**Homework 5 by Jingmin Sun**
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**ALL REFERENCE NUMBERS ARE CORRESPONDING TO THE TEXT**

1. **Page 78, 1**

   We can prove by contradiction, suppose dim range$(T) = m$, and $T(x) = Ax$.

   Assume there are $n$ nonzero entries in $A$, where $n < m$, so we have at most $n$ linearly independent rows of $A$, Then:

   $$\dim \operatorname{range} T = \dim(Ax) \leq n < m$$

   which contradicts with dim range$(T) = m$, so there are at least $m$ nonzero entries in $A$.

2. **Page 78, 3**

   We can construct such basis by first find the set of basis of null$(T) \subseteq V$, call them $\{v_1, v_2 \cdots v_m\}$, and assume $\dim V = n$, and we can extend the basis to span the whole $V$, which is $\{v_1, v_2 \cdots v_m, v_{m+1} \cdots v_n\}$. And we can apply the linear map $T$ to $v_{m+1}, \cdots v_n$, and $\{T(v_{m+1}), \cdots T(v_n)\}$ are linearly independent. so we can extend the basis to $\{T(v_{m+1}), \cdots T(v_n), w_{n+1}, \cdots w_p\}$, if $\dim(W) = p - m$. And we can get such basis satisfies $\mathcal{M}(T)$ is a sparse matrix with nonzero entries on the diagonal.

   Sketch of proof of independency:

   Consider

   $$v_{m+1} = \begin{bmatrix} v_{m+1,1} \\ v_{m+1,2} \\ \vdots \\ v_{m+1,n} \end{bmatrix} \quad \cdots \quad v_n = \begin{bmatrix} v_{n,1} \\ v_{n,2} \\ \vdots \\ v_{nn} \end{bmatrix}$$

   which can be written as

   $$\begin{aligned} v_{m+1} &= v_{m+1,1}e_1 + \cdots + v_{m+1,n}e_n \\ &\cdots \\ v_n &= v_{n,1}e_1 + \cdots + v_{n,n}e_n \end{aligned}$$

   And

   $$\begin{aligned} T(v_{m+1}) &= v_{m+1,1}T(e_1) + \cdots v_{m+1,n}T(e_n) \\ T(v_n) &= v_{n,1}T(e_1) + \cdots v_{n,n}T(e_n) \end{aligned}$$

   So if $\mathcal{M}(T)$ is diagonal, so if $v_{m+1} \cdots v_n$ are linearly independent, then $T(v_{m+1}) \cdots T(v_n)$ is linearly independent.

3. **Page 78, 4**

   $$Tv_k = A_{1,k}w_1 + \cdots A_{m,k}w_m$$

   If $Tv_1 = 0$, then for any basis of $w_1 \cdots w_m$, we can have let $A_{1,:} = 0$, and satisfy the above equation.

If $Tv_1 \neq \vec{0}$, we can set $w_1 = Tv_1$, so

$$Tv_1 = w_1 + 0 + 0 + \cdots = w_1$$

satisfy the above equation.

4. **Page 79, 6**

   $\Longrightarrow$ If dim range $(T) = 1$. Then we can have range$(T) = \alpha \cdot p$, so there must exists a basis of $W$, such that $p = \alpha_1 p_1 + \cdots \alpha_n p_n$, let $w_i = \alpha_i p_i$, so we have $p = \sum w_i$, and for all basis of $V$, $Tv_k = \sum w_i$, which means the element of $\mathcal{M}(T) = 1$

   $\Longleftarrow$ If all the element of $\mathcal{M}(T) = 1$, we can have $Tv_k = \sum w_i$ for all $v_k$, so dim range $(T) = 1$.

5. **Page 78, 12**

   Let $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$

$$AC = \begin{bmatrix} 1 & 4 \\ -3 & 0 \end{bmatrix}$$
$$CA = \begin{bmatrix} 1 & -3 \\ 4 & 0 \end{bmatrix}$$

6. **Page 88, 1**

   Suppose both $S$ and $T$ are invertible, which means both $S$ and $T$ are injective and surjective, since $S$ and $T$ are both injective, so we can have if $T$ is surjective, $T(u) = T(v)$ implies $u = v$, and if $S$ is surjective, $S(T(u)) = S(T(v))$ implies $T(u) = T(v)$ , and $u = v$ follows, so $ST$ is surjective. Since $T$ is injective, so we can have range$(T) = V$, and similarly, range $(S) = W$, since $S \in \mathcal{L}(V, W)$, so range $(ST)$ = range $S = W$. Thus, $ST$ is injective.

   Finally, we have

$$\begin{aligned} (ST)^{-1}ST &= I \\ (ST)^{-1}STT^{-1} &= T^{-1} \\ (ST)^{-1}S &= T^{-1} \\ (ST)^{-1}SS^{-1} &= T^{-1}S^{-1} \\ (ST)^{-1} &= T^{-1}S^{-1} \end{aligned}$$

7. **Page 88, 2**

   Since it's not close under addition,

   Suppose $T_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$ and $T_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$

   Obviously, both $T_1$ and $T_2$ are not invertible, but $T_1 + T_2 = I$ is invertible.

8. **Page 88, 7**

   (a) • $0 \in E$:
   Since for a matrix with all 0 entries, we can have $0 \cdot v = 0$ for all $v \in V$, so $0 \in E$.

- Closed under addition
  If $T_1 \in E$ and $T_2 \in E$, then we can have $T_1 v = 0$ and $T_2 v = 0$, then we can have $(T_1+T_2)v = 0$, which means $T_1 + T_2 \in E$.
- Closed under multiplication
  If $T_1 \in E$, then we can have $T_1 v = 0$, and $aT_1 v = 0$, which means $aT_1 \in E$ for any $a \in \mathbf{F}$.

(b) Let $v = v_1 \in V$ be an element of basis of $V$, and $v \neq 0$. So, we can extend this to the basis of $V$, $v_1 \cdots v_n$. And since $Tv = Tv_1 = 0$. Since

$$Tv_1 = A_{1,1}w_1 + \cdots A_{m,1}w_m = 0$$

So we can have $A_{:,1} = 0$.

Then we can have the set $D$ for such matrix $A$ , $\dim(D) = m(n-1)$. Since $A$ is a representation of $T$, so $E$ and $D$ are isomorphic, which means they have same dimension, which is $\dim W(\dim V - 1)$

9. **Page 89, 9**

If both matrix are both invertible , then $ST$ is invertible by the previous question.

If $ST$ is invertible, then we can have $ST$ is both surjective and injective, which means if $ST(v_1) = ST(v_2)$, it would imply $v_1 = v_2$. And range $ST = V$.

Suppose $T(v_1) = T(v_2)$, then we can have $ST(v_1) = ST(v_2)$, and $v_1 = v_2$ follows. So $T$ is injective. Suppose $S(v_1) = S(v_2)$, then we can let $T(v_3) = v_1$, $T(v_4) = v_2$, so $ST(v_3) = ST(v_4)$, and $v_3 = v_4$, which implies $v_1 = v_2$. And $S$ is injective.

Suppose $S$ is not surjective, then range $S \subset V$, so since $T \in \mathcal{L}(V)$, so we can have range $ST \subset$ range $S \subset V \neq V$, so there is an contradiction. Then we can have $S$ is surjective.

Suppose $V$ is not surjective, then range $T \subset V$, since $S$ is injective, so dim range $S(T) =$ dim range $T < \dim V$, range $ST \subset$ range $T \subset V \neq V$, so there is an contradiction. Then we can have $T$ is surjective.

10. **Page 89, 10**

By the uniqueness of inverse (Theorem 3.54), we can have this is true. Proof omit.

11. **Page 89, 18**

Suppose $A \in \mathcal{L}(\mathbb{F}, V)$, so $A\lambda = v$ for some $v \in V$

Suppose there is a linear map $M$ from $V$ to $\mathcal{L}(\mathbb{F}, V)$, which means $M(v) = A$, define a set of basis for $V$, which is $\{v_1 \cdots v_m\}$, and a set of basis for $\mathcal{L}(\mathbb{F}, V)$, which is $\{a_1, a_2 \cdots a_m\}$ (Since $\dim \mathcal{L}(\mathbb{F}, V) = \dim V$)

Let $\mathcal{M}(M) = M$, and

$$Mv_k = M_{1,k}a_1 + \cdots M_{m,k}a_m = a_k$$

So $M$ is identity matrix, which is invertible, (easy to proof by both surjective and injective). So $M$ is an isomorphism fromv$V$ to $\mathcal{L}(\mathbb{F}, V)$, which means they are isomorphic.