>1] = E[(X+1)^2]$ is correct.

#2 **(Chapter 5, Exercise 4)** Consider a post office with two clerks. Three people, A, B, and C, enter simultaneously. A and B go directly to the clerks, and C waits until either A or B leaves before he begins service. What is the probability that A is still in the post office after the other two have left when

(a) the service time for each clerk is exactly (nonrandom) ten minutes?

Let T_A, T_B, and T_C denote the service time of A, B, and C respectively.

$\because T_A = T_B = T_C = 10$ min

$\therefore P(T_A > T_B+T_C) = P(10>20) = 0$

Hence, the probability is 0.

(c) the service times are exponential with mean $1/\mu$?

$T_A, T_B, T_C \overset{i.i.d}{\sim} Exp(\mu)$

$P(T_A > T_B+T_C) = P(T_A > T_B+T_C \mid T_A > T_B)\cdot P(T_A>T_B)$

$= P(T_A > T_C)\cdot P(T_A > T_B)$

$= \frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

Hence, the probability is $\frac{1}{4}$

#3 **(Chapter 5, Exercise 50)** The number of hours between successive train arrivals at the station is uniformly distributed on $(0,1)$. Passengers arrive according to a Poisson process with rate 7 per hour. Suppose a train has just left the station. Let X denote the number of people who get on the next train. Find

(a) $E[X]$,

Let Y denote #hours before the next train arrive, $Y \sim U[0,1]$.

$\Rightarrow X \sim Poi(7Y)$

$E[X] = E[X|Y]\cdot E[Y] = 7\cdot\frac{1}{2} = \frac{7}{2}$

Hence, $E[X] = \frac{7}{2}$

(b) $Var(X)$.

$Var(X) = Var[E(X|Y)] + E[Var(X|Y)] = Var(7Y) + E(7Y) = 49Var(Y) + 7E(Y) = \frac{49}{12}+\frac{7}{2} = \frac{91}{12}$

Hence, $Var(X) = \frac{91}{12}$

#4 **(Chapter 5, Exercise 59)** Cars pass an intersection according to a Poisson process with rate λ. There are 4 types of cars, and each passing car is, independently, type i with probability p_i, $\sum_{i=1}^{4} p_i = 1$.

(a) Find the probability that at least one of each of car types 1, 2, 3 but none of type 4 have passed by time t.

type i car by time t $\sim Poi(P_i \lambda t)$

$$P(N_1(t) \geq 1, N_2(t) \geq 1, N_3(t) \geq 1, N_4(t) = 0) = P(N_1(t)\geq 1)\, P(N_2(t)\geq 1)\, P(N_3(t)\geq 1)\, P(N_4(t)=0)$$

$$= \mathrm{Exp}(-P_4 \lambda t) \cdot \prod_{i=1}^{3} [1 - \mathrm{Exp}(-P_i \lambda t)]$$

Hence, the probability is $\mathrm{Exp}(-P_4 \lambda t) \cdot \prod_{i=1}^{3} [1 - \mathrm{Exp}(-P_i \lambda t)]$

(b) Given that exactly 6 cars of type 1 or 2 passed by time t, find the probability that 4 of them were type 1.

$$P(N_1(t) = 4 \mid N_1(t) + N_2(t) = 6) = \frac{P(N_1(t) = 4,\ N_2(t) = 2)}{P(N_1(t) + N_2(t) = 6)}$$

$$= \frac{P(N_1(t) = 4)\, P(N_2(t) = 2)}{P(N_1(t) + N_2(t) = 6)}$$

$$= \frac{\frac{1}{4!}(P_1\lambda t)^4 \cdot \exp[-P_1\lambda t] \cdot \frac{1}{2!}(P_2\lambda t)^2 \cdot \exp[-P_2\lambda t]}{\frac{1}{6!}[P_1\lambda t + P_2\lambda t]^6 \cdot \exp[-(P_1\lambda t + P_2\lambda t)]}$$

$$= \frac{6!}{4!\,2!} \cdot \frac{P_1^4 \cdot P_2^2}{(P_1+P_2)^6}$$

$$= \binom{6}{4}\left(\frac{P_1}{P_1+P_2}\right)^4\left(\frac{P_2}{P_1+P_2}\right)^2 \sim \mathrm{Binomial}\left(6, \frac{P_1}{P_1+P_2}\right)$$

Hence, the probability is $\binom{6}{4}\left(\frac{P_1}{P_1+P_2}\right)^4\left(\frac{P_2}{P_1+P_2}\right)^2$

1. Write two functions `Poisson1` and `Poisson2` to simulate the sequence of arrival times of events on $[0, 1]$ for a Poisson process with rate λ, where λ is the input to your functions. In `Poisson1`, the simulation is based on generating i.i.d. inter-arrival times. In `Poisson2`, you first generate the total number of arrivals and then the conditional distribution of arrival times.

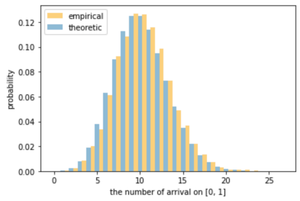
```python
def Poisson1(lam):
    '''
        Parameters:
        ----------
        lam : float
            rate

        Returns
        -------
        A list or an array
            arrival times on interval [0,time].
    '''
    n = 0
    list_of_T =[]
    inter_arrival_t = np.random.exponential(1/lam)
    sum_time = inter_arrival_t

    while sum_time < time:
        list_of_T.append(inter_arrival_t)
        n += 1
        inter_arrival_t = np.random.exponential(1/lam)
        sum_time += inter_arrival_t

    return list_of_T
```

```python
def Poisson2(lam):
    '''
        Parameters:
        ----------
        lam : float
            rate

        Returns
        -------
        A list or an array
            arrival times on interval [0,time].
    '''
    n_arrivals = np.random.poisson(lam=lam)
    if n_arrivals == 0:
        return [1]
    arrival_times = np.random.uniform(0, 1, n_arrivals)
    arrival_times.sort()

    return arrival_times
```

2. Set $\lambda = 10$. Run `Poisson1` for 10000 rounds, record the total number of arrivals in each round. Plot the empirical distribution of the simulated number of arrivals, and validate your codes by comparing the empirical distribution with the theoretic distribution.

```python
lam = 10
N = 10000
```

```python
numbers=[]
for i in range(N):
    numbers.append(len(Poisson1(lam)))
plot1(numbers,lam)
```



3. Set $\lambda = 10$. Run `Poisson2` for 10000 rounds, record the first arrival time. (What if there is no arrival on $[0, 1]$?) Plot the empirical distribution of the first arrival time, and validate your codes by comparing the empirical distribution with the theoretic distribution.

```python
lam = 10
N = 10000
```

```python
first_arrivals=[]
for i in range(N):
    first_arrivals.append(Poisson2(lam)[0])
plot2(first_arrivals, lam, bins=50)
```