HW5: MATH/CSCI-4800-02 Numerical Computing

Due by 2pm on March 21, 2019 (Thursday)

1. Given four points $(x_j, y_j), j = 1, 2, 3, 4$ as follows,

$$\begin{array}{c|cc} j & x_j & y_j \\ \hline 1 & -1 & 6 \\ 2 & 1 & 4 \\ 3 & 2 & 3 \\ 4 & 3 & -6 \end{array} \tag{1}$$

   1.) Find a cubic polynomial $p(x)$ interpolating all four points, that is $p(x_j) = y_j, j = 1, 2, 3, 4$.

   (i.) find $p(x)$ using the direct approach discussed in class. *One would need to solve a linear system. You can solve it either by hand or by Matlab.*

   (ii.) find $p(x)$ using the Lagrange approach; *It is fine to write your answer in the Lagrange form.*

   (iii.) find $p(x)$ using the Newton's divided difference approach.

   2.) Find a piecewise linear interpolant $g(x)$. That is $g(x) = g_j(x)$ when $x \in [x_j, x_{j+1}]$, satisfying $g_j(x_j) = y_j$, $g_j(x_{j+1}) = y_{j+1}$, $j = 1, 2, 3$.

   3.) Find a polynomial $q(x) = \sum_{k=0}^{6} a_k x^k$ of degree six, where $a_6 \neq 0$, such that $q(x)$ interpolates all four given points, that is $q(x_j) = y_j, j = 1, 2, 3, 4$.

2. Consider the cubic spline

$$g(x) = \begin{cases} g_1(x) = 6 - 2x + \frac{1}{2}x^3 & \text{on } [0, 2] \\ g_2(x) = 6 + 4(x-2) + c(x-2)^2 + d(x-2)^3 & \text{on } [2, 3] \end{cases} \tag{2}$$

   (i.) Find c;

   (ii.) Does there exist a number d such that the spline is natural? If so, find d.

3. Given the data points $(1, 0)$, $(2, \ln(2))$, $(4, \ln(4))$, which are sampled from the natural logarithm function $y = ln(x)$,

   1.) find a degree 2 interpolating polynomial $p(x)$.

   2.) based on the Theorem given in class on interpolating error (see Section 3.2 of the textbook), give an error bound for the approximation at $x = 3$. Namely get a bound for $|p(3) - \ln(3)|$ based on the theorem.

   3.) Compute $p(3)$, and use it to compute the actual error $|p(3) - \ln(3)|$.

4. Given a set of data points $\{(x_j, y_j)\}_{j=1}^n$, with $\{x_j\}_{j=1}^n$ being distinct and $n \geq 2$, there is one and only one polynomial $p(x)$ of degree at most $n-1$ interpolating these points, namely, $p(x_j) = y_j$, for $j = 1, \cdots, n$. Based on Newton's divided difference, we can write

$$p(x) = a_1 + \sum_{j=2}^{n} a_j (x - x_1)(x - x_2) \cdots (x - x_{j-1}), \tag{3}$$

where $a_j = g[x_1, x_2, \cdots, x_j]$ is the j-th divided difference related to the given data.

4.a) Using the idea of Horner's nested method, write down how to evaluate $p(x)$ with as few multiplications as possible.

4.b) Write a Matlab function, myPloyCoef, that computes $\mathbf{a} = (a_1, a_2, \cdots, a_n)^T$ in equation (3), where the input is the set of data (x_j, y_j), $j = 1, \cdots n$. (*You can organize the input data any way you want, such as a* $2 \times n$ *matrix, an* $n \times 2$ *matrix, or two vectors in* $\mathbb{R}^n$*.*)

4.c) Write a Matlab function, myPloyEval, that computes $p(x)$ at any given x based on your strategy in 4.a), where the inputs are $\mathbf{x} = (x_1, x_2, \cdots, x_n)^T$, $\mathbf{a} = (a_1, a_2, \cdots, a_n)^T$, and the given point x.

4.d) Before you apply the two functions you just build, how would you gain confidence that they might be correct? Propose some strategies (namely, some necessary properties) to validate these functions, try them out, and *briefly* report your experience.

4.e) Consider the function $f(x) = \frac{1}{1+4x^2}$ for $x \in [-2, 2]$, and take $\{(x_j, y_j)\}_{j=1}^n$ as a data set where $\{x_j\}_{j=1}^n$ are uniformly distributed points in $[-2, 2]$ (*satisfying* $x_1 = -2$ *and* $x_n = 2$*; one one use x=linespace(-2,2,n)*) and $y_j = f(x_j)$. Apply your Matlab function myPloyCoef with $n = 6$ and report the corresponding coefficient vector $\mathbf{a}$. Plot $f(x)$ and $p(x)$ (evaluated using your myPloyEval) on the same graph with many more points on $[-2, 2]$, for instance with x=linspace(-2,2,200). Summarize and comment your observations. Repeat your experiments and report your results for $n = 13$.

4.f) Redo 4.e) using the Chebyshev nodes $\{x_j\}_{j=1}^n$, where $x_j = 2\cos(\frac{(2j-1)\pi}{2n})$. Again, report the coefficient vector $\mathbf{a}$, and summarize and comment your observations from the plots.

Notes:

- Feel free to use codes in the textbook.
- Attach your codes (to the end of your report).