HW3: MATH/CSCI-4800-02 Numerical Computing

1. Text problem on p.43: problem 8.
   (*if you use 2nd edition of the book, it is problem 8 on p. 41*).

   Solutions: Compute $S = |g'(r)|$, where r is the fixed point being considered. Based on Theorem 1.6, if $S < 1$, the fixed point iteration (FPI) is locally convergent. If $S > 1$ (implied by the proof of Theorem 1.6), the FPI is not locally convergent.

   For a) $g'(x) = \frac{2-2x}{x^3}$, and $S = |g'(1)| = 0 < 1$. Hence the FPI is locally convergent.
   For b) $g'(x) = -\sin x$, and $S = |g'(\pi)| = 0 < 1$. Hence the FPI is locally convergent.
   For c) $g'(x) = 2e^{2x}$, and $S = |g'(0)| = 2 > 1$. Hence the FPI is not locally convergent.

2. Find each fixed point of $g(x) = x^2 - \frac{3}{2}x + \frac{3}{2}$ and decide whether fixed point iteration is locally convergent to it.

   Solution: To find the fixed point of g, we solve $x = g(x)$, and this gives

   $$x^2 - \frac{5}{2}x + \frac{3}{2} = 0, \quad \text{or equivalently} \quad 2x^2 - 5x + 3 = 0.$$

   Using the quadratic formula, we find two roots that are also the two fixed points of g,

   $$x = \frac{5 \pm \sqrt{(-5)^2 - 4 \times 2 \times 3}}{2 \times 2} = \frac{5 \pm 1}{2 \times 2} = \frac{3}{2} \text{ or } 1.$$

   On the other hand $g'(x) = 2x - \frac{3}{2}$.

   At the fixed point $x = \frac{3}{2}$, we have $|g'(\frac{3}{2})| = \frac{3}{2} > 1$, and the PFI is not locally convergent.

   At the fixed point $x = 1$, we have $|g'(1)| = \frac{1}{2} < 1$, and the PFI is locally convergent.

3. Express $2x^3 - x + e^x = 0$ as a fixed point problem $x = g(x)$ in three different ways.

   Solution: there are many many different answers. Here are few possible ones:

   $$x = 2x^3 + e^x, \quad x = \sqrt[3]{\frac{x - e^x}{2}}, \quad x = \ln(x - 2x^3), \quad x = \frac{3x - e^x}{2(x^2 + 1)}.$$

4. Text problem on p.44: problem 14 (refer to Definition 1.5 in the textbook for the linear convergence rate S, as well as Theorem 1.6).
   (*if you use 2nd edition of the book, it is problem 14 on p.41*).

   Solution: We know that if a FPI converges, then it will converge to a fixed point. So we first need to check for which problems $\sqrt{2}$ is a fixed point, then compute the convergence rate of the respective FPI.

   For A), we set $g(x) = \frac{1}{2}x + \frac{1}{x}$, and

   $$g(\sqrt{2}) = \frac{\sqrt{2}}{2} + \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2},$$

hence $\sqrt{2}$ is a fixed point of this function g. The convergence rate

$$S = |g'(x)|_{x=\sqrt{2}} = \left|\frac{1}{2} - \frac{1}{x^2}\right|_{x=\sqrt{2}} = 0.$$

For B), we set $g(x) = \frac{2}{3}x + \frac{2}{3x}$, and

$$g(\sqrt{2}) = \frac{2\sqrt{2}}{3} + \frac{2}{3\sqrt{2}} = \frac{2\sqrt{2}}{3} + \frac{\sqrt{2}}{3} = \sqrt{2},$$

hence $\sqrt{2}$ is a fixed point of this function g. The convergence rate

$$S = |g'(x)|_{x=\sqrt{2}} = \left|\frac{2}{3} - \frac{2}{3x^2}\right|_{x=\sqrt{2}} = \frac{1}{3}.$$

For C), we set $g(x) = \frac{3}{4}x + \frac{1}{2x}$, and

$$g(\sqrt{2}) = \frac{3\sqrt{2}}{4} + \frac{1}{2\sqrt{2}} = \frac{3\sqrt{2}}{4} + \frac{\sqrt{2}}{4} = \sqrt{2},$$

hence $\sqrt{2}$ is a fixed point of this function g. The convergence rate

$$S = |g'(x)|_{x=\sqrt{2}} = \left|\frac{3}{4} - \frac{1}{2x^2}\right|_{x=\sqrt{2}} = \frac{1}{2}.$$

For all three problems, the rate satisfies $S < 1$, and the corresponding FPIs converge locally. The smaller S is, the faster the convergence of the FPI is. With this, we can rank from the fastest to the slowest: A), B), C).

5. Text problem on p.121: problem 2(c).
(*if you use 2nd edition of the book, it is problem 2(c) on p.116.*).
Solution:

- We rearrange the system into the following

$$\begin{aligned} 4u + 3w &= 0 \\ u + 4v &= 5 \\ v + 2w &= 2. \end{aligned}$$

The coefficient matrix A is

$$A = \begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix}, \tag{1}$$

and it is strictly diagonally dominant.

- The update of the Jacobi method is
$$u^{(k+1)} = -\frac{3w^{(k)}}{4}, \quad v^{(k+1)} = \frac{5-u^{(k)}}{4}, \quad w^{(k+1)} = \frac{2-v^{(k)}}{2} = 1 - \frac{v^{(k)}}{2}.$$
With the initial being $u^{(0)} = v^{(0)} = w^{(0)} = 0$, we can compute
$$u^{(1)} = 0, \quad v^{(1)} = \frac{5}{4}, \quad w^{(1)} = 1,$$
and
$$u^{(2)} = -\frac{3}{4}, \quad v^{(2)} = \frac{5}{4}, \quad w^{(2)} = \frac{3}{8}.$$
- The update of the Gauss-Seidel method is
$$u^{(k+1)} = -\frac{3w^{(k)}}{4}, \quad v^{(k+1)} = \frac{5-u^{(k+1)}}{4}, \quad w^{(k+1)} = \frac{2-v^{(k+1)}}{2} = 1 - \frac{v^{(k+1)}}{2}.$$
With the initial being $u^{(0)} = v^{(0)} = w^{(0)} = 0$, we can compute
$$u^{(1)} = 0, \quad v^{(1)} = \frac{5}{4}, \quad w^{(1)} = 1 - \frac{5/4}{2} = \frac{3}{8},$$
and
$$u^{(2)} = -\frac{9}{32}, \quad v^{(2)} = \frac{5+9/32}{4} = \frac{169}{128}, \quad w^{(2)} = 1 - \frac{169/128}{2} = 1 - \frac{169}{256} = \frac{87}{256}.$$

6. This is to revisit some examples and theories discussed in class. To solve $Ax = b$, where
$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}. \tag{2}$$
Note that this A is strictly diagonally dominant. It is known that both Jacobi and Gauss-Seidel methods lead to iterative methods in the following form
$$x^{k+1} = Bx^k + d, \tag{3}$$
for some $B \in \mathbb{R}^{2\times 2}$ and $d \in \mathbb{R}^2$.

For each of Jacobi and Gauss-Seidel methods, answer the following. You can either calculate by hand or use Matlab.

- What is B?
- What is the spectral radius $\rho(B)$ of B. Here $\rho(B) = \max(|\lambda_1|, |\lambda_2|)$ where λ_1, λ_2 are the two eigenvalues of B. (If you want to calculate using matlab, you may find *eigs* useful.)
- From the value of $\rho(B)$, decide whether the corresponding scheme (3) converges or diverges.
- Repeat the three questions above for the following A,
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}. \tag{4}$$
Note that this A is not strictly diagonally dominant.

Solution: When

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}, \tag{5}$$

it can be split into as $A = D + L + U$, where

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{6}$$

<u>For Jacobi method:</u>

$$B = -D^{-1}(L+U) = -\begin{bmatrix} 1/3 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1/3 \\ 1/2 & 0 \end{bmatrix}. \tag{7}$$

The two eigenvalues of B are $\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$, hence the spectral radius $\rho(B) = \frac{1}{\sqrt{6}}$ (which is approximately 0.4082). Since $\rho(B) < 1$, the Jacobi method converges. This conclusion is also implied by the fact that A is strictly diagonally dominant. (Recommended reading to those who want to understand the underlying mathematics: proof of theorem 2.10 in Section 2.5.3.)

<u>For Gauss-Seidel method:</u>

$$B = -(L+D)^{-1}U = -\begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 1/3 & 0 \\ -1/6 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/3 \\ 0 & 1/6 \end{bmatrix}. \tag{8}$$

The two eigenvalues of B are $0, \frac{1}{6}$, hence the spectral radius $\rho(B) = \frac{1}{6}$. Since $\rho(B) < 1$, the Gauss-Seidel method converges. This conclusion is also implied by the fact that A is strictly diagonally dominant.

When

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \tag{9}$$

it can be split into $A = D + L + U$, where

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

<u>*For Jacobi method:*</u>

$$B = -D^{-1}(L+U) = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}. \tag{11}$$

The two eigenvalues of B are $\sqrt{6}, -\sqrt{6}$, hence the spectral radius $\rho(B) = \sqrt{6}$. Since $\rho(B) > 1$, the Jacobi method diverges.

For Gauss-Seidel method:

$$B = -(L+D)^{-1}U = -\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 6 \end{bmatrix}. \quad (12)$$

The two eigenvalues of B are $0, 6$, hence the spectral radius $\rho(B) = 6$. Since $\rho(B) > 1$, the Gauss-Seidel method diverges.

7. Recommended Reading: Section 2.5.2 about SOR method (this is not covered in the exam), and the comparison of three methods in Example 2.24.