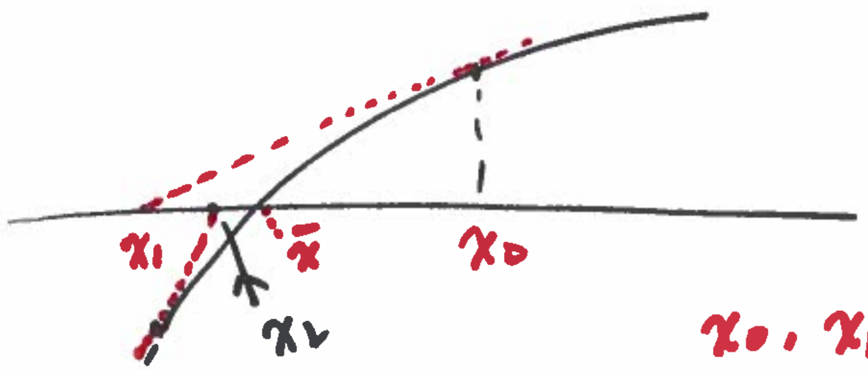
§1.2

Lecture 5

1.28.2019

Newton's method

(Tangent line method)



" a curve locally can be approximated by a straight line (recall Taylor Theorem) "

To solve $f(x)=0$. we start with an initial guess x_0, and consider the tangent line of $f(x)$ passing through $(x_0, f(x_0))$

$$\boxed{f(x_0) + f'(x_0)(x-x_0)}$$

Instead of solving $f(x)=0$
we solve

$$f(x_0)+f'(x_0)(x-x_0)=0$$

and the solution is denoted as x_1.

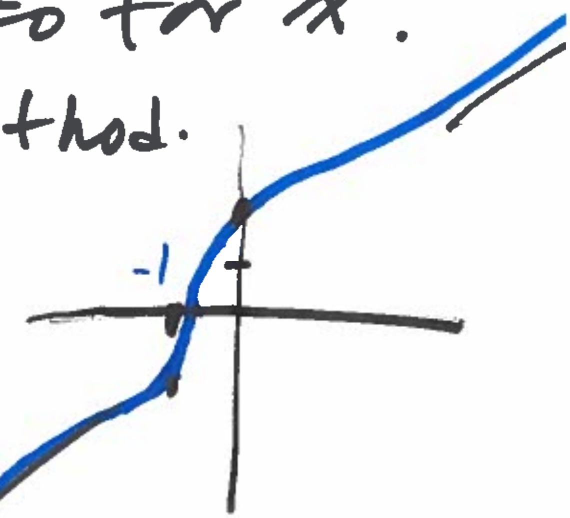
$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

Repeat the process, and we get Newton's method

$$\boxed{x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}} \quad \boxed{j=0,1,2,\dots}$$

Example:

$f(x) = x^3 + 2x + 2$, we want to solve $f(\bar{x}) = 0$ for $\bar{x}$. Apply Newton's method.



By sketching, we know $\bar{x} \in (-1, 0)$

Start with $x_0 = -\frac{1}{2}$

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}$$

$$= x_j - \frac{x_j^3 + 2x_j + 2}{3x_j^2 + 2}$$

$$= \frac{2x_j^3 - 2}{3x_j^2 + 2}$$

$$x_1 = \frac{2(-\frac{1}{2})^3 - 2}{3(-\frac{1}{2})^2 + 2} = -\frac{9}{11}$$

$x - 0.81818$

Apply Newton's method to

$$f(x) = x^3 + 2x + 2 = 0$$

$e_j = |x_j - \bar{x}|$

j	x_j	e_j	γ
0	$-0.5000\cdots00$	2.71×10^{-1}	
1	-0.81818181818182	4.73×10^{-2}	
2	$-\underbrace{0.77}225866916589$	1.34×10^{-3}	
3	$-\underbrace{0.770918097035}76$	1.10×10^{-6}	
4	$-\underbrace{0.7709169970 5}999$	7.40×10^{-13}	
5	$-\underbrace{0.7709169970592}5$	1.11×10^{-16}	
6	same	same	

Observation:

1) $e_{j+1} \approx c\, e_j^{\gamma}$ $\qquad \gamma = 2$

2) $j = 5, 6$:

Due to finite precision

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}$$

Theorem: Given $f \in C^2([a,b])$ with $f(\bar{x}) = 0$ for some $\bar{x} \in (a,b)$ and $f'(\bar{x}) \neq 0$. Start with $x_0 \in (a,b)$ that is sufficiently close to $\bar{x}$, then Newton's method converges to $\bar{x}$, namely

$$\lim_{j \to \infty} x_j = \bar{x}$$

And the error $e_j = |x_j - \bar{x}|$ satisfies

$$e_{j+1} = c_j e_j^{\gamma}$$

where $\gamma = 2$

and $\displaystyle \lim_{j \to \infty} c_j = \left| \frac{f''(\bar{x})}{2f'(\bar{x})} \right|$

convergence of Newton's method: quadratic

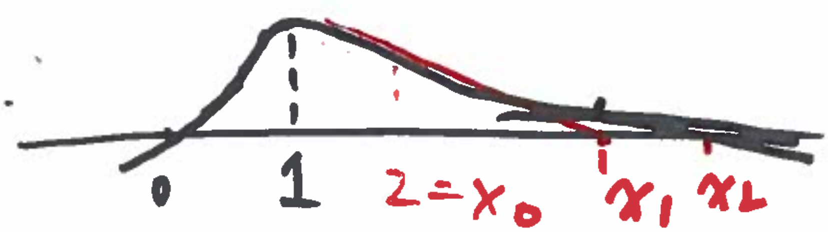
Example: (with a bad initial x_0)

Apply Newton method to

$$f(x) = \frac{x}{1+x^2} = 0$$

we know $\bar{x} = 0$

initial $x_0 = 2$



$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = \cdots = \frac{-2x_i^3}{1-x_i^2}$$

$$x_1 = \frac{16}{3} \approx 5.333$$

$$x_2 = \frac{8192}{741} \approx 11.055$$

Newton's method diverges with $x_0 = 2$

Instead: we can take $x_0 = \frac{1}{2}$.

Newton's method will converge.

Example: apply Newton's method

$$f(x) = x^2 = 0$$

Root:

$\bar{x} = 0 \qquad f'(\bar{x}) = 2\bar{x} = 0$

Theorem no longer holds

Newton's:

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} = x_j - \frac{x_j^2}{2x_j}$$

$$= x_j - \frac{x_j}{2} = \frac{x_j}{2}$$

$$\boxed{x_{j+1} = \frac{1}{2} x_j}$$

not sensitive to initial

$$e_j = |x_j - \bar{x}| = |x_j|$$

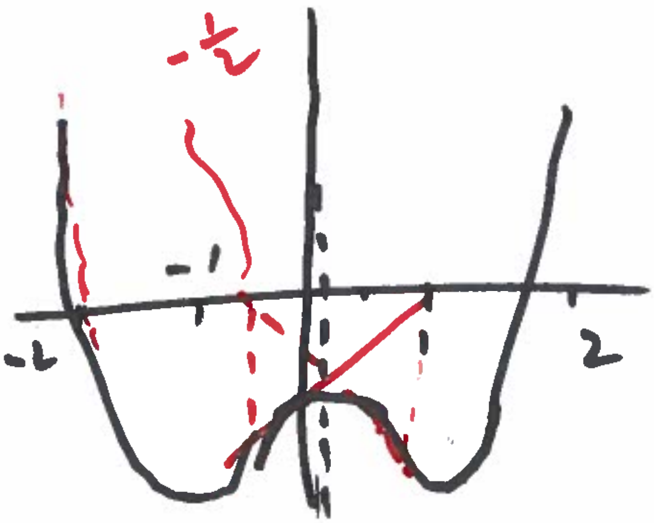
$$e_{j+1} = \frac{1}{2} e_j$$

linear convergence

Example: Apply Newton's method to

$$f(x) = 4x^4 - 6x^2 - \frac{11}{4}$$

with $x_0 = \frac{1}{2}$



$x_0 = \frac{1}{2}$

$x_1 = -\frac{1}{2}$

$x_2 = \frac{1}{2}$

$x_3 = -\frac{1}{2}$

divergent case

x_j, $j = 0, 1, 2, \ldots$ will oscillate between two values $\frac{1}{2}$, $-\frac{1}{2}$

Note: the method can work with a different initial

Stopping Criteria:

- $|x_{j+1} - x_j| < \mathrm{ToL}$ (absolute)

  ToL: choice of user

  or

  $$\frac{|x_{j+1} - x_j|}{\max(|x_j|, \varepsilon)} < \mathrm{ToL} \quad \text{(relative)}$$

  ε: to deal with the case of $\bar{x} = 0$,

  Here ε is some small number of user's choice

- $j < \underbrace{I_{max}}$

  maximum number of iteration

- $|x_j| < \underbrace{x_{max}}$

  a bound

Newton's method

Red: take into account storage

pick an initial guess x_0 ... x

error tolerance $tol > 0$

maximum iteration number $I_{max} > 0$

((possibly an upper bound of approximated root. $x_{max} > 0$))

Let $err = 10 \times tol$, $j = 0$

loop: while $(err > tol)$

$$z = \frac{f(x_j)}{f'(x_j)} \quad \cdots \quad \frac{f(x)}{f'(x)}$$

$$err = |z| = abs(z)$$

$$x_{j+1} = x_j - z \quad \cdots \quad x = x - z$$

$$j = j + 1$$

if $j > I_{max}$ (or $|x_{j+1}| > x_{max}$)

stop

End

Example: (Application.)

Given $a \neq 0$
we want to calculate $x = \frac{1}{a}$
by using only $+, -, *$

one try:

$$f(x) = ax - 1 = 0$$

Newton: $x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}$

$$= x_j - \frac{ax_j - 1}{a}$$

$$= \frac{1}{a}$$

(not good)

Another try: $f(x) = a - \frac{1}{x}$

$$f'(x) = \frac{1}{x^2}$$

Newton's: $\boxed{x_{j+1} =} x_j - \frac{f(x_j)}{f'(x_j)}$

$$= x_j - \left(a - \frac{1}{x_j}\right) x_j^2 = x_j(2 - ax_j)$$

The update $x_{j+1} = x_j(2 - ax_j)$ involves two "$*$" and one "$-$"

2 : $*$

1 : $-$

$a = 3$

$x_0 = 0.3$

$x_1 = 0.33$

$x_2 = 0.3333$

$x_3 = 0.33333333$

due to quadratic convergence