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ALL REFERENCE NUMBERS ARE CORRESPONDING TO THE TEXT

1. **Page 17, 1**

$$\begin{aligned}
-(-\mathbf{v}) &= -1 \cdot (-\mathbf{v}) && 1.31 \\
&= -1 \cdot (-1 \cdot \mathbf{v}) && 1.31 \\
&= (-1 \cdot -1)\mathbf{v} && \text{Associativity} \\
&= 1\mathbf{v} \\
&= \mathbf{v} && \text{Multiplicative Identity}
\end{aligned}$$

Or,

$$\begin{aligned}
-\mathbf{v} + \mathbf{v} &= 0 && \text{Additive Inverse} \\
-(-\mathbf{v}) + (-\mathbf{v}) &= 0 && \text{Additive Inverse} \\
\therefore -(-\mathbf{v}) &= \mathbf{v} && \text{1.26 Unique Additive Inverse}
\end{aligned}$$

2. **Page 17, 3** Firstly, we can solve the equation:

$$\begin{aligned}
\mathbf{v} + 3\mathbf{x} = \mathbf{w} \\
3\mathbf{x} &= \mathbf{w} - \mathbf{v} \\
\mathbf{x} &= \frac{1}{3}(\mathbf{w} - \mathbf{v})
\end{aligned}$$

Since $\mathbf{x}$ is a linear combination of $\mathbf{w}$ and $\mathbf{v} \in V$, so $\mathbf{x} \in V$.

Suppose there exists $\mathbf{x_1}$ and $\mathbf{x_2}$ satisfying the equation $\mathbf{v} + 3\mathbf{x} = \mathbf{w}$, so

$$\begin{aligned}
\mathbf{v} + 3\mathbf{x_1} &= \mathbf{w} \\
\mathbf{v} + 3\mathbf{x_2} &= \mathbf{w}
\end{aligned}$$

Subtracting (1) from (2), we can get

$$
\begin{aligned}
\mathbf{v} + 3\mathbf{x_2} - (\mathbf{v} + 3\mathbf{x_1}) &= \mathbf{w} - \mathbf{w} && \\
(\mathbf{v} - \mathbf{v}) + (3\mathbf{x_2} - 3\mathbf{x_1}) &= \mathbf{w} - \mathbf{w} && \text{Distributive property} \\
0 + (3\mathbf{x_2} - 3\mathbf{x_1}) &= 0 && \text{Additive Inverse} \\
3\mathbf{x_2} - 3\mathbf{x_1} &= 0 && 0 - 0 = 0 \\
3(\mathbf{x_2} - \mathbf{x_1}) &= 0 && \text{Distributive Property} \\
\frac{1}{3} \cdot 3(\mathbf{x_2} - \mathbf{x_1}) &= \frac{1}{3} \cdot 0 && \\
1(\mathbf{x_2} - \mathbf{x_1}) &= \frac{1}{3} \cdot 0 && \\
\mathbf{x_2} - \mathbf{x_1} &= 0 && \text{Multiplicative Identity} \\
\mathbf{x_2} - \mathbf{x_1} + \mathbf{x_1} &= 0 + \mathbf{x_1} && \\
\mathbf{x_2} + (-\mathbf{x_1} + \mathbf{x_1}) &= \mathbf{x_1} && \text{Associativity and Additive Identity} \\
\mathbf{x_2} + 0 &= \mathbf{x_1} && \text{Additive Inverse} \\
\mathbf{x_2} &= \mathbf{x_1} && \text{Additve Identity}
\end{aligned}
$$

Thus, there is only one $\mathbf{x}$ satisfying the equation $\mathbf{v} + 3\mathbf{x} = \mathbf{w}$.

3. **Page 17, 4**

   Additive Identity. Since the empty set does not have any element, so there does not exists $0 \in \emptyset$ such that $\mathbf{v} + 0 = \mathbf{v}$ for all $\mathbf{v} \in \emptyset$

4. **Page 24, 1**

   To examine a subspace, we need to show **Additive Identity**, **Closed Under Addition**, and **Closed under scalar multiplication** in the subspace.

   (a) $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$

   - Additive Identity

     $$
     \begin{aligned}
     0 + 2 \cdot 0 + 3 \cdot 0 = 0 \\
     \therefore 0 \in V
     \end{aligned}
     $$

   - Closed Under Addition
     Suppose $\mathbf{x} = (x_1, x_2, x_3) \in V$, $\mathbf{y} = (y_1, y_2, y_3) \in V$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
     And

     $$
     \begin{aligned}
     (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) &= x_1 + y_1 + 2x_2 + 2y_2 + 3x_3 + 3y_3 \\
     &= (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) \\
     &= 0 + 0 \\
     &= 0
     \end{aligned}
     $$

     Therefore, $\mathbf{x} + \mathbf{y} \in V$

   - Closed under scalar multiplication

Suppose $a \in \mathbf{F}$, $\mathbf{x} \in V$ and we can get

$$\begin{aligned}
a\mathbf{x} &= (ax_1, ax_2, ax_3) \\
ax_1 + 2(ax_2) + 3(ax_3) &= a(x_1 + 2x_2 + 3x_3) \\
&= a \cdot 0 \\
&= 0 \\
\therefore a\mathbf{x} &\in V
\end{aligned}$$

So, V is a subspace of $\mathbf{F}^3$.

(b) $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$

- Additive Identity

$$\begin{aligned}
0 + 2 \cdot 0 + 3 \cdot 0 = 0 \\
\therefore 0 \notin V
\end{aligned}$$

  So, V is not a subspace of $\mathbf{F}^3$.
  Or we can check

- Closed under scalar multiplication
  Suppose $a \in \mathbf{F}$, $\mathbf{x} \in V$ and we can get

$$\begin{aligned}
a\mathbf{x} &= (ax_1, ax_2, ax_3) \\
ax_1 + 2(ax_2) + 3(ax_3) &= a(x_1 + 2x_2 + 3x_3) \\
&= a \cdot 4 \\
&= 4a \\
\therefore a\mathbf{x} &\notin V
\end{aligned}$$

(c) $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$

- Additive Identity

$$\begin{aligned}
0 \cdot 0 \cdot 0 = 0 \\
\therefore 0 \in V
\end{aligned}$$

- Closed Under Addition
  Suppose $\mathbf{x} = (x_1, x_2, x_3) \in V$, $\mathbf{y} = (y_1, y_2, y_3) \in V$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  And

$$\begin{aligned}
(x_1 + y_1) \cdot (x_2 + y_2) \cdot (x_3 + y_3) &= (x_1 + y_1) \cdot (x_2x_3 + x_2y_3 + y_2x_3 + y_2y_3) \\
&= x_1x_2x_3 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + x_2x_3y_1 + x_2y_1y_3 + y_1y_2x_3 + y_1y_2y_3 \\
&= 0 + x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + x_2x_3y_1 + x_2y_1y_3 + y_1y_2x_3 + 0 \\
&= x_1x_2y_3 + x_1y_2x_3 + x_1y_2y_3 + x_2x_3y_1 + x_2y_1y_3 + y_1y_2x_3 \\
&\neq 0
\end{aligned}$$

  Therefore, $\mathbf{x} + \mathbf{y} \notin V$

So, V is not a subspace of $\mathbf{F}^3$.

(d) $V = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

- Additive Identity

$$0 = 5 \cdot 0$$
$$\therefore 0 \in V$$

- Closed Under Addition
  Suppose $\mathbf{x} = (x_1, x_2, x_3) \in V$, $\mathbf{y} = (y_1, y_2, y_3) \in V$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  And

$$\begin{aligned} x_1 + y_1 &= 5x_3 + 5y_3 \\ &= 5(x_3 + y_3) \end{aligned}$$

  Therefore, $\mathbf{x} + \mathbf{y} \in V$
- Closed under scalar multiplication
  Suppose $a \in \mathbf{F}$, $\mathbf{x} \in V$ and we can get

$$\begin{aligned} a\mathbf{x} &= (ax_1, ax_2, ax_3) \\ ax_1 &= a \cdot 5x_3 \\ &= 5 \cdot ax_3 \\ \therefore a\mathbf{x} &\in V \end{aligned}$$

So, V is a subspace of $\mathbf{F}^3$.

5. **Page 24,5**

   No.

   Firstly, $\mathbf{R}^2 \subset \mathbf{C}^2$ then we can examine three properties:

   - Additive Identity

$$\therefore 0 \in \mathbf{R}^2$$

   - Closed Under Addition
     Suppose $\mathbf{x} = (x_1, x_2) \in \mathbf{R}^2$, $\mathbf{y} = (y_1, y_2) \in \mathbf{R}^2$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)$
     $\mathbf{x} + \mathbf{y} \in \mathbf{R}^2$, since if $a, b \in \mathbf{R}$, $a + b \in \mathbf{R}$.
   - Closed under scalar multiplication
     Suppose $a \in \mathbf{F}$, $\mathbf{x} \in \mathbf{R}^2$ and we can get

$$a\mathbf{x} = (ax_1, ax_2, ax_3) \qquad \notin \mathbf{R}^2$$

     when $\mathbf{F}$ stands for $\mathbf{C}$, since the multiplication of reals and complex number may not be a real number.

   So, $\mathbf{R}^2$ is not subspace of $\mathbf{C}^2$.

6. **Page 24,6**

   (a) $V = \{(a, b, c) \in \mathbf{R}^3 : a^3 = b^3\}$

   - Additive Identity

$$0^3 = 0^3 \therefore 0 \in V$$

- Closed Under Addition
  Since for $a, b \in \mathbf{R}$, $a^3 = b^3$ iff $a = b$, so:
  Suppose $\mathbf{x} = (x_1, x_2, x_3) \in V$, $\mathbf{y} = (y_1, y_2, y_3) \in V$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  And

$$\begin{aligned}(x_1 + y_1)^3 &= x_1^3 + y_1^3 + 3x_1y_1(x_1 + y_1) \\ &= x_2^3 + y_2^3 + 3x_2y_2(x_2 + y_2) \\ &\neq (x_2 + y_2)^3\end{aligned}$$

  Therefore, $\mathbf{x} + \mathbf{y} \in V$
- Closed under scalar multiplication
  Suppose $a \in \mathbf{F}$, $\mathbf{x} \in V$ and we can get

$$\begin{aligned}(ax_1)^3 &= a^3x_1^3 \\ &= a^3x_2^3 \\ &= (ax_2)^3 \\ \therefore a\mathbf{x} &\in V\end{aligned}$$

So, V is a subspace of $\mathbf{R}^3$.

(b) $V = \{(a, b, c) \in \mathbf{C}^3 : a^3 = b^3\}$

- Additive Identity

$$0^3 = 0^3 \therefore 0 \in V$$

- Closed Under Addition
  Suppose $\mathbf{x} = (x_1, x_2, x_3) \in V$, $\mathbf{y} = (y_1, y_2, y_3) \in V$, then $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$
  Since $\sqrt[3]{1} = e^{2\pi/3i}, e^{4\pi/3i}, 1 = (\frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}, 1)$, so we can make $\mathbf{x} = (\frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}, 1)$
  and $\mathbf{y} = (1, \frac{-1-\sqrt{3}i}{2}, \frac{-1+\sqrt{3}i}{2})$
  and $\mathbf{x} + \mathbf{y} = (\frac{1+\sqrt{3}i}{2}, 2 \cdot \frac{-1-\sqrt{3}i}{2}, \frac{1+\sqrt{3}i}{2})$, and

$$\left(\frac{1+\sqrt{3}i}{2}\right)^3 = -1$$

$$\left(2 \cdot \frac{-1-\sqrt{3}i}{2}\right)^3 = 8 \neq 1$$

  Therefore, $\mathbf{x} + \mathbf{y} \notin V$

So, V is a subspace of $\mathbf{C}^3$.

7. **Page 24, 7**

$$\{(a, b) \in \mathbf{R}^2 : a \neq 0\}$$

8. **Page 24, 8**

$$\{(a, b) \in \mathbf{R}^2 : a = 0 \text{ (and/) or } b = 0\}$$

9. **Page 25,10**

   - Additive Identity
     Since U_1 and U_2 are subspaces of V, so $0 \in U_1$ and $0 \in U_2$ follows, which implies $0 \in U_1 \cap U_2$
   - Closed under Addition
     Suppose $\mathbf{x}, \mathbf{y} \in U_1 \cap U_2$, then $\mathbf{x} \in U_1$, $\mathbf{x} \in U_2$, $\mathbf{y} \in U_1$, and $\mathbf{y} \in U_2$. Since U_1 and U_2 are subspaces of V, $\mathbf{x} + \mathbf{y} \in U_1$, and $\mathbf{x} + \mathbf{y} \in U_2$. Thus, $\mathbf{x} + \mathbf{y} \in U_1 \cap U_2$
   - Closed under scalar multiplication
     Suppose $\mathbf{x} \in U_1 \cap U_2$, then $\mathbf{x} \in U_1$, $\mathbf{x} \in U_2$. Since U_1 and U_2 are subspaces of V, $a\mathbf{x} \in U_1$, $a\mathbf{x} \in U_2$, and $a\mathbf{x} \in U_1 \cap U_2$

   So, $U_1 \cap U_2$ is a subspace of V.

10. **Page 25,12**

    To prove that statement, we can show

    - If one subspace is contained in the other, then the union of two subspace of V is a subspace of V
      Obviously, if U_1 and U_2 are subspace of V : if $U_1 \subseteq U_2$, then $U_1 \cup U_2 = U_2$ and it's a subspace of V.
      If $U_1 \subseteq U_2$, then $U_1 \cup U_2 = U_1$ and it's a subspace of V.
    - If the union of two subspace of V is a subspace of V, then one subspace is contained in the other.
      Suppose U_1 and U_2 are subspace of V and $U_1 \nsubseteq U_2$ and $U_2 \nsubseteq U_1$, so there exists an $\mathbf{x} \in U_1 \backslash U_2$ and $\mathbf{y} \in U_2 \backslash U_1$.
      Suppose $U_1 \cup U_2$ is a subspace of V. And since $\mathbf{x} \in U_1 \subset U_1 \cup U_2$ and $\mathbf{y} \in U_2 \subset U_2 \cup U_1$, $\mathbf{x} + \mathbf{y} \in U_1 \cup U_2$.
      Thus
      $$\mathbf{x} + \mathbf{y} \in U_1$$
      or
      $$\mathbf{x} + \mathbf{y} \in U_2$$
      If $\mathbf{x} + \mathbf{y} \in U_1$, and from $\mathbf{x} \in U_1$, we can get $\mathbf{y} \in U_1$, which contradicts with $\mathbf{y} \in U_2 \backslash U_1$, and
      If $\mathbf{x} + \mathbf{y} \in U_2$, and from $\mathbf{y} \in U_2$, we can get $\mathbf{x} \in U_2$, which contradicts with $\mathbf{x} \in U_1 \backslash U_2$
      So, $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$.

11. **Page 25,19** Suppose $V = \mathbf{R}^2$
    If $U_1 = \{(0,0)\}$, $U_2 = \{(x_1, x_2) \in \mathbf{R}^2 : x_1 = x_2\}$, and $W = \mathbf{R}^2$.
    It's obvious that U_1, U_2 and W is a subspace of V, and $U_1 + W = U_2 + W = W = V$, but $U_1 \neq U_2$

12. **Page 25,20**

    We can express $U = x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

    To make $\mathbf{F}^4 = U \bigoplus W$, we need $W = a\mathbf{v} + b\mathbf{u}$, such that $\mathbf{v}, \mathbf{u}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ are linear independent.

For example, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

It's obvious that $W = a\mathbf{v} + b\mathbf{u}$ satisfies three properties of subspaces, and since four vectors are linearly independent, so for any $\mathbf{x} \in \mathbf{F}^4$, there is only one kind of combination of 4 vectors such that $\mathbf{x} = a\mathbf{v} + b\mathbf{u} + x \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. So $\mathbb{F}^4 = W + U$, and for all $\mathbf{x} \in \mathbf{F}^4$, there only one way as a sum of $\mathbf{f} \in U$ and $\mathbf{g} \in W$.