**Homework 2 by Jingmin Sun**
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**ALL REFERENCE NUMBERS ARE CORRESPONDING TO THE TEXT**

1. **Page 37, 1**

   Since $v_1, v_2, v_3, v_4$ spans $V$, we can get $span\{v_1, v_2, v_3, v_4\} = V$, which means for each $x \in V$, we can always find $a_1, a_2, a_3, a_4 \in \mathbf{F}$ such that $x = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4$.

   Since

   $$\begin{aligned}
   v_1 &= (v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4 \\
   v_2 &= (v_2 - v_3) + (v_3 - v_4) + v_4 \\
   v_3 &= (v_3 - v_4) + v_4 \\
   v_4 &= v_4
   \end{aligned}$$

   Thus, we can get

   $$\begin{aligned}
   x &= a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 \\
   &= a_1((v_1 - v_2) + (v_2 - v_3) + (v_3 - v_4) + v_4) + a_2((v_2 - v_3) + (v_3 - v_4) + v_4) + a_3((v_3 - v_4) + v_4) + a_4v_4 \\
   &= a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4
   \end{aligned}$$

   Let $c_1 = a_1$, $c_2 = a_1 + a_2$, $c_3 = a_1 + a_2 + a_3 + a_4$, $c_4 = a_1 + a_2 + a_3 + a_4$, so that the new list of the vectors spans $V$.

2. **Page 37, 3**

   Since we need three vectors are not linear independent, which means:

   $$a \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + b \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \\ t \end{bmatrix}$$

   From the first two rows, we can get

   $$\begin{aligned}
   3a + 2b &= 5 \\
   a - 3b &= 9
   \end{aligned}$$

   Solve the system and get

   $$\begin{aligned}
   a &= 3 \\
   b &= -2
   \end{aligned}$$

   And we can get $t = 4a + 5b = 12 - 10 = 2$

3. **Page 37, 7**

   Suppose there exists $a_1, a_2 \cdots a_m \in \mathbf{F}$ such that

   $$\begin{aligned}
   a_1(5v_1 - 4v_2) + a_2v_2 + a_3v_3 + \cdots a_mv_m &= 0 \\
   5a_1v_1 + (-4a_1 + a_2)v_2 + a_3v_3 + \cdots a_mv_m &= 0
   \end{aligned}$$

Since the set of vectors $v_1, v_2, v_3 \cdots v_m$ are linearly independent. so

$$\begin{aligned} 5a_1 &= 0 \\ -4a_2 + a_2 &= 0 \\ a_i &= 0 \qquad\qquad\qquad\qquad \text{for } i = 3, 4, \cdots m \end{aligned}$$

So we can get $a_i = 0$ for all $i = 1, 2, 3, \cdots m$, which means the new set is linearly independent.

4. **Page 37, 9**

   **Counterexample**:

   Let

   $$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   and

   $$w_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad w_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

   then we can get

   $$v_1 + w_1 = v_2 + w_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   so, they are not linearly independent.

5. **Page 37,10**

   If $v_1 + w$, $v_2 + w, \cdots v_m + w$ is linearly dependent, then for $a_1, a_2 \cdots a_m \in \mathbf{F}$, we can get

   $$\begin{aligned} a_1(v_1 + w) + a_2(v_2 + w) + \cdots a_m(v_m + w) &= 0 \\ a_1v_1 + a_2v_2 + \cdots + a_mv_m + (a_1 + a_2 + \cdots + a_m)w &= 0 \end{aligned}$$

   Since linearly dependency, we can get

   $$a_1, a_2, a_3 \cdots a_m \neq 0$$

   And we can get

   $$a_1 + a_2 + \cdots + a_m \neq 0$$

   Since if $a_1 + a_2 + \cdots + a_m = 0$, then $a_1v_1 + a_2v_2 + \cdots + a_mv_m = 0$,and $a_1 = a_2 = \cdots a_m = 0$ follows, which contradict with previous statement.

   Then,

   $$w = \frac{a_1v_1 + a_2v_2 + \cdots + a_mv_m}{a_1 + a_2 + \cdots + a_m} \in \text{ span } (v_1, v_2 \cdots v_m)$$

6. **Page 43,3**

   (a) We can express $U$ as $U = \{(3x_2, x_2, 7x_4, x_4, x_5) \in \mathbf{R}^5 : x_2, x_4, x_5 \in \mathbf{R}\}$

So, any $\mathbf{x} \in U$ can be expressed as

$$\mathbf{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

So we can get the following vectors spans $U$,

$$\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

And we can show that they are linearly independent: Suppose

$$a \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$a = b = c = 0$$

Thus, they are linearly independent. So, this is a basis of $U$

(b) So we can add other two linearly independent vector to independent to all three vectors above:

$$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

We can see that for all $x \in \mathbf{R}^5$, we can set

$$a \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 7 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + e \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$\begin{aligned} 3a &= x_1 \\ a + d &= x_2 \\ 7b + e &= x_3 \\ b &= x_4 \\ c &= x_5 \end{aligned}$$

And we can get

$$\begin{cases} a &= \dfrac{x_1}{3} \\ b &= x_4 \\ c &= x_5 \\ d &= x_2 - \dfrac{x_1}{3} \\ e &= x_3 - 7x_4 \end{cases}$$

So we can get the set spans $\mathbf{R}^5$, and we can prove it is linearly independent: when $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, we can get $a = n = c = d = e = 0$ is the only solution, so they are linearly independent.

(c) $W = \text{ span } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

As we have shown above, $\mathbf{R}^5 = U + W$, and we need to show that $U \cap W = \vec{0}$, it's obvious that $0 \in U \cap W$, and suppose $x \in U \cap W$, so $x \in U$ and $x \in W$, so $x \in U$ meas

$$x_1 = 3x_2$$
$$x_3 = 7x_4$$

And $x \in W$ means $x_1 = x_4 = x_5 = 0$, so we can get $x_1 = x_2 = x_3 = x_4 = x_5 = 0$.
So, $U \cap W = \vec{0}$, and $\mathbf{R}^5 = U \oplus W$.

7. **Page 43,5**

Since this is a there exists statement, so we can proof by an example:

Let

$$p_0 = 1$$
$$p_1 = x$$
$$p_2 = x^3 + x^2$$
$$p_3 = x^3$$

Then this is a basis of $\mathcal{P}_3(\mathbf{F})$.

Since for all $p \in \mathcal{P}_3(\mathbf{F})$, $p = a_1 + a_2x + a_3x^2 + a_4x^3 = a_1p_0 + a_2p_1 + a_3p_2 + (-a_3 + a_4)p_3$, and $a_i$ are unique for $i = 1, 2, 3, 4$.

8. **Page 43,6**

We need to prove the four vectors are linearly independent first, which is easy to prove:

Suppose $a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4v_4 = 0$, then we can get

$$a_1v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4 = 0$$

Since $v_1, v_2, v_3$ and $v_4$ are linearly independent, we can get

$$\begin{aligned} a_1 &= 0 \\ a_1 + a_2 &= 0 \\ a_2 + a_3 &= 0 \\ a_3 + a_4 &= 0 \end{aligned}$$

So that $a_1 = a_2 = a_3 = a_4 = 0$ , which means they are linearly independent.

Since $v_1, v_2, v_3$ and $v_4$ are basis of $V$, so for all $v \in V$,

$$\begin{aligned} v &= b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 \\ &= b_1(v_1 + v_2) + (-b_1 + b_2)(v_2 + v_3) + (-b_2 + b_3)(v_3 + v_4) + (-b_3 + b_4)v_4 \end{aligned}$$

Since $b_1, b_2, b_3, b_4$ are unique, then the coefficient for new set are unique as well. Thus the new set is a basis of $V$.

9. **Page 43,7**

   **Counterexample**:

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

   And $U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_4 = 0\}$

10. **Page 43,8**

   Suppose

$$a_1 u_1 + a_2 u_2 + \cdots a_m u_m + b_1 w_1 + b_2 w_2 \cdots b_n w_n = 0$$

   Since $V = U \oplus W$, so for $0 \in V$, there exists a unique combination $u \in U, w \in W$, such that $u + w = 0$, namely, $u = w = 0$. Thus

$$\begin{aligned} a_1 u_1 + a_2 u_2 + \cdots a_m u_m = 0 \\ b_1 w_1 + b_2 w_2 + \cdots b_n w_n = 0 \end{aligned}$$

   Since $u_1 \cdots u_m$ is a basis for $U$, so they are linear independent, and $w_1 \cdots w_n$ is a basis for $W$, and they are linearly independent as well. So $a_i = b_j = 0$ for $i = 1, 2 \cdots m; j = 1, 2, \cdots n$

   So, $u_1 \cdots u_m, w_1 \cdots w_n$ are linearly independent, and since $V = U \oplus W$, so for $v \in V$, there exists a unique combination $u \in U, w \in W$, such that $u + w = v$, namely,

$$v = a_1 u_1 + a_2 u_2 + \cdots a_m u_m + b_1 w_1 + b_2 w_2 \cdots b_n w_n$$

   for all $v \in V$

   Thus, it's a basis for $V$.