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Feb. 12 2020
ALL REFERENCE NUMBERS ARE CORRESPONDING TO THE TEXT

1. **Page 57,1**

   To show if and only if, firstly, let's suppose that T is linear, which means it satisfy the additivity and homogeneity.

   Suppose,

$$\begin{aligned}T(x_1, y_1, z_1) &= (2x_1 - 4y_1 + 3z_1 + b, 6x_1 + cx_1y_1z_1)\\ T(x_2, y_2, z_2) &= (2x_2 - 4y_2 + 3z_2 + b, 6x_2 + cx_2y_2z_2)\end{aligned}$$

   And due to the additivity, we can get

$$\begin{aligned}T(x_1 + x_2, y_1 + y_2, z_1 + z_2) &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + b, 6(x_1 + y_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2))\\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)\\ &= (2x_1 - 4y_1 + 3z_1 + b, 6x_1 + cx_1y_1z_1) + (2x_2 - 4y_2 + 3z_2 + b, 6x_2 + cx_2y_2z_2)\\ &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2) + 2b, 6(x_1 + y_2) + c(x_1y_1z_1 + x_2y_2z_2))\end{aligned}$$

   And we can get $b = 2b$, and $c(x_1y_1z_1 + x_2y_2z_2) = c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2)$, so $b = c = 0$ follows.

   The other direction is simple, set $b = c = 0$, and we can get that

$$\begin{aligned}T(x_1, y_1, z_1) &= (2x_1 - 4y_1 + 3z_1, 6x_1)\\ T(x_2, y_2, z_2) &= (2x_2 - 4y_2 + 3z_2, 6x_2)\end{aligned}$$

   Since

$$\begin{aligned}T(x_1 + x_2, y_1 + y_2, z_1 + z_2) &= (2(x_1 + x_2) - 4(y_1 + y_2) + 3(z_1 + z_2), 6(x_1 + y_2))\\ &= (2x_1 - 4y_1 + 3z_1, 6x_1) + (2x_2 - 4y_2 + 3z_2, 6x_2)\\ &= T(x_1, y_1, z_1) + T(x_2, y_2, z_2)\end{aligned}$$

   And

$$\begin{aligned}T(\lambda x_1, \lambda y, \lambda z) &= (2\lambda x - 4\lambda y + 3\lambda z, 6\lambda x)\\ &= \lambda(2x - 4y + 3z, 6x)\\ &= \lambda T(x, y, z)\end{aligned}$$

2. **Page 57,4**

$$a_1v_1 + a_2v_2 + \cdots a_mv_m = 0$$

Since T is a linear map, so we can get $T(0) = 0$, which means

$$\begin{aligned} T(a_1v_1 + a_2v_2 + \cdots a_mv_m) &= 0 \\ T(a_1v_1) + T(a_2v_2) + \cdots + T(a_mv_m) &= 0 && \text{Additivity} \\ a_1T(v_1) + a_2T(v_2) + \cdots + a_mT(v_m) &= 0 && \text{Homogeneity} \end{aligned}$$

Since $T(v_1) \cdots T(v_m)$ are linear independent, so we can get $a_1 = \cdots a_m = 0$, which means the set of the vectors $v_1 \cdots v_m$ are linearly independent.

3. **Page 58,7**

   Since T is a linear map from V to V, so for all $v \in V$, we can always get $Tv = w \in V$. Since $\dim(V) = 1$, we can get $w = \lambda v$, which means $Tv = \lambda v$.

4. **Page 58,8**

   Suppose

$$\psi(v_1, v_2) = \begin{cases} \dfrac{v_1^2}{v_2} & v_2 \neq 0 \\ 0 & v_2 = 0 \end{cases}$$

5. **Page 58,10**

   Suppose $u \in U$ and $v \in V \backslash U$, which means that $u + v \in V \backslash U$, so $T(u + v) = 0$, so

$$\begin{aligned} T(u) + T(v) &= Sv + 0 \\ &= Sv \neq 0 \end{aligned}$$

   which means T is not a linear map on V.

6. **Page 58,11**

   If we get the basis work, we will get the whole space work.

   Let's say, if a set of basis is $u_1 \cdots u_m$, and we can get there is a set of V that can be expressed as $u_1 \cdots u_m, v_{m+1} \cdots v_n$. So, let's define the linear map on the basis such that

$$\begin{aligned} T(u_i) &= S(u_i) && i = 1 \cdots m \\ T(v_j) &= v_j && j = m + 1 \cdots n \end{aligned}$$

   So, we can get for all $u \in U$,

$$\begin{aligned} u &= a_1u_1 + \cdots a_mu_m \\ T(u) &= a_1T(u_1) + \cdots a_mT(u_m) \\ &= a_1S(u_1) + \cdots a_mS(u_m) \\ &= S(u) \end{aligned}$$

7. **Page 58,14**

   Suppose

$$S(v) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} v_2 \\ v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$T(v) = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = \begin{bmatrix} 2v_1 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

And we can find that

$$S(T(v)) = \begin{bmatrix} v_2 \\ 2v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad T(S(v)) = \begin{bmatrix} 2v_2 \\ v_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which are different.

8. **Page 67,1**

$$\begin{aligned} T(v) &= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} \\ &= \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 + v_3 + v_4 + v_5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

And it can be represented as

$$\begin{bmatrix} x \\ y \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so rank $(T) = 2$, and null $(T) = 5 - 2 = 3$

9. **Page 67,2**

We can first expand $(ST)^2(v)$, which means

$$\begin{aligned} (ST)^2(v) &= (ST)(ST)(v) \\ &= (S(T(S(T(v))))) \end{aligned}$$

Since $S, T \in \mathcal{L}(V, V)$, so for all $v \in V$, we have $T(v) \in V$, so $S(T(v)) \in$ Range S follows. Since Range $S \subset$ null T, so $T(S(T(v))) = 0$. Since $T(0) = 0$, we can get $(S(T(S(T(v))))) = 0$.

10. **Page 67,4**

    To show that it's not a subspace, we can show that it violates one of the three properties for the subspace. Now, we can show that it is not closed under addition.

    Define T_1 and T_2 such that

    $$T_1(e_i) = \begin{cases} 0 & i = 1, 2, 3 \\ 1 & i = 4, 5 \end{cases}$$

    ; and

    $$T_2(e_i) = \begin{cases} 0 & i = 1, 2, 4 \\ 1 & i = 3, 5 \end{cases}$$

    We can easily find that

    $$(T_1 + T_2)(e_i) = \begin{cases} 0 & i = 1, 2 \\ 1 & i = 3, 4 \\ 2 & i = 5 \end{cases}$$

    which means null$(T_1 + T_2) = 2$, so $T_1 + T_2$ is not in the set, which means the set is not a subspace of all linear map from $\mathbb{R}^5$ to $\mathbb{R}^4$.

11. **Page 67,5**

    $$\begin{aligned} T(v) &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \\ &= \begin{bmatrix} v_1 + v_2 \\ v_1 + v_2 + v_3 + v_4 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

12. **Page 67,14**

    Since

    $$\begin{aligned} \dim(null(T)) + \dim(range(T)) &= 8 \\ \dim(null(T)) &= \dim(U) = 3 \\ \therefore \dim(range(T)) &= 5 \end{aligned}$$

    and $T \in \mathcal{L}(\mathbb{R}^8, \mathbb{R}^5)$, so range$(T) = \mathbb{R}^5$.

13. **Page 67,20**

    To prove if and only if, we need to prove in two direction, so we first assume T is injective, so if $T(u_1) = T(u_2)$, then $u_1 = u_2$ follows. Then we can define S in the following way: $S : range(T) \rightarrow T$, which is defined as $S(T(u_1)) = u_1$. So, we can always find S such that ST is identity.

    Then, if we assume ST is identity, which means $S(T(u)) = u$, so we can get if $T(u_1) = T(u_2)$, then

    $$\begin{aligned} S(T(u_1)) &= S(T(u_2)) \\ u_1 &= u_2 \end{aligned}$$