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ALL REFERENCE NUMBERS ARE CORRESPONDING TO THE TEXT

1. **Page 99, 2**

   We can prove this by contradiction. Assume there exists $j = 1, 2 \cdots m$, such that $\dim V_j = \infty$, then we can get that

$$\begin{aligned}\dim (V_1 \times V_2 \times \cdots \times V_m) &= \dim V_1 + \dim V_2 + \dim V_3 + \cdots \dim V_m \\ &= \infty\end{aligned}$$

   which is not finite, then we can get V_j is finite dimensional for each $j = 1, 2 \cdots m$

2. **Page 99, 4**

   Suppose $f \in \mathcal{L}(V_1 \times \cdots V_m, W)$, and $f_i \in \mathcal{L}(V_i, W)$ such that $i = 1, 2, \cdots m$

   So, let's define $\phi : \mathcal{L}(V_1 \times \cdots V_m, W) \to \mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W) \times \cdots \times \mathcal{L}(V_m, W)$, such that $\phi : f \mapsto f_1 \times f_2 \times f_3 \cdots \times f_m$, and

$$\phi(f(v_1 \times \cdots v_m)) = f_1(v_1) \times \cdots \times f_m(v_m)$$

   And we can easily see that $f(V_1 \times \cdots V_m) \in W$, since $f_i(V_i) \in W$. It is obvious that ϕ is linear, and we can define $\phi : \mathcal{L}(V_1, W) \times \mathcal{L}(V_2, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots V_m, W)$ and $\psi : f_1 \times f_2 \times f_3 \cdots \times f_m \mapsto f$ and

$$\psi(f_1(v_1) \times \cdots f_m(v_m)) = f(v_1 \times \cdots v_m)$$

   So we can calculate:

$$\begin{aligned}\phi(\psi(f_1(v_1) + \cdots f_m(v_m))) &= \phi(f(v_1 \cdots v_m)) \\ &= f_1(v_1) + \cdots f_m(v_m)\end{aligned}$$

   And similarly,

$$\begin{aligned}\psi(\phi(f(v_1 \cdots v_m))) &= \psi(f_1(v_1) + \cdots f_m(v_m)) \\ &= f(v_1 \cdots v_m)\end{aligned}$$

   So there is an isomorphism between two space, so they are isomorphic.

3. **Page 99,7**

   Since $v + U = x + W$, we can get that for all $u \in U$, we can get find $w \in W$ such that

$$v + u = x + w$$

   suppose $u = \vec{0}$, so

$$v = x + w_1$$

   for some $w_1 \in w$, and $v - x \in W$ follows, which means

$$u = x + w - v = -(v - x) + w \in W$$

   Thus, $U \subseteq W$. Similarly, $W \subseteq U$, so $U = W$.

4. **Page 99,11**

(a) To prove it is an affine subset, we need to show that A is closed under affine combination, which means if $a_1, a_2, a_3 \cdots a_k \in A$, there exists $\sum_{i=1}^{k} \sigma_i = 1$, and

$$\begin{aligned}\sum_i \sigma_i a_i &= \sigma_1 a_1 + \cdots \sigma_k a_k \\ &= \sigma_1(\lambda_{1,1} v_1 + \cdots \lambda_{1,m} v_m) + \cdots + \sigma_k(\lambda_{k,1} v_1 + \cdots \lambda_{k,m} v_m) \\ &= \sum_{i=1}^{k} \sigma_i \lambda_{i,1} v_1 + \cdots \sum_{i=1}^{k} \sigma_i \lambda_{i,m} v_m\end{aligned}$$

And we can get the sum of the coefficients are

$$\sum_{j=1}^{m} \sigma_m \lambda_{i,m} = \sum_{j=1}^{m} \sigma_m = 1$$

So, A is an affine subset of V.

(b) Let S be an affine subset of V, since it contains $v_1 \cdots v_m$, and by the definition, we can get that it contains any vector that $v = \sum_{i=1}^{m} \sigma_i v_i$ with $\sum_{i=1}^{m} \sigma_i = 1$.
Thus, every element of A contains in S. so S contains A.

(c) Since for all $a \in A$, we can write represent in this way:

$$\begin{aligned}a &= \lambda_1 v_1 + \cdots \lambda_m v_m \\ &= v_1 + \lambda_2(v_2 - v_1) + \cdots + \lambda_m(v_m - v_1)\end{aligned}$$

So, we can get $A \subseteq v_1 + U$, where $U = \operatorname{span}\{v_2 - v_2, \cdots v_m - v_1\}$
And the reverse is true **Proof Omitted**
And $\dim(U) \leq m - 1$ is obvious, since there are $m - 1$ vectors in the spanning set.

5. **Page 100.13**

Since $v_1 + U, \cdots v_m + U$ is a set of basis of V/U, so we can get

$$v + U = \sum_{i=1}^{m} \lambda_i (v_i + U)$$

for all $v \in V$, and we can have

$$v - \sum_{i=1}^{m} \lambda_i (v_i) \in U$$

Let $u_1 \cdots u_n$ be a set of basis of U, we can have

$$\begin{aligned}v - \sum_{i=1}^{m} \lambda_i (v_i) &= \sum_{j=1}^{n} \sigma_j u_j \\ v &= \sum_{j=1}^{n} \sigma_j u_j + \sum_{i=1}^{m} \lambda_i (v_i)\end{aligned}$$

So $V = \operatorname{span}\{u_1 \cdots u_n, v_1 \cdots v_m\}$

And we can prove they are linear independent by letting

$$\sum_i \lambda_i u_i + \sum_j \sigma_j v_j = 0$$

And this shows:

$$\sum_{i=1}^{m} \lambda_i (v_i + U) = 0$$

and $\lambda_i = 0$for all i follows. And $\sigma_j = 0$ for all j as well, so they are linearly independent.

6. **Page 100,15**

   Since

   $$\dim \operatorname{range} \phi + \dim \operatorname{null} \phi = \dim V$$

   And since $\dim V/(\operatorname{null} \phi) = \operatorname{range} \phi$, and since range $\phi \subseteq \mathbb{F}$, and $\phi \neq 0$ so dim range $\phi = 1$.

7. **Page 100,16**

   Since $\dim V/U = 1$, so there is an isomorphism between this set and $\mathbb{F}$.

   Let $f = \mathcal{L}(V/U, \mathbb{F})$, and $\pi(v) = v + U$, (null $\pi = U$)). So let's define $\phi(v) = f(\pi(v))$

   Since if $\pi(v) = 0$, $\phi(v) = 0$, and if $\pi(v) \neq 0$, $\phi(v) \neq 0$ (since f is isomorphism).

   Thus, we can have null ϕ = null $\pi = U$

8. **Page 100,17**

   Since $\dim(V/U) + \dim U = \dim V$, and we can extend the basis of U to V, since U is a subspace of V. **Details omitted.**

9. **Page 113,3**

   Let $v = v_1, v_2 \cdots v_n$ as a basis of V, then we can get the dual basis is $\phi_1 \cdots \phi_n$, and

   $$\phi_j(v_k) = \begin{cases} 1 & \text{if } k = j \\ 0 & \text{otherwise} \end{cases}$$

   So that we have $\phi(v) = 1$

10. **Page 113,5**

    Define $P_i \in \mathcal{L}(V_i, V_1 \times \cdots \times V_m)$ as

    $$P_i(x) = xe_i$$

    where e_i is defined as the vector of length m and

    $$e_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

    Let $\phi \in \mathcal{L}((V_1 \times \cdots V_m)', V_1' \times \cdots V_m')$, and

    $$\phi(f) = (P_1' f, \cdots P_m' f)$$

Injectivity: Suppose $(P_1'f, \cdots P_m'f) = (P_1'g, \cdots P_m'g)$, then we xan get

$$f(xe_1, xe_2 \cdots xe_m) = g(xe_1, xe_2 \cdots xe_m)$$
$$f = g$$

Hence injective.

Surjectivity: for any $(f_1 \cdots f_m) \in V_1' \times \cdots V_m'$, define $f \in (V_1 \times V_m)'$:

$$f(x_1 \cdots x_m) = \sum_{i=1}^{m} f_i(x_i)$$

And we can get $\phi f = (f_1 \cdots f_m)$

11. **Page 113,6**

   (a) If $v_! \cdots v_m$ spans V, then $\Gamma(\phi) =$ implies

   $$\phi(v_1) = \cdots = \phi(v_m) = 0$$

   Hence $\phi = 0$. For all $v \in V$, $v = \sum_{i=1}^{m} k_i v_i$
   Thus,

   $$\phi(v) - \sum_{i=1}^{m} k_i \phi_i(v_i) = 0$$

   implies $\phi = 0$, which implies Γ is injective.
   And if Γ is injective, and $span(v_1 \cdots v_m) \neq V$, but we can always find a $\phi \in V'$ and $\phi(span(v_1 \cdots v_m)) = 0$ with $\phi \neq 0$ (by problem 4, proof similar to problem 3).
   So Γ is not injective . Thus, $span(v_1 \cdots v_m) = V$.

   (b) If $v_1 \cdots v_m$ is linearly independent, then there always exists $\phi \in V'$, such that $\phi(v_i) = f_i$, for any $(f_1, f_2 \cdots f_m) \in \mathbb{F}^m$. So, this implies surjectivity.
   And if Γ is surjective, suppose $v_1 \cdots v_m$ is not linearly independent, which means $k_1 v_1 + \cdots + k_m v_m = 0$ and some of $k_i \neq 0$, so v_i can be written as a linear combination of $v_1 \cdots v_{i-1}, v_{i+1}, \cdots v_m$. So $(0, 0, 0 \cdots 0, 1, 0 \cdots 0)$ is not in range Γ. Otherwise we have $\phi \in \Gamma(\phi) = (0, 0, \cdots 0, 1, 0 \cdots 0)$. Then $\phi(v_j) = 0$ and $\phi(v_i) = 1$. And since v_i is a linear combination of other vectors in the basis, this will lead a contradiction that $0 = 1$. So, $v_1 \cdots v_m$ is linearly independent.

12. **Page 113,9**

   Since

   $$(\psi(v_i)\phi_1 + \cdots \psi(v_n)\phi_n)(v_i) = \psi(v_i)$$

   So, $\psi = \psi(v_i)\phi_1 + \cdots \psi(v_n)\phi_n$

13. **Page 114,12**

   The identity map is that $I : V \to V$, with $I(v) = v$ for all $v \in V$. And the dula map is :

   $$I'(\phi) = \phi I$$
   $$= \phi$$

   which is an identity map.

14. **Page 114,15**

If $T = 0$, then for any $f \in W'$, and for any $v \in V$m we have:

$$T'(f) = f(T(v)) = f(0) = 0$$

Thus $T' = 0$

Conversely, if $T' = 0$, then suppose $T \neq 0$

Then there exists $v \in V$ such that $Tv \neq 0$. So there exists $\phi \in W'$ such that $\phi(Tv \neq 0)$ (by Problem 3), and $T'(\phi) = \phi(T(v)) \neq 0$, and contradicts with $T' = 0$. Then $T = 0$.

15. **Page 114,16**

Let $\Gamma : \mathcal{L}(V, W) \to \mathcal{L}(W', V')$ and $\Gamma(T = T')$ By 3.60 and 3.95, we can get $\dim \mathcal{L}(V, W) = \dim \mathcal{L}(W', V')$.

Suppose $\Gamma(S) = 0$ for some $S \in \mathcal{L}(V, W)$, then $S' = 0$ Hence for any $\phi \in W'$ and $v \in V$, we have $S'(\phi)(v) = \phi(Sv) = 0$. And by Problem 3, this implies $Sv = 0$ then $S = 0$. So Γ is injective. By theorem 3.69, we can get that Γ is invertible, and it is an isomorphism from $\mathcal{L}(V, W)$ to $\mathcal{L}(W', V')$.

16. **Page 114,17**

Since $\phi(u) = 0$ for all $u \in U$ means $U \subseteq \text{null } \phi$.

17. **Page 114,20**

If $\phi \in W^0$, then $\phi(w) = 0$ for all $w \in W$. As $U \subset W$, we also have $\phi(u) = 0$ for all $u \in W$. Hence $\phi \in U^0$. Thus $W^0 \subset U^0$

18. **Page 114,22**

It is easy to get $(U + W)^0 \subset U^0$ and$(U + W)^0 \subset W^0$ from Problem 20, then $(U + W)^0 \subset U^0 \cap W^0$

And suppose $f \in U^0 \cap W^0$, then we have $f(u) = 0$ and $f(w) = 0$ for all $u \in U$ and $w\ inW$. So $f(u + w) = 0$. And we can get $f \in (U + W)^0$, Hence $U^0 \cap W^0 \subset (U + W)^0$.

Therefore $U + W^0 = U^0 \cap W^0$

19. **Page 115,23**

It's easy to get $U^0 \subset (U \cap W)^0$ and $W^0 \subset (U \cap W)^0$, which means $U^0 + W^0 \subset (U \cap W)^0$.

And

$$\begin{aligned}
\dim(U^0 + W^0) &= \dim U^0 + \dim W^0 - \dim(U^0 \cap W^0) \\
&= \dim V - \dim U + \dim V - \dim W - \dim((U + W)^0) && \text{by Problem 22 and 3.106} \\
&= \dim V - \dim U + \dim V - \dim W - \dim V + \dim(U + W) && \text{by 3.106} \\
&= \dim V - \dim U - \dim W + \dim(U) + \dim W - \dim(U \cap W) \\
&= \dim V - \dim(U \cap W) \\
&= \dim(U \cap W)^0
\end{aligned}$$

Thus, $U^0 + W^0 = (U \cap W)^0$.

20. **Page 115,25**

Let $u \in U$, so we can have $\phi(u) = 0$ for all $\phi \in U^0$, and since $U \subset V$, $u \in V$ then $U \subset \{v \in V : \phi(v) = 0 \text{ for all } \phi \in U^0\}$.

And if $v \in \{v \in V : \phi(v) = 0 \text{ for all } \phi \in U^0\}$. Let $W = \text{span } (v)$, then we can get $\phi(w) = 0$ for all $w \in W$. Thus, $U^0 \subset W^0$. And by Problem 21, $W \subset U$. And $v \in U$, which means $\{v \in V : \phi(v) = 0 \text{ for all } \phi \in U^0\} \subset U$.

Proof of **Problem 21**:

Since $W^0 \subset U^0$, then from Problem 22 we can have:

$$(U+W)^0 = U^0 \cap W^0 = W^0$$

And by 3.106 we can have

$$\dim(U+W)^0 = \dim V - \dim(U+W)$$
$$\dim(W)^0 = \dim V - \dim(W)$$

Thus, $\dim U + W = \dim W$, since $W \subset U + W$ then $U + W = W$, which means $U \subset W$.

Thus two sets equals.

21. **Page 115,29**

    Hint: if $T \in \mathcal{L}(V, W)$, then range $T' = (\text{ null } T)^0$

22. **Page 115,32**

    I'll only prove one of them:

    $a \implies b$:

    If T is invertible, then it is an isomorphism, and we can get $\mathcal{M}(T)$ is an isomorphism, which means by Problem 3.B.9, we can get $\mathcal{M}(T)(e_i)$ are linearly independent, and $\{e_i\}$ defined as above problem, and it's a set of basis of V. And $\mathcal{M}(T)(e_i) = \mathcal{M}(T)_{:,i}$, so the columns are linearly independent.