Homework 3
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1. (a)

$$\begin{aligned}
g(x) &= \frac{2x-1}{x^2} \\
&= (2x-1)\cdot x^{-2} \\
g'(x) &= 2\cdot x^{-2} - 2x^{-3}\cdot(2x-1) \\
&= 2\cdot x^{-2} - 4x^{-2} + 2x^{-3} \\
&= -2x^{-2} + 2x^{-3}
\end{aligned}$$

$$\text{At the point } x = r = 1,\ |g'(1)| = |0| < 1$$

$$\therefore g(x) \text{ is locally convergent.}$$

(b)

$$\begin{aligned}
g(x) &= \cos x + \pi + 1 \\
g'(x) &= -\sin x
\end{aligned}$$

$$\text{At the point } x = r = \pi,\ |g'(\pi)| = |-\sin\pi| = 0 < 1$$

$$\therefore g(x) \text{ is locally convergent.}$$

(c)

$$\begin{aligned}
g(x) &= e^{2x} - 1 \\
g'(x) &= 2e^{2x}
\end{aligned}$$

$$\text{At the point } x = r = 0,\ |g'(0)| = |2| > 1$$

$$\therefore g(x) \text{ is not locally convergent.}$$

2. To find the fixed point for $g(x) = x^2 - \frac{3}{2}x + \frac{3}{2}$, which means we need to solve the equation for $x = g(x)$, which means

$$
\begin{aligned}
x &= g(x) \\
x &= x^2 - \frac{3}{2}x + \frac{3}{2} \\
x^2 - \frac{5}{2}x + \frac{3}{2} &= 0 \\
2x^2 - 5x + 3 &= 0 \\
(x-1)(2x-3) &= 0 \\
x_1 = 1\ x_2 &= \frac{3}{2} \\
\because g'(x) &= 2x - \frac{3}{2} \\
|g'(1)| &= \left|2 - \frac{3}{2}\right| \\
&= \frac{1}{2} < 1 \\
\left|g'\left(\frac{3}{2}\right)\right| &= \left|3 - \frac{3}{2}\right| \\
&= \frac{3}{2} > 1
\end{aligned}
$$

$\therefore$ At the point $x = 1,\ g(x)$ is locally convergent.

At the point $x = \frac{3}{2},\ g(x)$ is not locally convergent.

3.

$$2x^3 - x + e^x = 0$$

- $x = 2x^3 + e^x$
- $$
\begin{aligned}
2x^3 &= -e^x + x \\
x^3 &= \frac{x - e^x}{2} \\
x &= \sqrt[3]{\frac{x - e^x}{2}}
\end{aligned}
$$
- $$
\begin{aligned}
e^x &= x - 2x^3 \\
x &= \ln(x - 2x^3)
\end{aligned}
$$

4. (a) Firstly, check the convergence or not:

$$\begin{aligned} g(x) &= \frac{1}{2}x + \frac{1}{x} \\ x &= \frac{1}{2}x + \frac{1}{x} \\ \frac{1}{2}x &= \frac{1}{x} \\ x^2 &= 2 \end{aligned}$$

And $x = \sqrt{2}$ is a solution follows, and we can check the convergence rate:

$$\begin{aligned} g'(x) &= \frac{1}{2} - \frac{1}{x^2} \\ g'(\sqrt{2}) &= \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

(b) Firstly, check the convergence or not:

$$\begin{aligned} g(x) &= \frac{2}{3}x + \frac{2}{3x} \\ x &= \frac{2}{3}x + \frac{2}{3x} \\ \frac{1}{3}x &= \frac{2}{3x} \\ x^2 &= 2 \end{aligned}$$

And $x = \sqrt{2}$ is a solution follows, and we can check the convergence rate:

$$\begin{aligned} g'(x) &= \frac{2}{3} - \frac{2}{3x^2} \\ g'(\sqrt{2}) &= \frac{2}{3} - \frac{2}{6} = \frac{1}{3} \end{aligned}$$

(c) Firstly, check the convergence or not:

$$\begin{aligned} g(x) &= \frac{3}{4}x + \frac{1}{2x} \\ x &= \frac{3}{4}x + \frac{1}{2x} \\ \frac{1}{4}x &= \frac{1}{2x} \\ x^2 &= 2 \end{aligned}$$

And $x = \sqrt{2}$ is a solution follows, and we can check the convergence rate:

$$g'(x) = \frac{3}{4} - \frac{1}{2x^2}$$
$$g'(\sqrt{2}) = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$$

So the rank is $A > B > C$

5. (a) The original system can be expressed as

$$\begin{bmatrix} 1 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

And we can change it to diagonally dominant by

$$\begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix}$$
$$D = \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}, D^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$
$$U = \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$
$$U + L = \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}, D + L = \begin{bmatrix} 5 & 0 \\ 1 & 3 \end{bmatrix}$$
$$(D+L)^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{1}{15} & \frac{1}{3} \end{bmatrix}$$

- Jacobi Method

$$\begin{aligned}
x^{(1)} &= D^{-1}(b-(U+L)x^{(0)}) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{6}{5} \\ -\frac{1}{3} \end{bmatrix} \\
x^{(2)} &= D^{-1}(b-(U+L)x^{(1)}) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{6}{5} \\ -\frac{1}{3} \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} -\frac{4}{3} \\ \frac{6}{5} \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{22}{3} \\ -\frac{11}{5} \end{bmatrix} \\
&= \begin{bmatrix} \frac{22}{15} \\ -\frac{11}{15} \end{bmatrix}
\end{aligned}$$

- Gauss - Seidel Method

$$\begin{aligned}
x^{(1)} &= (D+L)^{-1}(b-Ux^{(0)}) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{1}{15} & \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{6}{5} \\ -\frac{11}{15} \end{bmatrix} \\
x^{(2)} &= (D+L)^{-1}(b-Ux^{(1)}) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{1}{15} & \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{6}{5} \\ -\frac{11}{15} \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{1}{15} & \frac{1}{3} \end{bmatrix} \left(\begin{bmatrix} 6 \\ -1 \end{bmatrix} - \begin{bmatrix} -\frac{44}{15} \\ 0 \end{bmatrix} \right) \\
&= \begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{1}{15} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{134}{15} \\ -1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{134}{75} \\ -\frac{209}{225} \end{bmatrix}
\end{aligned}$$

(b) The original problem can be expressed as

$$\begin{bmatrix} 1 & -8 & -2 \\ 1 & 1 & 5 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

And we can change it to diagonally dominant by

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

And we can get

$$\begin{aligned} u &= \frac{-2 + v - w}{3} \\ v &= \frac{u - 2w - 1}{8} \\ w &= \frac{4 - u - v}{5} \end{aligned}$$

- Jacobi Method

$$\begin{aligned} x^{(1)} &= \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{8} \\ \frac{4}{5} \end{bmatrix} \\ x^{(2)} &= \begin{bmatrix} \frac{-2+v^1-w^1}{3} \\ \frac{u^1-2w^1-1}{8} \\ \frac{4-u^1-v^1}{5} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{-2 + \frac{-1}{8} - \frac{4}{5}}{3} \\ \dfrac{\frac{-2}{3} - 2 \cdot \frac{4}{5} - 1}{8} \\ \dfrac{4 - \frac{-2}{3} - \frac{-1}{8}}{5} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{117}{120} \\ -\frac{49}{120} \\ \frac{23}{24} \end{bmatrix} \end{aligned}$$

- Gauss - Seidel Method

$$x^{(1)} = \begin{bmatrix} \frac{-2+v^0-w^0}{3} \\ \frac{u^1-2w^0-1}{8} \\ \frac{4-u^1-v^1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} \\ \frac{-\frac{2}{3}-1}{8} \\ \frac{4+\frac{2}{3}-v^1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} \\ -\frac{5}{24} \\ \frac{4+\frac{2}{3}+\frac{5}{24}}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2}{3} \\ -\frac{5}{24} \\ \frac{39}{40} \end{bmatrix}$$

$$x^{(2)} = \begin{bmatrix} \frac{-2+v^1-w^1}{3} \\ \frac{u^2-2w^1-1}{8} \\ \frac{4-u^2-v^2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2-\frac{5}{24}-\frac{39}{40}}{3} \\ \frac{u^2-2\frac{39}{40}-1}{8} \\ \frac{4-u^2-v^2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{191}{180} \\ \frac{-\frac{191}{180}-2\frac{39}{40}-1}{8} \\ \frac{4+\frac{191}{180}-v^2}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{191}{180} \\ -\frac{361}{720} \\ \frac{4+\frac{191}{180}+\frac{361}{720}}{5} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{191}{180} \\ -\frac{361}{720} \\ \frac{89}{80} \end{bmatrix}$$

(c) The original problem can be expressed as

$$\begin{bmatrix} 1 & 4 & 0 \\ 0 & 1 & 2 \\ 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 0 \end{bmatrix}$$

And we can change it to diagonally dominant by

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

And we can get

$$\begin{aligned} u &= -\frac{3w}{4} \\ v &= \frac{5-u}{4} \\ w &= \frac{2-v}{2} \end{aligned}$$

- Jacobi Method

$$\begin{aligned} x^1 &= \begin{bmatrix} -\frac{3w^0}{4} \\ \frac{5-u^0}{4} \\ \frac{2-v^0}{2} \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \frac{5}{4} \\ 1 \end{bmatrix} \\ x^2 &= \begin{bmatrix} -\frac{3w^1}{4} \\ \frac{5-u^1}{4} \\ \frac{2-v^1}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3\cdot 1}{4} \\ \frac{5}{4} \\ \frac{2-\frac{5}{4}}{2} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{4} \\ \frac{5}{4} \\ \frac{3}{8} \end{bmatrix} \end{aligned}$$

- Gauss - Seidel Method

$$x^1 = \begin{bmatrix} -\frac{3w^0}{4} \\ \frac{5-u^1}{4} \\ \frac{2-v^1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{4} \\ \frac{2-\frac{5}{4}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{4} \\ \frac{3}{8} \end{bmatrix}$$

$$x^2 = \begin{bmatrix} -\frac{3w^1}{4} \\ \frac{5-u^2}{4} \\ \frac{2-v^2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3\cdot\frac{3}{8}}{4} \\ \frac{5-u^2}{4} \\ \frac{2-v^2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{9}{32} \\ \frac{5+\frac{9}{32}}{4} \\ \frac{2-v^2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{9}{32} \\ \frac{169}{128} \\ \frac{2-\frac{169}{128}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{9}{32} \\ \frac{169}{128} \\ \frac{87}{256} \end{bmatrix}$$

6. (a) Jacobi Method:

- $$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} x = b$$
$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x^k + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x^{k-1} = b$$
$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x^k = b - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x^{k-1}$$
$$\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x^k = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(b - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x^{k-1} \right)$$
$$x^k = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} b - \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix} x^{k-1}$$
$$B = -\begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{2} & 0 \end{bmatrix}$$

- $$\begin{vmatrix} -\lambda & -\frac{1}{3} \\ -\frac{1}{2} & -\lambda \end{vmatrix} = 0$$
$$\lambda^2 = \frac{1}{6}$$
$$\rho(B) = \frac{\sqrt{6}}{6}$$

- Since $\rho(B) = \frac{\sqrt{6}}{6} < 1$, so converge.

- $$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} x = b$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^k + \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} x^{k-1} = b$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x^k = b - \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} x^{k-1}$$

$$x^k = b - \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix} x^{k-1}$$

$$B = -\begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 3 \\ 2 & -\lambda \end{vmatrix} = 0$$

$$\lambda^2 = 6$$

$$\rho(B) = \sqrt{6}$$

Since $\rho(B) = \sqrt{6} > 1$, so diverge.

(b) Gauss - Seidel Method

- $$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} x = b$$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} x^k + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^{k-1} = b$$

$$\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} x^k = -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^{k-1} + b$$

$$\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} x^k = -\begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x^{k-1} + \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix} b$$

$$x^k = \begin{bmatrix} 0 & -\frac{1}{3} \\ 0 & \frac{1}{6} \end{bmatrix} x^{k-1} + \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{1}{6} & \frac{1}{2} \end{bmatrix} b$$

$$B = \begin{bmatrix} 0 & -\frac{1}{3} \\ 0 & \frac{1}{6} \end{bmatrix}$$

- $$\begin{vmatrix} -\lambda & -\frac{1}{3} \\ 0 & \frac{1}{6}-\lambda \end{vmatrix} = 0$$
$$\lambda = 0, \frac{1}{6}$$
$$\rho(B) = \frac{1}{6}$$

- Since $\rho(B) = \frac{1}{6} < 1$, so converge.

- $$\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} x = b$$
$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x^k + \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x^{k-1} = b$$
$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x^k = -\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x^{k-1} + b$$
$$\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} x^k = -\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} x^{k-1} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} b$$
$$x^k = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix} x^{k-1} + \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} b$$
$$B = \begin{bmatrix} 0 & 2 \\ 0 & -6 \end{bmatrix}$$
$$\begin{vmatrix} -\lambda & 2 \\ 0 & -6-\lambda \end{vmatrix} = 0$$
$$\lambda = 0, -6$$
$$\rho(B) = 6$$

  Since $\rho(B) = 6 > 1$, so diverge.