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1. **Page 160,1**

   Since T is diagonalizable, so by 5.41 d, we have

   $$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$

   So, if T is invertible, then all $\lambda_i \neq 0$, so null$T = 0$, we can get $V =$ range $T =$ null $T \oplus$ range T.

   And if T is not invertible, then there exists j such that $\lambda_j = 0$, for all $j \in J$ and it corresponds a set of eigenvectors v_j, such that $T(v_j) = 0$ for all $j \in J$. And

   $$\begin{aligned} V &= E(\lambda_1, T) \oplus \cdots E(\lambda_i, T) \oplus E(\lambda_m, T) \\ &= E(\lambda_I, T) \oplus E(\lambda_J, T) \\ &= E(\lambda_I, T) \end{aligned}$$

   and index I represents the union of all index not in J. So, we can get $V =$ range $T =$ null $T \oplus$ range T in this case as well.

2. **Page 160,2**

   Let V be any infinite dimensional vector space. And T be any matrix with full rank in dimension V.

3. **Page 160,3**

   $(a) \implies (b)$ is obvious.

   $(b) \implies (c)$:

   Since $V =$ null $T +$ range T, we can get $\dim V = \dim$ null $T + \dim$ range T.

   But we have $\dim V = \dim$ null $T + \dim$ range $T - \dim($ null $T \cap$ range $T)$.

   So we can get (c).

   $(c) \implies (a)$ Since $\dim V = \dim$ null $T + \dim$ range $T - \dim($ null $T \cap$ range $T)$. and $\dim($ null $T \cap$ range $T) = 0$, we can have $V =$ null $T +$ range T, so together with (c), we can get a.

4. **Page 160,5**

   Since λI is diagonal matrix, so if T is diagonalizable, then $T - \lambda I$ is diagnolizable. so by Problem 1, we can get the answer.

   Conversely, since V is finite dimensional, T has only finite many eigenvalues. $\lambda_i : i = 1 \cdots m$

   Since $V =$ null $(T - \lambda_1 I) \oplus$ range $(T - \lambda_1 I) =$ null $(T - \lambda_2 I) \oplus$ range $(T - \lambda_2 I)$

   Since null $(T - aI) \subset$ range $(T - bI)$ for any $a \neq b$. (This can be shown by let $v \in$ null $(T - aI)$, and $(T - bI)\left(\dfrac{v}{a-b}\right) = v$.

   So we can get range$(T - \lambda_1) =$ null $(T - \lambda_2 I) \oplus$ range $(T - \lambda_2 I) \cap$ range$(T - \lambda_1)$.

   And by induction we can have $V =$ null $(T - \lambda_1 I) \oplus \cdots \oplus$ null $(T - \lambda_m I) + \bigcap_i$ range $(T - \lambda_i I)$

   If there exists $x \in X = \bigcap_i$ range $(T - \lambda_i I)$, then we can have $(T - \lambda_i I)T = T(T - \lambda_i I)$, which means range $(T - \lambda_i I)$ is in an invariant subspace under T, and so does X. So there exists an eigenvalue $\tilde{\lambda}$ and corresponding eigenvector ϕ in X. Since $\tilde{\lambda}$ is also an eigenvalue of T, so $\phi \in$null $(T - \lambda_i I)$ for some i, and there is an contradiction, so $X = \{0\}$, which means $V =$ null $(T - \lambda_1 I) \oplus \cdots \oplus$ null $(T - \lambda_m I)$, and it is diagnolizable.

5. **Page 160,6**

   Suppose the common eigenvectors are $v_1 \cdots v_m$, and for any $v \in V$, we can get $v = \sum_{i=1}^{m} a_i v_i$

   Suppose the eigenvalue of T are λ_i^T, and those for S are λ_i^S, we can get

$$\begin{aligned} TS(v) &= T(S(\sum_{i=1}^{m} a_i v_i)) \\ &= T(\sum_{i=1}^{m} a_i \lambda_i^S v_i) \\ &= \sum_{i=1}^{m} a_i \lambda_i^S \lambda_i^T v_i \\ &= S(\sum_{i=1}^{m} a_i \lambda_i^T v_i) \\ &= S(T(\sum_{i=1}^{m} a_i v_i)) \\ &= S(T(v)) \end{aligned}$$

6. **Page 160,7**

   If λ appears n times, and if we write T into its diagonal form, then there will be n zeros on the diagonal of $T - \lambda I$, so that is $n = \text{dim null } T - \lambda I$.

7. **Page 160,8**

   Since $\dim(\mathbb{F}^5) = 5$, and $\mathbb{F}^5 = \sum E(\lambda_i, T)$, (direct sum), then we can have $\dim \mathbb{F}^5 = \sum \dim E(\lambda_i, T)$, so there is at most one more eigenvalue.

8. **Page 161,10**

   If T is diagonalizable, we can get it easily, since range $T = V$, and the equation holds at equality.

   But if T is not diagonalizable, so there is not invertible matrix P such that $PDP^{-1} = A$, so P is not invertible, so we do not have complete set of eigenvectors, which means we can get LHS is less than RHS.

$$\begin{aligned} \dim \operatorname{range} T &= \dim V - \dim \operatorname{null} T \\ &= \dim V - \dim E(0, T) \\ &\geq E(0, T) + \sum_{i=1}^{m} E(\lambda_i, T) - E(0, T) \\ &= \sum_{i=1}^{m} E(\lambda_i, T) \end{aligned}$$

9. **Page 161,13**

$$R = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix} \quad T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 1 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

10. **Page 175,2**

    It violates definiteness, since $\langle(0,1,0),(0,1,0)\rangle = 0$.

11. **Page 175,4**

    (a)

    $$\begin{aligned}\langle u+v, u-v\rangle &= \langle u, u-v\rangle + \langle v, u-v\rangle \\ &= \langle u,u\rangle - \langle u,v\rangle + \langle v,u\rangle - \langle v,v\rangle \\ &= ||u||^2 - ||v||^2\end{aligned}$$

    (b) if they have the same norm,

    $$\langle u+v, u-v\rangle = ||u||^2 - ||v||^2 = 0$$

    so, $u+v$ and $u-v$ are orthogonal.

    (c) As picture in page 174, and $||u|| = ||v||$, so we can get the result.

12. **Page 175,6**

    If $\langle u, v\rangle = 0$, then

    $$\begin{aligned}||u+av||^2 &= ||u||^2 + a^2||v||^2 \\ &\geq ||u||^2 \\ ||u|| &\leq ||u+av||\end{aligned}$$

    And if $||u|| \leq ||u+av||$, we have

    $$\begin{aligned}||u+av||^2 &\geq ||u||^2 \\ ||u||^2 + 2a\langle u,v\rangle + a^2||v||^2 &\geq ||u||^2 \\ 2a\langle u,v\rangle + a^2||v||^2 &\geq 0\end{aligned}$$

    so we can get $\langle u, v\rangle = 0$,.

13. **Page 176,11**

    According to Cauchy-Schwartz:

    $$\begin{aligned}(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) &\geq \left(\sqrt{\frac{a}{a}}+\sqrt{\frac{b}{b}}+\sqrt{\frac{c}{c}}+\sqrt{\frac{d}{d}}\right) \\ &= 16\end{aligned}$$

14. **Page 176,15**

    $$\begin{aligned}\left(\sum_{j=1}^{n} ja_j^2\right)\left(\sum_{j=1}^{n}\frac{b_j^2}{j}\right) &\geq \left|\sum_{j=1}^{n}\sqrt{j}a_j\frac{b_j}{\sqrt{j}}\right|^2 \\ &= \left(\sum_{i=1}^{n} a_jb_j\right)^2\end{aligned}$$