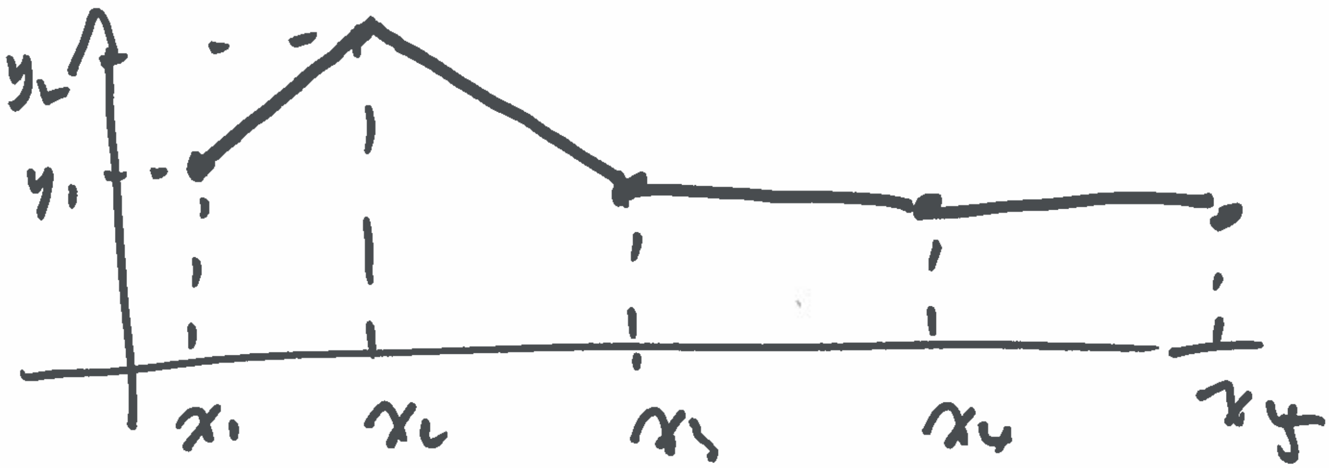
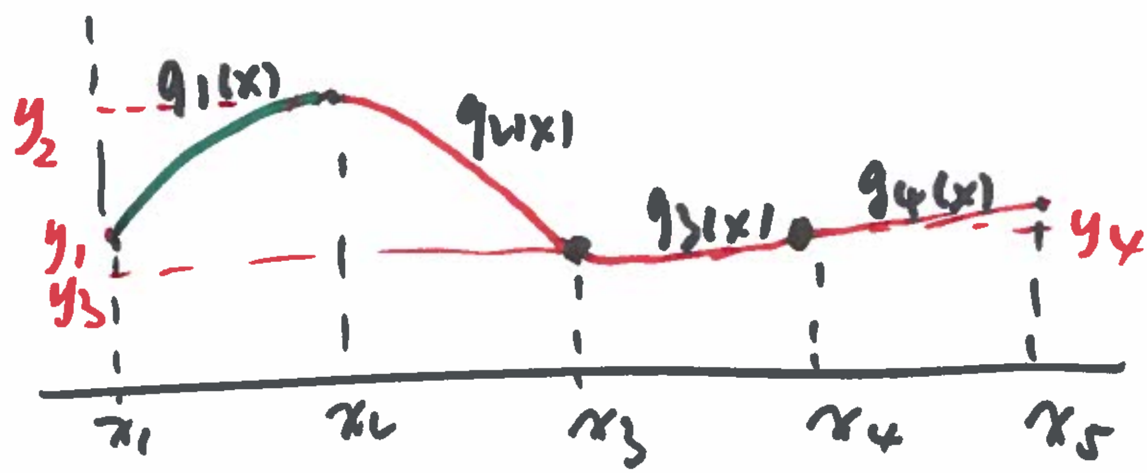
§ 3-2 Interpolation : local

Lecture 15
3.14.2019

§ 3-2-1 piecewise linear



§ 3.2-2 Cubic splines : better smoothness.



$g(x) = g_j(x) \quad x \in [x_j, x_{j+1}]$ $\quad (x_j, y_j)$, $j = 1, \ldots n+1$

property 1 : $\begin{cases} g_j(x_j) = y_j . \\ g_j(x_{j+1}) = y_{j+1} \end{cases}$

property 2 : $g_j'(x_j) = g_{j-1}'(x_j) \quad j = 2, \cdots n$

property 3 $\quad g_j''(x_j) = g_{j-1}''(x_j) \quad j = 2 \cdots n$

$g(x)$, $g'(x)$, $g''(x)$ are all continuous

unique existence

$$g_j(x) = a_j + b_j(x - x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3 \Leftrightarrow (\tilde{a_j} + \tilde{b_j} x + \tilde{c_j} x^2 + \tilde{d_j} x^3)$$

n intervals $[x_j, x_{j+1}]$ $j = 1, \dots n$

$4n$ unknowns

conditions. property 1 $\Leftrightarrow$ $2n$

property 2: $n-1$

property 3: $n-1$

2 condition short: no unique ness.

Two more conditions can be imposed at both ends of the data interval

Choice 1: Natural splines

$$g_1''(x_1) = 0, \quad g_n''(x_{n+1}) = 0$$

Choice 2: Clamped splines

$$g_1'(x_1) = \alpha$$

$$g_n'(x_{n+1}) = \beta$$

α, β are given

Choice 3: Not-a-knot spline

$$g_1'''(x_2) = g_2'''(x_2)$$

$$g_{n-1}'''(x_n) = g_n'''(x_n)$$

With any of the choices above, the cubic spline interpolation can be uniquely determined.

$\hookrightarrow$ existence

$$A\underline{x} = \underline{b}$$

$\uparrow$ square and invertible

Example given 3 data points

j	x_j	y_j
1	0	1 $(= y_1)$
2	$\frac{1}{2}$	-1 $(= y_2)$
3	1	2 $(= y_3)$

Find the natural cubic spline interpolation

on $[x_1, x_2] = [0, \frac{1}{2}]$

$$g_1(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1 (x - x_1)^2$$

$$= a_1 + b_1 x + c_1 x^2 + d_1 x^2$$

on $[x_2, x_3] = [\frac{1}{2}, 1]$

$$g_2(x) = a_2 + b_2(x - x_2) + c_2(x - x_2)^2 + d_2(x - x_2)^3$$

$$= a_2 + b_2(x - \tfrac{1}{2}) + c_2(x - \tfrac{1}{2})^2 + d_2(x - \tfrac{1}{2})^3$$

property 1 $g_1(x_1) = y_1, \quad g_1(x_2) = y_2$

$$\Leftrightarrow \begin{cases} a_1 = y_1 = 1 \\ a_1 + \frac{1}{2} b_1 + \frac{1}{4} c_2 + \frac{1}{8} d_1 = -1 \end{cases}$$

$$g_2(\overset{\frac{1}{2}}{x_2}) = y_2, \quad g_2(\overset{1}{x_3}) = y_3$$

$$\boxed{\begin{aligned} a_2 &= y_2 = -1 \\ a_2 + \tfrac{1}{2} b_2 + \tfrac{1}{4} c_2 + \tfrac{1}{8} d_2 &= 2 \end{aligned}}$$

property 2: $g_1'(x_2) = g_2'(x_2)$

$$\Leftrightarrow \quad b_1 + 2c_1 x + 3d_1 x^2 \Big|_{x=\frac{1}{2}}$$

$$= b_2 + 2c_1\left(x - \tfrac{1}{2}\right) + 3d_2\left(x - \tfrac{1}{2}\right)^2 \Big|_{x=\frac{1}{2}}$$

$$\Leftrightarrow \quad \boxed{b_1 + c_1 + \tfrac{3}{4} d_1 = b_2}$$

property 3: $g_1''(x_2) = g_2''(x_2)$

$$\Leftrightarrow \quad 2c_1 + 6d_1 x \Big|_{x=\frac{1}{2}}$$

$$= 2c_2 + 6d_2\left(x - \tfrac{1}{2}\right) \Big|_{x=\frac{1}{2}}$$

$$\Leftrightarrow \quad \boxed{c_1 + \tfrac{3}{2} d_1 = c_2}$$

Natural spline boundary conditions

$$g_1''(x_1) = 0 \Rightarrow 2c_1 + 6d_1 x \Big|_{x=0} = \boxed{2c_1 = 0}$$

$$g_2''(x_3) = 0 \Rightarrow 2c_2 + 6d_2\left(x - \tfrac{1}{2}\right)\Big|_{x=1}$$

$$= \boxed{2c_2 + 3d_2 = 0}$$

$\vdots$

$a_1 = 1,\ a_2 = -1,\ c_1 = 0$

$\vdots$

$d_2 = -10 \quad d_1 = 10,\ b_1 = -6.5$

$b_2 = 1 \quad c_2 = 15$

$$\Rightarrow g(x) = \begin{cases} g_1(x) = 1 - 6.5x + 10x^3 & x \in [0, \frac{1}{2}] \\ g_2(x) = -1 + (x - \frac{1}{2}) + 15(x - \frac{1}{2})^2 \\ \qquad -10(x - \frac{1}{2})^3 \end{cases}$$

Remark: In general, with more data, all the unknowns can be expressed in terms of $\{c_j\}_{j=1}^{n}$, and $\underset{\sim}{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_{n} \end{bmatrix}$ satisfies a tri-diagonal system.

$$A\underset{\sim}{c} = \text{given}$$

Can be solved with $\rightarrow O(n)$

$O(n)$ computational complexity

§ 3-2-3 Interpolating errors

"use piecewise linear interpolation as an example"

Assume $\{(x_j, y_j)\}_{j=1}^{n+1}$ are from a given function $f(x)$, namely $y_j = f(x_j)$, we want to bound $|f(x) - g(x)|$, here $g(x)$ is a piecewise linear interpolation

Recall $g(x) = g_j(x)$ on $[x_j, x_{j+1})$

$g_j(x)$ is a polynomial of degree 1

$g_j(x_j) = y_j$, $g_j(x_{j+1}) = y_{j+1}$

Consider $[x_j, x_{j+1})$, $g_j(x)$ is a global interpolant for $\boxed{f(x) \text{ based on}}$ (x_j, y_j) (x_{j+1}, y_{j+1})

Based on the error of the global interpolant.

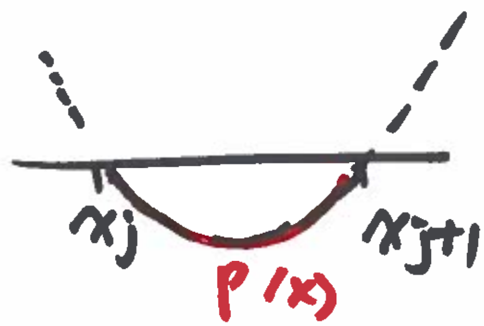
we know for $x \in [x_j, x_{j+1}]$

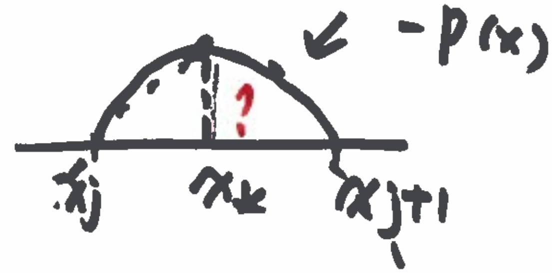
$$f(x) - g_j(x) = \overbrace{\frac{(x-x_j)(x-x_{j+1})}{2}}^{p(x)} f''(c)$$

c is some number from (x_j, x_{j+1})

$$\Leftrightarrow |f(x) - g_j(x)| \le \max_{x_j \le x \le x_{j+1}} |p(x)| \max_{s \in [x_j, x_{j+1}]} |f''(s)|$$

we next take a closer look at $p(x)$





$$(-p(x))' = 0 \Rightarrow x = x_* = \frac{x_j + x_{j+1}}{2}$$

$$(-p(x))''\Big|_{x=x_*} < 0$$

$$-p(x)\Big|_{x=x_*} = \frac{(\overbrace{x_{j+1} - x_j}^{h_j})^2}{8}$$

$$\Rightarrow \boxed{\begin{array}{l} \text{for } x \in [x_j, x_{j+1}] \\ |f(x) - g_j(x)| \le \dfrac{h_j^2}{8} \max\limits_{s \in [x_j, x_{j+1}]} |f''(s)| \end{array}}$$

Theorem: Given f on $[a,b]$, f, f', f'' are continuous, consider

$a = x_1 < x_2 < \cdots < x_{n+1} = b$

and the piecewise linear interpolant $g(x)$ of $f(x)$ $\quad h = \max_{1 \le j \le n} |x_{j+1} - x_j|$

then

$$\boxed{|f(x) - g(x)| \le \frac{1}{8} h^2 \max_{s \in [a,b]} |f''(s)| \quad \text{for } x \in [a,b]}$$

2nd order accuracy

$h \to h/2 \to h/4$

$\text{error} \to \text{error}/4 \to \frac{\text{error}}{16}$

Discussion: for what $f(x)$, the error in g is zero?

Answer: when $f(x)$ is linear (polynomial of degree 1) $\Leftarrow f''(x) \equiv 0$

§ 3-3 Least squares Solution and data fitting

Motivation:

1) $A \in \mathbb{R}^{m \times n}$, $m > n$ $\quad \underline{b} \in \mathbb{R}^{m}$

to solve $A\underline{x} = \underline{b}$ for $\underline{x} \in \mathbb{R}^{n}$

2) To overcome the possible issue of global interpolation (Runge phenomenon)

$\{(x_j, y_j)\}_{j=1}^{n+1}$

approximate data by $P_m(x)$, $m < n$

3) Represent or analyze a more scattered data set

Review

- matrix-vector multiplication

Given $A = (a_{ij}) \in \mathbb{R}^{m \times n}$

$$\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

we define $A\underline{x} = \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m$

as $b_1 = \sum_{j=1}^{n} a_{1j} x_j$

$b_2 = \sum_{j=1}^{n} a_{2j} x_j$

$\vdots$

A new view point

$$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \sum_{j=1}^{n} a_{2j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{mj} x_j \end{pmatrix}$$

$$= \sum_{j=1}^{n} x_j \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix} \leftarrow j\text{th column of } A$$

let $\underline{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$ be the jth column of A

then $\underline{b} = \boxed{A\underline{x}} = \sum_{j=1}^{n} x_j \underline{a}_j$

$= x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n$

That is $A\underline{x}$ is a linear combination of column vectors of A, with the coefficients being the entries of $\underline{x}$.

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notation

$A = [\underline{a}_1, \underline{a}_2, \cdots \underline{a}_n]$

$A\underline{x} = (\underline{a}_1, \underline{a}_2, \cdots, \underline{a}_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$= x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n$

Definition

Range of A, also denoted as range(A)

$$\text{range}(A) = \{x_1 \underline{a_1} + x_2 \underline{a_2} + \cdots + x_n \underline{a_n} : \forall x_i \in \mathbb{R},\ i = 1, \cdots n\}.$$

$$\subset \mathbb{R}^m$$

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linearly independent:

$\underline{a_1} \cdots \quad \underline{a_n}$

when none of them is a linear combination of the other

Or $\displaystyle\sum_{i=1}^{n} x_i \underline{a_i} = 0 \Leftrightarrow x_1 = x_2 \cdots = x_n = 0$

(no redundancy)