CIVIL-557
Decision-aid methodologies in transportation
Review II
Branch-and-Price

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VRP Set-partitioning formulation

Master problem (MP)

$$Minimize \sum_{r \in \Omega} c_r \lambda_r \tag{1}$$

$$s.t. \sum_{r \in \Omega} a_{ir} \lambda_r = 1 \qquad \forall i \in N, \tag{2}$$

$$\lambda_r \in \{0, 1\} \qquad \forall r \in \Omega.$$

Restricted Master Problem

What is the **RMP**?
optimal if reduced cost are non negative

- The set Ω is too big ($\approx (|N|-1)!$).
- Lets consider a smaller set of variables, e.g., $P \subset \Omega$
- Where $|P|$ is a small number, i.e., not exponential.
- The following model is the Restricted Master Problem (RMP):

$$
\text{(RMP)} \qquad
\begin{aligned}
& Min \sum_{r \in P} c_r \lambda_r & \\
s.t. \; & \sum_{r \in P} a_{ir} \lambda_r = 1 & \forall i \in N, \\
& 1 \geq \lambda_r \geq 0 & \forall r \in P.
\end{aligned}
$$

Optimal solution of the MP

When does the **MP = RMP** ?

Optimal if the reduced costs of NB varibles is non negative

- Recall that a basic solution is optimal if there are no non-basic variables (λ_{NB}) with a negative reduced cost.
- We must find non-basic variables (i.e., routes) that have a negative reduced cost to add to the set P and reoptimize.
- If we can proof that all non-basic variables have a non-negative reduced cost, the current basic solution of the RMP is also optimal for the MP and we do not have to search for more variables.

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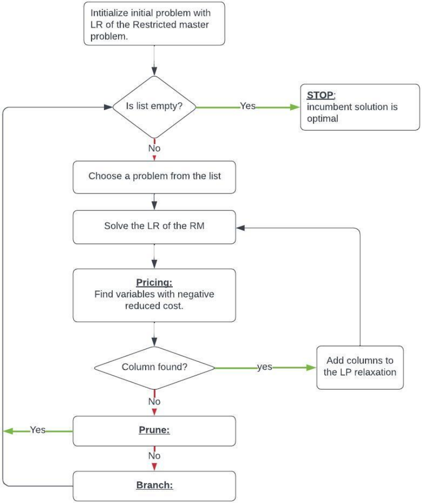
Minimum reduced cost route (Pricing problem)

Elemetary Shortest Path Problem with Resource Constraints (ESPPRC)

$$
\begin{aligned}
& Min \sum_{i \in V} \sum_{j \in V} \hat{c}_{ij} x_{ij} && \\
& s.t. \sum_{i \in N} q_i \sum_{j \in V} x_{ij} \leq Q, && \\
& \sum_{j \in N} x_{0j} = 1, && \\
& \sum_{i \in V} x_{i0} = 1, && \\
& \sum_{i \in N} x_{ih} - \sum_{j \in N} x_{hj} = 0 && \forall h \in N, \\
& T_i + t_{ij} - M_{ij}(1 - x_{ij}) \leq T_j && \forall i, j \in V, \\
& a_i \leq T_i \leq b_i && \forall i \in N, \\
& x_{ij} \in \{0, 1\} && \forall i, j \in N.
\end{aligned}
$$

B&P



Intitialize initial problem with LR of the Restricted master problem.
Is list empty?
Yes
STOP:
incumbent solution is optimal
No
Choose a problem from the list
Solve the LR of the RM
Pricing:
Find variables with negative reduced cost.
Column found?
yes
Add columns to the LP relaxation
No
Prune:
Yes
No
Branch:

Feasible solutions

Dummy variable

$$\begin{aligned} \min\ & M\lambda_D \\ s.t.\ & \lambda_D = 1 && \forall i \in N, \\ & 1 \geq \lambda_D \geq 0 \end{aligned}$$

Add more variables that satisfy constraints to the model to make $\lambda_D = 0$.
If $\lambda_D = 0$ is not possible, then, the problem is infeasible.

Branching

Branching on the variables of the master problem (λ) makes the pricing problem more difficult to solve at each branch.

$\lambda^j = 0$

ESPPRC and Forbidden Paths. Every time we set a variable λ to 0, we must make sure the the path is not produced by the pricing problem

$\lambda^j = 1$

The path "j" has to be in the solution, thus, we can eliminate the customers that are visited in j and have an easier problem to solve. The ESPPRC becomes smaller since the number of customers to visit is reduced.

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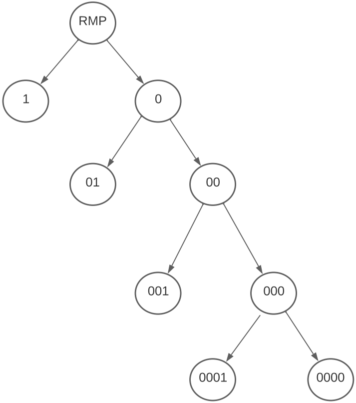
Branching

Unbalanced binary tree: Branching on variables makes the binary tree unbalanced since most paths are not in the optimal solution. Setting $\lambda = 0$ is not significant.



Pricing problem

ESPPRC and Forbidden Paths.

$$
\begin{aligned}
& Min \sum_{i\in V}\sum_{j\in V} \hat{c}_{ij}x_{ij} \\
& s.t. \sum_{i\in N} q_i \sum_{j\in V} x_{ij} \leq Q, \\
& \sum_{j\in N} x_{0j} = 1, \\
& \sum_{i\in V} x_{i0} = 1, \\
& \sum_{i\in N} x_{ih} - \sum_{j\in N} x_{hj} = 0 && \forall h \in N, \\
& \sum_{(i,j)\in\mathcal{P}} x_{i,j} \leq |\mathcal{P}| - 1 \leftarrow Forbiddenpath && \forall \mathcal{P} \in \mathcal{B}, \\
& T_i + t_{ij} - M_{ij}(1 - x_{ij}) \leq T_j && \forall i,j \in V, \\
& a_i \leq T_i \leq b_i && \forall i \in N, \\
& x_{ij} \in \{0,1\} && \forall i,j \in N.
\end{aligned}
$$

Branching

Option

Branch: On x_{ij} variables instead.

- At each node of the branch-and-bound tree set the corresponding variables to 1 or 0, and solve the pricing problem.
- The pricing problem will not produce routes that contain the variables that are set to 0.