Lecture 7, 2.4.2019

§1.5 Revisit Newton's method

Fixed point Iteration (FPI)

To solve $f(x)=0$

Recall Newton's method:

x_0: initial

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} \quad j=0,1,2, \ldots$$

...

If $x_0 \approx \bar{x}$ (a solution of $f(x)=0$)

$x_j \to \bar{x}$ as $j \to \infty$ (condition $f'(\bar{x}) \neq 0$)

Define $g(x) = x - \dfrac{f(x)}{f'(x)}$

then Newton's method is also

$$\boxed{x_{j+1} = g(x_j)}$$

On the other hand, $\bar{x}$ also satisfies

$$x = g(x) = x - \frac{f(x)}{f'(x)}$$

$$(\underbrace{x = g(x)})$$

Definition

z is a fixed point of the function G if

$$\boxed{z = G(z)}.$$

Fixed point iteration (FPI)

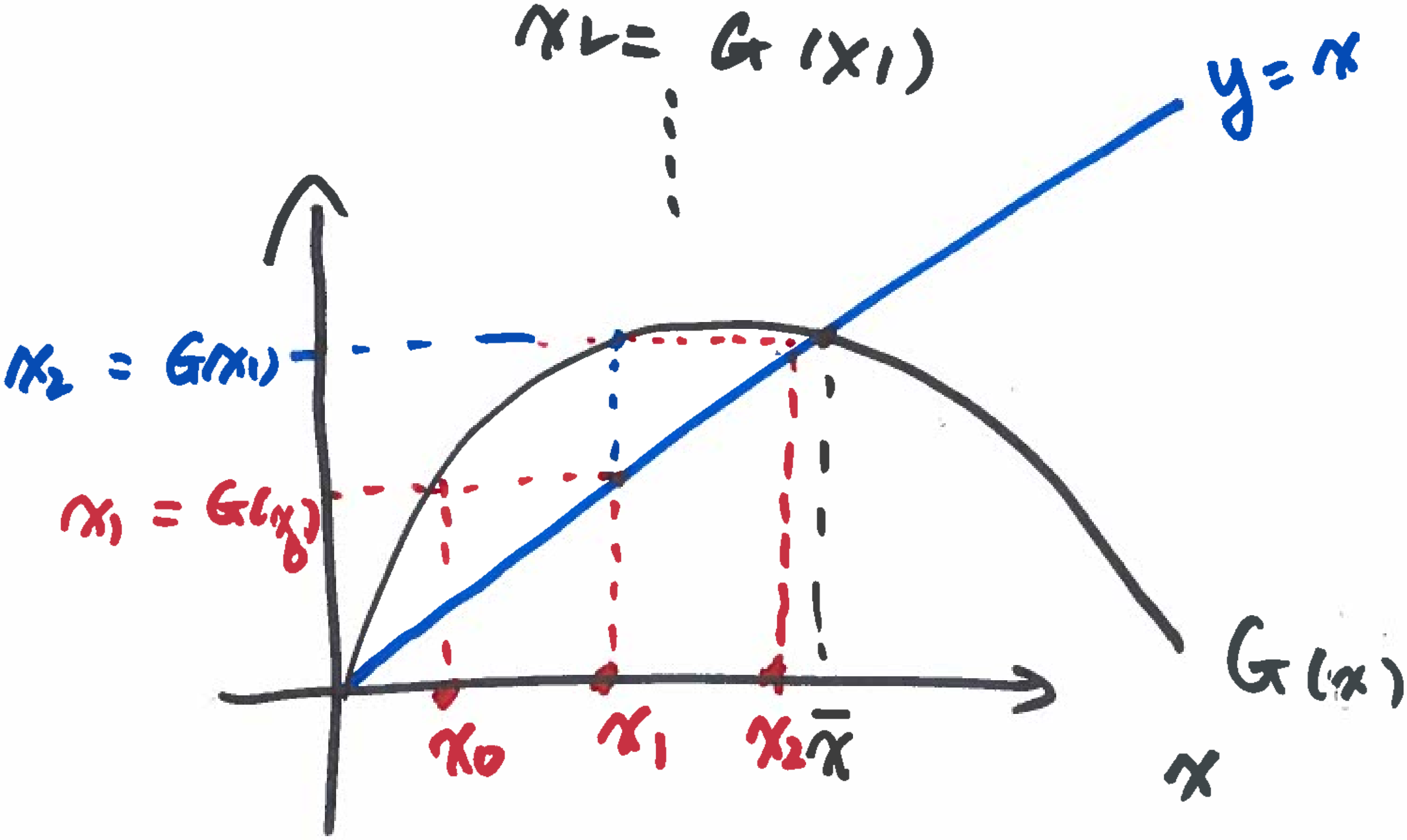
let x_0 be an initial guess

$$x_{j+1} = G(x_j) \quad j = 0, 1, 2 \ldots$$

x_0 ✓ $\qquad x_1 = G(x_0)$

$\qquad\qquad x_2 = G(x_1)$

$\qquad\qquad \vdots$



How are fixed point problem/iteration related?

If the FPI converges.

$$x_j \to z \quad \text{as } j \to \infty$$

if G is continuous

$$G(x_j) \to G(z) \quad \text{as } j \to \infty$$

Recall $\quad x_{j+1} = G(x_j)$

let $j \to \infty$

$$\boxed{z = G(z)}$$

That is, the limit z of x_j is a fixed point of G.

Quick summary:

1) The solution $\bar{x}$ of $f(x)=0$ is also a fixed point of
$$g(x) = x - \frac{f(x)}{f'(x)}.$$

2) Newton's method is a fixed point iteration to find a fixed point of g.

Remarks:

1) fixed point problems are a general class of problems

2) FPI is to try to capture a fixed point. The method may converge (hence work) or may diverge (hence fail)

3) A root finding problem can be converted to a fixed point problem:

$$f(x) = 0$$

$\Rightarrow$ Newton's $\quad x = g(x) := x - \dfrac{f(x)}{f'(x)}$

$$x = x - 100 f(x)$$

$$x = x + f(x)$$

$$\cos x = 2 \sin x$$

$\rightarrow$ add x

$$x = \underbrace{2\sin x - \cos x + x}_{g(x)}$$

Example:

↑ two messages
root finding ⇒ many fixed point problems
some FPIs work
some don't

To solve $x^3 + x - 1 = 0$

converted to fixed point problems.

FPI

- $x = g_1(x) = 1 - x^3$ ✗
- $x = g_2(x) = \sqrt[3]{1-x}$ ✓
- $x = g_3(x) = \dfrac{1+2x^3}{1+3x^2}$ ✓

matlab demo : demoFPI.m

next demo: demoSlope.m

$$g(x) = S x + T$$

S, T two constants

fixed point problem

$$x = g(x) = S x + T$$

$$x = \frac{T}{1-S} \quad (S \neq 1)$$

Indeed:

when $|S| > 1$, the scheme diverges

when $|S| < 1$, ... converges

$\boxed{\text{Some analysis}}$: $x = g(x) = S x + T$

here S, T are constant

FPI $\begin{cases} x_j = S(x_{j-1}) + T \\ \bar{x} = S \bar{x} + T \end{cases}$ ($\bar{x}$ is a fixed point of g)

$\Rightarrow (x_j - \bar{x}) = S(x_{j-1} - \bar{x})$

(linear convergence)

$= S^2 (x_{j-2} - \bar{x})$

$\vdots$

$= S^j (x_0 - \bar{x})$

For $x_j \to \bar{x}$ as $j \to \infty$

$\Longleftrightarrow |S| < 1$

iff

Theorem (Fixed point Iteration)

Assume g, g' are continuous.

Let $\bar{x}$ be a fixed point of g

and $S = |g'(\bar{x})| < 1$. Then the

fixed point iteration $x_{j+1} = g(x_j)$

converges **linearly**, namely

$$x_j \to \bar{x} \quad \text{as } j \to \infty$$

when x_0 is sufficiently close to $\bar{x}$.

That is $\quad e_j = c_j \, e_{j-1}$

and $\quad c_j \to S \quad \text{as } j \to \infty$.

Here $\quad e_j = |x_j - \bar{x}|$

Finally, we go back to Newton's method, which is a FPI, with

$$g(x) = x - \frac{f(x)}{f'(x)}$$

Assume $f(\bar{x}) = 0$

$$g'(x) = 1 - \left(\frac{f'(x)f'(x) - f(x)f''(x)}{(f'(x))^2} \right)$$

$$= \frac{f(x)f''(x)}{(f'(x))^2}$$

$$S = |g'(\bar{x})| = \left| \frac{f(\bar{x})f''(\bar{x})}{(f'(\bar{x}))^2} \right| = 0$$

$\Rightarrow$ Newton's method is a locally convergent FPI.

Note: quadratic convergent rate of Newton's method need to be proved separately key: (for fixed point iteration works)

contractive property

there exists a constant r: $0 < r < 1$

such that

$$\boxed{|g(x_1) - g(x_2)| \le r\,|x_1 - x_2|}$$

for any relevant x_1, x_2 (say in a neighborhood of a fixed point $\bar{x}$)

§2 Solving systems of equations

'To solve more than one equation together'

we start with systems of linear equations

In matrix-vector form:

to solve

$$A \underline{x} = \underline{b} \quad \text{for } \underline{x} \in \mathbb{R}^n$$

where $A \in \mathbb{R}^{n \times n}$

$\underline{b} \in \mathbb{R}^n$ are given

and A is invertible.

$$(\Rightarrow) \quad \underline{x} = \underbrace{A^{-1}}_{\text{inverse}} \underline{b}$$

Recall

$$\mathbb{R}^n = \left\{ \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad x_j \in \mathbb{R}, \quad j = 1, 2 \cdots n \right\}$$

real number

by default, vectors are column vectors.

$$\mathbb{R}^{n\times n} = \left\{ A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \quad a_{ij} \in \mathbb{R}, \quad i,j = 1, \cdots n \right\}$$

A is invertible.

if there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = BA = I_{n\times n}$

$$= \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \in \mathbb{R}^{n\times n}$$

B is denoted as A^{-1}, called the inverse of A

§2.1 Iterative methods

$$A\underline{x} = \underline{b}$$

$$\underline{x}^{(k)} \xrightarrow{?} \underline{x}^{(k+1)}$$

superscript

Example:

$$\begin{cases} 3u + v = 5 \\ u + 2v = 5 \end{cases}$$

first

2nd

matrix-vector form

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

To solve $\underline{x}$ from $A\underline{x} = \underline{b}$

Start with an initial

$$\underline{x}^{(0)} = \begin{pmatrix} u^{(0)} \\ v^{(0)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Jacobi method :

idea : solve the i th unknown from the i th equation

$$u = \frac{5-v}{3} \quad : \quad u^{(1)} = \frac{5-v^{(0)}}{3} = \frac{5-0}{3} = \frac{5}{3}$$

$$v = \frac{5-u}{2} \quad : \quad v^{(1)} = \frac{5-u^{(0)}}{2} = \frac{5}{2}$$

$$\Rightarrow \underline{x}^{(2)} = \begin{pmatrix} u^{(1)} \\ v^{(1)} \end{pmatrix} = \begin{pmatrix} \frac{5}{3} \\ \frac{5}{2} \end{pmatrix}$$