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ALL REFERENCE NUMBERS ARE CORRESPONDING TO THE TEXT

1. **Page 48, 1**

   (a) Since U is a subspace of V, then $U \subseteq V$, and the basis of U: $u_1, u_2 \cdots u_m$ are linearly independent as well. So we can get for all $v \in V$, $v = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n-m} b_j v_j$, where $n = \dim V$. And since $\dim V = \dim U = m$, $n - m = 0$, so $v = \sum_{i=1}^{m} a_i u_i$, which means $V = \operatorname{span}\{u_1, u_2 \cdots u_m\} = U$.

2. **Page 48, 4**

   (a) Since we need $p(6) = 0$, it's the same as $p(0) = 0$, but now, we move the origin to $x = 6$.
   So, we can consider a basis of $x-6, (x-6)^2, (x-6)^3, (x-6)^4$.
   And for every $p \in U$, we can get $p = a(x-6) + b(x-6)^2 + c(x-6)^3 + d(x-6)^4$.
   It's obvious that $p(6) = 0$.
   And we need to prove the independency that assume $a(x-6)+b(x-6)^2+c(x-6)^3+d(x-6)^4 = 0$, and we can easily get $a = b = c = d = 0$

   (b) If we consider 6 as the origin, we can see that we are absent of order 0, which is the constant term, so we claim that the basis to add is $\{1\}$. It's clear that they are linearly independent, and we can show that for all $p \in \mathcal{P}_4$, we can represent p as

   $$p = e + a(x-6) + b(x-6)^2 + c(x-6)^3 + d(x-6)^4$$

   You can expand and show, proof omitted.

   (c) $W = \{c : c \in \mathbf{F}\}$, as in b, we show that $\mathcal{P}_4 = U + W$. And the only thing we remain to proof is that the only intersection of U and W is zero. It is obvious since they contains the polynomial basis in different orders.

3. **Page 48, 9**

   Firstly, we can do some reformation of $\operatorname{span}\{v_1 + w \cdots v_m + w\}$, it's obvious that

   $$\begin{aligned} \operatorname{span}\{v_1 + w \cdots v_m + w\} &= \operatorname{span}\{v_1 + w, v_2 + w - (v_1 + w), \cdots v_m + w - (v_1 + w)\} \\ &= \operatorname{span}\{v_1 + w, v_2 - v_1, \cdots v_m - v_1\} \end{aligned}$$

   Since $v_1, \cdots v_m$ are linearly independent, so $\{v_2 - v_1, \cdots v_m - v_1\}$ are linearly independent, and

   $$\dim \operatorname{span}\{v_2 - v_1, \cdots v_m - v_1\} = m - 1$$

   so that

   $$\dim \operatorname{span}\{v_1 + w, v_2 - v_1, \cdots v_m - v_1\} \geq m - 1$$

   which means

   $$\dim \operatorname{span}\{v_1 + w \cdots v_m + w\} \geq m - 1$$

4. **Page 48, 11**

   $$\begin{aligned} \dim(U \cap W) &= \dim U + \dim W - \dim(U + W) \\ &= 3 + 5 - 8 = 0 \end{aligned}$$

Therefore, since U and W are both subspace of V, so that $U \cap W$ is also a subspace, so $U \cap W = 0$, and we can get that $\mathbb{R}^8 = U \oplus W$

5. **Page 48, 12**

$$\begin{aligned}
\dim(U+W) &\leq 9 \\
\dim(U \cap W) &= \dim U + \dim W - \dim(U+W) \\
&= 5 + 5 - \dim(U+W) \\
&\geq 10 - 9 \\
&\geq 1
\end{aligned}$$

Therefore, $U \cap W \neq \{0\}$.

6. **Page 48, 13**

$$\begin{aligned}
\dim(U+W) &\leq 6 \\
\dim(U \cap W) &= \dim U + \dim W - \dim(U+W) \\
&= 4 + 4 - \dim(U+W) \\
&\geq 8 - 6 \\
&\geq 2
\end{aligned}$$

So, the dimension of the intersection is at least 2, which means there exists at least two independent vector $a, b \in U \cap W$, since they are independent, so neither of these vectors is a scalar multiple of the other.

7. **Page 48, 14**

$$\begin{aligned}
\dim(U_1 + \cdots + U_m) &= \dim U_1 + \dim(U_2 + \cdots + U_m) - \dim(U_1 \cap (U_2 + \cdots + U_m)) \\
&\leq \dim U_1 + \dim(U_2 + \cdots + U_m) \\
&= \dim U_1 + \dim U_2 + \dim(U_3 + \cdots + U_m) - \dim(U_2 \cap (U_3 + \cdots + U_m)) \\
&\leq \dim U_1 + \dim(U_2) + \dim(U_3) + \dim(U_4 \cdots + U_m) \\
&\leq \dim U_1 + \cdots + \dim(U_m)
\end{aligned}$$