Homework 1
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1. Since we need to use the least manipulation as possible, we can use Horner's method:

$$\begin{aligned}p(x) &= a_7x^7 + a_{12}x^{12} + a_{17}x^{17} + a_{22}x^{22} + a_{27}x^{27}\\ &= x^7(a_7 + a_{10}x^5 + a_{17}x^{10} + a_{22}x^{15} + a_{27}x^{20})\\ &= x^7(a_7 + x^5(a_{10} + a_{15}x^5 + a_{22}x^{10} + a_{27}x^{15}))\\ &= x^7(a_7 + x^5(a_{10} + x^5(a_{15} + a_{22}x^5 + a_{27}x^{10})))\\ &= x^7(a_7 + x^5(a_{10} + x^5(a_{15} + x^5(a_{22} + a_{27}x^5))))\end{aligned}$$

We can easily observe that inside the outer-most parenthesis, it's a formula with typical Horner's method with x^5 instead of x, so besides x^5, there is 4 multiplications and 4 additions, and we need to multiple it as whole with x^7, so one more multiplication involved besides x^7 and x^5.

When we calculate x^7, we can use the "storage" method, and on the way, we calculate x^5.

$$\begin{aligned}x^2 &= x \cdot x\\ x^3 &= x \cdot x^2\\ x^5 &= x^2 \cdot x^3\\ x^7 &= x^5 \cdot x^2\end{aligned}$$

So that there is 4 more multiplications involved to carry out both x^5 and x^7, thus there are 9 multiplications and 4 additions in total.

2. Firstly, we can use myPolyEval function to evaluate $p(x)$ as:

```
vec = ones(100,1);
vec(1:2:100) = -1;
a = myPolyEval(1.00001, vec ,99)

a =

  -5.0025e-04
```

We can observe that it actually a sum of geometric sequence and $p(x) = \frac{1\cdot(1-(-x)^{100})}{1-(-x)} = \frac{1-x^{100}}{1+x}$, with $x = 1.00001$, so that we can calculate the answer with matlab, and get the answer of

```
b = (1-(1.00001)^(100))/(1+1.00001)

b =
```

```
  -5.0025e-04
```

And the error follows to be:

```
error = a - b

error =

  -1.7130e-16
```

3. (a)

$$\begin{aligned}(101.101)_2 &= 1\cdot 2^2 + 0\cdot 2^1 + 1\cdot 2^{-1} + 0\cdot 2^{-2} + 1\cdot 2^{-3}\\ &= 4+0+1+\frac{1}{2}+0+\frac{1}{8}\\ &= 5+\frac{5}{8}\\ &= \boxed{\frac{45}{8}}\end{aligned}$$

(b) $(10.\overline{101})_2$:

Let $x = (0.\overline{101})_2$, $2^3\cdot x = (101.\overline{101})_2$, so that $(2^3-1)x = (101)_2 = 1\cdot 2^2+1 = 5$, and $x=\frac{5}{7}$ follows.

$$\begin{aligned}\therefore (10.\overline{101})_2 &= (10)_2 + (0.\overline{101})_2\\ &= 2+\frac{5}{7}\\ &= \boxed{\frac{19}{7}}\end{aligned}$$

4. (a) In double precision computer arithmetic, the first step is $2^{-51}+2^{-52}$, which is : $(1.\boxed{10\cdots 0})_2\times 2^{-51}$, and now, we add 2^{-54} to it, which is

$$\begin{array}{rr} & (0.\boxed{000\cdots 00011}00)_2\times 2^0\\ + & (0.\boxed{000\cdots 00000}01)_2\times 2^0\\ \hline & (0.\boxed{000\cdots 00011}01)_2\times 2^0\end{array}$$

Since the 53^{th} bit in the sum is 0, according to the rounding rule, the sum is $(0.\boxed{000\cdots 00011})_2\times 2^0$, and now, we add 1 to the sum:

$$\begin{array}{rr} & (1.\boxed{000\cdots 00000})_2\times 2^0\\ + & (0.\boxed{000\cdots 00011})_2\times 2^0\\ \hline & (1.\boxed{000\cdots 00011})_2\times 2^0\end{array}$$

which is $1+3\times 2^{-52}$, and finally, we subtract 1 and get 3×2^{-52}, which is $3\epsilon_{\text{mach}}$.

And we can check with Matlab:

```
(1+(2^(-51)+2^(-52)+2^(-54)))-1

ans =

   6.6613e-16

ans-3*2^(-52)

ans =

     0

diary off
```

5. Firstly, we need to convert 4.9 and 3.9 into binary form, since $4.9 = 4 + 0.9$, $3.9 = 3 + 0.9$, we need to find the binary form of $3, 4, 0.9$:

   it's easy to observe that $4 = 2^2 = (100)_2$, $3 = 2 + 1 = (11)_2$, and as for 0.9:

$$\begin{aligned}
0.9 \times 2 &= 0.8 + 1 \\
0.8 \times 2 &= 0.6 + 1 \\
0.6 \times 2 &= 0.2 + 1 \\
0.2 \times 2 &= 0.4 + 0 \\
0.4 \times 2 &= 0.8 + 0 \\
0.8 \times 2 &= 0.6 + 1 \\
&\cdots
\end{aligned}$$

   Thus, $0.9 = 0.1\overline{1100}$

   So that $4.9 = (100.1\overline{1100})_2$, $3.9 = (11.1\overline{1100})_2$, in double precision format,

$$\begin{aligned}
(100.1\overline{1100})_2 &= (1.\boxed{00111001100 \cdots 110011001}\,\overline{1001})_2 \times 2^2 \\
&= (1.\boxed{00111001100 \cdots 110011010})_2 \times 2^2 \\
(11.1\overline{1100})_2 &= (1.\boxed{11110011001 \cdots 100110011}\,\overline{0011})_2 \times 2 \\
&= (1.\boxed{11110011001 \cdots 100110011})_2 \times 2
\end{aligned}$$

So that

$$
\begin{aligned}
4.9 - 3.9 &= (1.\boxed{00111001100\cdots110011010})_2 \times 2^2 \\
&\quad - (1.\boxed{11110011001\cdots100110011})_2 \times 2 \\
&= (1.\boxed{00111001100\cdots110011010}0)_2 \times 2^2 \\
&\quad - (0.\boxed{111110011001\cdots10011001}1)_2 \times 2^2 \\
&= (0.\boxed{0100\cdots00}1)_2 \cdot 2^2 \\
&= (1.\boxed{00\cdots0010})_2 \\
&= 1 + 2^{-51} \\
\therefore (4.9 - 3.9) - 1 &= 2^{-51}
\end{aligned}
$$

And we can check with Matlab:

```
(4.9 - 3.9) -1

ans =

   4.4409e-16

ans - 2^(-51)

ans =

     0

diary off
```

6. Propose the alternative form of $f(h)$:

$$
\begin{aligned}
f(h) &= \frac{x^4 - (x-h)^4}{h} \\
&= \frac{(x^2 - (x-h)^2)(x^2 + (x-h)^2)}{h} \\
&= \frac{x^2 - x^2 + 2xh - h^2}{h}(x^2 + (x-h)^2) \\
&= (2x - h)(x^2 + (x-h)^2)
\end{aligned}
$$

And we can use the following code and get the following data:

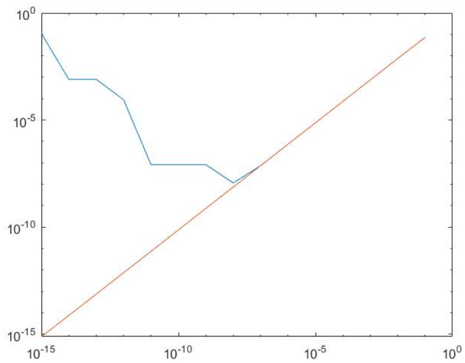
Listing 1: derivative.m

```matlab
%% This program aims to produce a table of input value h
%% and the answer for two ways to calculate its derivative
%% and the error for them between the analytical answer

hr = 1*(10^(-1)).^(1:15); % Create the vector of h

hrt = transpose(hr); % Transpose it to make it convinient to look up in table

[yr1, yr2] = derivative1(2,hrt); % The answer by two methods


% Calculate the error from the analytical answer
error1 = abs(yr1 - 32)/32;

error2 = abs(yr2 -32)/32;

loglog(hrt, error1);
hold on;
loglog(hrt, error2);

function [y1, y2] = derivative1(x, h)

y1 = (x ^4 - (x - h).^4) .* h .^(-1);

y2 = (2 * x + (-h)) .* ( x^2 + (x -h).^2);

end
```

and the following data:

```
 derivative
 T = table(hrt, yr1, yr2, error1, error2) % Display the table

T =

  15 5  table

     hrt       yr1       yr2        error1         error2
    ______    ______    ______    __________    __________

       0.1    29.679    29.679      0.072531      0.072531
      0.01    31.761    31.761      0.007475      0.007475
     0.001    31.976    31.976    0.00074975    0.00074975
    0.0001    31.998    31.998    7.4997e-05    7.4998e-05
     1e-05        32        32       7.5e-06       7.5e-06
     1e-06        32        32    7.5009e-07       7.5e-07
```

1e-07	32	32	7.4356e-08	7.5e-08
1e-08	32	32	1.1629e-08	7.5e-09
1e-09	32	32	8.274e-08	7.5e-10
1e-10	32	32	8.274e-08	7.5e-11
1e-11	32	32	8.274e-08	7.5e-12
1e-12	32.003	32	8.8901e-05	7.5007e-13
1e-13	31.974	32	0.00079928	7.494e-14
1e-14	31.974	32	0.00079928	7.5495e-15
1e-15	35.527	32	0.11022	7.7716e-16



Since analytically $\lim_{h\to 0} f(h) = \dfrac{dx^4}{dx} = 4x^3 = 32$ when $x = 2$, and from the figure and the table above, we see that the new method is getting closer to 32 as h approaches 0, but as for the original method, the value of computing value firstly getting closer to 32, but when $h < 10^{-8}$, the error become increasing, and finally get an error of 0.1102, this is caused by the subtraction between two similar value (x^4 and $(x-h)^4$ as $h \to 0$).

7. (a)

$$
\begin{aligned}
b^2 &= (1.234 \times 10^5)^2 \\
&= 1.522756 \times 10^{10} \\
\text{myfl}(b^2) &= 1.523 \times 10^{10} \\
\text{myfl}(4a) &= 4 \\
\text{myfl}(4ac) &= \text{myfl}(\text{myfl}(4a) \times c) \\
&= \text{myfl}(4 \cdot 4.567 \times 10^3) \\
&= \text{myfl}(1.8268 \times 10^4) \\
&= 1.827 \times 10^4
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{myfl}(b^2 - 4ac) &= \mathrm{myfl}(\mathrm{myfl}(b^2) - \mathrm{myfl}(4ac)) \\
&= \mathrm{myfl}(1.523 \times 10^{10} - 1.824 \times 10^4) \\
&= \mathrm{myfl}(1.522998173 \times 10^{10}) \\
&= 1.523 \times 10^{10} \\
\mathrm{myfl}(\sqrt{b^2 - 4ac}) &= \mathrm{myfl}(\sqrt{\mathrm{myfl}(b^2 - 4ac)}) \\
&= \mathrm{myfl}(\sqrt{1.523 \times 10^{10}}) \\
&= \mathrm{myfl}(1.23409 \cdots \times 10^5) \\
&= 1.234 \times 10^5 \\
\mathrm{myfl}(-b) &= -1.234 \times 10^5 \\
\mathrm{myfl}(-b - \sqrt{b^2 - 4ac}) &= -1.234 \times 10^5 - 1.234 \times 10^5 \\
&= -2.468 \times 10^5 \\
\mathrm{myfl}(2a) &= 2 \\
\mathrm{myfl}\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) &= -1.234 \times 10^5 \\
\mathrm{myfl}(-b + \sqrt{b^2 - 4ac}) &= -1.234 \times 10^5 + 1.234 \times 10^5 \\
&= 0 \\
\mathrm{myfl}\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right) &= 0 \\
\therefore \hat{x_1} = -1.234 \times 10^5, \hat{x_2} = 0
\end{aligned}
$$

And we can use Matlab to get the "exact" roots and compare the error:

```
a = 1;
b = 1.234*10^5;
c = 4.567*10^3;
x1 = (-b - sqrt(b^2 - 4*a*c))/(2*a);
x2 = (-b + sqrt(b^2 - 4*a*c))/(2*a);
x11 = -1.234*10^5;
x22 = 0;
error1 = abs(x1 -x11)/abs(x1)

error1 =

   2.9992e-07
error2 = abs(x2 -x22)/abs(x2)

error2 =

     1
diary off
```

We can observe that $\hat{x_1}$ is approximately accurate, but $\hat{x_2}$ is not so accurate.

(b) Since $\hat{x_2}$ is the one with larger error, we can use

$$\begin{aligned} x_2 &= \frac{-b+\sqrt{b^2-4ac}}{2a} \\ &= \frac{-2c}{b+\sqrt{b^2-4ac}} \end{aligned}$$

Since $\text{myfl}(b) = 1.234 \times 10^5$, $\text{myfl}(\sqrt{b^2-4ac}) = 1.234 \times 10^5$, so that $\text{myfl}(b+\sqrt{b^2-4ac}) = 2.468 \times 10^5$.

Since $\text{myfl}(-2c) = \text{myfl}(-2 \times 4.567 \times 10^3) = -9.134 \times 10^3$,

$$\begin{aligned} \therefore \frac{\text{myfl}(-2c)}{\text{myfl}(b+\sqrt{b^2-4ac})} &= \frac{-9.134 \times 10^3}{2.468 \times 10^5} \\ &= -3.7009\cdots \times 10^{-2} \\ \therefore \hat{x_2} &= -3.701 \times 10^{-2} \end{aligned}$$

And we can check with Matlab that the error reduces a lot:

```
x2 = (-2*c)/(b+sqrt(b^2-4*a*c));
x22 = -3.701 *10^(-2);
error2 = abs(x2 -x22)/abs(x2)

error2 =

   7.1448e-06

diary off
```