Lecture 10 : 2.17.2019

§ 2-2 Direct method

Gaussian Elimination (GE) and LU factorization.

Given $A \in \mathbb{R}^{n \times n}$, $\underline{b} \in \mathbb{R}^n$

to solve $A\underline{x} = \underline{b}$ for $x \in \mathbb{R}^n$.

$\uparrow$ invertible

Direct method

special case. A : triangular

$$\begin{pmatrix} \times & \times & \times \\ & \times & \times \\ & & \times \end{pmatrix} \qquad \begin{bmatrix} \times & & \\ \times & \times & \\ \times & \times & \times \end{bmatrix}$$

$\uparrow$ back substitution n^2

$\uparrow$ forward sub... n^2

general case $\qquad A\underline{x} = \underline{b}$

$$\underbrace{\begin{pmatrix} x & x & x \\ x & x & x \\ x & x & x \end{pmatrix}}_{A} \longrightarrow \begin{pmatrix} x & x & x \\ 0 & \boxed{x} & \boxed{x} \\ x & x & x \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} x & x & x \\ 0 & \boxed{x} & \boxed{x} \\ 0 & \boxed{x} & \boxed{x} \end{pmatrix}$$

$$\downarrow$$

$$\underbrace{\begin{pmatrix} x & x & x \\ 0 & \boxed{x} & \boxed{x} \\ 0 & 0 & x \end{pmatrix}}_{U}$$

$$U\underline{x} = \underline{\hat{b}}$$

$$A \longrightarrow U \qquad (A = LU) \Leftarrow$$

next, to understand this

$$\text{cost} \approx \frac{2}{3}n^3$$

Revisit matrix multiplication

Given $A = (a_{ij}) \in \mathbb{R}^{n\times n}$

$B = (b_{ij}) \in \mathbb{R}^{n\times n}$

we know $C = AB = (c_{ij}) \in \mathbb{R}^{n\times n}$

where $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$, next to establish a new view-point

$$B = \begin{array}{c} \\ \\ \\ \\ \end{array}\left[\begin{array}{cccc} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{array}\right]\begin{array}{l} \text{first row } \underline{\beta}_1^T \\ \text{2nd row } \underline{\beta}_2^T \\ \\ \text{nth row } \underline{\beta}_n^T \end{array}$$

$$\underline{\beta}_j = \begin{bmatrix} \beta_{j1} \\ \beta_{j2} \\ \vdots \\ \beta_{jn} \end{bmatrix}$$

Take a closer look at the ith row of C

$$\boxed{[c_{i1}, c_{i2}, \cdots c_{in}]}$$

$$= \left[\sum_{k=1}^{n} a_{ik} b_{k1}, \sum_{k=1}^{n} a_{ik} b_{k2}, \cdots \sum_{k=1}^{n} a_{ik} b_{kn}\right]$$

$$= \sum_{k=1}^{n} a_{ik} \underbrace{[b_{k1}, b_{k2}, \cdots b_{kn}]}_{\text{kth row of } B \text{, or } \underline{\beta}_k^T}$$

$$\boxed{= a_{i1}\underline{\beta}_1^T + a_{i2}\underline{\beta}_2^T + \cdots + a_{in}\underline{\beta}_n^T}$$

C = AB : "a new viewpoint"

ith row of C is a linear combination of the rows of B, with the coefficients from ith row of A

Example :

$$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 9 & 10 \end{bmatrix}$$

A B C

1st row of C

$1 \cdot [3, 5] + 0 \cdot [1, 0] = [3, 5]$

2nd row of C

$2 \cdot [3, 5] + 3 \cdot [1, 0]$

$= (9, 10)$

Exercises. $AB = C$,

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$ ← fixed. To compute C, when

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c & 1 \end{bmatrix}$$

more exercises

- $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix} \overset{?}{=}$

- $\begin{bmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ c & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & d & 1 \end{bmatrix} \overset{?}{=}$

- $$\begin{bmatrix} & ? & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} b_{21}-b_{11} & b_{22}-b_{12} & b_{23}-b_{13} \\ b_{31} & b_{32} & b_{33} \\ 2b_{21} & 2b_{22} & 2b_{23} \end{bmatrix}$$

How to use matrix multiplication to exchange row 1 and row 2 of $B \in \mathbb{R}^{4\times 4}$.

$\boxed{\text{Answers}}$: $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = B \qquad C = AB$

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad C = \begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4-c & 5-2c & 6-3c \\ 7 & 8 & 9 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & & \\ 0 & 1 & \\ -c & 1 & \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7-c & 8-2c & 9-3c \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -c, & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7-4c & 8-5c & 9-6c \end{pmatrix}$$

$\boxed{\text{Answers}}$ to "more exercise"

$$\begin{bmatrix} 1 & & \\ 0 & 1 & \\ -c & & 1 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ c & & 1 \end{bmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

they are inverse to each other

identity

$$\begin{pmatrix} 1 & & \\ b & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ c & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & 1 & \\ & d & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ b & 1 & \\ c & d & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} B = \begin{pmatrix} b_{21}-b_{11} & b_{22}-b_{12} & b_{23}-b_{13} \\ b_{31} & b_{32} & b_{33} \\ 2b_{21} & 2b_{22} & 2b_{23} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ to exchange row 1, row 2 of $B \in \mathbb{R}^{4\times 4}$

To understand G E

go back to the example $Ax = b$ where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 5 & -1 & -1 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$x = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\text{system} \Leftrightarrow \begin{cases} u + v + w = 1 & \cdots (1) \\ 2u + 4v + w = 0 & \cdots (2) \\ 5u - v - w = 2 & \cdots (3) \end{cases}$$

$(2) - 2 \cdot (1)$

$$\Rightarrow 2v - w = -2 \qquad \cdots (2)'$$

$(3) - 5 \cdot (1)$

$$\Rightarrow -6v - 6w = -3 \qquad \cdots (3)'$$

finally $\quad (3)' - (-3)(2)'$

$$-9w = -9$$

$$\Rightarrow \begin{cases} u + v + w = 1 \\ 2v - w = -2, \quad -9w = -9 \end{cases}$$

Tableau form

A

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 4 & 1 & 0 \\ 5 & -1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_2 - 2R_1 \to R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -2 \\ 5 & -1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_3 - 5R_1 \to R_3}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & -6 & -6 & -3 \end{array}\right]$$

$$\xrightarrow{R_3 - (-3)R_2 \to R_3}$$

upper-triangular

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -2 \\ 0 & 0 & -9 & -9 \end{array}\right)$$

U

Express every action using matrix notation

$$\begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\-5&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}A$$

$$= U = \begin{pmatrix}1&1&1\\0&2&-1\\0&0&-9\end{pmatrix}$$

$\Leftrightarrow$

$$A = \begin{pmatrix}1&0&0\\-2&1&0\\0&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\-5&0&1\end{pmatrix}^{-1}\begin{pmatrix}1&0&0\\0&1&0\\0&3&1\end{pmatrix}^{-1}U$$

$$= \underbrace{\begin{pmatrix}1&0&0\\2&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\5&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&-3&1\end{pmatrix}}U$$

$$= \underbrace{\begin{pmatrix}1&0&0\\2&1&0\\5&-3&1\end{pmatrix}}_{L}U$$

That is GE $\Leftrightarrow$ $A = LU$

Once we have LU factorization of A

namely $A = LU$

then $A\underline{x} = \underline{b}$

$\Leftrightarrow \quad L\underbrace{U\underline{x}}_{\underline{y}} = \underline{b}$ ---- Cost $\frac{2}{3}n^3$

$\Leftrightarrow \begin{cases} \text{Solve } L\underline{y} = \underline{b} \text{ for } \underline{y} & n^2 \\ \text{Solve } U\underline{x} = \underline{y} \text{ for } \underline{x}. & n^2 \end{cases}$

GE method does not always work

typical step

$$\text{Row } i - \frac{a_{ij}}{a_{jj}} \text{Row } j \to \text{Row } i$$

a_{jj} → pivot

when pivot $= 0$, the method fails.

Example
$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

To solve $A\underline{x} = \underline{b}$ $\qquad \underline{x} = \begin{pmatrix} b_2 \\ b_1 \end{pmatrix}$

GE fails at the first step.

Remedy: "pivoting" by exchanging equations

exchanging rows.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underline{x} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

Exchange rows, by left multiplying $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

or $PA\,\underline{x} = P\underline{b}$

Apply GE to this and the new pivot $= 1$

Example:

$$A = \begin{bmatrix} 0 & 5 & 6 \\ 1 & 2 & 3 \\ -7 & 8 & 9 \end{bmatrix}$$ To solve $A\underline{x} = \underline{b}$ by GE.

$a_{11} = 0$, GE fails.

"pivoting!"

Strategy 1: exchange row 1 and row 2.

$$PA\underline{x} = P\underline{b}$$

where $P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightarrow PA = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ -7 & 8 & 9 \end{bmatrix}$

Strategy 2: exchange row 1 and row 3

$$PA\underline{x} = P\underline{b}$$

$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $PA = \begin{pmatrix} -7 & 8 & 9 \\ 1 & 2 & 3 \\ 0 & 5 & 6 \end{pmatrix}$

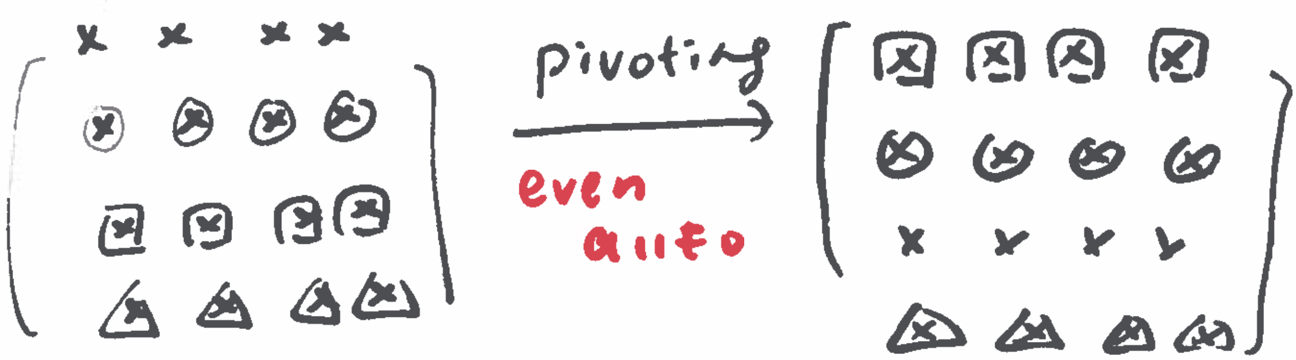
Strategy 2 is preferred in practice

pivot: is preferred to have the largest possible magnitude.

Overall, the reason to choose a pivot with largest possible magnitude:

to ensure the algorithm to be **Less sensitive** to rounding error ((to be illustrated next)).

GE with pivoting (In practice)



$$\xrightarrow[\text{for 1st column}]{GE} \begin{pmatrix} \boxed{x} & \boxed{x} & \boxed{x} & \boxed{x} \\ 0 & + & + & + \\ 0 & + & + & + \\ 0 & + & + & + \end{pmatrix}$$

repeat the process to the part in the box.

$\{$ pivoting $\rightarrow$ GE for 2nd column $\}$