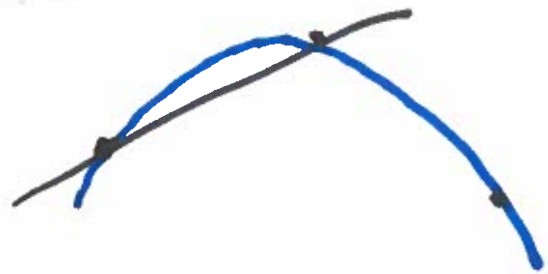
Lecture 14

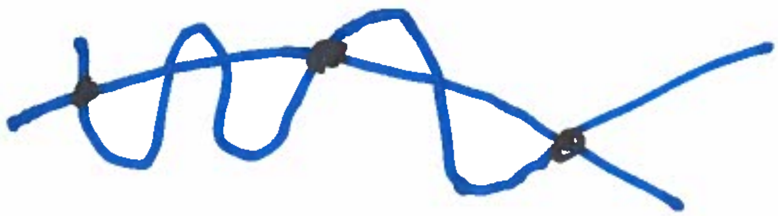
3.11.2019

Recap

→ (x_j, y_j)

"Interpolation"





(x_j, y_j) $j = 1, 2 \ldots n+1$

distinct

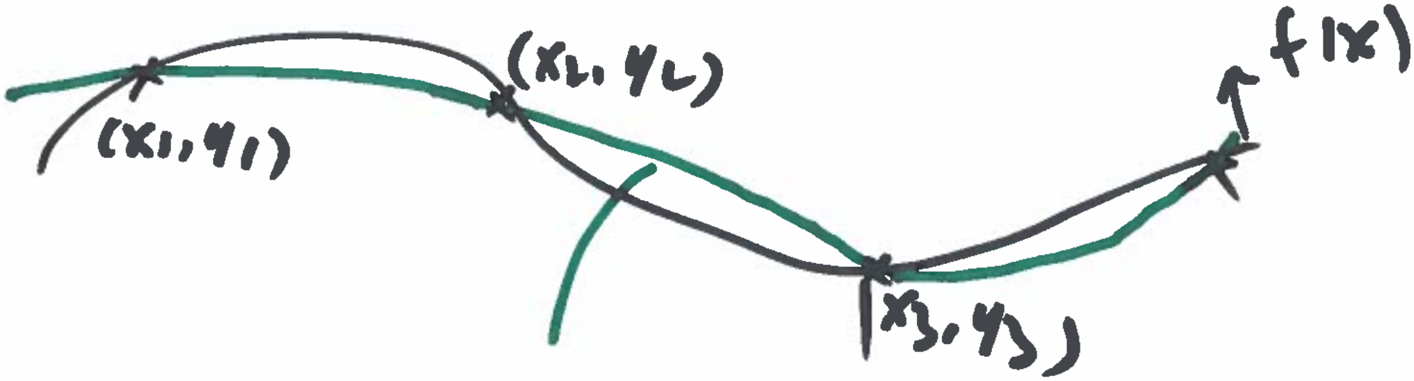
$P_n(x)$: $P_n(x_j) = y_j$

↓ polynomial of degree n.

interpolation

- Direct: $a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$
- Lagrange approach:
- Newton's divided difference

Interpolation error



$p_n(x)$

$y_j = f(x_j)$

Suppose $\{(x_j, y_j)\}_{j=1}^{n+1}$ come from sampling $y = f(x)$

$\{x_j\}_{j=1}^{n+1}$ are distinct, $y_j = f(x_j)$

$p_n(x)$: interpolating polynomial of degree n.

interpolating error :

$$|f(x) - p_n(x)| = ?$$

Theorem:

$$f(x) - p_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_{n+1})}{(n+1)!} \cdot f^{(n+1)}(c) \qquad c=c(x)$$

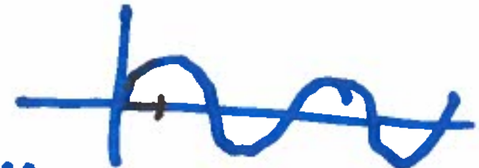
Here c is some number from

$[\min(x, x_1, x_2, \cdots x_{n+1}),$
$\max(x, x_1, x_2, \cdots x_{n+1})]$

Runge's Phenomenon (Wikipedia)

- related to global interpolation, based on equi-distanced points,
- non-equal distanced sampling points, "Chebyshev nodes",
  local interpolation
  ⋮

Example $y = f(x) = \sin x$



we sample 4 points (x_j, y_j) $y_j = f(x_j)$

$x_1 = 0$ $x_2 = \frac{\pi}{6}$, $x_3 = \frac{\pi}{3}$, $x_4 = \frac{\pi}{2}$

we consider the cubic interpolation $P_3(x)$, estimate the error at $x = 1$, 0.2

Solution: $f(x) - P_3(x)$ $\boxed{n=3}$

$$= \frac{(x-0)(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})}{4!} \cdot \underbrace{f^{(4)}(c)}$$

$f'(x) = \cos x$

$f''(x) = -\sin x$, $f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$|f^{(4)}(c)| \le 1$

$$\Rightarrow |f(x) - P_3(x)| \le \frac{|x(x-\frac{\pi}{6})(x-\frac{\pi}{3})(x-\frac{\pi}{2})|}{4!}$$

at $x = 1$, upper bound $\approx 5.35 \times 10^{-4}$

$x = 0.2$ ... $\approx 3.13 \times 10^{-3}$

§ 3-2 Interpolation : local.

Given $\{(x_j, y_j)\}_{j=1}^{n+1}$

with $x_1 < x_2 < \cdots < x_n < x_{n+1}$

Recall: an interpolant $y = p(x)$ of the data:

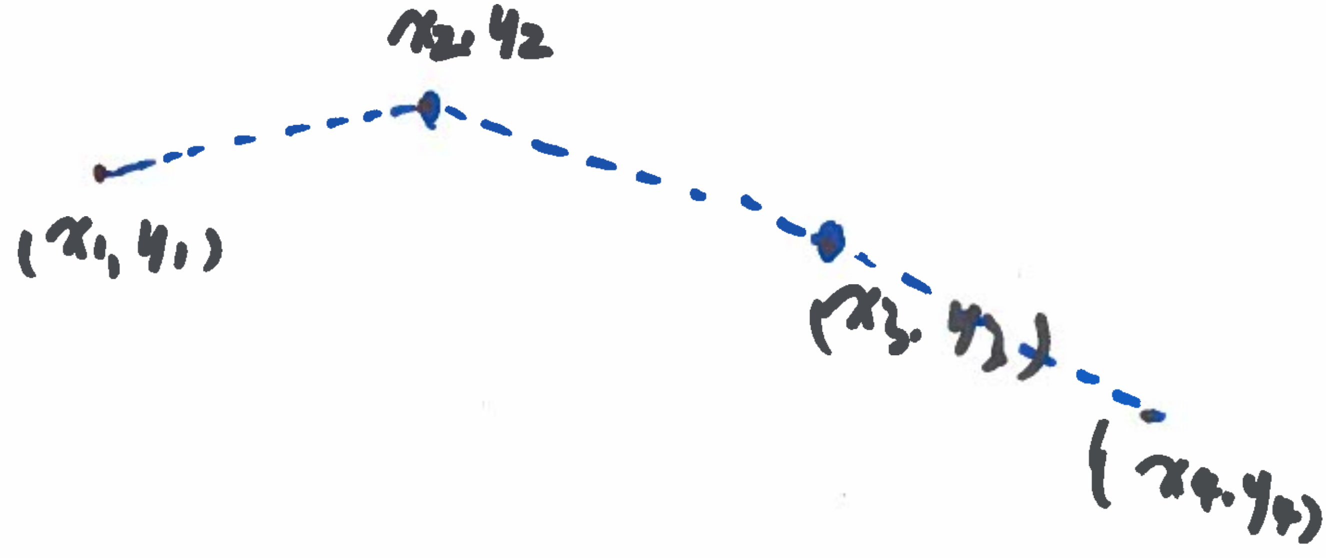
$$p(x_j) = y_j \quad j = 1, \ldots, n+1$$

So far: $p(x)$ — polynomial of degree n.

'global'

we here consider local interpolation

§ 3-2-1 : piecewise linear interpolation



look for $g(x)$, with $g(x) = g_j(x)$ on $[x_j, x_{j+1}]$

such that $g_j(x)$ is linear

(or equivalent, a polynomial of degree 1)

and
$$\begin{cases} g_j(x_j) = y_j \\ g_j(x_{j+1}) = y_{j+1} \end{cases}$$

To find $g(x)$, by construction

$$\begin{cases} g_j(x) = y_j + \dfrac{y_{j+1} - y_j}{x_{j+1} - x_j}(x - x_j) \\ \qquad j = 1, 2 \ldots n \end{cases}$$

(existence, uniqueness : ✓)

In practice, Lagrange-type basis is used to represent $g(x)$

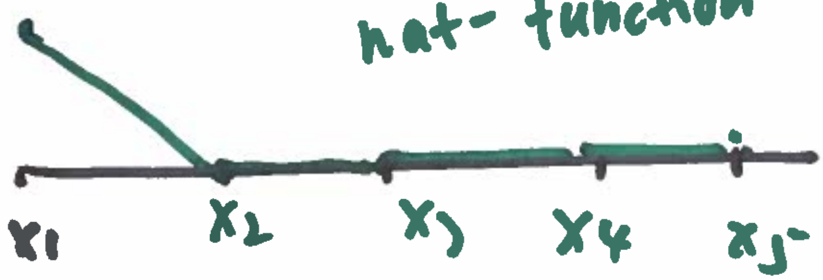
(useful in finite element methods)

$\phi_j(x)$: · piecewise linear

· $\phi_j(x_i) = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} = \delta_{ij}$

hat-function

$j=1,2,\ldots n+1$



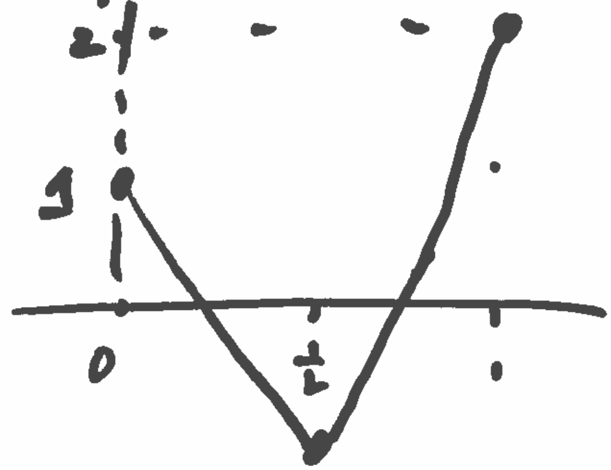
$\phi_1(x)$

$\phi_2(x)$

specific meaning

$$\Rightarrow \boxed{g(x) = \sum_{j=1}^{n+1} y_j \, \phi_j(x)}$$

Example: Given 3 data points



j	(x_j, y_j)
1	$(0, 1)$
2	$(\frac{1}{2}, -1)$
3	$(1, 2)$

the piecewise linear interpolant:

$$g(x) = \begin{cases} g_1(x) & x \in [0, \frac{1}{2}] \\ g_2(x) & x \in [\frac{1}{2}, 1] \end{cases}$$

$$g_1(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$= 1 + \frac{-2}{1/2}(x - 0) = 1 - 4x$$

$$g_2(x) = y_2 + \frac{y_3 - y_2}{x_3 - x_2}(x - x_2)$$

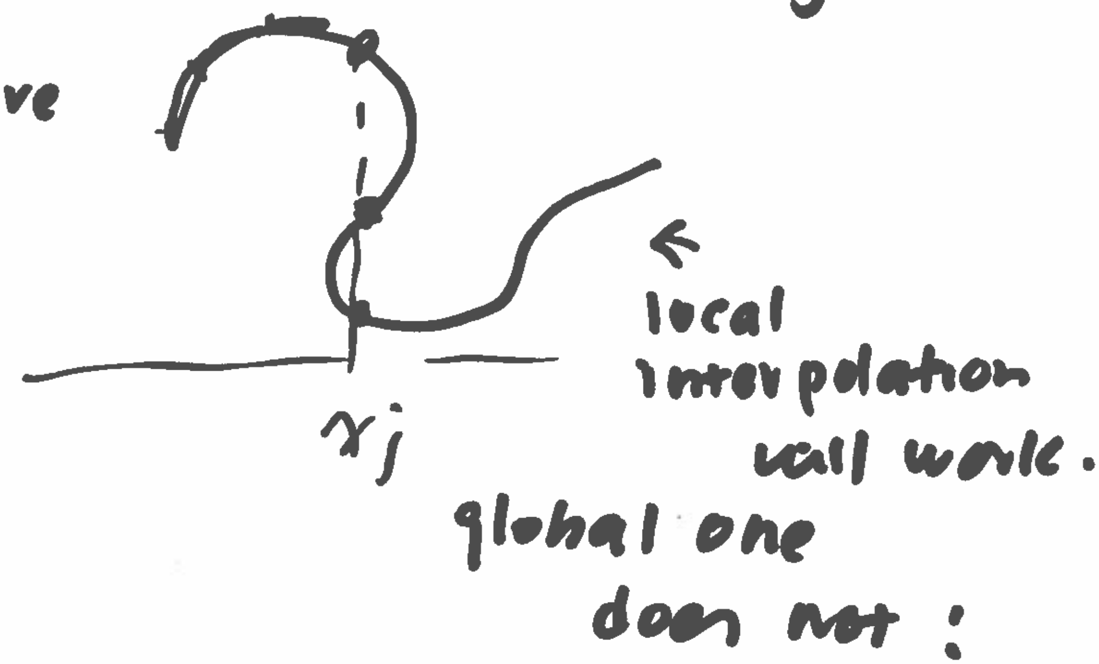
$$= -1 + \frac{3}{1/2}\left(x - \frac{1}{2}\right)$$

$$= -4 + 6x$$

Discussion:

1) $g(x)$ is continuous, it is not differentiable at x_j

2) Curve



§ 3-2-2 Cubic splines

goal: to get better smoothness at x_j

Again, given $\{(x_j, y_j)\}_{j=1}^{n+1}$

$x_1 < x_2 < x_3 < \ldots < x_{n+1}$

cubic spline:

we look for $g(x)$ with

$$g(x) = g_j(x) \text{ on } x \in [x_j, x_{j+1}]$$

and $g_j(x)$ is a cubic polynomial

property 1: $g_j(x_j) = y_j$

$$g_j(x_{j+1}) = y_{j+1} \quad j = 1, 2 \ldots \cancel{n+1}\ n$$

property 2: $g'_{j-1}(x_j) = g_j'(x_j)$

$$j = \underbrace{2, 3 \ldots n}_{\text{interior}}$$

property 3: $g''_{j-1}(x_j) = g_j''(x_j)$

$$j = 2, 3, \ldots n$$