Homework 6
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1. - If $b \in \text{span}(A)$, then the system is solvable.
   - When the column of A is linear independent.
   - For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \\ 1 & 5 & 6 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 6 \end{bmatrix}$, and we can get $x = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, or $x = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

2. Quadratic fitting: Look for $f(x) = a_0 + a_1 x + x_2 x^2$, which means

$$A \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

$$\text{where } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}$$

Since we want to solve this system in the least square sense, we would like to solve the norm equation instead, which is

$$A^T A \tilde{a} = A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 \\ 1 & 9 & 16 & 36 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 14 & 62 \\ 14 & 62 & 308 \\ 62 & 308 & 1634 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 \\ 1 & 9 & 16 & 36 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 30 \\ 144 \end{bmatrix}$$

$$\therefore \tilde{a} = \begin{bmatrix} \frac{77}{26} \\ -\frac{79}{78} \\ \frac{1}{6} \end{bmatrix}$$

Since RMSE= $\sqrt{\frac{SE}{m}}$, so

$$\begin{aligned}
\sqrt{\frac{SE}{m}} &= \sqrt{\frac{||r||^2}{m}} \\
&= \sqrt{\frac{(Aa-b)^T(Aa-b)}{m}} \\
Aa-b &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 6 & 36 \end{bmatrix} \begin{bmatrix} \frac{77}{26} \\ -\frac{79}{78} \\ \frac{1}{6} \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{26} \\ -\frac{15}{26} \\ \frac{15}{26} \\ -\frac{3}{26} \end{bmatrix} \\
\sqrt{\frac{SE}{m}} &= \sqrt{\frac{9/13}{4}} \\
&= 0.416
\end{aligned}$$

3. We need to find $z = c_1 + c_2 x + c_3 y$, so we can have the linear system

$$Ac = z$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} z = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 5 \\ 6 \end{bmatrix}$$

Since in least square sense, so

$$\begin{aligned}
A^T A c &= A^T z \\
A^T A &= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 5 & 3 & 4 \\ 3 & 3 & 3 \\ 4 & 3 & 6 \end{bmatrix}
\end{aligned}$$

$$A^T z = \begin{bmatrix} 19 \\ 14 \\ 19 \end{bmatrix}$$

$$c = \begin{bmatrix} 2 \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

$$\therefore z = 2 + \frac{5}{3}x + y$$

4. We need to solve $f(t) = c_1 + c_2 \cos 2\pi t + c_3 \sin 2\pi t$, which is equivalent to solve

$$Ac = y$$

$$A = \begin{bmatrix} 1 & \cos(0) & \sin(0) \\ 1 & \cos(\pi) & \sin(\pi) \\ 1 & \cos(2\pi) & \sin(2\pi) \\ 1 & \cos(3\pi) & \sin(3\pi) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} y = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

Since we need to solve it in the least square sense, which means

$$A^T Ac = A^T y$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^T y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ 3 \\ 0 \end{bmatrix}$$

$$c = \begin{bmatrix} \frac{9}{4} \\ \frac{3}{4} \\ c_3 \end{bmatrix}$$

$$\therefore f(t) = \frac{9}{4} + \frac{3}{4}\cos 2\pi t + c_3 \sin 2\pi t$$

The error is

$$\begin{aligned} r &= y - Ac \\ &= \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{9}{4} \\ \frac{3}{4} \\ c_3 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 1 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ \frac{3}{2} \\ 3 \\ \frac{3}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \end{aligned}$$

And 2-norm error is

$$\begin{aligned} ||r||_2 &= \sqrt{||r||^2} \\ &= \sqrt{\frac{1}{2}} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

RMSE :

$$\begin{aligned} \sqrt{\frac{||r||^2}{4}} &= \sqrt{\frac{\frac{1}{2}}{4}} \\ &= \frac{\sqrt{2}}{4} \end{aligned}$$