Exam 2: MATH/CSCI-4800-02 Numerical Computing

March 28, 2019

NAME:

Instruction: (i) Write your names on the first and the last pages;
(ii) There are in total 120 points. The points one gets above 100 will be counted as extra credit;
(iii) The parameter n or m in this exam is some positive integer.
(iv) One-page one-side hand-written crib sheet can be used, and its size should be no larger than standard printing paper. Calculators are not allowed.
(v) Justify your answers with sufficient details to avoid partial or no points.

1. (30 points, with 5 points for each sub-problem) For each of the following statements, state it is true (or T) or false (or F). No proof or any argument is needed to support your answer.

   1.a) We apply Gaussian elimination (GE) to a matrix $A \in \mathbb{R}^{n\times n}$ to get its LU factorization. If A is invertible, then GE can not fail. However, if A is not invertible, then GE may fail.

   1.b) Given an invertible matrix $A \in \mathbb{R}^{n\times n}$ and a vector $b \in \mathbb{R}^n$. Suppose $x \in \mathbb{R}^n$ be the exact solution to $Ax = b$. Apply either a direct method discussed in class or the Matlab backslash, one can solve $Ax = b$ numerically and get a solution $x_c \in \mathbb{R}^n$. Then it is possible to have $\|x - x_c\|_2 \approx 10^{-4}$. For instance, this can happen when the condition number of A is very large. *Here it is assumed that the computation is carried out following the IEEE double precision floating point arithmetic.*

   1.c) Let $(x_j, y_j), j = 1, 2, \cdots, n+1$ be given. One can then always find a polynomial $p(x)$ of degree no bigger than n, such that $p(x_j) = y_j$, $j = 1, 2, \cdots, n+1$. This interpolating polynomial is also unique.

   1.d) Given a matrix $A \in \mathbb{R}^{m\times n}$ and a vector $b \in \mathbb{R}^m$. Assume $m > n$. We know $Ax = b$ is not always solvable. But if one instead looks for a solution in the least squares sense, then the problem is always solvable.

   1.e) Let $\{(x_j, y_j), j = 1, 2, \cdots, n+1\}$ be sampled from a given function $f(x)$, with $y_j = f(x_j)$ and $x_j \in [a, b]$ for each j. Let $F(x)$ be an interpolating polynomial of degree n, satisfying $F(x_j) = y_j, j = 1, 2, \cdots, n+1$. Let $G(x)$ be the least squares fitted polynomial of degree m with $m < n$. Then the interpolating error $|f(x) - F(x)|$ is zero yet $|f(x) - G(x)|$ is likely nonzero for $x \in [a, b]$.

   1.f) The following relation holds

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & 6 \\ 5 & 6 & 7 & 8 \\ 7 & 8 & 9 & 10 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 2 & 4 & 6 & 8 \\ 7 & 8 & 9 & 10 \\ 4 & 6 & 8 & 10 \end{bmatrix}.$$

Your answers for this problem: True (T) or False (F)

(1.a) Not graded (1.b) T (1.c) F (1.d) T (1.e) F (1.f) T

2. (15 points) **Choose ONE and ONLY ONE of the following two problems to work. If by accident you work on both problems, your scores will be averaged, not be summed up.**

Include the steps and details to show how you find the factorization. Simply giving L & U, or R without details can only earn you partial points.

- Find the LU factorization of the following matrix A,

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 7 \\ 2 & 1 & \frac{1}{3} \end{bmatrix}.$$

- Find the Cholesky factorization for the following matrix $B = R^T R$,

$$B = \begin{bmatrix} 4 & -2 & 0 \\ -2 & 6 & -1 \\ 0 & -1 & 3 \end{bmatrix}.$$

Here R is upper triangular, with the diagonal entries being positive:

3. (15 points) Find ALL values of the real number d such that the symmetric matrix A given below is positive-definite. You need to state clearly based on what condition or property such d is identified.

$$A = \begin{bmatrix} 1 & -2 \\ -2 & d \end{bmatrix}$$

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problem 1

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 6 & 4 & 7 \\ 2 & 1 & 1/3 \end{bmatrix} \xrightarrow{R_2 - 2R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 3 \\ 2 & 1 & 1/3 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{2}{3}R_1} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 1/3 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{6}R_2} \begin{bmatrix} 3 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & -3/2 \end{bmatrix} = U$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ \frac{2}{3} & \frac{1}{6} & 1 \end{bmatrix}$$

2. problem 2)
(Cont'd)

let $R = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$ $\quad a, d, f > 0$

$$R^T R = \begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} a^2 & & * \\ ba & b^2+d^2 & \\ ca & cb+de & c^2+e^2+f^2 \end{pmatrix}$$

symmetric

Compare $B = R^T R$ entry-wise,

column 1: $a^2 = 4 \Rightarrow a = 2$

$ab = -2 \Rightarrow b = -1$

$ca = 0 \Rightarrow c = 0$

Column 2: $b^2 + d^2 = 6 \Rightarrow d = \sqrt{5}$

$de = -1 \Rightarrow e = -\frac{1}{\sqrt{5}}$

column 3: $e^2 + f^2 = 3 \Rightarrow f = \sqrt{3 - \frac{1}{5}} = \sqrt{\frac{14}{5}}$

$$\Rightarrow R = \begin{bmatrix} 2 & -1 & 0 \\ 0 & \sqrt{5} & -\frac{1}{\sqrt{5}} \\ 0 & 0 & \sqrt{\frac{14}{5}} \end{bmatrix}$$

Remark: if one starts with assuming R is bidiagonal, with $c=0$, it is fine.

3 One criterion is based on the following fact:

A is SPD $\Leftrightarrow$ all eigenvalues of A are positive.

To get the eigenvalues λ of A,

set $\det(\lambda I - A) = \det\begin{pmatrix} \lambda-1 & 2 \\ 2 & \lambda-d \end{pmatrix} = 0$

$\Leftrightarrow (\lambda-1)(\lambda-d) - 4 = 0 \Leftrightarrow \lambda^2 - (1+d)\lambda + d - 4 = 0$

The two roots λ_1, λ_2 are eigenvalues of A.

λ_1, λ_2 both being positive $\Leftrightarrow$ $\lambda_1\lambda_2 = d-4 > 0$

$\lambda_1 + \lambda_2 = 1 + d > 0$

$\Leftrightarrow d > 4$.

Alternatively. one can calculate eigenvalues directly

solve $\lambda^2 - (1+a)\lambda + a - 4 = 0$

we get $\lambda_1 = \dfrac{1+d+\sqrt{(1+d)^2 - 4(d-4)}}{2}$

$= \dfrac{1+d+\sqrt{(d-1)^2+16}}{2}$

$\lambda_2 = \dfrac{1+d-\sqrt{(d-1)^2+16}}{2}$

First $(d-1)^2+16 > 0$ always hold, hence both λ_1, λ_2 are real

$\lambda_2 > 0 \iff 1+d > \sqrt{(d-1)^2+16}$

$\iff \begin{cases} 1+d > 0 \\ (1+d)^2 > (d-1)^2+16 \iff d^2+2d+1 > d^2-2d+1+16 \end{cases}$

$\iff d > 4$

This requires $d > 4$

when $d > 4$, $\lambda_1 > 0$ also holds

Hence all possible values of d to make A SPD are:

$d > 4$

A different approach

Combination of the following two: Given A is symmetric,

$$\begin{cases} A \text{ is SPD then } \det(A) > 0 & \text{---- (1)} \\ A \text{ is SPD} \iff \underline{x}^T A \underline{x} > 0, \ \forall \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \underline{0} & \text{--- (2)} \end{cases}$$

From (1), one gets $d > 4$, a necessary condition ---- (3)

From (2), assume $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \underline{0}$ be arbitrary, then

$$\underline{x}^T A \underline{x} = x_1^2 - 4x_1x_2 + d x_2^2$$

$$= (x_1 - 2x_2)^2 + (d-4)x_2^2$$

$\underline{x} \neq \underline{0} \Rightarrow$ at least one of the following holds, $(x_1 - 2x_2)^2 > 0$
$x_2^2 > 0$

This, in combination with (3), implies $\underline{x}^T A \underline{x} > 0, \ \forall \underline{x} \neq \underline{0}$ as long as $d > 4$. In other words "$d > 4$" is also sufficient.

4. (60 points in total, with 12 points for each sub-problem) For a given data set $\{(x_j, y_j)\}_{j=1}^4$:

j	x_j	y_j
1	-1	-5
2	0	-1
3	2	1
4	3	11

Note: your answers for the following questions are expected to be reasonably simplified (This has been illustrated in class or in homework reference solutions). On the other hand, you would not want to oversimplify your answers to cost your time unnecessarily.

4.a) Use Lagrange interpolation to find a polynomial $p(x)$ that passes through all the points. Here the degree of $p(x)$ is no higher than 3. Express your polynomial in its Lagrangian form, namely, $p(x) = \sum_{j=1}^4 y_j L_j(x)$, such that $L_j(x)$ is cubic, satisfying $L_j(x_i) = \delta_{ij}$. ($\delta_{ij} = 0$, when $i \neq j$; and $= 1$ when $i = j$.)

4.b) Use Newton's divided difference to find a polynomial $q(x)$ that passes through all the points. Here the degree of $q(x)$ is no higher than 3. Express your polynomial in its Newton's divided difference form, namely,

$$q(x) = \sum_{j=1}^{4} a_j(x - x_1)\cdots(x - x_{j-1}) = a_1 + \sum_{j=2}^{4} a_j(x - x_1)\cdots(x - x_{j-1})$$

with some a_j you find explicitly.

4.c) Find the piecewise linear interpolation $s(x)$ of the data. *Make sure to specify the intervals on which each piece is defined.*

4.d) Find the linear fit $f(x)$ of the data in the least squares sense. *The linear fit refers to f being a linear polynomial.*

4.e) Find a polynomial of degree exactly equal to $r = 5$ which passes through all the given points.

4a) $p(x) = -5 \cdot L_1(x) - L_2(x) + L_3(x) + 11 \cdot L_4(x)$

where $L_1(x) = \dfrac{(x-x_2)(x-x_3)(x-x_4)}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} = \dfrac{x(x-2)(x-3)}{(-1)(-3)(-4)} = -\dfrac{x(x-2)(x-3)}{12}$

$L_2(x) = \dfrac{(x-x_1)(x-x_3)(x-x_4)}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} = \dfrac{(x+1)(x-2)(x-3)}{6}$

$L_3(x) = \dfrac{(x-x_1)(x-x_2)(x-x_4)}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} = -\dfrac{x(x+1)(x-3)}{6}$

$L_4(x) = \dfrac{(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} = \dfrac{x(x+1)(x-2)}{12}$

(Cont'd)

4.b) following the standard recursive formula, one can calculate a_1, a_2, a_3, a_4

$$
\begin{array}{c|cccc}
-1 & \boxed{-5} & & & \\
0 & -1 & \boxed{4} & & \\
2 & 1 & 1 & \boxed{-1} & \\
3 & 11 & 10 & 3 & \boxed{1}
\end{array}
$$

$\Rightarrow \quad q(x) = -5 + 4(x-x_1) - (x-x_1)(x-x_2) + (x-x_1)(x-x_2)(x-x_3)$

$\qquad = -5 + 4(x+1) - x(x+1) + x(x+1)(x-2)$

4.c) $\quad S(x) = S_j(x) \quad x \in [x_j, x_{j+1}), \quad j = 1, 2, 3$

where $\quad S_1(x) = y_1 + \dfrac{(y_2-y_1)}{(x_2-x_1)}(x-x_1)$

$\qquad = -5 + 4(x+1) = 4x-1 \qquad$ on $[-1, 0)$

$S_2(x) = y_2 + \dfrac{y_3-y_2}{x_3-x_2}(x-x_2)$

$\qquad = -1 + (x-0) = x-1 \qquad$ on $[0, 2)$

$S_3(x) = y_3 + \dfrac{y_4-y_3}{x_4-x_3}(x-x_3)$

$\qquad = 1 + 10(x-2) = 10x-19 \qquad$ on $[2, 3]$

4.e) the answer is not unique

one example is

$Q(x) = P(x) + x^2(x+1)(x-2)(x-3)$, where $P(x)$ is the solution to 4.a)

This is a polynomial of degree 5

and $\quad Q(x_j) = P(x_j) + x^2(x+1)(x-1)(x-3)\big|_{x=x_j}$

$\qquad = P(x_j) = y_j, \quad j = 1, 2, 3, 4$

(Cont'd)

4.d) let $f(x) = c_1 + c_2 x$

then $f(x_j) = y_j \quad j = 1, 2, 3, 4$

In matrix-vector form, this is $A\underline{x} = \underline{b}$

where $A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{pmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$

$\underline{b} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 1 \\ 11 \end{bmatrix} \quad \underline{x} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

The least squares solution is equivalent to the normal equation:

$$A^T A \hat{\underline{x}} = A^T \underline{b}, \text{ where}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 14 \end{bmatrix}$$

$$A^T \underline{b} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -5 \\ -1 \\ 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 6 \\ 40 \end{bmatrix}$$

That is if $\hat{\underline{x}} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, then it satisfies

$$\begin{cases} 4c_1 + 4c_2 = 6 & \cdots (1) \\ 4c_1 + 14c_2 = 40 & \cdots (2) \end{cases}$$

(2) − (1) $\Rightarrow 10c_2 = 34 \Rightarrow c_2 = 3.4$

(1) $\Rightarrow c_1 = \frac{3}{2} - c_2 = 1.5 - 3.4 = -1.9$

$\Rightarrow$ The linear fit is

$f(x) = -1.9 + 3.4x$.