Homework 7 by Jingmin Sun
Mar. 31 2020

1. **Page 139,4**

   Suppose $u \in U_1 + U_2 + \cdots U_m$, so we can express u as

   $$u = u_1 + u_2 \cdots u_m$$

   such that $u_i \in U_i$ for $i = 1 \cdots m$

   Since T is a linear map, we can get:

   $$\begin{aligned} T(u) &= T(u_1 + u_2 + \cdots u_m) \\ &= T(u_1) + T(u_2) + \cdots + T(u_m) \end{aligned}$$

   Since $U_1 \cdots U_m$ are invariant under T, which means $T(u_i) \in U_i$ for all i. Thus, $T(u) \in U_1 + U_2 + \cdots U_m$. So $U_1 + U_2 + \cdots U_m$ is invariant.

2. **Page 139,5**

   Suppose U_1 and U_2 are invariant under T, let $u \in U_1 \cap U_2$, we can get $u \in U_1$ and $u \in U_2$, then $T(u) \in U_1$ and $T(u) \in U_2$ follows, which means $T(u) \in U_1 \cap U_2$, and $U_1 \cap U_2$ is invariant under T.

3. **Page 139,7**

   $$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} 0 & -3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} -\lambda & -3 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= 0 \\ \lambda^2 &= -3 \end{aligned}$$

   If $x = 0$, $y = 0$; and if $y = 0$, $x = 0$. So there is no eigenvectors and eigenvalues for T.

4. **Page 139,8**

   $$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= \lambda \begin{bmatrix} x \\ y \end{bmatrix} \\ \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} &= 0 \\ \lambda^2 &= 1 \\ \lambda &= \pm 1 \end{aligned}$$

   So, when $\lambda = 1$, $x = y$, and $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

And when $\lambda = -1$, $x = -y$, and $\begin{bmatrix} x \\ -y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

5. **Page 139,10**

   (a)

$$T(x_1, x_2 \cdots x_n) = (x_1, 2x_2, 3x_3 \cdots nx_n) = \lambda(x_1, x_2, x_3 \cdots x_n)$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\begin{bmatrix} 1-\lambda & 0 & 0 & \cdots & 0 \\ 0 & 2-\lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & n-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0$$

   Thus, the eigenvalues $\lambda_i = i$, with $i = 1, 2, \cdots n$, and with corresponding eigenvectors $e_i = \begin{bmatrix} 0 & 0 & \cdots 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T$, and the ith element is 1.

   (b) Let e_i defined as above, and we get to know that $\{e_i\}_{i=1}^n$ is a set of basis of $\mathbb{F}^n$. Let U be a subspace of $\mathbb{F}^n$, and we set $U \cap \{e_i\}_{i=1}^n = \{e_{ij}\}_{j=1}^k$

   So, there exist $v = \sum_{j=1}^k a_j e_{ij} \in U$ with nonzero $a_j s$.

   And we assume that there exists $m \in U \setminus$ span $\{e_{ij}\}$, so we can express m as

$$m = \sum_{j=1}^k b_j e_{ij} + \sum_{r=1}^q c_r e_q$$

   This contradicts with t $U \cap \{e_i\}_{i=1}^n = \{e_{ij}\}_{j=1}^k$, so that $U =$ span $\{e_{ij}\}$. Thus, the invariant subspaces of T are spanning set of $\{e_{ij}\}_{j=1}^k$, $0 \leq k \leq n$.

6. **Page 140,14**

   Write $v = u + w$ for all $v \in V$, since we want to find eigenvalues and eigenvectors of P, so we can write

$$Pv = \lambda v$$

   and we can get

$$u = \lambda(u + w)$$
$$(\lambda - 1)u + \lambda w = 0$$

   If $\lambda_1 = 1$, the corresponding eigenvectors are $w = 0$, which is $v_1 = u$.

   And if $\lambda_2 = 0$, so $u = 0$, and the corresponding eigenvectors are $v_2 = w$.

7. **Page 140,21**

(a) If λ is an eigenvalue of T, so for some $v \in V$, we can get $T(v) = \lambda v$, which means

$$\begin{aligned} T^{-1}(\lambda v) &= v \\ w &= \lambda v \\ T^{-1}(w) &= \frac{1}{\lambda} w \end{aligned}$$

Thus, $\frac{1}{\lambda}$ is an eigenvalue for T^{-1}

And if λ^{-1} is an eigenvalue of T^{-1}, so for some $v \in V$, we can get $T^{-1}(v) = \frac{1}{\lambda} v$, which means

$$\begin{aligned} T(\frac{1}{\lambda} v) &= v \\ w &= \frac{1}{\lambda} v \\ T(w) &= \lambda w \end{aligned}$$

Thus, λ is an eigenvalue for T

(b) As above, suppose λ is an eigenvalue of T, so for some $v \in V$, we can get $T(v) = \lambda v$, v is an eigenvectors of T, and since $w = \lambda v$ is an eigenvector of T^{-1}, so v and w are differ by a constant factor, so they have the same eigenvectors, i.e. all kv are both eigenvectors for T and T^{-1} for eigenvector v for T.