**Homework 7**
**Jingmin Sun**
**661849071**

1. Based on Taylor expansion, we can get

$$
\begin{aligned}
f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + O(h^4) \\
f(x-2h) &= f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{3!}f'''(x) \\
&= f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4h^3}{3}f'''(x) + O(h^4) \\
\therefore 4f(x+h) - 3f(x) - f(x-2h) &= 4f(x) + 4hf'(x) + 2h^2f''(x) + \frac{2h^3}{3}f'''(x) - 3f(x) \\
&\quad - f(x) + 2hf'(x) - 2h^2f''(x) + \frac{4h^3}{3}f'''(x) + O(h^4) \\
&= 6hf'(x) + 2h^3f'''(x) + O(h^4) \\
\therefore \frac{4f(x+h) - 3f(x) - f(x-2h)}{6h} &= f'(x) + \frac{h^2}{3}f'''(x)
\end{aligned}
$$

Therefore, the error term is $\frac{h^2}{3}f'''(x)$, and it's second order with respect to h.

2. Based on Taylor expansion,

$$
\begin{aligned}
f(x-2h) &\approx f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) \\
&= f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4h^3}{3}f'''(x) + \frac{2h^4}{3}f^{(4)}(x) \\
f(x-h) &\approx f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) \\
&= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) \\
f(x+h) &\approx f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) \\
&= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) \\
\therefore af(x-2h) &= af(x) - 2ahf'(x) + 2ah^2f''(x) - \frac{4ah^3}{3}f'''(x) + \frac{2ah^4}{3}f^{(4)}(x) \\
bf(x-h) &= bf(x) - bhf'(x) + \frac{bh^2}{2}f''(x) - \frac{bh^3}{6}f'''(x) + \frac{bh^4}{24}f^{(4)}(x) \\
cf(x) &= cf(x) \\
df(x+h) &= df(x) + dhf'(x) + \frac{dh^2}{2}f''(x) + \frac{dh^3}{6}f'''(x) + \frac{dh^4}{24}f^{(4)}(x) \\
\therefore af(x-2h) + bf(x-h) + cf(x) + df(x+h) &= (a+b+c+d)f(x) + (d-2a-b)hf'(x) + \frac{4a+b+d}{2}h^2f''(x) \\
&\quad + \frac{d-b-8a}{6}h^3f'''(x) + \frac{16a+b+d}{24}h^4f^{(4)}(x)
\end{aligned}
$$

To require nth-order accuracy, for $n \in \mathbb{R}_+$, we need $a+b+c+d=0$.

$$\begin{aligned}
\therefore \frac{af(x-2h)+bf(x-h)+cf(x)+df(x+h)}{(d-2a-b)h} &= f'(x) + \frac{4a+b+d}{2(d-2a-b)}hf''(x) \\
&+ \frac{d-b-8a}{6(d-2a-b)}h^2 f'''(x) + \frac{16a+b+d}{24(d-2a-b)}h^3 f^{(4)}(x)
\end{aligned}$$

To require first order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, and $4a+b+d \neq 0$ we have infinitely many solution for $a, b, c$ and $d$.

To require second order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, $4a+b+d=0$, and $d-b-8a \neq 0$ we have infinitely many solution for $a, b, c$ and $d$.

To require third order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, $4a+b+d=0$, $d-b-8a=0$ and $16a+b+d \neq 0$ and we can get

$$\begin{aligned}
d &= b+8a \\
4a+b+b+8a &= 0 \\
12a &= -2b \\
b &= -6a \\
d &= 2a \\
a+b+c+d &= 0 \\
a-6a+c+2a &= 0 \\
c &= 3a \\
\therefore \frac{af(x-2h)-6af(x-h)+3af(x)+2af(x+h)}{(2a-2a+6a)h} &= f'(x) + \frac{16a-6a+2a}{24(2a-2a+6a)}h^3 f^{(4)}(x) \\
\frac{f(x-2h)-6f(x-h)+3f(x)+2f(x+h)}{6h} &= f'(x) + \frac{h^3}{12}f^{(4)}(x)
\end{aligned}$$

To require forth order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, $4a+b+d=0$, $d-b-8a=0$ and $16a+b+d=0$, since $a \neq 0$, so we can not make $4a+b+d=0$ and $16a+b+d=0$ at the same time, so the third order accuracy is the highest accuracy we can get based on $f(x-2h)$, $f(x-h)$, $f(x)$, $f(x+h)$.

3.

Listing 1: richardson.m

```
%% An example of applying Richardson Extrapolation
% To approximate the second derivative of sin(x)+x*exp(-x)
% with F2 knwon as (f(x+h)-2*f(x)+f(x-h))/h^2 to get F4

f = @(x) sin(x)+x*exp(-x);
ffirst = @(x) cos(x)+ exp(-x)*(1-x);
fsecond = @(x) x*exp(-x) - sin(x) - 2*exp(-x);
F2fun = @(x,h) (f(x+h)-2*f(x)+f(x-h))/h^2;
```

```
F4fun = @(x,h) (2^2*F2fun(x,h/2)-F2fun(x,h))/(2^2-1);
%F4fun = @(x,h) (-f(x-h)+16*f(x - h/2)-30*f(x)+16*f(x+h/2)-f(x+h))/(3*h^2);
e2fun = @(h) abs(F2fun(pi/3,h) - fsecond(pi/3));
e4fun = @(h) abs(F4fun(pi/3,h) - fsecond(pi/3));
hvec = 0.1*0.5 .^(1:6);
hvec = hvec';
F2 = zeros(6,1);
e2 = zeros(6,1);
redfac2 = zeros(6,1);
F4 = zeros(6,1);
e4 = zeros(6,1);
redfac4  = zeros(6,1);
for i = 1:6
    h = hvec(i);
   %fprintf("h%d = %5.4f; F2(h%d) = %5.4f; e2(h%d) = %5.4e\n", i,h,i,F2(pi/3,h),i,e2(h));
   F2(i) = F2fun(pi/3,h);
   e2(i) = e2fun(h);
   F4(i) = F4fun(pi/3,h);
   e4(i) = e4fun(h);
   if( i ~= 1)
       redfac2(i) = e2(i-1)/e2(i);
       redfac4(i) = e4(i-1)/e4(i);
   end
end
format
T = table(hvec, F2, e2, redfac2, F4, e4, redfac4)
loglog(hvec,e2,'b-');
hold on
loglog(hvec, e4,'r-');
xlabel("h");
ylabel("Error");
legend("Second Order", "Fourth Order");
```

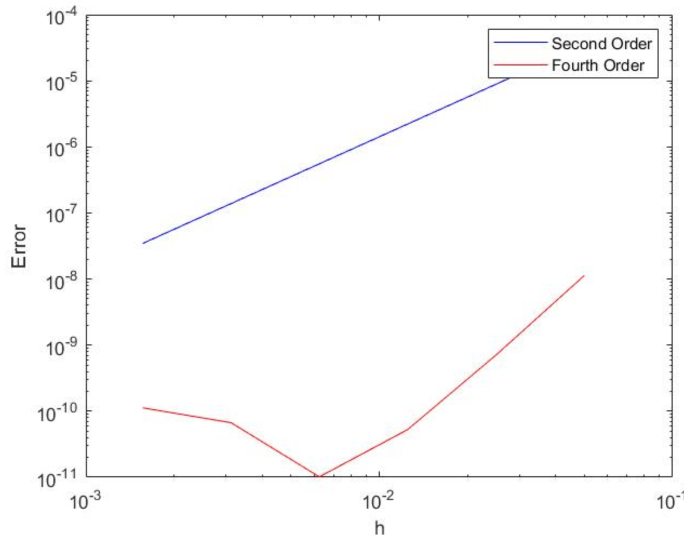
Listing 2: output

(a)

```
>> richardson
T =
  6 7 table
      hvec        F2          e2        redfac2     F4          e4        redfac4

    _________   _______   __________   _______   _______   __________   _______
         0.05   -1.2004   3.5498e-05         0   -1.2004   1.1303e-08         0
        0.025   -1.2004   8.8659e-06    4.0038   -1.2004   7.0744e-10    15.977
       0.0125   -1.2004   2.2159e-06     4.001   -1.2004   5.2798e-11    13.399
      0.00625   -1.2004   5.5395e-07    4.0003   -1.2004   1.0166e-11    5.1938
     0.003125   -1.2004   1.3848e-07    4.0002   -1.2004   6.7009e-11   0.15171
    0.0015625   -1.2004    3.457e-08    4.0058   -1.2004   1.1248e-10   0.59572
```

(b) The graph is:




(c) Since we expect that $F_2(h)$ generate an error of second order, and $\dfrac{2^2 \cdot F_2(h/2) - F_2(h)}{2^2 - 1}$, which generally generate $F_3(h)$, which has an error of third error, but due to the symmetry here, we can get an error of fourth order here. And the expected reduction factor of second order and forth order is that

Listing 3: expected

```
%% Expected redunction factor for the second order
(1/0.5)^2

ans =

     4

%% Expected redunction factor for the forth order
(1/0.5)^4

ans =

    16

diary off
```

And compared with the column of $redfac2$ and $redfac4$, we found that the $redfac4$ seems incorrect when $h < 0.025$, and this may cause by the rounding error when we subtract two nearly same value, namely, since $h$ is small, so that $4 \times F_2(x + h/2)$ and $F_2(x + h)$ are close, and cause a large rounding error, but we do not have a good remedy here, so we can examine it by change the scale of h, so when we set h to

```
hvec = 0.1 *0.9.^(1:6);
```

And the expected reduction factor of second order and forth order is that

Listing 4: expected_0.9

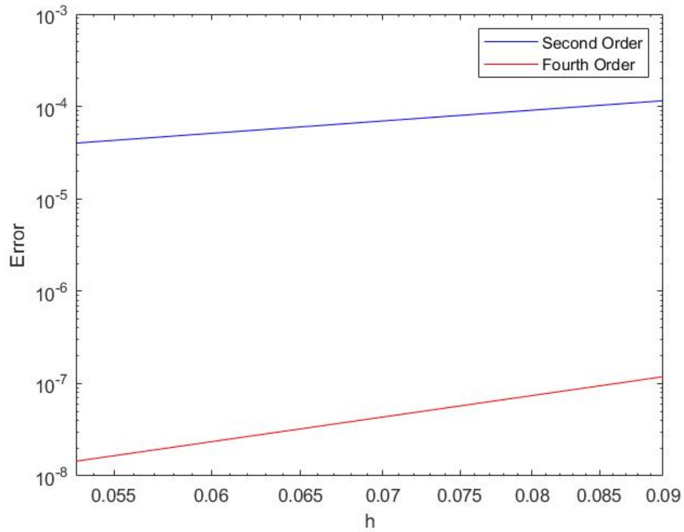
```
%% Expected redunction factor for the second order
(1/0.9)^2
ans =
   1.2346
%% Expected redunction factor for the forth order
(1/0.9)^4
ans =
   1.5242
diary off
```

And our table output is

Listing 5: output_0.9

```
>> richardson
T =
  6 7  table
     hvec       F2          e2        redfac2     F4          e4        redfac4
   --------   -------   ----------   -------   -------   ----------   -------
       0.09   -1.2005   0.00011534        0    -1.2004   1.1866e-07        0
      0.081   -1.2005   9.3353e-05   1.2355    -1.2004   7.7852e-08   1.5242
     0.0729   -1.2005   7.5568e-05   1.2354    -1.2004   5.1078e-08   1.5242
    0.06561   -1.2004   6.1178e-05   1.2352    -1.2004   3.3512e-08   1.5242
   0.059049   -1.2004   4.9534e-05   1.2351    -1.2004   2.1987e-08   1.5242
   0.053144   -1.2004   4.0109e-05    1.235    -1.2004   1.4426e-08   1.5241
```

which support my expectation, and we can observe that of the graph, which is




we can observe that the graph make more sense that when h decreases, the error always decreases, and when error is in second order, we know that

$$\begin{aligned} e2 &= O(h^2) \\ \ln(e2) &= 2\ln(h) + c \\ e4 &= O(h^4) \\ \ln(e4) &= 4\ln(h) + c \end{aligned}$$

and the slope approximate shows the pattern of +2 and +4 respectively.

Thus the $F_4$ here actually exhibit an error of forth order.

4. (a)

$$\begin{aligned} \int_0^2 x\cos(x)dx &= \int_{x=0}^2 xd\sin(x) \\ &= x\sin(x)|_{x=0}^2 - \int_{x=0}^2 \sin(x)dx \\ &= (2\sin(2) - 0) + \cos(x)|_{x=0}^2 \\ &= 2\sin(2) + \cos(2) - 1 \end{aligned}$$

- m =1

$$\begin{aligned} \int_0^2 x\cos(x)dx &\approx 2 \cdot (f(0) + f(2))/2 \\ &= f(0) + f(2) \\ &= 2\cos(2) \end{aligned}$$

And we can calculate the error is

$$\begin{aligned} e &= |2\cos(2) - 2\sin(2) - \cos(2) - 1| \\ &= 2\sin(2) - \cos(2) - 1 \\ &\approx 1.23 \end{aligned}$$

- m=2

$$\begin{aligned} \int_0^2 x\cos(x)dx &\approx (f(0) + 2f(1) + f(2))/2 \\ &= (f(0) + 2f(1) + f(2))/2 \\ &= (2\cos(1) + 2\cos(2))/2 \\ &= \cos(1) + \cos(2) \end{aligned}$$

And we can calculate the error is

$$\begin{aligned} e &= |\cos(1) + \cos(2) - 2\sin(2) - \cos(2) + 1| \\ &= 2\sin(2) - \cos(1) - 1 \\ &\approx 0.28 \end{aligned}$$

- m=4

$$\begin{aligned} \int_0^2 x\cos(x)dx &\approx \frac{1}{2} \cdot \frac{f(0) + 2f(1/2) + 2f(1) + 2f(3/2) + f(2)}{2} \\ &= \frac{\cos(1/2) + 2\cdot\cos(1) + 3\cdot\cos(3/2) + 2\cos(2))}{4} \\ &= \frac{\cos(1/2)}{4} + \frac{\cos(1)}{2} + \frac{3\cdot\cos(3/2)}{4} + \frac{\cos(2)}{2} \end{aligned}$$

And we can calculate the error is

$$\begin{aligned} e &= \left|\frac{\cos(1/2)}{4} + \frac{\cos(1)}{2} + \frac{3\cdot\cos(3/2)}{4} + \frac{\cos(2)}{2} - 2\sin(2) - \cos(2) + 1\right| \\ &\approx 0.068 \end{aligned}$$

(b)

$$
\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &= \int_0^1 \frac{1}{1+tan^2\theta} d\tan\theta \\
&= \int_0^1 \frac{\sec^2\theta}{1+tan^2\theta} d\theta \\
&= \int_0^1 \frac{\sec^2\theta}{1+\frac{\sin^2\theta}{\cos^2\theta}} d\theta \\
&= \int_0^1 \frac{1}{\cos^2\theta+\sin^2\theta} d\theta \\
&= \theta|_{x=0}^1 \\
&= \arctan x|_0^1 \\
&= \frac{\pi}{4}
\end{aligned}
$$

- m =1

$$
\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &\approx 1 \cdot \frac{f(0)+f(1)}{2} \\
&= \frac{1+\frac{1}{2}}{2} \\
&= \frac{3}{4}
\end{aligned}
$$

And the error is

$$
\begin{aligned}
e &= \left|\frac{3}{4}-\frac{\pi}{4}\right| \\
&\approx 0.035
\end{aligned}
$$

- m=2

$$
\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &\approx \frac{1}{2} \cdot \frac{f(0)+2f(1/2)+f(1)}{2} \\
&= \frac{1+\frac{2}{1+1/4}+1/2}{4} \\
&= \frac{1+\frac{8}{5}+1/2}{4} \\
&= \frac{31}{40}
\end{aligned}
$$

And the error is

$$
\begin{aligned}
e &= \left|\frac{31}{40}-\frac{\pi}{4}\right| \\
&\approx 0.0104
\end{aligned}
$$

- m=4

$$
\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &\approx \frac{1}{4}\frac{f(0)+2f(1/4)+2f(1/2)+2f(3/4)+f(1)}{2} \\
&= \frac{1+\dfrac{2}{1+1/16}+\dfrac{2}{1+1/4}+\dfrac{2}{1+9/16}+1/2}{8} \\
&= \frac{5323}{6800}
\end{aligned}
$$

And the error is

$$
\begin{aligned}
e &= \left|\frac{5323}{6800}-\frac{\pi}{4}\right| \\
&\approx 0.002604
\end{aligned}
$$

5. (a)

$$
\begin{aligned}
\int_0^2 \frac{dx}{\sqrt{2-x}} &= \int_0^2 (2-x)^{-1/2}dx \\
&= -\int_0^2 (2-x)^{-1/2}d(2-x) \\
&= -2(2-x)^{1/2}|_{x=0}^2 \\
&= 2\sqrt{2}
\end{aligned}
$$

- m =1

$$
\begin{aligned}
\int_0^2 \frac{dx}{\sqrt{2-x}} &\approx 2f(1) \\
&= 2\cdot\frac{1}{\sqrt{2-1}} \\
&= 2
\end{aligned}
$$

And the error is

$$
\begin{aligned}
e &= |2-2\sqrt{2}| \\
&= 2(\sqrt{2}-1) \\
&\approx 0.828
\end{aligned}
$$

- m=2

$$\begin{aligned}\int_0^2 \frac{dx}{\sqrt{2-x}} &\approx f(1/2)+f(3/2)\\ &= \frac{1}{\sqrt{2-1/2}}+\frac{1}{\sqrt{2-3/2}}\\ &= \frac{1}{\sqrt{3/2}}+\frac{1}{\sqrt{1/2}}\\ &= \sqrt{2/3}+\sqrt{2}\\ &= \left(\frac{\sqrt{3}}{3}+1\right)\sqrt{2}\end{aligned}$$

And the error is

$$\begin{aligned}e &= |\sqrt{2/3}+\sqrt{2}-2\sqrt{2}|\\ &= |\sqrt{2/3}-\sqrt{2}|\\ &\approx 0.598\end{aligned}$$

- m=4

$$\begin{aligned}\int_0^2 \frac{dx}{\sqrt{2-x}} &\approx \frac{1}{2}(f(1/4)+f(3/4)+f(5/4)+f(7/4))\\ &= \frac{1}{2}\left(\frac{1}{\sqrt{2-1/4}}+\frac{1}{\sqrt{2-3/4}}+\frac{1}{\sqrt{2-5/4}}+\frac{1}{\sqrt{2-7/4}}\right)\\ &= \frac{1}{2}\left(\frac{1}{\sqrt{7/4}}+\frac{1}{\sqrt{5/4}}+\frac{1}{\sqrt{3/4}}+\frac{1}{\sqrt{1/4}}\right)\\ &= \frac{1}{2}\left(\sqrt{4/7}+\sqrt{4/5}+\sqrt{4/3}+\sqrt{4}\right)\\ &= \sqrt{1/7}+\sqrt{1/5}+\sqrt{1/3}+1\end{aligned}$$

And the error is

$$\begin{aligned}e &= |\sqrt{1/7}+\sqrt{1/5}+\sqrt{1/3}+1-2\sqrt{2}|\\ &\approx 0.426\end{aligned}$$

(b) 
- m =1

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x} dx &\approx \pi/2 f(\pi/4)\\ &= \pi/2 \cdot \frac{\cos(\pi/4)}{\pi/4}\\ &= 2\cos(\pi/4)\\ &= \sqrt{2}\end{aligned}$$

- m=2

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x} dx &\approx \pi/4(f(\pi/8) + f(3\pi/8))\\ &= \pi/4 \cdot \left(\frac{\cos(\pi/8)}{3\pi/8} + \frac{\cos(3\pi/8)}{\pi/8}\right)\\ &= \frac{\cos(\pi/8)}{3/2} + \frac{\cos(3\pi/8)}{1/2}\\ &= \frac{\sqrt{\sqrt{2}+2}}{3} + \sqrt{2-\sqrt{2}}\end{aligned}$$

- m=4

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x} dx &\approx \pi/8(f(\pi/16) + f(3\pi/16) + f(5\pi/16) + f(7\pi/16))\\ &= \pi/8 \cdot \left(\frac{\cos(\pi/16)}{7\pi/16} + \frac{\cos(3\pi/16)}{5\pi/16} + \frac{\cos(5\pi/16)}{3\pi/16} + \frac{\cos(7\pi/16)}{\pi/16}\right)\\ &= \frac{\cos(\pi/16)}{7/2} + \frac{\cos(3\pi/16)}{5/2} + \frac{\cos(5\pi/16)}{3/2} + \frac{\cos(7\pi/16)}{1/2}\end{aligned}$$

6. 
- $f(x) = 1$

$$\begin{aligned}LHS &= \int_{x_0}^{x_4} 1 dx\\ &= x_4 - x_0\\ RHS &= \frac{2h}{45}(7 + 32 + 12 + 32 + 7)\\ &= 4h\\ &= x_4 - x_0\\ &= LHS\end{aligned}$$

- $f(x) = x$

$$
\begin{aligned}
LHS &= \int_{x_0}^{x_4} x dx \\
&= \frac{1}{2} x^2 |_{x_0}^{x_4} \\
&= \frac{x_4^2 - x_0^2}{2} \\
&= \frac{(x_4 - x_0)(x_4 + x_0)}{2} \\
&= \frac{4h(x_0 + 4h + x_0)}{2} \\
&= 2h(2x_0 + 4h) \\
&= 4hx_0 + 8h^2
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0 + 32x_1 + 12x_2 + 32x_3 + 7x_4) \\
&= \frac{2h}{45}(7x_0 + 32(x_0 + h) + 12(x_0 + 2h) + 32(x_0 + 3h) + 7(x_0 + 4h)) \\
&= \frac{2h}{45}(90x_0 + 180h) \\
&= 4hx_0 + 8h^2 \\
&= LHS
\end{aligned}
$$

- $f(x) = x^2$

$$
\begin{aligned}
LHS &= \frac{x^3}{3} |_{x_0}^{x_4} \\
&= \frac{x_4^3 - x_0^3}{3} \\
&= \frac{(x_4 - x_0)(x_4^2 + x_4 x_0 + x_0^2)}{3} \\
&= \frac{4h((x_0 + 4h)^2 + (x_0 + 4h) \cdot x_0 + x_0^2)}{3} \\
&= \frac{4h(x_0^2 + 8x_0 h + 16h^2 + x_0^2 + 4hx_0 + x_0^2)}{3} \\
&= \frac{4h}{3}(3x_0^2 + 12x_0 h + 16h^2)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^2 + 32x_1^2 + 12x_2^2 + 32x_3^2 + 7x_4^2) \\
&= \frac{2h}{45}(7x_0^2 + 32(x_0+h)^2 + 12(x_0+2h)^2 + 32(x_0+3h)^2 + 7(x_0+4h)^2) \\
&= \frac{2h}{45}(90x_0^2 + 360x_0h + 480h^2) \\
&= 4x_0^2h + 16x_0h^2 + \frac{64h^3}{3} \\
&= \frac{4h}{3}(3x_0^2 + 12x_0h + 16h^2) \\
&= LHS
\end{aligned}
$$

- $f(x) = x^3$

$$
\begin{aligned}
LHS &= \frac{x^4}{4}|_{x_0}^{x_4} \\
&= \frac{x_4^4 - x_0^4}{4} \\
&= \frac{(x_4-x_0)(x_4+x_0)(x_4^2+x_0^2)}{4} \\
&= \frac{4h(2x_0+4h)(x_0^2+(x_0+4h)^2)}{4} \\
&= h(2x_0+4h)(x_0^2+x_0^2+8hx_0+16h^2) \\
&= 2h(2x_0+4h)(x_0^2+4hx_0+8h^2) \\
&= 4h(x_0+2h)(x_0^2+4hx_0+8h^2) \\
&= 4h(x_0^3+4hx_0^2+8h^2x_0+2hx_0^2+8h^2x_0+16h^3) \\
&= 4h(x_0^3+6hx_0^2+16h^2x_0+16h^3)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^3 + 32x_1^3 + 12x_2^3 + 32x_3^3 + 7x_4^3) \\
&= \frac{2h}{45}(7x_0^2 + 32(x_0+h)^3 + 12(x_0+2h)^3 + 32(x_0+3h)^3 + 7(x_0+4h)^3) \\
&= \frac{2h}{45}(90x_0^3 + 540x_0^2h + 1440x_0h^2 + 1440h^3) \\
&= 4h(x_0^3 + 6x_0^2h + 16x_0h^2 + 16h^3) \\
&= LHS
\end{aligned}
$$

- $f(x) = x^4$

$$
\begin{aligned}
LHS &= \frac{x^5}{5}\Big|_{x_0}^{x_4} \\
&= \frac{(x_4 - x_0)(x_4^4 + x_4^3 x_0 + x_4^2 x_0^2 + x_4 x_0^3 + x_0^4)}{5} \\
&= \frac{4h}{5}(5x_0^4 + 40x_0^3 h + 160x_0^2 h^2 + 320x_0 h^3 + 256h^4)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^4 + 32x_1^4 + 12x_2^4 + 32x_3^4 + 7x_4^4) \\
&= \frac{2h}{45}(7x_0^4 + 32(x_0 + h)^4 + 12(x_0 + 2h)^4 + 32(x_0 + 3h)^4 + 7(x_0 + 4h)^4) \\
&= \frac{2h}{45}(90x_0^4 + 720x_0^3 h + 2880x_0^2 h^2 + 5760x_0 h^3 + 4608h^4) \\
&= \frac{4h}{5}(5x_0^4 + 40x_0^3 h + 160x_0^2 h^2 + 320x_0 h^3 + 256h^4) \\
&= LHS
\end{aligned}
$$

- $f(x) = x^5$

$$
\begin{aligned}
LHS &= \frac{x^6}{6}\Big|_{x_0}^{x_4} \\
&= \frac{(x_0 - x_4)(x_0 + x_4)(x_0^2 - x_4 x_0 + x_4^2)(x_0^2 + x_4 x_0 + x_4^2)}{6} \\
&= \frac{4h}{6}(1024h^5 + 1536x_0 h^4 + 960x_0^2 h^3 + 320x_0^3 h^2 + 60x_0^4 h + 6x_0^5) \\
&= \frac{2h}{3}(1024h^5 + 1536x_0 h^4 + 960x_0^2 h^3 + 320x_0^3 h^2 + 60x_0^4 h + 6x_0^5)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^5 + 32x_1^5 + 12x_2^5 + 32x_3^5 + 7x_4^5) \\
&= \frac{2h}{45}(7x_0^5 + 32(x_0 + h)^5 + 12(x_0 + 2h)^5 + 32(x_0 + 3h)^5 + 7(x_0 + 4h)^5) \\
&= \frac{2h}{45}(15360h^5 + 23040x_0 h^4 + 14400x_0^2 h^3 + 4800x_0^3 h^2 + 900x_0^4 h + 90x_0^5) \\
&= \frac{2h}{3}(1024h^5 + 1536x_0 h^4 + 960x_0^2 h^3 + 320x_0^3 h^2 + 60x_0^4 h + 6x_0^5) \\
&= LHS
\end{aligned}
$$

- $f(x) = x^6$

$$
\begin{aligned}
LHS &= \frac{x^7}{7}|_{x_0}^{x_4} \\
&= \frac{-(x_0 - x_4)(x_0^6 + x_4x_0^5 + x_4^2x_0^4 + x_4^3x_0^3 + x_4^4x_0^2 + x_4^5x_0 + x_4^6)}{7} \\
&= \frac{4h}{7}(4096h^6 + 7168h^5x_0 + 5376h^4x_0^2 + 2240h^3x_0^3 + 560h^2x_0^4 + 84hx_0^5 + 7x_0^6) \\
&= 4h\left(\frac{4096}{7}h^6 + 1024h^5x_0 + 768h^4x_0^2 + 320h^3x_0^3 + 80h^2x_0^4 + 12hx_0^5 + x_0^6\right)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^6 + 32x_1^6 + 12x_2^6 + 32x_3^6 + 7x_4^6) \\
&= \frac{2h}{45}(7x_0^6 + 32(x_0 + h)^6 + 12(x_0 + 2h)^6 + 32(x_0 + 3h)^6 + 7(x_0 + 4h)^6) \\
&= \frac{2h}{45}(52800h^6 + 92160h^5x_0 + 69120h^4x_0^2 + 28800h^3x_0^3 + 7200h^2x_0^4 + 1080hx_0^5 + 90x_0^6) \\
&= 4h\left(\frac{1760h^6}{3} + 1024h^5x_0 + 768h^4x_0^2 + 320h^3x_0^3 + 80h^2x_0^4 + 12hx_0^5 + x_0^6\right) \\
&\because \frac{4096}{7} \neq \frac{1760}{3} \\
&\therefore RHS \neq LHS
\end{aligned}
$$

Therefore, the degree of precision is 5.