Homework 7 by Jingmin Sun
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1. Let

$$\begin{aligned} B_U &= \{e_i^u\} & i &= 1 \cdots n \\ B_V &= \{e_j^v\} & j &= 1 \cdots m \end{aligned}$$

Let $\ell \in BL(U \times V, F)$, and $u = \sum_{i=1}^{n} a_i e_i^u \in U$, $v = \sum_{j=1}^{m} b_j e_j^v \in V$, thus

$$\begin{aligned} \ell(u, v) &= \ell(\sum_{i=1}^{n} a_i e_i^u, \sum_{j=1}^{m} b_j e_j^v) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \ell(e_i^u, e_j^v) \end{aligned}$$

Let $\phi_i(e_j^u, v) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$ and $\psi_i(u, e_j^v) = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases}$.

And

$$\ell(e_i^u, e_j^v) = \ell_1 \phi_i(e_i^u, e_j^v) + \ell_2 \psi_j(e_i^u, e_j^v) = \ell_1 + \ell_2$$

So $\{\phi_i, \psi_j\} : i = 1 \cdots n, j = 1 \cdots m$ is a set of basis for $BL(U \times V, F)$.

2. Let $u_1^1, u_1^2 \in U_1$, then we can get

$$\begin{aligned} \Phi(au_1^1 + bu_1^2, u_2) &= T_1(au_1^1 + bu_1^2) \otimes T_2(u_2) \\ &= aT_1(u_1^1) + bT_1(u_1^2) \otimes T_2(u_2) \\ &= a(T_1(u_1^1) \otimes T_2(u_2)) + b(T_1(u_1^2) \otimes T_2(u_2)) \\ &= a\Phi(u_1^1, u_2) + b\Phi(u_1^2, u_2) \end{aligned}$$

Similarly,

$$\Phi(u_1, au_2^1 + bu_2^2) = a\Phi(u_1, u_2^1) + b\Phi(u_1, u_2^2)$$

3. **Additivity**:

$$\begin{aligned} \alpha_\beta(x_1 \otimes y_1) + \alpha_\beta(x_2 \otimes y_1) &= \beta(x_1, y_1) + \beta(x_2, y_1) \\ &= \beta(x_1 + x_2, y_1) \\ &= \alpha_\beta((x_1 + x_2) \otimes (y_1)) \end{aligned}$$

Similarly,

$$\begin{aligned} \alpha_\beta(x_1 \otimes y_1) + \alpha_\beta(x_1 \otimes y_2) &= \beta(x_1, y_1) + \beta(x_1, y_2) \\ &= \beta(x_1, y_1 + y_2) \\ &= \alpha_\beta(x_1 \otimes (y_1 + y_2)) \end{aligned}$$

Homoegeneity:

$$\begin{aligned}\alpha_\beta(c(x \otimes y)) &= \alpha_\beta(cx \otimes y) \\ &= c\beta(x, y) \\ &= c\alpha_\beta(x \otimes y)\end{aligned}$$

4.

$$\Psi(u_1 \otimes u_2) = \alpha_\Phi(u_1 \otimes u_2) = \Phi(u_1, u_2)$$

And we can get

$$\begin{aligned}\Psi((u_1^1 \otimes u_2^1)) + \Psi((u_1^2 \otimes u_2^1)) &= \Phi(u_1^1, u_2^1) + \Phi(u_1^2, u_2^1) \\ &= \Phi(u_1^1 + u_1^2, u_2^1) \\ &= \Psi((u_1^1 + u_1^2) \otimes u_2^1))\end{aligned}$$

$$\begin{aligned}\Psi((u_1^1 \otimes u_2^1)) + \Psi((u_1^1 \otimes u_2^2)) &= \Phi(u_1^1, u_2^1) + \Phi(u_1^1, u_2^2) \\ &= \Phi(u_1^1, u_2^1 + u_2^2) \\ &= \Psi((u_1^1 \otimes (u_2^1 + u_2^2))\end{aligned}$$

And

$$\begin{aligned}\Psi(a(u_1 \otimes u_2)) &= \Phi(au_1, u_2) \\ &= a\Phi(u_1, u_2) \\ &= a\Psi(u_1 \otimes u_2)\end{aligned}$$