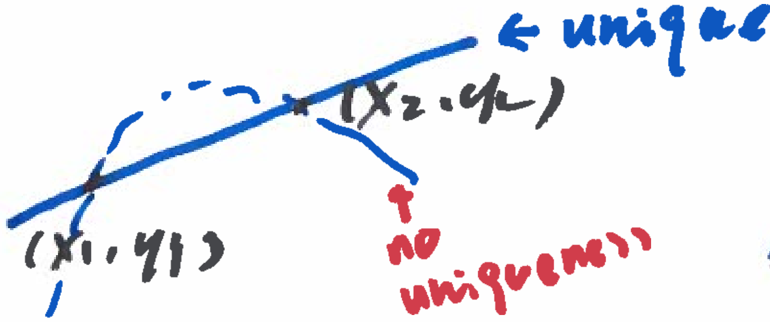
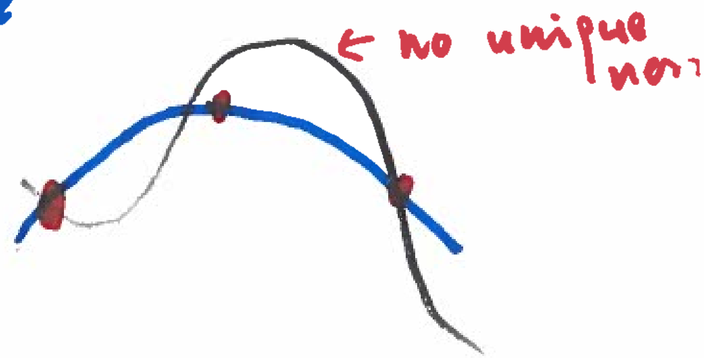
§3 Interpolation and Data fitting

Lecture 13 2-28-2019

§3-1 Interpolation : global





Problem Given

$\{(x_i, y_i)\}_{i=1}^{n+1}$

look for $P_n(x)$ $P_n(x_i) = y_i$ $i = 1, 2 \ldots n+1$

↑ degree up to n

"interpolant" ("$P_n(x)$ interpolates these $(n+1)$ points")

Approach: direct method

$$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$$

$$P_n(x_i) = y_i \quad i = 1, \ldots n+1$$

$$A \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}$$

$$\det(A) = \prod_{1 \le i < j \le n+1} (x_j - x_i)$$

$\Rightarrow$ $P_n(x)$ uniquely exist,

$\Leftrightarrow$ $\{x_i\}_{i=1}^{n+1}$ are distinct.

if and only if

Discussion:

1) Given $\{(x_i, y_i)\}_{i=1}^{n+1}$, $\{x_i\}_{i=1}^{n+1}$ are distinct, $P_n(x)$ is the unique polynomial interpolant of degree n.

$$P_{n+1}(x) = P_n(x) + c\,(x-x_1)(x-x_2)\cdots(x-x_{n+1})$$

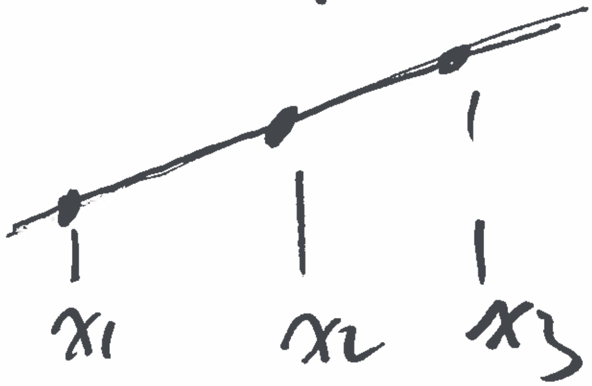
$$\underbrace{P_{n+1}(x_i)}_{} = \underbrace{P_n(x_i)}_{= y_i} + c\,\underbrace{(\qquad)\Big|_{x=x_i}}_{= 0}$$

any constant.

infinitely many interpolants of polynomial of higher degree.

Given 3 points

2)



polynomial of degree 1.

Special polynomials of degree 2

$$a_0 + a_1 x + a_2 x^2$$

can be zero

can be zero

7) In practice Vandermonde matrix can be ill-conditioned.

example: $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n+1} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \\ \frac{1}{3} \\ \vdots \\ \frac{1}{n} \end{pmatrix}$

$A = \text{Vander}(\underline{x})$ ← matlab

$\text{Cond}(A)$	≈ 753	$n=4$
	$\approx 2.4 \times 10^5$	$n=6$
	$\approx 1.52 \times 10^8$	$n=8$
	$\approx 1.59 \quad 10^{11}$	$n=10$

↳ "ill-conditioned"

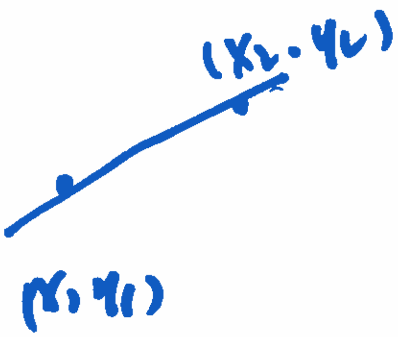
Approach 2: Lagrange approach.

problem
$$\begin{cases} \{(x_i, y_i)\}_{i=1}^{n+1}, & \text{look for } P_n(x) \\ \{x_i\}_{i=1}^{n+1} \text{ distinct} \\ P_n(x_i) = y_i \end{cases}$$

$n=1$, two points.

$$P_1(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$= y_1\left(1 - \frac{x - x_1}{x_2 - x_1}\right) + y_2\left(\frac{x - x_1}{x_2 - x_1}\right)$$



$$= y_1 \underbrace{\left(\frac{x - x_2}{x_1 - x_2}\right)}_{\ell^{(1)}(x)} + y_2 \underbrace{\left(\frac{x - x_1}{x_2 - x_1}\right)}_{\ell^{(2)}(x)}$$

$$= y_1 \ell^{(1)}(x) + y_2 \ell^{(2)}(x)$$

Features of $\ell^{(1)}(x)$, $\ell^{(2)}(x)$

- polynomial of degree 1
- $$\underbrace{\begin{cases} \ell^{(1)}(x_1) = 1 \\ \ell^{(1)}(x_2) = 0 \end{cases} \quad \begin{cases} \ell^{(2)}(x_1) = 0 \\ \ell^{(2)}(x_2) = 1 \end{cases}}$$

$$\ell^{(j)}(x_i) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

δ_{ij}: Kronecker delta.

For general n.

$$\boxed{P_n(x)} = y_1 \ell^{(1)}(x) + y_2 \ell^{(2)}(x) + \cdots + y_{n+1} \ell^{(n+1)}(x)$$

$$= \boxed{\sum_{j=1}^{n+1} y_j \ell^{(j)}(x)}$$

- $\ell^{(j)}(x)$: polynomial of degree n
- $\ell^{(j)}(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

uniquely exists, as it interpolates:

$(x_1, 0)$
$(x_2, 0)$
$\vdots$
$(x_{i-1}, 0)$
$(x_i, 1)$
$(x_{i+1}, 0)$
$\vdots$
$(x_{n+1}, 0)$

$$\ell^{(j)}(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{j-1})(x-x_{j+1})(x-x_{j+2})\cdots(x-x_{n+1})}{(x_j-x_1)(x_j-x_2)\cdots(x_j-x_{j-1})(x_j-x_{j+1})(x_j-x_{j+2})\cdots(x_j-x_{n+1})}$$

(Exercise: to verify)

"Lagrange polynomial"

Remarks:

in approach 1

$p_n(x) = a_0 + a_1 x + \dots + a_n x^n$.

$1, x, x^2, \dots x^n$: monomial basis of polynomial space of degree up to n

in approach 2:

a different basis 'Lagrange basis'

$\ell^{(1)}(x), \ell^{(2)}(x) \dots \ell^{(n+1)}(x)$

Also if

$$p_n(x) = \sum_{j=1}^{n+1} y_j \ell^{(j)}(x)$$

then $p_n(x_i) = \sum_{j=1}^{n+1} y_j \underbrace{\ell^{(j)}(x_i)}_{\delta_{ij}} = y_i$

$$\Rightarrow \boxed{y_i = p_n(x_i)}$$

coefficients of Lagrange representation give function values at x_j, $j = 1, 2 \dots$

Example: given 3 data points

j	x_j	y_j
1	0	1
2	1/2	-1
3	1	2

find $P_2(x)$ which interpolates these 3 points.

$\Downarrow$

uniquely exists.

Approach 1. $P_2(x) = a_0 + a_1 x + a_2 x^2$

$P_2(x_j) = y_j \quad j = 1, 2, 3$

$$\Leftrightarrow \quad A \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & (\frac{1}{2})^2 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} a_0 = 1 \\ a_0 + \frac{1}{2} a_1 + \frac{1}{4} a_2 = -1 \\ a_0 + a_1 + a_2 = 2 \end{cases} \Rightarrow a_0 = 1$$

$$\text{and } \begin{cases} a_1 + \frac{1}{2} a_2 = -4 \\ a_1 + a_2 = 1 \end{cases} \Rightarrow \begin{cases} \frac{1}{2} a_2 = 5 \Rightarrow a_2 = 10 \\ a_1 = -9. \end{cases}$$

$$\Rightarrow P_2(x) = 10x^2 + -9x + 1.$$

By approach 2:

$$P_2(x) = \sum_{j=1}^{3} y_j \, \ell^{(j)}(x)$$

$$= \underset{=1}{y_1} \ell^{(1)}(x) + \underset{=-1}{y_2} \ell^{(2)}(x) + \underset{=2}{y_3} \ell^{(3)}(x)$$

$\ell^{(1)}(x) =$

$$\frac{(x-x_2)(x-x_3)}{(x_1-x_2)(x_1-x_3)} = \frac{(x-\frac{1}{2})(x-1)}{(0-\frac{1}{2})(0-1)}$$

$$= 2(x-\tfrac{1}{2})(x-1)$$

$$\ell^{(2)}(x) = \frac{(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_3)} = \frac{(x-0)(x-1)}{(\frac{1}{2}-0)(\frac{1}{2}-1)}$$

$$= -4x(x-1)$$

$$\ell^{(3)}(x) = \frac{(x-x_1)(x-x_2)}{(x_3-x_1)(x_3-x_2)} = \frac{(x-0)(x-\frac{1}{2})}{(1-0)(1-\frac{1}{2})}$$

$$= 2x(x-\tfrac{1}{2})$$

$$\Rightarrow P_2(x) = \ell^{(1)}(x) - \ell^{(2)}(x) + 2\ell^{(3)}(x)$$

Approach 3: Newton's divided difference.

Recall Given 1 point (x_1, y_1)

then $P_0(x) = y_1$

Given 2 points (x_1, y_1) (x_2, y_2)

$$P_1(x) = \underbrace{y_1}_{\overset{\shortparallel}{P_0(x)}} + \underbrace{\frac{y_2 - y_1}{x_2 - x_1}(x - x_1)}_{a_1 (x - x_1)}$$

$$P_1(x) = \underline{P_0(x)} + \underbrace{a_1 (x - x_1)}_{\text{correction}}$$

To verify:

$$P_1(x_1) = P_0(x_1) + a_1 \underbrace{(x_1 - x_1)}_{\overset{\shortparallel}{0}}$$

$$= P_0(x_1)$$

$$P_2(x_2) = \underbrace{P_0(x_2)}_{y_1} + a_1 (x_2 - x_1) = y_2$$

$$\Rightarrow a_1 = \frac{y_2 - y_1}{x_2 - x_1}$$

now we add one more point (x_3, y_3)

$$P_2(x) = P_1(x) + \underbrace{a_2(x-x_1)(x-x_2)}$$

we can see

$$P_2(x_1) = P_1(x_1) + 0 = y_1$$

$$P_2(x_2) = P_1(x_2) + 0 = y_2$$

$$P_2(x_3) = P_1(x_3) + a_2(x_3-x_1)(x_3-x_2) = y_3$$

$$\Rightarrow a_2 = \frac{y_3 - P_1(x_3)}{(x_3-x_1)(x_3-x_2)}$$

$\vdots$

$$= \frac{\frac{y_3-y_2}{x_3-x_2} - \frac{y_2-y_1}{x_2-x_1}}{x_3-x_1}$$

$\leftarrow$ Exercise: to verify.

related to "divided difference"

Given $\{(x_j, y_j)\}_{j=1}^{n}$ n points,

$\{x_j\}_{j=1}^{n}$: distinct

$P_{n-1}(x)$: unique interpolating polynomial of degree $n-1$

Now suppose there is an extra point (x_{n+1}, y_{n+1}), ($x_{n+1} \neq x_j$, $j=1,\dots n$)

$$P_n(x) = P_{n-1}(x) + a_n (x-x_1)(x-x_2)\cdots(x-x_n)$$

One can check

$$P_n(x_i) = P_{n-1}(x_i) + 0 \qquad i=1,\dots n$$
$$= y_i$$

$$P_n(x_{n+1}) = P_{n-1}(x_{n+1}) + \overset{?}{a_n} (x_{n+1}-x_1)(x_{n+1}-x_2)\cdots(x_{n+1}-x_n)$$
$$= y_{n+1}$$

$$\Rightarrow a_n = \frac{y_{n+1} - P_{n-1}(x_{n+1})}{(x_{n+1}-x_1)(x_{n+1}-x_2)\cdots(x_{n+1}-x_n)}$$

Introduce some notation.

Given $\{(x_i, y_i)\}_{i=1}^{n+1}$

Define **divided difference**, recursively

$$g[x_j] = y_j \quad , \quad \forall j$$

$$g[x_j, x_{j+1}] = \frac{g[x_{j+1}] - g[x_j]}{x_{j+1} - x_j} \quad \forall j$$

$$g[x_j, x_{j+1}, x_{j+2}]$$

$$= \frac{g[x_{j+1}, x_{j+2}] - g[x_j, x_{j+1}]}{x_{j+2} - x_j}.$$

$\vdots$

$$g[x_j, x_{j+1}, \dots x_{j+k}]$$

$$= \frac{g[x_{j+1}, x_{j+2}, \dots x_{j+k}] - g[x_j, x_{j+1}, \dots x_{j+k-1}]}{x_{j+k} - x_j}$$

what we got so far $P_0(x)$, $P_1(x)$, $P_2(x)$ ... can be written in terms of divided difference

$$P_0(x) = y_1 = g[x_1]$$

$$P_1(x) = P_0(x) + g[x_1, x_2](x - x_1)$$

$$= g[x_1] + g[x_1, x_2](x - x_1)$$

$$P_2(x) = P_1(x) + g[x_1, x_2, x_3](x - x_1)(x - x_2)$$

$$= g[x_1] + g[x_1, x_2](x - x_1)$$

$$+ g[x_1, x_2, x_3](x - x_1)(x - x_2)$$

In general

$$P_n(x) = \sum_{j=1}^{n+1} g[x_1, x_2, \ldots x_j](x - x_1)(x - x_2) \cdots (x - x_{j-1})$$

Divided differences can be organized and computed via a table.

Given (x_j, y_j) $j = 1, 2, 3$

x_1	$g[x_1]$			
x_2	$g[x_2]$	$g[x_1, x_2]$		
x_3	$g[x_3]$	$g[x_2, x_3]$	$g[x_1, x_2, x_3]$	
x_4	$g[x_4]$	$g[x_3, x_4]$	$g[x_2, x_3, x_4]$	$g[x_1, x_2, x_3, x_4]$

- To add one new point (x_4, y_4), we just need to add one more row in this table

- The diagonal values contribute to the coefficients of $P_2(x)$ (or $P_3(x)$) in current approach 3.