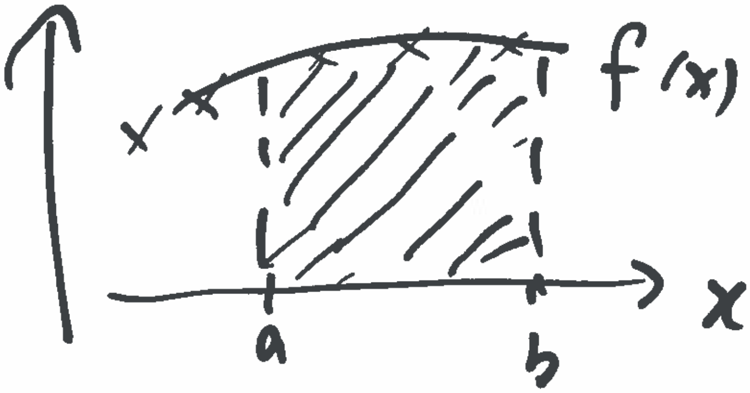
Lecture 19 4.1.2019

§4-2 Numerical integration



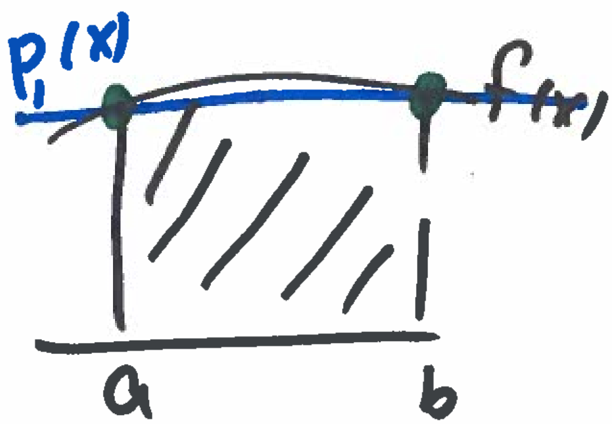
$\int_a^b f(x)\,dx$ = area of the shaded area.

$f(x) = x^x$

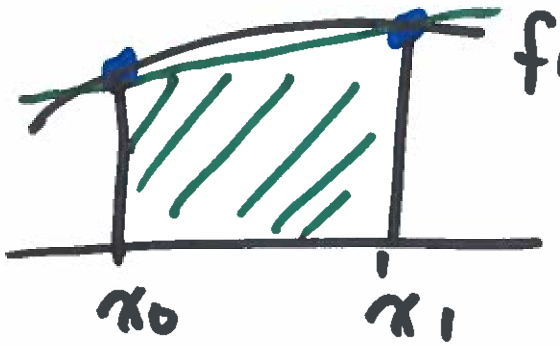
$f(x) \sim p(x)$ interpolation

$$\int_a^b f(x)\,dx \approx \int_a^b p(x)\,dx$$

Newton - Cotes : when $p(x)$ is an interpolation based on equally spaced points.



linear interpolation



$$\int_{x_0}^{x_1} f(x)\,dx$$

$$y_i = f(x_i)$$

$$i = 0, 1$$

$$P_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$f(x) = P_1(x) + E(x)$$

$\downarrow$ error

$$E(x) = \frac{(x - x_0)(x - x_1)}{2} f''(c)$$

$c = c(x)$

$c \in (x_0, x_1)$

$$\int_{x_0}^{x_1} f(x)\,dx = \int_{x_0}^{x_1} P_1(x)\,dx + \int_{x_0}^{x_1} E(x)\,dx$$

$$\approx \int_{x_0}^{x_1} P_1(x)\,dx$$

$$= \int_{x_0}^{x_1} \left(y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \right) dx$$

$$= \frac{y_0 + y_1}{2} \cdot h \qquad h := x_1 - x_0$$

Error

$$\int_{x_0}^{x_1} E(x)\,dx = -\frac{h^3}{12} f''(\hat{c})$$

$\hat{c}$ is between x_0 and x_1

Trapezoidal rule

$$\int_{x_0}^{x_1} f(x)\,dx \approx \frac{f(x_1)+f(x_0)}{2} h$$

$$\text{Error} = -\frac{h^3}{12} f''(\hat{c})$$

$\hat{c} \in [x_0, x_1]$

$h = x_1 - x_0$

Question: For what type f the rule is exact?

$f'' \equiv 0 \Leftrightarrow f(x) \doteq a + bx$

$\forall\, a, b$ constant.

Composite numerical quadrature

Consider an equally spaced grid

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < \underset{=b}{x_N}$$

$$x_{j+1} - x_j = h = \frac{b-a}{N}$$

On each subinterval (panel)

$$\int_{x_j}^{x_{j+1}} f(x) = \frac{h}{2}\left(f(x_{j+1}) + f(x_j)\right) - \frac{h^3}{12} f''(c_j)$$

$$c_j \in [x_j, x_{j+1}]$$

Sum up in j

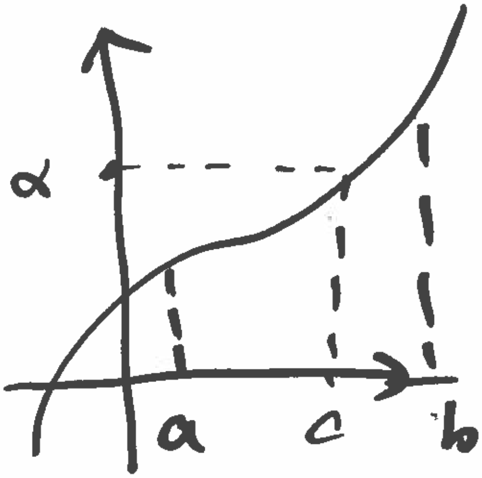
$$\int_a^b f(x)\,dx = \sum_{j=0}^{N-1} \frac{h}{2}\left(f(x_{j+1}) + f(x_j)\right) - \frac{h^3}{12} \underbrace{\sum_{j=0}^{N-1} f''(c_j)}_{N f''(c)}$$

$Nh = b-a$

some $c \in [a,b]$

Composite Trap.

$$= \boxed{\frac{h}{2}\left(f(a) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{N-1}) + f(b)\right)} - \frac{h^2}{12}(b-a) f''(c)$$



$$\min_{x\in[a,b)} g(x) < \alpha < \max_{x\in[a,b)} g(x)$$

g continuous

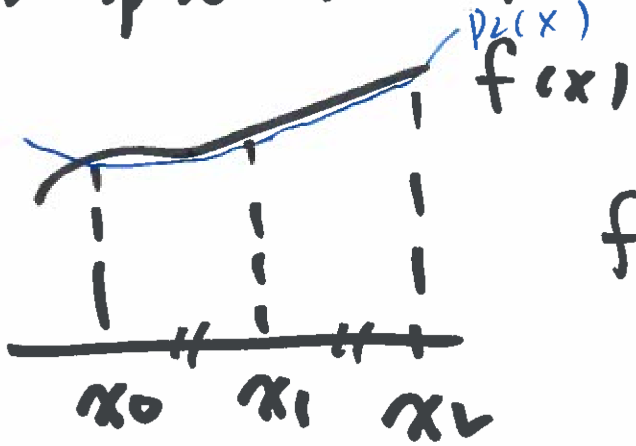
$\exists$ c s.t $f(c) = \alpha$

$$\frac{N \min_{x\in[a,b)} f''(x)}{N} \le \frac{\sum_{j=0}^{N-1} f''(c_j)}{N} \le \frac{N \max_{x\in[a,b)} f''(x)}{N}$$

$$\| \; f''(c)$$

hence $\sum_{j=0}^{N-1} f''(c_j) = N f''(c)$ for some $c \in [a,b)$

Simpson's rule:



$x_2 - x_1 = x_1 - x_0 = h$

$f(x) \approx p_2(x)$ ↳ quadratic polynomial

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} p_2(x)\,dx$$

$$= \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$$

$$\text{Error} = -\frac{h^5}{90} f^{(4)}(c)$$ for some $c \in [x_0, x_2]$

Composite Simpson's Rule.

$a = x_0 < x_1 < x_2 < \cdots < x_{2N} = b$

on each panel $[x_{2j}, x_{2j+2}]$

$$\int_{x_{2j}}^{x_{2j+2}} f(x)\,dx = \frac{h}{3}\left(f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})\right) - \frac{h^5}{90} f^{(4)}(c_j)$$

$$\int_a^b f(x)\,dx = \underbrace{\frac{h}{3}\Big(f(a)+f(b) + 4\sum_{j=1}^{N} f(x_{2j+1}) + 2\sum_{j=1}^{N-1} f(x_{2j}) \Big)}_{\text{composite simpson's rule}} \underbrace{- \frac{b-a}{180} h^4 f^{(4)}(\hat{c})}_{\text{error}}$$

$\hat{c} \in (a, b)$

Question: when will the rule exact?

when $f^{(4)} \equiv 0 \iff$

$$f(x) = a + bx + cx^2 + ax^3$$

Degree of precision (D. O. P).

of a numerical integration formula is the largest integer k, such that the formula is exact when the integrand is any polynomial of degree up to k.

So far

	D. O. P.
Trap.	1
Simpson	3.

next Question: How to find D. O. P.?

Change of varriable

reference element $[0,1]$ $[-1,1]$

Suppose we have

quadrature weight quadrature node

$$\int_0^1 f(x)\,dx \approx \sum_{j=1}^{n} w_j f(x_j)$$

on a physical interval $[a, b)$

$$\int_a^b f(y)\,dy \qquad \underline{\underline{\frac{y-a}{b-a} = x}}$$

$$\underbrace{\int_0^1 \underbrace{f(a+(b-a)x)}\,dx}_{F(x)}\ (b-a)$$

$$\approx \sum_{j=1}^{n} w_j\, F(x_j)\ (b-a)$$

given on reference element $[0,1]$

$$= \sum_{j=1}^{n} \underbrace{\{(b-a)\,w_j]}_{\hat{w}_j} f(\underbrace{a+(b-a)x_j}_{\hat{x}_j}))$$

$$= \sum_{j=1}^{n} \hat{w}_j\, f(\hat{x}_j) \quad \begin{cases} \hat{w}_j = (b-a)w_j \\ \hat{x}_j = a + (b-a)x_j \end{cases}$$

D.O.P: preserved under the <u>linear</u> change of variable

Exercise: Trap.

$$\int_0^1 f(x)\,dx \approx \frac{1}{2}\left(f(0) + f(1)\right)$$

we want to find D.O.P.

$f(x) = 1$ LHS $= \int_0^1 1\,dx = 1$

RHS $= \frac{1}{2}(1+1) = 1$

$f(x) = x$ LHS $= \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

RHS $= \frac{1}{2}\left(\underbrace{f(0)}_{=0} + \underbrace{f(1)}_{=1}\right) = \frac{1}{2}$

$f(x) = x^2$, LHS $= \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$

RHS $= \frac{1}{2}\left(\underbrace{f(0)}_{=0} + \underbrace{f(1)}_{=1}\right) = \frac{1}{2}$

$\Rightarrow$ D.O.P of Trap. is 1.

Example:

Find D.O.P of the following numerical integrations

$$\int_{x_0}^{x_1} f(x)\,dx \approx \begin{cases} f(x_0)\,h & \text{left-Rectangle} \\ f\left(\frac{x_0+x_1}{2}\right)h & \text{mid-point} \\ f(x_1)\,h & \text{Right-rectangle.} \end{cases}$$

On a reference element $(0,1)$

formulas become

$$\int_0^1 f(x)\,dx \approx \begin{cases} f(0) & \text{left-R} \\ f(\frac{1}{2}) & \text{mid point} \\ f(1) & \text{right-R} \end{cases}$$

$$LHS = \int_0^1 x^k\,dx = \frac{1}{k+1} \quad k=0,1,2\ldots$$

	$\int_0^1 f(x)dx$	$f(0)$	$f(\frac{1}{2})$	$f(1)$
$k=0$	1	1 ✓	1 ✓	1 ✓
$k=1$	$\frac{1}{2}$	0 ✗	$\frac{1}{2}$ ✓	1 ✗
$k=2$	$\frac{1}{3}$		$\frac{1}{4}$ ✗	

$\Rightarrow$ rectangle rules : d.o.p = 0

mid-point rule : d.o.p = 1.

Remark. Composite midpoint method

$a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$

$x_j - x_{j-1} = h.$

$$\int_a^b f(x)\,dx \approx h \sum_{j=0}^{N-1} f(w_j)$$

$$w_j = \frac{x_j + x_{j+1}}{2}$$

Error : $\frac{b-a}{24} h^2 f''(c)$

$c \in [a, b]$

$$h = \frac{b-a}{N}$$