Homework 7
Jingmin Sun
661849071

1. Based on Taylor expansion, we can get

$$\begin{aligned}
f(x+h) &\approx f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) \\
f(x-2h) &\approx f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{3!}f'''(x) \\
&= f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4h^3}{3}f'''(x) \\
\therefore 4f(x+h) - 3f(x) - f(x-2h) &= 4f(x) + 4hf'(x) + 2h^2f''(x) + \frac{2h^3}{3}f'''(x) - 3f(x) - f(x) + 2hf'(x) - 2h^2f''(x) + \frac{4h^3}{3}f'''(x) \\
&= 6hf'(x) + 2h^3f'''(x) \\
\therefore \frac{4f(x+h) - 3f(x) - f(x-2h)}{6h} &= f'(x) + \frac{h^2}{3}f'''(x)
\end{aligned}$$

Therefore, the error term is $\frac{h^2}{3}f'''(x)$, and it's second order with respect to h.

2. Based on Taylor expansion,

$$\begin{aligned}
f(x-2h) &\approx f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{3!}f'''(x) + \frac{16h^4}{4!}f^{(4)}(x) \\
&= f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4h^3}{3}f'''(x) + \frac{2h^4}{3}f^{(4)}(x) \\
f(x-h) &\approx f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) \\
&= f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) \\
f(x+h) &\approx f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(x) \\
&= f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(x) \\
\therefore af(x-2h) &= af(x) - 2ahf'(x) + 2ah^2f''(x) - \frac{4ah^3}{3}f'''(x) + \frac{2ah^4}{3}f^{(4)}(x) \\
bf(x-h) &= bf(x) - bhf'(x) + \frac{bh^2}{2}f''(x) - \frac{bh^3}{6}f'''(x) + \frac{bh^4}{24}f^{(4)}(x) \\
cf(x) &= cf(x) \\
df(x+h) &= df(x) + dhf'(x) + \frac{dh^2}{2}f''(x) + \frac{dh^3}{6}f'''(x) + \frac{dh^4}{24}f^{(4)}(x) \\
\therefore af(x-2h) + bf(x-h) + cf(x) + df(x+h) &= (a+b+c+d)f(x) + (d-2a-b)hf'(x) + \frac{4a+b+d}{2}h^2f''(x) \\
&+ \frac{d-b-8a}{6}h^3f'''(x) + \frac{16a+b+d}{24}h^4f^{(4)}(x)
\end{aligned}$$

To require nth-order accuracy, for $n \in \mathbb{R}_+$, we need $a+b+c+d=0$.

$$\therefore \frac{af(x-2h)+bf(x-h)+cf(x)+df(x+h)}{(d-2a-b)h} = f'(x) + \frac{4a+b+d}{2(d-2a-b)}hf''(x) + \frac{d-b-8a}{6(d-2a-b)}h^2f'''(x) + \frac{16a+b+d}{24(d-2a-b)}h^3f^{(4)}(x)$$

To require first order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, and $4a+b+d \neq 0$ we have infinitely many solution for a, b, c and d.

To require second order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, $4a+b+d=0$, and $d-b-8a \neq 0$ we have infinitely many solution for a, b, c and d.

To require third order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, $4a+b+d=0$, $d-b-8a=0$ and $16a+b+d \neq 0$ and we can get

$$\begin{aligned} d &= b+8a \\ 4a+b+b+8a &= 0 \\ 12a &= -2b \\ b &= -6a \\ d &= 2a \\ a+b+c+d &= 0 \\ a-6a+c+2a &= 0 \\ c &= 3a \\ \therefore \frac{af(x-2h)-6af(x-h)+3af(x)+2af(x+h)}{(2a-2a+6a)h} &= f'(x) + \frac{16a-6a+2a}{24(2a-2a+6a)}h^3f^{(4)}(x) \\ \frac{f(x-2h)-6f(x-h)+3f(x)+2f(x+h)}{6h} &= f'(x) + \frac{h^3}{12}f^{(4)}(x) \end{aligned}$$

To require forth order accuracy, we need $a+b+c+d=0$, $d-2a-b \neq 0$, $4a+b+d=0$, $d-b-8a=0$ and $16a+b+d=0$, since $a \neq 0$, so we can not make $4a+b+d=0$ and $16a+b+d=0$ at the same time, so the third order accuracy is the highest accuracy we can get based on $f(x-2h)$, $f(x-h)$, $f(x)$, $f(x+h)$.

3.

Listing 1: richardson.m

```


f = @(x) sin(x)+x*exp(-x);
ffirst = @(x) cos(x)+ exp(-x)*(1-x);
fsecond = @(x) x*exp(-x) - sin(x) - 2*exp(-x);
F2fun = @(x,h) (f(x+h)-2*f(x)+f(x-h))/h^2;
F4fun = @(x,h) (2^2*F2fun(x,h/2)-F2fun(x,h))/(2^2-1);
%F4fun = @(x,h) (-f(x-h)+16*f(x - h/2)-30*f(x)+16*f(x+h/2)-f(x+h))/(3*h^2);
```

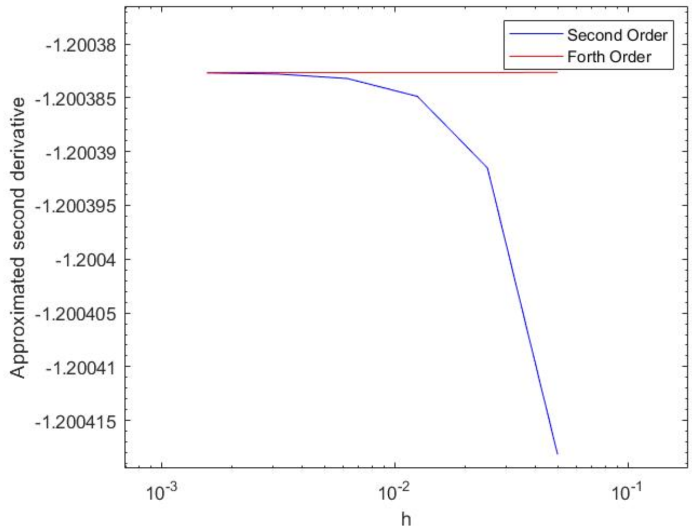
```
e2fun = @(h) abs(F2fun(pi/3,h) - fsecond(pi/3));
e4fun = @(h) abs(F4fun(pi/3,h) - fsecond(pi/3));
hvec = 0.1*0.5.^(1:6);
hvec = hvec';
F2 = zeros(6,1);
e2 = zeros(6,1);
redfac2 = zeros(6,1);
F4 = zeros(6,1);
e4 = zeros(6,1);
redfac4  = zeros(6,1);
for i = 1:6
    h = hvec(i);
   %fprintf("h%d = %5.4f; F2(h%d) = %5.4f; e2(h%d) = %5.4e\n", i,h,i,F2(pi/3,h),i,e2(h));
   F2(i) = F2fun(pi/3,h);
   e2(i) = e2fun(h);
   F4(i) = F4fun(pi/3,h);
   e4(i) = e4fun(h);
   if( i ~= 1)
       redfac2(i) = e2(i-1)/e2(i);
       redfac4(i) = e4(i-1)/e4(i);
   end
end
format
T = table(hvec, F2, e2, redfac2, F4, e4, redfac4)
loglog(hvec,F2,'b-');
hold on
loglog(hvec, F4,'r-');
xlabel("h");
ylabel("Approximated second derivative");
legend("Second Order", "Forth Order");
```

(a) Listing 2: output

```
>> richardson
T =
  67 table
      hvec         F2          e2         redfac2      F4          e4         redfac4

    _________    ________    __________    _______    ________    __________    _______
         0.05    -1.2004    3.5498e-05          0    -1.2004    1.1303e-08          0
        0.025    -1.2004    8.8659e-06     4.0038    -1.2004    7.0744e-10     15.977
       0.0125    -1.2004    2.2159e-06      4.001    -1.2004    5.2798e-11     13.399
      0.00625    -1.2004    5.5395e-07     4.0003    -1.2004    1.0166e-11     5.1938
     0.003125    -1.2004    1.3848e-07     4.0002    -1.2004    6.7009e-11    0.15171
    0.0015625    -1.2004     3.457e-08     4.0058    -1.2004    1.1248e-10    0.59572
```

(b) The graph is:



(c) Since we expect that $F_2(h)$ generate an error of second order, and $\dfrac{2^2 \cdot F_2(h/2) - F_2(h)}{2^2 - 1}$, which generally generate $F_3(h)$, which has an error of third error, but due to the symmetry here, we can get an error of fourth order here. And the expected reduction factor of second order and forth order is that

Listing 3: expected

```
%% Expected reednuction factor for the second order
(1/0.5)^2

ans =

     4

%% Expected redunction factor for the forth order
(1/0.5)^4

ans =

    16

diary off
```

And compared with the column of $redfac2$ and $redfac4$, we found that the $redfac4$ seems incorrect when $h < 0.025$, and this may cause by the rounding error when we subtract two nearly same value, namely, since h is small, so that $4 \times F_2(x + h/2)$ and $F_2(x + h)$ are close, and cause a large rounding error, but we do not have a good remedy here, so we can examine it by change the scale of h, so when we set h to

```
hvec = 0.1 *0.9.^(1:6);
```

And the expected reduction factor of second order and forth order is that

Listing 4: expected_0.9

```
%% Expected redunction factor for the second order
(1/0.9)^2
ans =
    1.2346
%% Expected redunction factor for the forth order
(1/0.9)^4
ans =
    1.5242
diary off
```

And our table output is

Listing 5: output_0.9

```
>> richardson
T =
  6 7  table
      hvec       F2          e2        redfac2     F4          e4        redfac4

    --------    -------    ----------    -------    -------    ----------    -------
        0.09    -1.2005    0.00011534          0    -1.2004    1.1866e-07          0
       0.081    -1.2005    9.3353e-05     1.2355    -1.2004    7.7852e-08     1.5242
      0.0729    -1.2005    7.5568e-05     1.2354    -1.2004    5.1078e-08     1.5242
     0.06561    -1.2004    6.1178e-05     1.2352    -1.2004    3.3512e-08     1.5242
    0.059049    -1.2004    4.9534e-05     1.2351    -1.2004    2.1987e-08     1.5242
    0.053144    -1.2004    4.0109e-05      1.235    -1.2004    1.4426e-08     1.5241
```

which support my expectation, so the F_4 here actually exhibit an error of forth order.

4. (a)

$$\begin{aligned}\int_0^2 x\cos(x)dx &= \int_{x=0}^2 x d\sin(x)\\ &= x\sin(x)|_{x=0}^2 - \int_{x=0}^2 \sin(x)dx\\ &= (2\sin(2)-0)+\cos(x)|_{x=0}^2\\ &= 2\sin(2)+\cos(2)-1\end{aligned}$$

- m =1

$$
\begin{aligned}
\int_0^2 x\cos(x)dx &\approx 2\cdot(f(0)+f(2))/2 \\
&= f(0)+f(2) \\
&= 2\cos(2)
\end{aligned}
$$

And we can calculate the error is

$$
\begin{aligned}
e &= |2\cos(2)-2\sin(2)-\cos(2)-1| \\
&= 2\sin(2)-\cos(2)-1 \\
&\approx 1.23
\end{aligned}
$$

- m=2

$$
\begin{aligned}
\int_0^2 x\cos(x)dx &\approx (f(0)+2f(1)+f(2))/2 \\
&= (f(0)+2f(1)+f(2))/2 \\
&= (2\cos(1)+2\cos(2))/2 \\
&= \cos(1)+\cos(2)
\end{aligned}
$$

And we can calculate the error is

$$
\begin{aligned}
e &= |\cos(1)+\cos(2)-2\sin(2)-\cos(2)+1| \\
&= 2\sin(2)-\cos(1)-1 \\
&\approx 0.28
\end{aligned}
$$

- m=4

$$
\begin{aligned}
\int_0^2 x\cos(x)dx &\approx \frac{1}{2}\cdot\frac{f(0)+2f(1/2)+2f(1)+2f(3/2)+f(2)}{2} \\
&= \frac{\cos(1/2)+2\cdot\cos(1)+3\cdot\cos(3/2)+2\cos(2))}{4} \\
&= \frac{\cos(1/2)}{4}+\frac{\cos(1)}{2}+\frac{3\cdot\cos(3/2)}{4}+\frac{\cos(2)}{2}
\end{aligned}
$$

And we can calculate the error is

$$
\begin{aligned}
e &= \left|\frac{\cos(1/2)}{4}+\frac{\cos(1)}{2}+\frac{3\cdot\cos(3/2)}{4}+\frac{\cos(2)}{2}-2\sin(2)-\cos(2)+1\right| \\
&\approx 0.068
\end{aligned}
$$

(b)

$$
\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &= \int_0^1 \frac{1}{1+tan^2\theta}d\tan\theta \\
&= \int_0^1 \frac{\sec^2\theta}{1+tan^2\theta}d\theta \\
&= \int_0^1 \frac{\sec^2\theta}{1+\frac{\sin^2\theta}{\cos^2\theta}}d\theta \\
&= \int_0^1 \frac{1}{\cos^2\theta+\sin^2\theta}d\theta \\
&= \theta|_{x=0}^1 \\
&= \arctan x|_0^1 \\
&= \frac{\pi}{4}
\end{aligned}
$$

- m =1

$$
\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &\approx 1\cdot\frac{f(0)+f(1)}{2} \\
&= \frac{1+\frac{1}{2}}{2} \\
&= \frac{3}{4}
\end{aligned}
$$

And the error is

$$
\begin{aligned}
e &= \left|\frac{3}{4}-\frac{\pi}{4}\right| \\
&\approx 0.035
\end{aligned}
$$

- m=2

$$
\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &\approx \frac{1}{2}\cdot\frac{f(0)+2f(1/2)+f(1)}{2} \\
&= \frac{1+\frac{2}{1+1/4}+1/2}{4} \\
&= \frac{1+\frac{8}{5}+1/2}{4} \\
&= \frac{31}{40}
\end{aligned}
$$

And the error is

$$
\begin{aligned}
e &= \left|\frac{31}{40}-\frac{\pi}{4}\right| \\
&\approx 0.0104
\end{aligned}
$$

- m=4

$$\begin{aligned}
\int_0^1 \frac{1}{1+x^2}dx &\approx \frac{1}{4}\frac{f(0)+2f(1/4)+2f(1/2)+2f(3/4)+f(1)}{2} \\
&= \frac{1+\dfrac{2}{1+1/16}+\dfrac{2}{1+1/4}+\dfrac{2}{1+9/16}+1/2}{8} \\
&= \frac{5323}{6800}
\end{aligned}$$

And the error is

$$\begin{aligned}
e &= \left|\frac{5323}{6800}-\frac{\pi}{4}\right| \\
&\approx 0.002604
\end{aligned}$$

5. (a)

$$\begin{aligned}
\int_0^2 \frac{dx}{\sqrt{2-x}} &= \int_0^2 (2-x)^{-1/2}dx \\
&= -\int_0^2 (2-x)^{-1/2}d(2-x) \\
&= -2(2-x)^{1/2}|_{x=0}^2 \\
&= 2\sqrt{2}
\end{aligned}$$

- m =1

$$\begin{aligned}
\int_0^2 \frac{dx}{\sqrt{2-x}} &\approx 2f(1) \\
&= 2\cdot\frac{1}{\sqrt{2-1}} \\
&= 2
\end{aligned}$$

And the error is

$$\begin{aligned}
e &= |2-2\sqrt{2}| \\
&= 2(\sqrt{2}-1) \\
&\approx 0.828
\end{aligned}$$

- m=2

$$\begin{aligned}
\int_0^2 \frac{dx}{\sqrt{2-x}} &\approx f(1/2) + f(3/2) \\
&= \frac{1}{\sqrt{2-1/2}} + \frac{1}{\sqrt{2-3/2}} \\
&= \frac{1}{\sqrt{3/2}} + \frac{1}{\sqrt{1/2}} \\
&= \sqrt{2/3} + \sqrt{2} \\
&= \left(\frac{\sqrt{3}}{3} + 1\right)\sqrt{2}
\end{aligned}$$

And the error is

$$\begin{aligned}
e &= |\sqrt{2/3} + \sqrt{2} - 2\sqrt{2}| \\
&= |\sqrt{2/3} - \sqrt{2}| \\
&\approx 0.598
\end{aligned}$$

- m=4

$$\begin{aligned}
\int_0^2 \frac{dx}{\sqrt{2-x}} &\approx \frac{1}{2}(f(1/4) + f(3/4) + f(5/4) + f(7/4)) \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{2-1/4}} + \frac{1}{\sqrt{2-3/4}} + \frac{1}{\sqrt{2-5/4}} + \frac{1}{\sqrt{2-7/4}}\right) \\
&= \frac{1}{2}\left(\frac{1}{\sqrt{7/4}} + \frac{1}{\sqrt{5/4}} + \frac{1}{\sqrt{3/4}} + \frac{1}{\sqrt{1/4}}\right) \\
&= \frac{1}{2}\left(\sqrt{4/7} + \sqrt{4/5} + \sqrt{4/3} + \sqrt{4}\right) \\
&= \sqrt{1/7} + \sqrt{1/5} + \sqrt{1/3} + 1
\end{aligned}$$

And the error is

$$\begin{aligned}
e &= |\sqrt{1/7} + \sqrt{1/5} + \sqrt{1/3} + 1 - 2\sqrt{2}| \\
&\approx 0.426
\end{aligned}$$

(b)
- m =1

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x} dx &\approx \pi/2 f(\pi/4)\\ &= \pi/2 \cdot \frac{\cos(\pi/4)}{\pi/4}\\ &= 2\cos(\pi/4)\\ &= \sqrt{2}\end{aligned}$$

- m=2

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x} dx &\approx \pi/4(f(\pi/8) + f(3\pi/8))\\ &= \pi/4 \cdot \left(\frac{\cos(\pi/8)}{3\pi/8} + \frac{\cos(3\pi/8)}{\pi/8}\right)\\ &= \frac{\cos(\pi/8)}{3/2} + \frac{\cos(3\pi/8)}{1/2}\\ &= \frac{\sqrt{\sqrt{2}+2}}{3} + \sqrt{2-\sqrt{2}}\end{aligned}$$

- m=4

$$\begin{aligned}\int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x} dx &\approx \pi/8(f(\pi/16) + f(3\pi/16) + f(5\pi/16) + f(7\pi/16))\\ &= \pi/8 \cdot \left(\frac{\cos(\pi/16)}{7\pi/16} + \frac{\cos(3\pi/16)}{5\pi/16} + \frac{\cos(5\pi/16)}{3\pi/16} + \frac{\cos(7\pi/16)}{\pi/16}\right)\\ &= \frac{\cos(\pi/16)}{7/2} + \frac{\cos(3\pi/16)}{5/2} + \frac{\cos(5\pi/16)}{3/2} + \frac{\cos(7\pi/16)}{1/2}\\ &= \frac{\sqrt{\sqrt{2}+2}}{3} + \sqrt{2-\sqrt{2}}\end{aligned}$$

6.
- $f(x) = 1$

$$\begin{aligned}LHS &= \int_{x_0}^{x_4} 1 dx\\ &= x_4 - x_0\\ RHS &= \frac{2h}{45}(7 + 32 + 12 + 32 + 7)\\ &= 4h\\ &= x_4 - x_0\\ &= LHS\end{aligned}$$

- $f(x) = x$

$$
\begin{aligned}
LHS &= \int_{x_0}^{x_4} x dx \\
&= \frac{1}{2}x^2|_{x_0}^{x_4} \\
&= \frac{x_4^2 - x_0^2}{2} \\
&= \frac{(x_4 - x_0)(x_4 + x_0)}{2} \\
&= \frac{4h(x_0 + 4h + x_0)}{2} \\
&= 2h(2x_0 + 4h) \\
&= 4hx_0 + 8h^2
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0 + 32x_1 + 12x_2 + 32x_3 + 7x_4) \\
&= \frac{2h}{45}(7x_0 + 32(x_0 + h) + 12(x_0 + 2h) + 32(x_0 + 3h) + 7(x_0 + 4h)) \\
&= \frac{2h}{45}(90x_0 + 180h) \\
&= 4hx_0 + 8h^2 \\
&= LHS
\end{aligned}
$$

- $f(x) = x^2$

$$
\begin{aligned}
LHS &= \frac{x^3}{3}|_{x_0}^{x_4} \\
&= \frac{x_4^3 - x_0^3}{3} \\
&= \frac{(x_4 - x_0)(x_4^2 + x_4x_0 + x_0^2)}{3} \\
&= \frac{4h((x_0 + 4h)^2 + (x_0 + 4h) \cdot x_0 + x_0^2)}{3} \\
&= \frac{4h(x_0^2 + 8x_0h + 16h^2 + x_0^2 + 4hx_0 + x_0^2)}{3} \\
&= \frac{4h}{3}(3x_0^2 + 12x_0h + 16h^2)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^2 + 32x_1^2 + 12x_2^2 + 32x_3^2 + 7x_4^2) \\
&= \frac{2h}{45}(7x_0^2 + 32(x_0+h)^2 + 12(x_0+2h)^2 + 32(x_0+3h)^2 + 7(x_0+4h)^2) \\
&= \frac{2h}{45}(90x_0^2 + 360x_0h + 480h^2) \\
&= 4x_0^2h + 16x_0h^2 + \frac{64h^3}{3} \\
&= \frac{4h}{3}(3x_0^2 + 12x_0h + 16h^2) \\
&= LHS
\end{aligned}
$$

- $f(x) = x^3$

$$
\begin{aligned}
LHS &= \frac{x^4}{4}|_{x_0}^{x_4} \\
&= \frac{x_4^4 - x_0^4}{4} \\
&= \frac{(x_4 - x_0)(x_4 + x_0)(x_4^2 + x_0^2)}{4} \\
&= \frac{4h(2x_0 + 4h)(x_0^2 + (x_0+4h)^2)}{4} \\
&= h(2x_0 + 4h)(x_0^2 + x_0^2 + 8hx_0 + 16h^2) \\
&= 2h(2x_0 + 4h)(x_0^2 + 4hx_0 + 8h^2) \\
&= 4h(x_0 + 2h)(x_0^2 + 4hx_0 + 8h^2) \\
&= 4h(x_0^3 + 4hx_0^2 + 8h^2x_0 + 2hx_0^2 + 8h^2x_0 + 16h^3) \\
&= 4h(x_0^3 + 6hx_0^2 + 16h^2x_0 + 16h^3)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^3 + 32x_1^3 + 12x_2^3 + 32x_3^3 + 7x_4^3) \\
&= \frac{2h}{45}(7x_0^2 + 32(x_0+h)^3 + 12(x_0+2h)^3 + 32(x_0+3h)^3 + 7(x_0+4h)^3) \\
&= \frac{2h}{45}(90x_0^3 + 540x_0^2h + 1440x_0h^2 + 1440h^3) \\
&= 4h(x_0^3 + 6x_0^2h + 16x_0h^2 + 16h^3) \\
&= LHS
\end{aligned}
$$

- $f(x) = x^4$

$$\begin{aligned} LHS &= \frac{x^5}{5}\Big|_{x_0}^{x_4} \\ &= \frac{(x_4 - x_0)(x_4^4 + x_4^3 x_0 + x_4^2 x_0^2 + x_4 x_0^3 + x_0^4)}{5} \\ &= \frac{4h}{5}(5x_0^4 + 40x_0^3 h + 160x_0^2 h^2 + 320x_0 h^3 + 256h^4) \end{aligned}$$

$$\begin{aligned} RHS &= \frac{2h}{45}(7x_0^4 + 32x_1^4 + 12x_2^4 + 32x_3^4 + 7x_4^4) \\ &= \frac{2h}{45}(7x_0^4 + 32(x_0 + h)^4 + 12(x_0 + 2h)^4 + 32(x_0 + 3h)^4 + 7(x_0 + 4h)^4) \\ &= \frac{2h}{45}(90x_0^4 + 720x_0^3 h + 2880x_0^2 h^2 + 5760x_0 h^3 + 4608h^4) \\ &= \frac{4h}{5}(5x_0^4 + 40x_0^3 h + 160x_0^2 h^2 + 320x_0 h^3 + 256h^4) \\ &= LHS \end{aligned}$$

- $f(x) = x^5$

$$\begin{aligned} LHS &= \frac{x^6}{6}\Big|_{x_0}^{x_4} \\ &= \frac{(x_0 - x_4)(x_0 + x_4)(x_0^2 - x_4 x_0 + x_4^2)(x_0^2 + x_4 x_0 + x_4^2)}{6} \\ &= \frac{4h}{6}(1024h^5 + 1536x_0 h^4 + 960x_0^2 h^3 + 320x_0^3 h^2 + 60x_0^4 h + 6x_0^5) \\ &= \frac{2h}{3}(1024h^5 + 1536x_0 h^4 + 960x_0^2 h^3 + 320x_0^3 h^2 + 60x_0^4 h + 6x_0^5) \end{aligned}$$

$$\begin{aligned} RHS &= \frac{2h}{45}(7x_0^5 + 32x_1^5 + 12x_2^5 + 32x_3^5 + 7x_4^5) \\ &= \frac{2h}{45}(7x_0^5 + 32(x_0 + h)^5 + 12(x_0 + 2h)^5 + 32(x_0 + 3h)^5 + 7(x_0 + 4h)^5) \\ &= \frac{2h}{45}(15360h^5 + 23040x_0 h^4 + 14400x_0^2 h^3 + 4800x_0^3 h^2 + 900x_0^4 h + 90x_0^5) \\ &= \frac{2h}{3}(1024h^5 + 1536x_0 h^4 + 960x_0^2 h^3 + 320x_0^3 h^2 + 60x_0^4 h + 6x_0^5) \\ &= LHS \end{aligned}$$

- $f(x) = x^6$

$$
\begin{aligned}
LHS &= \frac{x^7}{7}\Big|_{x_0}^{x_4} \\
&= \frac{-(x_0 - x_4)(x_0^6 + x_4 x_0^5 + x_4^2 x_0^4 + x_4^3 x_0^3 + x_4^4 x_0^2 + x_4^5 x_0 + x_4^6)}{7} \\
&= \frac{4h}{7}(4096h^6 + 7168h^5 x_0 + 5376h^4 x_0^2 + 2240h^3 x_0^3 + 560h^2 x_0^4 + 84h x_0^5 + 7x_0^6) \\
&= 4h\left(\frac{4096}{7}h^6 + 1024h^5 x_0 + 768h^4 x_0^2 + 320h^3 x_0^3 + 80h^2 x_0^4 + 12h x_0^5 + x_0^6\right)
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{2h}{45}(7x_0^6 + 32x_1^6 + 12x_2^6 + 32x_3^6 + 7x_4^6) \\
&= \frac{2h}{45}(7x_0^6 + 32(x_0 + h)^6 + 12(x_0 + 2h)^6 + 32(x_0 + 3h)^6 + 7(x_0 + 4h)^6) \\
&= \frac{2h}{45}(52800h^6 + 92160h^5 x_0 + 69120h^4 x_0^2 + 28800h^3 x_0^3 + 7200h^2 x_0^4 + 1080h x_0^5 + 90x_0^6) \\
&= 4h\left(\frac{1760h^6}{3} + 1024h^5 x_0 + 768h^4 x_0^2 + 320h^3 x_0^3 + 80h^2 x_0^4 + 12h x_0^5 + x_0^6\right)
\end{aligned}
$$

$$
\because \frac{4096}{7} \neq \frac{1760}{3}
$$

$$
\therefore RHS \neq LHS
$$

Therefore, the degree of precision is 5.