Homework 7 by Jingmin Sun
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1. **Page 153,1**

   (a) Since

   $$\begin{aligned}(I-T)(I+T+\cdots+T^{n-1}) &= I+T+\cdots+T^{n-1}-\left(T+T^2+\cdots T^n\right)\\ &= I-T^n\\ &= I\end{aligned}$$

   we get

   $$(I-T)^{-1}=(I+T+\cdots+T^{n-1})$$

   (b) Since we know $T^n=0$, so we can find B such that

   $$(I-T)B=1-T^n$$

   And since the formula:

   $$1-x^n=(1-x)(1+x+\cdots x^{n-1})$$

   we can have our guess.

2. **Page 153,3** For any $v\in V$, we have

   $$v=\frac{1}{2}(v-Tv)+\frac{1}{2}(v+Tv)$$

   Since $T^2=I$:

   $$(T+I)\left(\frac{1}{2}(v-Tv)\right)=-\frac{1}{2}(T^2-I)v=0$$

   So we can get $\frac{1}{2}(v-Tv)\in$ null $(T-I)$, and similarly, we can get $\frac{1}{2}(v+Tv)\in$ null $(T+I)$.
   So that we can get $V=$ null $(T-I)+$ null $(T+I)$.
   Since -1 is not an eigenvalue of T, null$(T+I)=\{0\}$. So $V=$ null $(T-I)$. So we have

   $$\begin{aligned}(T-I)v&=0\\ v&=Tv\end{aligned}$$

   for all $v\in V$, so $T=I$

3. **Page 153,4** For any $v\in V$,

   $$v=Pv+(v-Pv)$$

   Since $Pv\in$ range P and from $P^2=P$, we can get $P(v-Pv)=Pv-P^2v=0$, so $(v-Pv)\in$ null P.
   So $V=$ range $P+$ null P.
   Suppose $u\in$ range $P\cap$ null P. So there exists $Pb=u$ and $Pu=0$, so we can get

   $$0=Pu=P(Pb)=P^2b=Pb=u$$

   So range $P\cap$ null $P=\{0\}$, and $V=$ range $P\oplus$ null P.

4. **Page 153,6**

   Since for any $u \in U$, $T(u) \in U$, so we can easily prove by induction:

   For any $p \in \mathcal{P}(\mathbb{F})$, suppose the degree of p is n:

   when $n = 1$:

$$p(T(u)) = \lambda T(u) \in U$$

   when $n > 1$ we can get

$$\begin{aligned} p(T)u &= \left(\sum_{k=1}^{n} a_k T^k\right) u \\ &= \left(\sum_{k=1}^{n-1} a_k T^k(u)\right) + a_n T^n(u) \\ &\in U \end{aligned}$$

5. **Page 153,7**

   If 9 is an eigenvalue of T^2, then there exists u such that

$$\begin{aligned} T^2 u &= 9u \\ (T^2 - 9I)u &= 0 \\ (T + 3I)(T - 3I)u &= 0 \end{aligned}$$

   So that $(T + 3I)(T - 3I)$ is not injective, which means one of $(T + 3I)$ and $(T - 3I)$ or both are not injective. So eigenvalue of T is 3 or -3.

   And If 3 is an eigenvalue of T then there exists v such that

$$\begin{aligned} (T - 3I)v &= 0 \\ (T + 3I)(T - 3I)v &= 0 \\ (T^2 - 9I)v &= 0 \end{aligned}$$

   So, 9 is an eigenvalue of T^2. similar for -3 case.

6. **Page 154,13**

   Suppose U is a subspace of W and is invariant under T, and U is finite dimensional. If $U \neq \{0\}$, we have some $v \in U$ such that $Tv = \lambda v$, and this contradict that T does not have eigenvalues, so $U = \{0\}$ or infinite dimensional.

7. **Page 154,15**

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

8. **Page 154,18**

   Side: Definition of continuous function:

continuous at point p: Let $\epsilon > 0$ be given, there exists $\delta > 0$ such that if $|x - p| < \delta$, we can get $|f(x) - f(p)| < \epsilon$

And if the function is continuous at any point p on domain of $f(x)$, then the function $f(x)$ is continuous.

So, since we know at eigenvalues $\lambda = \lambda'$, we can get $T - \lambda' I$ is not surjective, which means $\dim(T - \lambda' I) < \dim V$, and at each other non-eigenvalue point, $\dim(T - \lambda I) = \dim V$, since there is no element $u \in V$ such that $(T - \lambda I)u = \lambda u$ if λ is not an eigenvalue.

So near point λ', we can get for all $|\lambda - \lambda'| < \min \Delta\lambda'$, where $\Delta\lambda'$ is the difference between two eigenvalues, $|f(\lambda) - f(\lambda')| \geq 1$, so it is not continuous.

9. **Page 154,20**

   Since by Theorem 5.27, we can have that T has an upper triangular matrix with respect to some basis of V, so let this basis to be $v_1 \cdots v_n$, and by the theorem 5.26, we can get $\operatorname{span}(v_1 \cdots v_j)$ is invariant under T for each $j = 1 \cdots n$