§2.2 Direct method:

Gaussian elimination (GE) and LU factorization.

Given $A \in \mathbb{R}^{n\times n}$. $\underline{b} \in \mathbb{R}^n$.

A is invertible, we want to solve $A\underline{x} = \underline{b}$ for $\underline{x} \in \mathbb{R}^n$

Direct method:

they are methods that involve finitely many operations

A special case: when A is upper triangular, that is

$a_{ij} = 0$ for any $i > j$

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

$$B = \begin{bmatrix} \pi & 1 \\ 0 & -10 \end{bmatrix}$$

?

Example: Solve $A\underline{x} = \underline{b}$, with A given above and

$$\underline{b} = \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$$

Let $\underline{x} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$, the system $A\underline{x} = \underline{b}$ is

$$\begin{cases} u + v + w = 0 \\ \quad 2v - w = 5 \\ \qquad 3w = 3 \end{cases}$$

(no u)

(no u and v)

Solve w from the last eqn

$$w = 1$$

Solve v from 2nd last eqn

$$v = \frac{5 + w}{2} = 3$$

Solve u from the first eqn.

$$u = -v - w = -4.$$

$$\Rightarrow \underline{x} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

The algorithm / process:

is back substitution.

↳ to solve upper-triangular system

In general, $\underline{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n = b_1 \\ \quad a_{22} x_2 + \cdots + a_{2n} x_n = b_2 \\ \quad \vdots \\ \quad a_{n-1\, n-1} x_{n-1} + a_{n-1\, n} x_n = b_{n-1} \\ \quad a_{nn} x_n = b_n \end{cases}$$

To solve using back substitution

Cost

$$x_n = \frac{b_n}{a_{nn}} \quad 1$$

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n} x_n}{a_{n-1,n-1}} \quad 3$$

$\vdots$ 5

$$x_2 = \frac{b_2 - a_{23}x_3 - \cdots - a_{2n}x_n}{a_{22}} \quad 2(n-1) - 1$$

$$x_1 = \frac{b_1 - a_{12}x_2 \cdots - a_{1n}x_n}{a_{11}} \quad 2n-1$$

Computational complexity of B.S.

$$1 + 3 + 5 + \cdots + (2n-1)$$

$$= \sum_{j=1}^{n} (2j-1) = \left(2\sum_{j=1}^{n} j\right) - n$$

$$= 2\,\frac{n(n+1)}{2} - n = n^2$$

Back Substitution

$A = (a_{ij}) \in \mathbb{R}^{n \times n}$: upper-triangular

$(a_{ij} = 0 \quad \forall\, i > j)$

$\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$

to solve $A\underline{x} = \underline{b}$ for $\underline{x} \in \mathbb{R}^n$

```
For i = n : -1 : 1
    For j = i+1 : n
        b(i) = b(i) - a(i,j) * x(j)
    End
    x(i) = b(i) / a(i,j)
End
```

Another special case: A is lower-triangular, that is:

$a_{ij} = 0$ for any $i < j$

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 5 & -9 & 0 \\ 10 & 0 & 100 \end{pmatrix}$

$A\underline{x} = b$: 'forward substitution'

Solve 1st equation for 1st unknown

2nd : 2nd :

⋮

in forward direction

Cost: n^2

General Case

Example: to solve $A\underline{x} = \underline{b}$

where $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 5 & -1 & -1 \end{pmatrix}$ $\quad \underline{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Let $\underline{x} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$$\text{system} \Leftrightarrow \begin{cases} u + v + w = 1 & (1) \\ 2u + 4v + w = 0 & (2) \\ 5u - v - w = 2 & (3) \end{cases}$$

we hope to convert this problem to an upper-triangular system.

$(2) - 2 \cdot (1)$

$$\Rightarrow 2v - w = -2 \qquad (2)'$$

$(3) - 5 \cdot (1)$

$$\Rightarrow -6v - 6w = -3 \qquad (3)'$$

So far we have

$$\begin{cases} u+v+w=1 & (1)\\ \quad 2v-w=-2 & (2)'\\ \quad -6v-6w=-3 & (3)' \end{cases}$$

$$(3)' - (-3)\cdot(2)'$$

$$= (3)' + 3\cdot(2)'$$

$$\Rightarrow \quad -9w = -9 \quad \cdots (3)''$$

now we have

$$\begin{cases} u+v+w=1 & \cdots (1)\\ \quad 2v-w=-2 & \cdots (2)'\\ \qquad -9w=-9 & \cdots (3)'' \end{cases}$$

$\downarrow$ upper-triangular

$\uparrow$

Solved with Back substitution

Sketch

$$\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix} \to \begin{bmatrix} x & x & x \\ 0 & \boxed{x} & \boxed{x} \\ x & x & x \end{bmatrix}$$

$$\to \begin{bmatrix} x & x & x \\ 0 & \boxed{x} & \boxed{x} \\ 0 & \boxed{x} & \boxed{x} \end{bmatrix}$$

$$\to \begin{bmatrix} x & x & x \\ 0 & \boxed{x} & \boxed{x} \\ 0 & 0 & \boxed{x} \end{bmatrix}$$

matrix-form of the process.

$[A \,\vdots\, \underline{b}]$ tableau form (augmented matrix)

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right]$$

we want to zero out the entries below the main diagonal.

Start with first column:

$$\text{Row } i - \frac{a_{i1}}{a_{11}} \text{Row } 1 \rightarrow \text{Row } i$$

$\Leftrightarrow$

$$a_{i1} - \frac{a_{i1}}{a_{11}} a_{11} = 0, \quad a_{i2} - \frac{a_{i1}}{a_{11}} a_{12}, \quad \cdots$$

↑ no need to compute

$i = 2, 3, \cdots n$ $\quad a_{in} - \frac{a_{i1}}{a_{11}} a_{1n}, \quad b_i - \frac{a_{i1}}{a_{11}} b_1$

now we have

$$\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ & \hat{a}_{22} & \cdots & \hat{a}_{2n} & \hat{b}_2 \\ & \vdots & & & \\ & \hat{a}_{n2} & \cdots & \hat{a}_{nn} & \hat{b}_n \end{array}$$

Repeat the process for the part in blue.

cost

$$\begin{pmatrix} 0 & & & & & \\ 2n+1 & 0 & & & & \\ 2n+1 & 2(n-1)+1 & 0 & & & \\ \vdots & \vdots & & \ddots & & \\ & & & & 0 & \\ & & & 2\cdot 3+1 & 0 & \\ 2n+1 & 2(n-1)+1 & & 2\cdot 3+1 & 2\cdot 2+1 & 0 \end{pmatrix}$$

$$\sum_{j=1}^{m} j^2 = \frac{m(m+1)(2m+1)}{6}$$

$$\sum_{j=1}^{m} j = \frac{m(m+1)}{2}$$

Total cost will be.

$$\# = \sum_{j=1}^{n-1} \big(\underbrace{2(j+1)+1}_{(2j+3)j}\big)\cdot j$$

$$= 2\sum_{j=1}^{n-1} j^2 + 3\sum_{j=1}^{n-1} j$$

$$= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{7}{6}n. \approx \frac{2}{3}n^3$$

$$= O(n^3)$$

This leads to the algorithm

Gaussian Elimination (GE)

that converts a full system to an upper-triangular system. We then call back substitution. This altogether solves $A\underline{x} = \underline{b}$.

Cost GE $\approx \frac{2}{3}n^3$

back substitution $= n^2$

Add-up: $\approx \frac{2}{3}n^3$.

Gaussian Elimination

$A = (a_{ij}) \in \mathbb{R}^{n \times n}$. $\underline{b} \in \mathbb{R}^n$

```
For j = 1 : n-1
  If |a(j,j)| < eps,
    error('       ');
  End
  For i = j+1 : n
    z = a(i,j) / a(j,j)
    For k = j+1 : n
      a(i,k) = a(i,k) - z*a(j,k)
    End
    b(i) = b(i) - z * b(j)
  End
```

To reduce A to an upper-triangular matrix U, namely

$$A \to U,$$

essentially is to find the LU factorization of A,

$$A = LU$$

U: upper-triangular

L: lower-triangular, where the main diagonal entries are 1.

Revisit the example (3x3 example of GE).

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 5 & -1 & -1 \end{bmatrix} =$$

$$\underbrace{\begin{bmatrix} 1 & & \\ 2 & 1 & \\ 5 & -3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -9 \end{bmatrix}}_{U}$$