HW7: MATH/CSCI-4800-02 Numerical Computing

Due by 2pm on April 11, 2019 (Thursday)

1. Using Taylor series expansion, find the error term (or the leading term in the error term) and order (*with respect to* h) of the following approximation for $f'(x)$:
$$\frac{4f(x+h) - 3f(x) - f(x-2h)}{6h}.$$

2. Using Taylor series expansion, find an approximation for $f'(x)$ based on the data $f(x-2h)$, $f(x-h)$, $f(x)$, $f(x+h)$, that has the highest approximation order (*with respect to* h).

3. (Computer problem) Given a smooth function $f(x)$, and it is known that
$$F_2(h) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{1}$$
provides a second order approximation for $f''(x)$ (with respect to the parameter h). Apply Richardson extrapolation to the formula. The resulted approximation for $f''(x)$, denoted as $F_4(h)$, turns out to be a fourth order approximation instead of a third order one for $f''(x)$. Demonstrate the performance of the original second order and the new fourth order formula by approximating $f''(\pi/3)$, where $f(x) = \sin(x) + xe^{-x}$, with $h = h_j = 0.1 * 0.5^j$, with $j = 1, 2, 3, 4, 5, 6$.

   a.) Tabulate your results and errors, with $j-th$ row ($j \geq 2$) including the following
   $$h_j, \quad F_2(h_j), \quad e_2(h_j), \quad \frac{e_2(h_{j-1})}{e_2(h_j)}, \quad F_4(h_j), \quad e_4(h_j), \quad \frac{e_4(h_{j-1})}{e_4(h_j)},$$
   while the first row includes ($j = 1$)
   $$h_j, \quad F_2(h_j), \quad e_2(h_j), \quad -, \quad F_4(h_j), \quad e_4(h_j), \quad -.$$
   Here the errors are $e_2(h) = |F_2(h) - f''(\pi/3)|$ and $e_4(h) = |F_4(h) - f''(\pi/3)|$.
   It is suggested that $F_2(h_j)$, $F_4(h_j)$ are shown with sufficiently many digits after decimal points, so you can see the change when h decreases.

   b.) Plot h_j versus $F_2(h_j), j = 1, \cdots, 6$ in loglog scale; on the same figure, plot h_j versus $F_4(h_j), j = 1, \cdots, 6$ in loglog scale;

   c.) Discussion: How do your results from a.) b.) confirm /support /contradict the claim that $F_2(h)$ is a second order approximation for $f''(x)$, where $F_4(h)$ is a fourth order approximation for $f''(x)$?

4. Apply the composite Trapezoid rule with $m = 1$, 2 and 4 panels to approximate the following integrals. Compute the error by comparing with the exact value from calculus.
$$a) \int_0^2 x\cos(x)dx, \qquad b) \int_0^1 \frac{1}{1+x^2}dx. \tag{2}$$

5. Apply the composite midpoint rule with $m = 1$, 2 and 4 panels to approximate the following integrals.
$$a) \int_0^2 \frac{dx}{\sqrt{2-x}}, \qquad b) \int_0^{\pi/2} \frac{\cos(x)}{\pi/2 - x}dx \tag{3}$$
For the integral in a), also compute the error by comparing with the exact value from calculus.

6. Find the degree of precision of the following degree four Newton-Cotes rule

$$\int_{x_0}^{x_4} f(x)dx \approx \frac{2h}{45}(7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4).$$

Here $x_j = x_0 + jh$, $y_j = f(x_j)$, $j = 1, 2, 3, 4$. And h is a fixed positive parameter.