

Homework 1 by Jingmin Sun

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ALL REFERENCE NUMBERS ARE CORRESPONDING TO THE TEXT

1. Page 17, 1

$$\begin{aligned} -(-\mathbf{v}) &= -1 \cdot (-\mathbf{v}) && 1.31 \\ &= -1 \cdot (-1 \cdot \mathbf{v}) && 1.31 \\ &= (-1 \cdot -1)\mathbf{v} && \text{Associativity} \\ &= 1\mathbf{v} \\ &= \mathbf{v} && \text{Multiplicative Identity} \end{aligned}$$

Or,

$$\begin{aligned} -\mathbf{v} + \mathbf{v} &= 0 && \text{Additive Inverse} \\ -(-\mathbf{v}) + (-\mathbf{v}) &= 0 && \text{Additive Inverse} \\ \therefore -(-\mathbf{v}) &= \mathbf{v} && 1.26 \text{ Unique Additive Inverse} \end{aligned}$$

2. Page 17, 3 Firstly, we can solve the equation:

$$\begin{aligned} \mathbf{v} + 3\mathbf{x} &= \mathbf{w} \\ 3\mathbf{x} &= \mathbf{w} - \mathbf{v} \\ \mathbf{x} &= \frac{1}{3}(\mathbf{w} - \mathbf{v}) \end{aligned}$$

Since  $\mathbf{x}$  is a linear combination of  $\mathbf{w}$  and  $\mathbf{v} \in V$ , so  $\mathbf{x} \in V$ .

Suppose there exists  $\mathbf{x}_1$  and  $\mathbf{x}_2$  satisfying the equation  $\mathbf{v} + 3\mathbf{x} = \mathbf{w}$ , so

$$\begin{aligned} \mathbf{v} + 3\mathbf{x}_1 &= \mathbf{w} \\ \mathbf{v} + 3\mathbf{x}_2 &= \mathbf{w} \end{aligned}$$

Subtracting (1) from (2), we can get

$$\begin{aligned}
 \mathbf{v} + 3\mathbf{x}_2 - (\mathbf{v} + 3\mathbf{x}_1) &= \mathbf{w} - \mathbf{w} \\
 (\mathbf{v} - \mathbf{v}) + (3\mathbf{x}_2 - 3\mathbf{x}_1) &= \mathbf{w} - \mathbf{w} && \text{Distributive property} \\
 0 + (3\mathbf{x}_2 - 3\mathbf{x}_1) &= 0 && \text{Additive Inverse} \\
 3\mathbf{x}_2 - 3\mathbf{x}_1 &= 0 && 0 - 0 = 0 \\
 3(\mathbf{x}_2 - \mathbf{x}_1) &= 0 && \text{Distributive Property} \\
 \frac{1}{3} \cdot 3(\mathbf{x}_2 - \mathbf{x}_1) &= \frac{1}{3} \cdot 0 \\
 1(\mathbf{x}_2 - \mathbf{x}_1) &= \frac{1}{3} \cdot 0 \\
 \mathbf{x}_2 - \mathbf{x}_1 &= 0 && \text{Multiplicative Identity} \\
 \mathbf{x}_2 - \mathbf{x}_1 + \mathbf{x}_1 &= 0 + \mathbf{x}_1 \\
 \mathbf{x}_2 + (-\mathbf{x}_1 + \mathbf{x}_1) &= \mathbf{x}_1 && \text{Associativity and Additive Identity} \\
 \mathbf{x}_2 + 0 &= \mathbf{x}_1 && \text{Additive Inverse} \\
 \mathbf{x}_2 &= \mathbf{x}_1 && \text{Additive Identity}
 \end{aligned}$$

Thus, there is only one  $\mathbf{x}$  satisfying the equation  $\mathbf{v} + 3\mathbf{x} = \mathbf{w}$ .

### 3. Page 17, 4

**Additive Identity.** Since the empty set does not have any element, so there does not exist  $0 \in \emptyset$  such that  $\mathbf{v} + 0 = \mathbf{v}$  for all  $\mathbf{v} \in \emptyset$

### 4. Page 24, 1

To examine a subspace, we need to show **Additive Identity**, **Closed Under Addition**, and **Closed under scalar multiplication** in the subspace.

(a)  $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

- Additive Identity

$$\begin{aligned}
 0 + 2 \cdot 0 + 3 \cdot 0 &= 0 \\
 \therefore 0 &\in V
 \end{aligned}$$

- Closed Under Addition

Suppose  $\mathbf{x} = (x_1, x_2, x_3) \in V$ ,  $\mathbf{y} = (y_1, y_2, y_3) \in V$ , then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

And

$$\begin{aligned}
 (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) &= x_1 + y_1 + 2x_2 + 2y_2 + 3x_3 + 3y_3 \\
 &= (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) \\
 &= 0 + 0 \\
 &= 0
 \end{aligned}$$

Therefore,  $\mathbf{x} + \mathbf{y} \in V$

- Closed under scalar multiplication

Suppose  $a \in \mathbf{F}$ ,  $\mathbf{x} \in V$  and we can get

$$\begin{aligned} a\mathbf{x} &= (ax_1, ax_2, ax_3) \\ ax_1 + 2(ax_2) + 3(ax_3) &= a(x_1 + 2x_2 + 3x_3) \\ &= a \cdot 0 \\ &= 0 \\ \therefore a\mathbf{x} &\in V \end{aligned}$$

So,  $V$  is a subspace of  $\mathbf{F}^3$ .

(b)  $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

- Additive Identity

$$\begin{aligned} 0 + 2 \cdot 0 + 3 \cdot 0 &= 0 \\ \therefore 0 &\notin V \end{aligned}$$

So,  $V$  is not a subspace of  $\mathbf{F}^3$ .

**Or we can check**

- Closed under scalar multiplication  
Suppose  $a \in \mathbf{F}$ ,  $\mathbf{x} \in V$  and we can get

$$\begin{aligned} a\mathbf{x} &= (ax_1, ax_2, ax_3) \\ ax_1 + 2(ax_2) + 3(ax_3) &= a(x_1 + 2x_2 + 3x_3) \\ &= a \cdot 4 \\ &= 4a \\ \therefore a\mathbf{x} &\notin V \end{aligned}$$

(c)  $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1x_2x_3 = 0\}$

- Additive Identity

$$\begin{aligned} 0 \cdot 0 \cdot 0 &= 0 \\ \therefore 0 &\in V \end{aligned}$$

- Closed Under Addition

Suppose  $\mathbf{x} = (x_1, x_2, x_3) \in V$ ,  $\mathbf{y} = (y_1, y_2, y_3) \in V$ , then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$   
And

$$\begin{aligned} (x_1 + y_1) \cdot (x_2 + y_2) \cdot (x_3 + y_3) &= (x_1 + y_1) \cdot (x_2x_3 + x_2y_3 + y_2x_3 + y_2y_3) \\ &= x_1x_2x_3 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + x_2x_3y_1 + x_2y_1y_3 + y_1y_2x_3 + y_1y_2y_3 \\ &= 0 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + x_2x_3y_1 + x_2y_1y_3 + y_1y_2x_3 + 0 \\ &= x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + x_2x_3y_1 + x_2y_1y_3 + y_1y_2x_3 \\ &\neq 0 \end{aligned}$$

Therefore,  $\mathbf{x} + \mathbf{y} \notin V$

So,  $V$  is not a subspace of  $\mathbf{F}^3$ .

(d)  $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

- Additive Identity

$$0 = 5 \cdot 0$$

$$\therefore 0 \in V$$

- Closed Under Addition

Suppose  $\mathbf{x} = (x_1, x_2, x_3) \in V$ ,  $\mathbf{y} = (y_1, y_2, y_3) \in V$ , then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$   
And

$$x_1 + y_1 = 5x_3 + 5y_3$$

$$= 5(x_3 + y_3)$$

Therefore,  $\mathbf{x} + \mathbf{y} \in V$

- Closed under scalar multiplication

Suppose  $a \in \mathbf{F}$ ,  $\mathbf{x} \in V$  and we can get

$$a\mathbf{x} = (ax_1, ax_2, ax_3)$$

$$ax_1 = a \cdot 5x_3$$

$$= 5 \cdot ax_3$$

$$\therefore a\mathbf{x} \in V$$

So,  $V$  is a subspace of  $\mathbf{F}^3$ .

#### 5. Page 24,5

No.

Firstly,  $\mathbf{R}^2 \subset \mathbf{C}^2$  then we can examine three properties:

- Additive Identity

$$\therefore 0 \in \mathbf{R}^2$$

- Closed Under Addition

Suppose  $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2$ ,  $\mathbf{y} = (y_1, y_2) \in \mathbf{R}^2$ , then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)$   
 $\mathbf{x} + \mathbf{y} \in \mathbf{R}^2$ , since if  $a, b \in \mathbf{R}$ ,  $a + b \in \mathbf{R}$ .

- Closed under scalar multiplication

Suppose  $a \in \mathbf{F}$ ,  $\mathbf{x} \in \mathbf{R}^2$  and we can get

$$a\mathbf{x} = (ax_1, ax_2) \notin \mathbf{R}^2$$

when  $\mathbf{F}$  stands for  $\mathbf{C}$ , since the multiplication of reals and complex number may not be a real number.

So,  $\mathbf{R}^2$  is not subspace of  $\mathbf{C}^2$ .

#### 6. Page 24,6

$$(a) V = \{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$$

- Additive Identity

$$0^3 = 0^3 \therefore 0 \in V$$

- Closed Under Addition

Since for  $a, b \in \mathbf{R}$ ,  $a^3 = b^3$  iff  $a = b$ , so:

Suppose  $\mathbf{x} = (x_1, x_2, x_3) \in V$ ,  $\mathbf{y} = (y_1, y_2, y_3) \in V$ , then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$   
And

$$\begin{aligned}(x_1 + y_1)^3 &= x_1^3 + y_1^3 + 3x_1y_1(x_1 + y_1) \\ &= x_2^3 + y_2^3 + 3x_2y_2(x_2 + y_2) \\ &\neq (x_2 + y_2)^3\end{aligned}$$

Therefore,  $\mathbf{x} + \mathbf{y} \in V$

- Closed under scalar multiplication  
Suppose  $a \in \mathbf{F}$ ,  $\mathbf{x} \in V$  and we can get

$$\begin{aligned}(ax_1)^3 &= a^3x_1^3 \\ &= a^3x_2^3 \\ &= (ax_2)^3 \\ \therefore a\mathbf{x} &\in V\end{aligned}$$

So,  $V$  is a subspace of  $\mathbf{R}^3$ .

(b)  $V = \{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$

- Additive Identity

$$0^3 = 0^3 \therefore 0 \in V$$

- Closed Under Addition

Suppose  $\mathbf{x} = (x_1, x_2, x_3) \in V$ ,  $\mathbf{y} = (y_1, y_2, y_3) \in V$ , then  $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

Since  $\sqrt[3]{1} = e^{2\pi/3i}, e^{4\pi/3i}, 1 = (\frac{-1 + \sqrt{3}i}{2}, \frac{-1 - \sqrt{3}i}{2}, 1)$ , so we can make  $\mathbf{x} = (\frac{-1 + \sqrt{3}i}{2}, \frac{-1 - \sqrt{3}i}{2}, 1)$

and  $\mathbf{y} = (1, \frac{-1 - \sqrt{3}i}{2}, \frac{-1 + \sqrt{3}i}{2})$

and  $\mathbf{x} + \mathbf{y} = (\frac{1 + \sqrt{3}i}{2}, 2 \cdot \frac{-1 - \sqrt{3}i}{2}, \frac{1 + \sqrt{3}i}{2})$ , and

$$\begin{aligned}\left(\frac{1 + \sqrt{3}i}{2}\right)^3 &= -1 \\ \left(2 \cdot \frac{-1 - \sqrt{3}i}{2}\right)^3 &= 8 \neq 1\end{aligned}$$

Therefore,  $\mathbf{x} + \mathbf{y} \notin V$

So,  $V$  is a subspace of  $\mathbf{C}^3$ .

7. Page 24, 7

$$\{(a, b) \in \mathbf{R}^2 : a \neq 0\}$$

8. Page 24, 8

$$\{(a, b) \in \mathbf{R}^2 : a = 0 \text{ (and/) or } b = 0\}$$

9. Page 25,10

- Additive Identity

Since  $U_1$  and  $U_2$  are subspaces of  $V$ , so  $0 \in U_1$  and  $0 \in U_2$  follows, which implies  $0 \in U_1 \cap U_2$

- Closed under Addition

Suppose  $\mathbf{x}, \mathbf{y} \in U_1 \cap U_2$ , then  $\mathbf{x} \in U_1$ ,  $\mathbf{x} \in U_2$ ,  $\mathbf{y} \in U_1$ , and  $\mathbf{y} \in U_2$ . Since  $U_1$  and  $U_2$  are subspaces of  $V$ ,  $\mathbf{x} + \mathbf{y} \in U_1$ , and  $\mathbf{x} + \mathbf{y} \in U_2$ . Thus,  $\mathbf{x} + \mathbf{y} \in U_1 \cap U_2$

- Closed under scalar multiplication

Suppose  $\mathbf{x} \in U_1 \cap U_2$ , then  $\mathbf{x} \in U_1$ ,  $\mathbf{x} \in U_2$ . Since  $U_1$  and  $U_2$  are subspaces of  $V$ ,  $a\mathbf{x} \in U_1$ ,  $a\mathbf{x} \in U_2$ , and  $a\mathbf{x} \in U_1 \cap U_2$

So,  $U_1 \cap U_2$  is a subspace of  $V$ .

10. Page 25,12

To prove that statement, we can show

- If one subspace is contained in the other, then the union of two subspace of  $V$  is a subspace of  $V$   
Obviously, if  $U_1$  and  $U_2$  are subspace of  $V$  : if  $U_1 \subseteq U_2$ , then  $U_1 \cup U_2 = U_2$  and it's a subspace of  $V$ .

If  $U_1 \subseteq U_2$ , then  $U_1 \cup U_2 = U_2$  and it's a subspace of  $V$ .

- If the union of two subspace of  $V$  is a subspace of  $V$ , then one subspace is contained in the other.  
Suppose  $U_1$  and  $U_2$  are subspace of  $V$  and  $U_1 \not\subseteq U_2$  and  $U_2 \not\subseteq U_1$ , so there exists an  $\mathbf{x} \in U_1 \setminus U_2$  and  $\mathbf{y} \in U_2 \setminus U_1$ .

Suppose  $U_1 \cup U_2$  is a subspace of  $V$ . And since  $\mathbf{x} \in U_1 \subset U_1 \cup U_2$  and  $\mathbf{y} \in U_2 \subset U_1 \cup U_2$ ,  $\mathbf{x} + \mathbf{y} \in U_1 \cup U_2$ .

Thus

$$\mathbf{x} + \mathbf{y} \in U_1$$

or

$$\mathbf{x} + \mathbf{y} \in U_2$$

If  $\mathbf{x} + \mathbf{y} \in U_1$ , and from  $\mathbf{x} \in U_1$ , we can get  $\mathbf{y} \in U_1$ , which contradicts with  $\mathbf{y} \in U_2 \setminus U_1$ , and

If  $\mathbf{x} + \mathbf{y} \in U_2$ , and from  $\mathbf{y} \in U_2$ , we can get  $\mathbf{x} \in U_2$ , which contradicts with  $\mathbf{x} \in U_1 \setminus U_2$

So,  $U_1 \subseteq U_2$  or  $U_2 \subseteq U_1$ .

11. Page 25,19 Suppose  $V = \mathbf{R}^2$

If  $U_1 = \{(0, 0)\}$ ,  $U_2 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = x_2\}$ , and  $W = \mathbf{R}^2$ .

It's obvious that  $U_1$ ,  $U_2$  and  $W$  is a subspace of  $V$ , and  $U_1 + W = U_2 + W = W = V$ , but  $U_1 \neq U_2$

12. Page 25,20

We can express  $U = x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

To make  $\mathbf{F}^4 = U \oplus W$ , we need  $W = a\mathbf{v} + b\mathbf{u}$ , such that  $\mathbf{v}, \mathbf{u}$ ,  $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$  are linear independent.

For example,  $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ ,  $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

It's obvious that  $W = a\mathbf{v} + b\mathbf{u}$  satisfies three properties of subspaces, and since four vectors are linearly independent, so for any  $\mathbf{x} \in \mathbf{F}^4$ , there is only one kind of combination of 4 vectors such that

$\mathbf{x} = a\mathbf{v} + b\mathbf{u} + x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ . So  $\mathbb{F}^4 = W + U$ , and for all  $\mathbf{x} \in \mathbf{F}^4$ , there only one way as a sum of  $\mathbf{f} \in U$  and  $\mathbf{g} \in W$ .