Recap: Lecture 6 1.31.2019

§1 To solve $f(x)=0$ for x

§1-1 Bisection

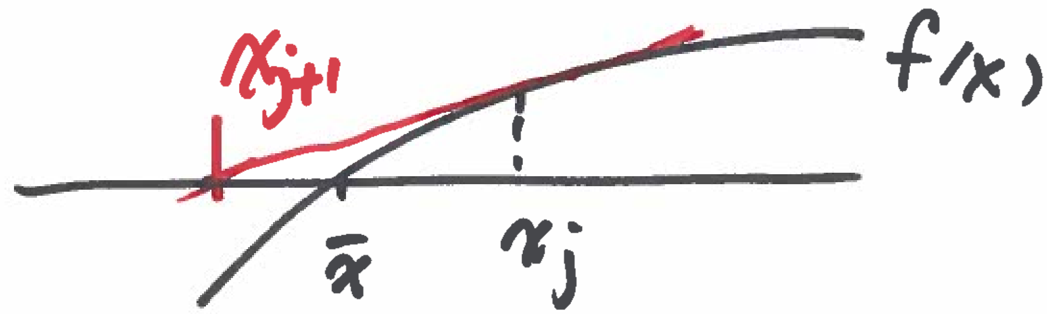
$$\begin{cases} f(a)f(b)<0 \\ f \in C[a,b] \end{cases}$$

- linear convergence

§1.2 Newton's method

Start with x_0

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}$$



quadratic convergence

$$e_{j+1} = \square \, e_j^2$$

$$\begin{cases} e_j = |x_j - \bar{x}| \\ \text{"locally convergence"} \end{cases}$$

Example: (Application)

Given $a \neq 0$
we want to calculate $x = \frac{1}{a}$
by using only $+, -, *$

one try:

$$f(x) = ax - 1 = 0$$

Newton: $x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}$

$$= x_j - \frac{ax_j - 1}{a}$$

$$= \frac{1}{a}$$ (not good)

Another try: $f(x) = a - \frac{1}{x}$

$$f'(x) = \frac{1}{x^2}$$

Newton's: $\boxed{x_{j+1} =} x_j - \frac{f(x_j)}{f'(x_j)}$

$$= x_j - (a - \frac{1}{x_j}) x_j^2 = x_j(2 - a x_j)$$

To solve $f(x) = a - \frac{1}{x} = 0$, $a = 3$

Newton's method

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)} = x_j \cdot (2 - a \cdot x_j)$$

Initial: $x_0 = \underbrace{0.3}$

$$\left(\begin{array}{c} \text{operation:} \\ 2 \quad * \\ 1 \quad - \end{array} \right)$$

$x_1 = 0.3(2 - 3 \times 0.3)$

$= 0.3(2 - 0.9) = 0.3 \times 1.1 = \underbrace{0.33}$

$x_2 = 0.33(2 - 3 \times 0.33)$

$= 0.33(2 - 0.99)$

$= 0.33 \times 1.01$

$= 0.33 \times (1 + 0.01) = 0.33 + 0.0033 = \underbrace{0.3333}$

$x_3 = 0.3333(2 - 3 \times 0.3333)$

$= 0.3333 \times 1.0001$

$= 0.3333 \times (1 + 0.0001)$

$= 0.3333 + 0.00003333$

$= \underbrace{0.33333333}$

A question from some student

How to compute
$$e_j = |x_j - \bar{x}| \ ?$$

- For general examples, $\bar{x}$ is not available, e_j cannot be calculated

- To verify your code, one can use examples that have exact solutions, namely $\bar{x}$ is known.

- e_j can be approximated:

$$e_j \approx |x_j - \bar{x}_{ref}|$$

$$\bar{x}_{ref} = x_J \quad J \gg j$$

§1-3 Secant method : To solve $f(x)=0$

Newton: x_j

$$x_{j+1} = x_j - \frac{f(x_j)}{f'(x_j)}$$

There are applications where f' is expensive or non-trivial to compute. Instead, we use

$$f'(x_j) \approx \frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}}$$

$\Rightarrow$ Secant method

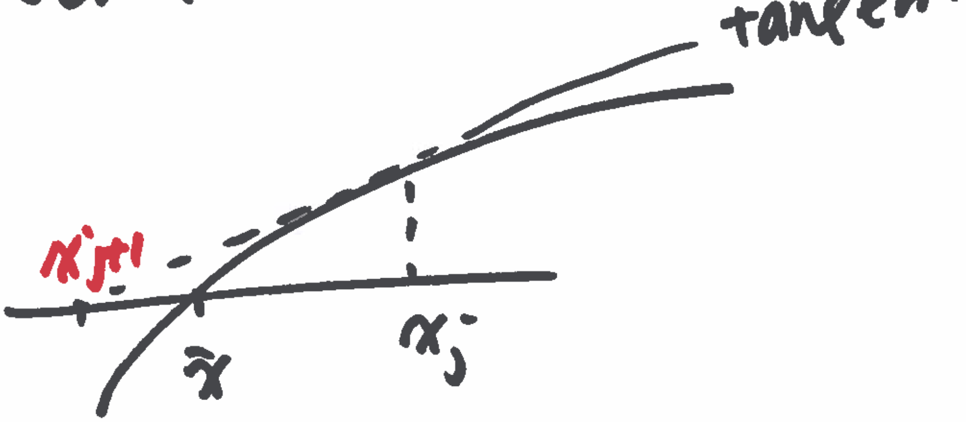
$$x_{j+1} = x_j - \frac{f(x_j)(x_j - x_{j-1})}{f(x_j) - f(x_{j-1})}$$

$j \geq 1$

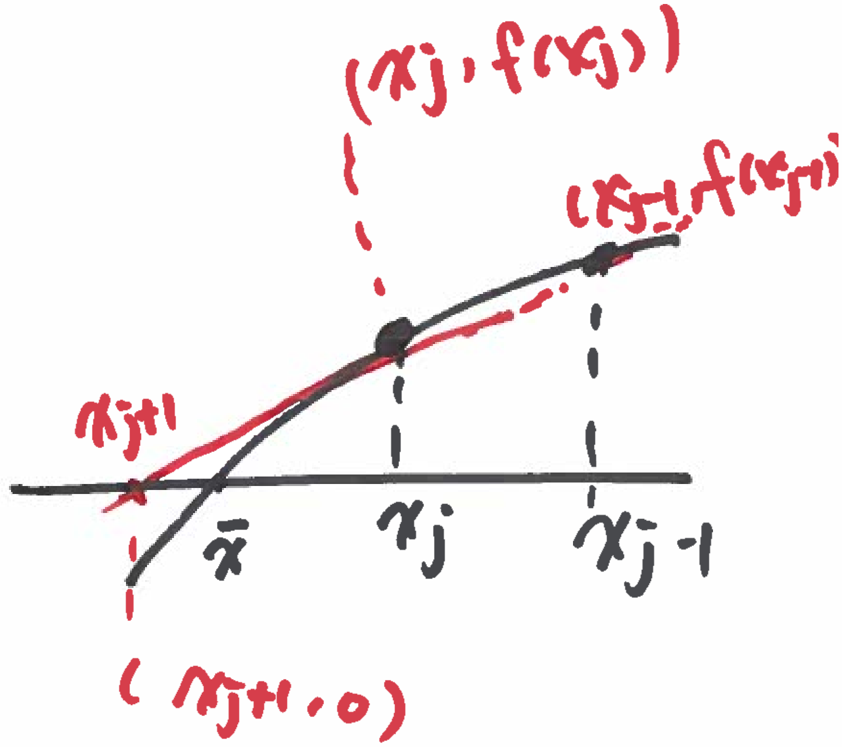
with x_0, x_1 given

geometrically:

Newton



secant method



To verify:

$$\frac{f(x_j) - f(x_{j-1})}{x_j - x_{j-1}} = \frac{f(x_j) - 0}{x_j - x_{j+1}}$$

↓ given the secant method update

Secant method

Red : with x, w, is to take into account storage

pick x_0, x_1, $tol > 0$, $I_{max} > 0$

w x

(and possibly $x_{max} > 0$)

let $err = 10 \times tol$, $j = 0$

loop: while $(err > tol)$

$$z = \frac{f(x_j)(x_j - x_{j-1})}{f(x_j) - f(x_{j-1})}$$

$$z = \frac{f(x)(x-w)}{f(x) - f(w)}$$

$err = abs(z)$ ($= |z|$)

$x_{j+1} = x_j - z$

$w = x$

$x = x - z$

$j = j + 1$

if $j > I_{max}$ (or $|x_{j+1}| > x_{max}$)

STOP

End

$$x_{j+1} = x_j - \frac{f(x_j)(x_j - x_{j-1})}{f(x_j) - f(x_{j-1})}$$

Theorem: Given $f \in C^2([a,b])$ with $f(\bar{x}) = 0$ for some $\bar{x} \in (a,b)$ and $f'(\bar{x}) \neq 0$. Start with x_0, x_1 that are sufficiently close to $\bar{x}$, then the secant method converges to $\bar{x}$, namely

$$\lim_{j \to \infty} x_j = \bar{x}$$

And the error $e_j = |x_j - \bar{x}|$ satisfies

$$e_{j+1} = D_j \, e_j^{\gamma}$$

where $\gamma = \frac{\sqrt{5}+1}{2} \approx 1.618$

and $\lim_{j \to \infty} D_j = \left| \frac{f''(\bar{x})}{2 f'(\bar{x})} \right|^{\gamma - 1}$

Convergence of Secant method: superlinear

i	x_i	$\|x_i - \bar{x}\|$	γ
0	−0.33333333333333	4.38e−01	
1	−0.66666666666667	1.04e−01	
2	−0.80000000000000	2.91e−02	
3	−0.76904176904177	1.88e−03	1.7749
4	−0.77088382152809	3.32e−05	1.6426
5	−0.77091703510617	3.80e−08	1.6565
6	−0.77091699705848	7.71e−13	1.6325
7	−0.77091699705925	1.11e−16	1.3172
8	−0.77091699705925	1.11e−16	1.0000

Table 2.7 Solving $x^3 + 2x + 2 = 0$ using the secant method formula given in (2.25). Also given is the error $e_i = |x_i - \bar{x}|$, and the approximate order of convergence γ as determined from (2.15).

§ 1.4

Sensitivity of root-finding problems

Demo: (matlab)

what is effectively solved on computer (or by matlab) is a perturbed problem.

$$\hat{f}(x) = f(x) + h(x) = 0$$

(original: $f(x) = 0$)

due to finite-precision arithmetic

Consider solving $f(x)=0$,

the solution is $\bar{x}$

perturbed problem:

$$\hat{f}(x) = f(x) + \varepsilon h(x)$$

$$0 < \varepsilon \ll 1$$

its root is $\hat{x} = \bar{x} + \Delta x$

$$|\Delta x| \ll 1$$

Try to establish "relation of ε and Δx"

$$\hat{f}(\hat{x}) = 0$$

$$\Leftrightarrow f(\underbrace{\bar{x}+\Delta x}_{\hat{x}}) + \varepsilon h(\bar{x}+\Delta x) = 0$$

$$\Leftrightarrow \underbrace{\boxed{f(\bar{x})}}_{0} + \Delta x f'(\bar{x}) + \cancel{O(\Delta x^2)} + \varepsilon\left(h(\bar{x}) + \Delta x h'(\bar{x}) + \cancel{O(\Delta x^2)}\right) = 0$$

big-O

(drop $O(\Delta x^2)$)

$$\Rightarrow \Delta x \left(f'(\bar{x}) + \varepsilon h'(\bar{x})\right) + \varepsilon h(\bar{x}) \approx 0$$

$$\Rightarrow \Delta x \approx - \frac{\varepsilon h(\bar{x})}{f'(\bar{x}) + \underbrace{\varepsilon h'(\bar{x})}_{\text{small}}}$$

$$\approx - \varepsilon \frac{h(\bar{x})}{f'(\bar{x})}$$

$f(\bar{x}) = 0$

$\hat{f}(\hat{x}) = 0$ $\quad$ $\hat{x} = \bar{x} + \Delta x$

$\hat{f}(x) = f(x) + \varepsilon h(x)$

$$\Delta x \approx - \varepsilon \frac{h(\bar{x})}{f'(\bar{x})}$$

This tells the sensitivity of $\bar{x}$, with the perturbation of $\varepsilon h(x)$

Example: Revisit the 2nd example

$$f(x) = (x-1)(x-2)\cdots(x-7)$$
$$= \prod_{n=1}^{7}(x-n)$$

$$h(x) = x^7$$

$$\hat{f}(x) = f(x) + \varepsilon h(x)$$

To capture a root $\bar{x} = 6$ of $\underset{=0}{f(x)}$

The root of $\hat{f}$ is $\hat{x} = \bar{x} + \Delta x$

From analysis:

$$\Delta x \approx -\varepsilon \frac{h(\bar{x})}{f'(\bar{x})}$$

To calculate $h(\bar{x}) = 6^7$

$f'(\bar{x}) = -5!$

$$\frac{\Delta x}{\varepsilon} \approx -\frac{h(\bar{x})}{f'(\bar{x})} = -\frac{6^7}{-5!} =$$
$$= 2332.8$$

Some calculation: $\boxed{\text{to get } f'(\bar{x})}$

$$f(x) = (x-1)(x-2)\cdots(x-7)$$

$$f'(x) = (x-2)(x-3)\cdots(x-7)$$

$$+ (x-1)(x-3)(x-4)\cdots(x-7)$$

$$+ (x-1)(x-2)(x-4)\cdots(x-7)$$

$$\vdots$$

$$(x-1)(x-2)(x-3)\cdots(x-6)$$

$$= \sum_{m=1}^{7} \prod_{\substack{n=1 \\ n \neq m}}^{7} (x-n)$$

$$f'(\bar{x}) = (x-1)(x-2)(x-3)(x-4)(x-5) \cdot (x-7)\Big|_{x=\bar{x}=6}$$

$$= 5 \cdot 4 \cdot 3 \cdot 2\ 1 \cdot (-1)$$

$$= -5!$$

In demo:

$\varepsilon = 10^{-6}$ $\qquad$ $\frac{\Delta x}{\varepsilon} \approx 2.3322 \times 10^{3}$

$\varepsilon = 10^{-10}$ $\qquad$ $\frac{\Delta x}{\varepsilon} \approx 2.3328 \times 10^{3}$

more sensitive: illed-conditioned

less sensitive: well-conditioned