Homework 4
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1. Firstly, we can get U first, which is

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 4 & 3 & 2 \\ 0 & 2 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_3 - 2R_2 \to R_3} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 2 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_4 - R_2 \to R_4} \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} = U$$

From the operations above, we can observe that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = U$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A = U$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} A = U$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

We can check by matlab that $A = LU$

Listing 1: Check1

```
L=[1,0,0,0;0,1,0,0;1,2,1,0;0,1,0,1];
U =[1,-1,1,2;0,2,1,0;0,0,1,2;0,0,0,-1];
A= L*U

A =

     1    -1     1     2
     0     2     1     0
     1     3     4     4
     0     2     1    -1

diary off
```

Now, we can solve $Ly = b$ by forward substitution that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} y = \begin{bmatrix} 3 \\ 2 \\ 11 \\ 1 \end{bmatrix}$$
$$\begin{aligned} y_1 &= 3 \\ y_2 &= 2 \\ y_1 + 2y_2 + y_3 &= 11 \\ 3 + 4 + y_3 &= 11 \\ y_3 &= 4 \\ y_2 + y_4 &= 1 \\ 2 + y_4 &= 1 \\ y_4 &= -1 \\ \therefore y &= \begin{bmatrix} 3 \\ 2 \\ 4 \\ -1 \end{bmatrix} \end{aligned}$$

Next, we can solve the system that $Ux = y$, which is

$$\begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix} x = \begin{bmatrix} 3 \\ 2 \\ 4 \\ -1 \end{bmatrix}$$

$$\begin{aligned} -x_4 &= -1 \\ x_4 &= 1 \\ x_3 + 2x_4 &= 4 \\ x_3 + 2 &= 4 \\ x_3 &= 2 \\ 2x_2 + x_3 &= 2 \\ 2x_2 + 2 &= 2 \\ x_2 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 &= 3 \\ x_1 + 4 &= 3 \\ x_1 &= -1 \\ x &= \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

And we can check the answer by Matlab:

Listing 2: Check2

```
x=[-1;0;2;1];
b=A*x

b =

     3
     2
    11
     1

diary off
```

2. • To change row 1 and row 4 and negate row 2, we can multiply B by $P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, and PB is the answer we deserved.

- To get C, we need multiply B by the matrix $Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ -2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, which means $C = QB$

3. •

$$
\begin{aligned}
\because A &= \begin{bmatrix} 1 & 2 \\ 1+\delta & 2 \end{bmatrix} \\
\therefore A^{-1} &= \begin{bmatrix} -\frac{1}{\delta} & \frac{1}{\delta} \\ \frac{1+\delta}{2\delta} & -\frac{1}{2\delta} \end{bmatrix} \\
\mathrm{cond}(A, \infty) &= ||A||_\infty ||A^{-1}||_\infty \\
&= |3+\delta| \cdot |\frac{2}{\delta}| \\
&= \left| \frac{2(3+\delta)}{\delta} \right| \\
&= \left| \frac{6+2\delta}{\delta} \right| \\
\text{When } \delta &= 10^{-2} \\
\mathrm{cond}(A, \infty) &= \left| \frac{6.02}{0.01} \right| \\
&= 602 \\
\text{When } \delta &= 10^{-6} \\
\mathrm{cond}(A, \infty) &= \left| \frac{6 + 2 \cdot 10^{-6}}{10^{-6}} \right| \\
&= 6000002 \\
\text{When } \delta &= 10^{-8} \\
\mathrm{cond}(A, \infty) &= \left| \frac{6 + 2 \cdot 10^{-8}}{10^{-8}} \right| \\
&= 600000002
\end{aligned}
$$

- We can write a code to calculate x_c and relative error:

Listing 3: compare.m

```
%% Function: Compare the relationship between relationship of computed error and
    condition number of matrix A
del_vec =[10^(-2), 10^(-6), 10^(-8)];
relationship_error(del_vec);

function relationship_error(del)
x =[7;3];
  for delta = del
    %Create A
    A=[1,2;1+delta,2];
```

```
    %Compute b
    b=A*x;

    %Give the computed root
    xc = A\b;

    %Compute the relative error
    err = norm(x-xc,inf)/norm(x,inf);

    %Compute the condition number
    condition = cond(A,inf);
    fprintf('delta = %10.8f solution = %15.10f %15.10f relative error = %e cond(A) =
        %5.4e\n', delta,xc,err,condition);
  end
end
```

and the output follows:

Listing 4: Output

```
compare
delta = 0.01000000 solution =  7.0000000000  3.0000000000 relative error = 3.172066e-15
    cond(A) = 6.0200e+02
delta = 0.00000100 solution =  7.0000000005  2.9999999997 relative error = 7.404046e-11
    cond(A) = 6.0000e+06
delta = 0.00000001 solution =  6.9999999189  3.0000000406 relative error = 1.158603e-08
    cond(A) = 6.0000e+08
diary off
```

so by comparing the exponential part of the relative error and condition number of A, we can observe that relative error is approximately equals to $\alpha \cdot \text{cond}(A)$, where $\alpha \approx 10^{-16}$. Since

Listing 5: Epsilon

```
eps

ans =

    2.220446049250313e-16

diary off
```

so we can conclude that $\frac{||\vec{x} - \vec{x_c}||}{||\vec{x}||} \approx \epsilon_{mach} \cdot \text{cond}(A)$

4. • Firstly, we need to show that A is symmetric, which is

$$A^T = \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix} = A$$

So that A is symmetric.

Second, we need to show that A is positive definite that

$$
\begin{aligned}
x^T A x &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 3 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 + 3x_2 & 3x_1 + 10x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= x_1^2 + 6x_1x_2 + 10x_2^2 \\
&= x_1^2 + 6x_1x_2 + 9x_2^2 + x_2^2 \\
&= (x_1 + 3x_2)^2 + x_2^2
\end{aligned}
$$

Since $(x_1 + 3x_2)^2 + x_2^2 \geq 0$, and $(x_1 + 3x_2)^2 + x_2^2 = 0 \iff x_1 = x_2 = 0$, which means $x = \vec{0}$, so that A is positive definite.

- Firstly, we need to show that A is symmetric, which is

$$A^T = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} = A$$

So that A is symmetric.

But we can observe that $|a_{12}| = 1, |a_{22}| = 0$, and $|a_{12}| > |a_{22}|$ follows, so the greatest element is off diagonal, so A is not positive definite.

- Firstly, we need to show that A is symmetric, which is

$$A^T = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix} = A$$

So that A is symmetric.

But we can observe that $\det(A) = 2 \cdot 3 - 4 \cdot 4 = 12 - 16 = -4 < 0$, so A is not positive definite.

5.

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

We can firstly check whether it is positive definite:

$$x^TAx = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1 - x_2 & -x_1 + 2x_2 - x_3 & -x_2 + 2x_3 - x_4 & -x_3 + 2x_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= 2x_1^2 - x_1x_2 - x_1x_2 + 2x_2^2 - x_2x_3 - x_2x_3 + 2x_3^2 - x_3x_4 - x_3x_4 + 2x_4^2$$

$$= x_1^2 + (x_1^2 - 2x_1x_2 + x_2^2) + (x_2^2 - 2x_2x_3 + x_2^3) + (x_3^2 - 2x_3x_4 + x_4^2) + x_4^2$$

$$= x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_4)^2 + x_4^2 > 0$$

Therefore, A is positive definite, and we can easily observe that A is symmetric, so A is a spd matrix.

$$A = \begin{bmatrix} r_{11} & 0 & 0 & 0 \\ r_{12} & r_{22} & 0 & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11}^2 & r_{11}r_{12} & r_{11}r_{13} & r_{11}r_{14} \\ r_{11}r_{12} & r_{12}^2 + r_{22}^2 & r_{12}r_{13} + r_{22}r_{23} & r_{12}r_{14} + r_{22}r_{24} \\ r_{11}r_{13} & r_{12}r_{13} + r_{22}r_{23} & r_{13}^2 + r_{23}^2 + r_{33}^2 & r_{13}r_{14} + r_{23}r_{24} + r_{33}r_{34} \\ r_{11}r_{14} & r_{12}r_{14} + r_{22}r_{24} & r_{13}r_{14} + r_{23}r_{24} + r_{33}r_{34} & r_{14}^2 + r_{24}^2 + r_{34}^2 + r_{44}^2 \end{bmatrix}$$

$$\begin{aligned} \therefore r_{11}^2 &= 2 \\ r_{11} &= \sqrt{2} \\ r_{11}r_{12} &= -1 \\ \therefore r_{12} &= -\frac{\sqrt{2}}{2} \\ r_{11}r_{13} &= r_{11}r_{14} = 0 \\ r_{13} &= r_{14} = 0 \\ r_{12}^2 + r_{22}^2 &= 2 \\ r_{22} &= \frac{\sqrt{6}}{2} \\ r_{12}r_{13} + r_{22}r_{23} &= -1 \\ r_{23} &= -\frac{\sqrt{6}}{3} \\ r_{12}r_{14} + r_{22}r_{24} &= 0 \end{aligned}$$

$$
\begin{aligned}
r_{24} &= 0 \\
r_{13}^2 + r_{23}^2 + r_{33}^2 &= 2 \\
r_{33}^2 &= \frac{4}{3} \\
r_{33} &= \frac{2\sqrt{3}}{3} \\
r_{13}r_{14} + r_{23}r_{24} + r_{33}r_{34} &= -1 \\
r_{34} &= -\frac{\sqrt{3}}{2} \\
r_{14}^2 + r_{24}^2 + r_{34}^2 + r_{44}^2 &= 2 \\
r_{44}^2 &= \frac{5}{4} \\
r_{44} &= \frac{\sqrt{5}}{2} \\
\therefore A &= \begin{bmatrix} r_{11} & 0 & 0 & 0 \\ r_{12} & r_{22} & 0 & 0 \\ r_{13} & r_{23} & r_{33} & 0 \\ r_{14} & r_{24} & r_{34} & r_{44} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ 0 & r_{22} & r_{23} & r_{24} \\ 0 & 0 & r_{33} & r_{34} \\ 0 & 0 & 0 & r_{44} \end{bmatrix} \\
&= \begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{3} & \frac{2\sqrt{3}}{3} & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{5}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & \frac{2\sqrt{3}}{3} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{5}}{2} \end{bmatrix} \\
\therefore R &= \begin{bmatrix} \sqrt{2} & -\frac{\sqrt{2}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{6}}{2} & -\frac{\sqrt{6}}{3} & 0 \\ 0 & 0 & \frac{2\sqrt{3}}{3} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{5}}{2} \end{bmatrix}
\end{aligned}
$$