Lecture 20 4.4.2019

Demo

NumQ-demo.m

$n = \frac{b-a}{h}$

r-th order

$h =$

$$\text{err}(n) \approx c h^r = c\left(\frac{b-a}{n}\right)^r$$

$n \sim \text{err}(n)$

$$\log(\text{err}(n)) \approx \log c + \log\left(\frac{b-a}{n}\right)^r$$

$$\downarrow$$

$$r\left(\log(b-a) - \log n\right)$$

$$\log(\text{err}(n)) \approx r\log(b-a) + \log c \boxed{-r} \log n$$

↓ slope

$c\, h^m f^{(n)}(c)$

§ 4-3 Romberg integration.

(Richardson extrapolation).

Consider composite Trap. rule.

$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left(f(a) + f(b) + 2\sum_{j=1}^{N-1} f(x_j)\right)$$

$$+ C_2 h^2 + C_4 h^4 + C_6 h^6 + \cdots$$

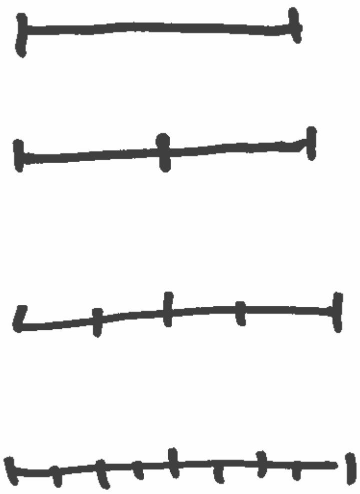
Taylor series expansion

$C_2, C_4 \ldots$ can be derived analytically

$$C_2 = \frac{f'(a) - f'(b)}{2}$$

$$N = \frac{b-a}{h}$$

Consider a sequence of meshes



$$h_1 = \frac{b-a}{1}$$

$$h_2 = \frac{b-a}{2^1}$$

$$h_3 = \frac{b-a}{2^2}$$

$$\vdots$$

$$h_j = \frac{b-a}{2^{j-1}}$$

Apply composite Trap. method to $f(x)$ on each mesh, and get an numerical integral, R_{j1}

$j = 1, 2, \ldots$

Recall Richardson extrapolation

$$\begin{cases} Q = \boxed{F_n(h)}^{\text{computable}} + K h^n + O(h^{n+r}) \\ Q = \boxed{F_n\left(\frac{h}{2}\right)} + K\left(\frac{h}{2}\right)^n + O\left(\left(\frac{h}{2}\right)^{n+r}\right) \end{cases}$$

$\to$ nth order

$$\Rightarrow Q = \underbrace{\frac{2^n F_n\left(\frac{h}{2}\right) - F_n(h)}{2^n - 1}}_{(n+r)\text{-th order accurate.}} + O(h^{n+r})$$

Based on this. based on R_{j1}

which is 2nd order accurate

$n=2$

$$\frac{2^n R_{j,1} - R_{j-1,1}}{2^n - 1} = \frac{4R_{j,1} - R_{j-1,1}}{3}$$

$$=: R_{j,2}$$

This provides a 4th order approximation for $\int_a^b f(x)\,dx$

Once we have a 4th order approximation R_{j2}.

Apply extrapolation $(n = 4)$

$$\frac{2^n R_{j,2} - R_{j-1,2}}{2^n - 1}$$

$$= \frac{16 R_{j,2} - R_{j-1,2}}{15} = R_{j,3}$$

This gives a sixth order approximation.

Romberg Tableau.

$$R_{jk} = \frac{4^{k-1} R_{j,k-1} - R_{j-1,k-1}}{4^{k-1} - 1}$$

R_{11}

$R_{21} > R_{22}$

$R_{31} > R_{32} > R_{33}$

$R_{41} > R_{42} > R_{43} > R_{44}$ | best

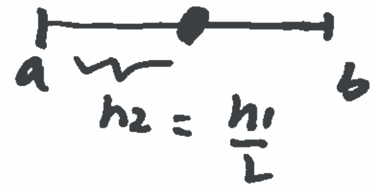
$R_{51} > R_{52} > R_{53} > R_{54} > R_{55}$

$\downarrow$ 2nd $\quad \downarrow$ 4th $\quad \downarrow$ 6th $\quad \downarrow$ 8th $\quad \downarrow$ 10th

To compute R_{j1} recursively

$$R_{11} = \frac{h_1}{2}\left(f(a) + f(b)\right)$$

$$R_{21} = \frac{h_2}{2}\Big(f(a) + f(b) + 2f(a+h_2)\Big)$$



$$= \frac{h_2}{2}\left(f(a) + f(b)\right) + \underbrace{h_2 f(a+h_2)}_{\text{new}}$$

$$= \frac{1}{2} R_{11} + h_2 f(a + h_2)$$

$$R_{31} = \frac{h_3}{2}\Big(f(a) + f(b) + \underbrace{2f(a+h_3)} + 2f(a+\underbrace{2h_3}_{h_2}) + \underbrace{2f(a+3h_3)}\Big)$$

$$R_{31} = \frac{1}{2} R_{21} + h_3\Big(f(a+h_3) + f(a+3h_3)\Big)$$

In general

$$R_{j1} = \frac{1}{2} R_{j-1,1} + h_j \sum_{i=1}^{2^{j-2}} f(a + (2i-1)h_j)$$

To control error, one can run till the level m, when

$$|R_{mm} - R_{m-1,m-1}| \le \text{MyTol}$$

$\boxed{\text{Demo}}$ "romberg.m"

$$f(x) = \log x$$

To compute $\int_1^2 \log x \, dx$

$$\vdots$$

$$= 2\log 2 - 1$$

§ 4-4 Adaptive quadrature.

To compute $\int_a^b f(x)\,dx$. numerically

recall on a single panel.

$$\int_{x_j}^{x_{j+1}} f(x)\,dx \approx \text{computed value}$$

$$+ \underbrace{\boxed{c\,h^m f^{(n)}(c)}}_{\text{error}}$$

$$x_{j+1} - x_j = h$$

To achieve a given level of error. the larger $f^{(n)}$ is, the small h should be

In practice. $f^{(n)}$ is often unknown.

Goal: to compute $\int_a^b f(x)\,dx$ with a given level of error, effecienty. using 'adaptive quadrature'

To see the main ingriedient,

given $f(x)$ on $[a, b)$

let $S[x_L, x_R]$ be a numerical strategy to compute $\int_{x_L}^{x_R} f(x)dx$

"building block"

Goal: compute $\int_a^b f(x)dx$ (error unknown)
with a given error tolerance.

First

: $S[a,b]$

(the error is unknown).

: $S[a,c] + S[c,b]$

Based on these two approximations
design an error indicator

Second Based on error indicator

decide whether you want to accept $S[a,c] + S(c,b]$

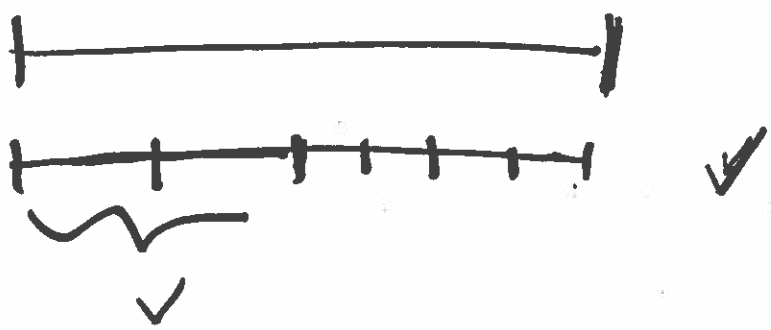
Yes

$INT = S[a,c] + S[c,b]$;

stop

NO

repeatedly treat $(a,c]$, $[c,b)$ as a starting interval



△1 design error indicator.

△2 { accept the result ? or not

△3 how to track or organize multiple subintervals.

First we want to design an error indicator

use trap. rule as an example.

computable

$$\underbrace{\int_a^b f(x)\,dx} = \boxed{S[a,b]} - \frac{h^3}{12} f''(c_0)$$

a b

$$S[a,b] = \frac{(f(a)+f(b))(b-a)}{2}$$

error unknown;

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

a c b

$$= S[a,c] - \left(\frac{h}{2}\right)^3 \frac{f''(c_1)}{12}$$

$$+ S[c,b] - \left(\frac{h}{2}\right)^3 \frac{f''(c_2)}{12}$$

$$= \boxed{S[a,c] + S[c,b]} - \frac{h^3}{4} \frac{f''(c_3)}{12}.$$

computable

Consider

$$S[a,b] - (S[a,c] + S[c,b])$$

$$= h^3 \frac{f''(c_0)}{12} - \frac{h^3}{4} \frac{f''(c_3)}{12}$$

$$\approx 3 \Bigg(\underbrace{\frac{h^3}{4} \frac{f''(c_3)}{12}}_{\text{error in } S[a,c] + S[c,b]} \Bigg) \qquad c_0 \approx c_3$$

Error indicator (must be computable)

$$\text{err } I = |S[a,b] - (S[a,c] + S[c,b])|$$