A Spectral Method Approach to 2D Viscous Flow Passing a Sphere

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Abstract

This report documents the project for the *Honors IV: Physics Mathematics* course in Spring 2018 and is to be understood as the final course submission.

Within this project, we explore variants of spectral method and simulate a variety of fluid dynamics phenomena, ranging from the 2D flow around a sphere, which entitles the project, to Kelvin−Helmholtz instability. In particular, we offer two extensions of the classical pseudo-spectral method, which we term the "damping" and "pushing" methods, that approximate the viscous flow passing obstacles. Future works will target at the theoretical and technical specifics of the methods we offer, such as the convergence and robustness of our approximation, the effect of obstacle with irregular geometry, and generalization of our methods to 3D.

Contents

1 Origin: Fluid Dynamics

1.1 Background

Fluid dynamics contains the richest results of combining physics and math. Its experimental and engineering value has long been explored by physicists and engineers, and mathematicians have been striving to explain its unpredictable complexity(or complex unpredictability).

Amazingly, it's still an approachable topic even for undergraduates because of its underlying familiarity in daily life, and the developments of computers and MATLAB, which allow us to easily implement well-developed methods to simulate it.

We start from the question of how to simulate 2D flow through a sphere, which simplifies the scenario of simulating 3D flow through a building in Manhattan.

1.2 Governing Equations

We consider the 2D incompressible flow, so we have Navier-Stokes with free divergence of the velocity field

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p - v\Delta \boldsymbol{u} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

moreover, to lay up for the Spectral Method, let's introduce vorticity $\boldsymbol{\omega}$

$$\omega = \nabla \times \mathbf{u} \tag{3}$$

and just take the curl of 8 and use 3 we have

$$\frac{d\boldsymbol{\omega}}{dt} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \boldsymbol{\omega} \cdot (\nabla \mathbf{u}) + v\Delta\boldsymbol{\omega} \tag{4}$$

the term $\boldsymbol{\omega} \cdot (\nabla \boldsymbol{u})$ represents the effects of vortex stretching(17, SMFD). This is zero in 2D flow, which will simplify our equation to

$$\frac{d\boldsymbol{\omega}}{dt} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = v\Delta\boldsymbol{\omega} \tag{5}$$

This will be the ideal equation we want to solve in our future cases, but we need the notion of stream function(2D) to further describe the fluid field and impose important conditions. It is defined by

$$\begin{aligned} u &= \boldsymbol{\psi_y} \\ v &= -\boldsymbol{\psi_x} \end{aligned} \tag{6}$$

then we have the relation between stream function and vorticity as following

$$\Delta\boldsymbol{\psi} = -\boldsymbol{\omega} \tag{7}$$

Notably, this result is achieved in 2D by having $\mathbf{u} = (u, v, 0)$ and $\boldsymbol{\psi} = \psi\hat{\boldsymbol{k}}$ and $\boldsymbol{\omega} = (0, 0, \omega)$. From now, we are all set to see the spectral method in fluid dynamics.

2 Spectral Method

2.1 Settings

According to the previous analysis, we know that the following incompressible Navier-Stokes equation

$$\begin{cases} \nabla \cdot \overrightarrow{u} = 0 \\ \frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = \mu \nabla^2 u \end{cases}$$

is what we need to solve. First, lets simplify the second equation. Expressing the equation as identities in components, we have

$$\begin{cases} u_t + uu_x + vu_y + P_x = \mu(u_{xx} + u_{yy}) \\ v_t + uv_x + vv_y + P_y = \mu(v_{xx} + v_{yy}) \end{cases}$$

To cancel out the term P, we take the derivative of the first equation with respect to y and the second equation with respect to x, we get

$$\begin{cases} (u_y)_t + u_y u_x + uu_{xy} + v_y u_y + vu_{yy} + P_{xy} = \mu(u_{xxy} + u_{yyy}) \\ (v_x)_t + u_x v_x + uv_{xx} + v_x v_y + vv_{xy} + P_{xy} = \mu(v_{xxx} + v_{xyy}) \end{cases}$$

Subtracting the two equation yields the equation of vorticity that we are able to solve:

$$w_t + wu_x + uw_x + v_y w + vw_y = \mu(w_{xx} + w_{yy})$$

$$\Rightarrow \frac{d\omega}{dt} + (u \cdot \nabla)\omega = \mu \triangle \omega \tag{8}$$

The method we use is the pseudo-spectral transform method in Fourier space. First we discretize both space and time. For space discretization we use equidistant square grid in a 2D periodic domain. Then by taking Fourier transform of the vorticity equation that we want to solve, we get

$$\hat{\omega_t} + (\widehat{u \cdot \nabla})\omega = \mu(k_x^2 + k_y^2)\hat{\omega}$$

which yields to

$$\hat{\omega_t} = \mu(k_x^2 + k_y^2)\hat{\omega} - \hat{u_x} * k_x \hat{\omega} - \hat{u_y} * k_y \hat{\omega}$$

where k_x and k_y are wave numbers and the is the convolution. The right hand side of the equation can be solved using the discrete Fourier transform on the grid points. This method can considerably speed up the calculation by using the fast Fourier transform algorithm. So this provides our iteration on time.

$$\begin{tikzcd}
\omega^t \arrow[d, "FFT"'] & (\hat{u}, \hat{v}) \arrow[r, "IFFT"] & (u^t, v^t) \arrow[d, "damping"] \\
\hat{\omega} \arrow[r, "(k_x^2+k_y^2)^{-1}"] \arrow[d] & \hat{\psi} \arrow[u, "{ik_y, -ik_x}"'] & (u^{t+1}, v^{t+1}) \arrow[d, "conv"] \\
\Delta\omega \arrow[rr, "Crank-Nicholson"] & & \omega^{t+1}
\end{tikzcd}$$

2.2 Crank-Nicholson method

For time stepping, we use the Crank-Nicolson method. For linear evolution PDEs, this method is unconditionally stable. Hence, it is also thought to be a good method for some non-linear PDEs. Crank-Nicolson method is an average of Forward Euler and Backward Euler methods, and we can write the method in the explicit form as follows:

$$\begin{aligned}
\frac{\widehat{\omega^{n+1}} - \widehat{\omega^{n}}}{\Delta t} &= \frac{1}{2}(\widehat{\omega_t^{n+1}} + \widehat{\omega_t^{n}}) \\
&= \mu(k_x^2 + k_y^2)\widehat{\omega^{n+1}} - \hat{u_x} * k_x\widehat{\omega^{n+1}} - \hat{u_y} * k_y\hat{\omega} \\
&- \mu(k_x^2 + k_y^2)\widehat{\omega^{n}} - \hat{u_x} * k_x\widehat{\omega^{n}} - \hat{u_y} * k_y\hat{\omega} \\
\widehat{\omega^{n+1}} &= \frac{1}{\frac{1}{\Delta t} - \frac{1}{2}\mu(k_x^2 + k_y^2)}\left(\frac{1}{\Delta t} + \frac{1}{2}\mu\left((k_x^2 + k_y^2)\right)\omega^n - (\widehat{u^n \cdot \nabla})\omega^n\right.
\end{aligned}$$

2.3 Pseudo-spectral de-aliasing

Our numerical solution uses discrete Fourier transform to simplify the computation of derivatives. The computation of the convective term $(\overrightarrow{u} \cdot \nabla)\omega$ introduces aliases, which are extra terms corresponding to out-of-range wavenumbers k ($k \geq N/2$ or $k < -N/2$). For example, the wavenumbers below initially range in $[-N/2, N/2-1]$, but after inverse Fourier, multiplication and then Fourier, the wavenumbers have an extended range.

$$\begin{cases}
u_{i,j} &= \sum_{k_x,k_y=-N/2}^{N/2-1} \hat{u}_{i,j} e^{i(k_x x_i + k_y y_j)} = \sum_{k_x,k_y=-N/2}^{N/2-1} \hat{w}_{k_x,k_y} \frac{ik_y}{k_x^2+k_y^2} e^{i(k_x x_i + k_y y_j)} \\
(w_x)_{i,j} &= \sum_{k_x,k_y=-N/2}^{N/2-1} (\hat{w_x})_{i,j} e^{i(k_x x_i + k_y y_j)} = \sum_{k_x,k_y=-N/2}^{N/2-1} ik_x \hat{w}_{k_x,k_y} e^{i(k_x x_i + k_y y_j)}
\end{cases}$$

$$X_{i,j} := u_{i,j}(w_x)_{i,j}$$

$$\hat{X}_{k_x,k_y} = \frac{1}{N^2}\sum_{i,j=0}^{N-1} X_{i,j} e^{-i(k_x x_i + k_y y_j)}$$

$$\Rightarrow \hat{X}_{k_x,k_y} = \sum_{\substack{a+b=k_x \\ c+d=k_y}} \hat{u}_{a,c}(\hat{w_x})_{b,d} + \sum_{\substack{a+b=k_x \pm N \\ c+d=k_y \pm N}} \hat{u}_{a,c}(\hat{w_x})_{b,d}$$

To de-alias the unwanted terms in the second summation, we apply the 2/3 rule to expand the wavenumbers' range into $[-M/2, M/2-1]$ where $M = 3N/2$, and set the additional terms to zero. For example,

$$u_{i,j} = \sum_{k_x,k_y=-N/2}^{N/2-1} \hat{u}_{i,j}, \text{ where } \hat{u}_{i,j} = 0 \text{ for } i,j \notin [-N/2, N/2-1]$$

Since $\forall k \; \nexists i,j \in [-N/2, N/2-1], i+j = k \pm M$, there will be no extra terms.

2.4 Recover velocity field

Another important problem we meet is that every round of the FFT and inverse FFT will remove the linear terms from velocity field. So when we implement the pseudo-spectral method, we extract these components and add them after each cycle to recover the velocity field.

3 Numerical Scheme

3.1 Initialization of simulation

There are two ways to initialize the simulation. The most convenient way is to specify an initial vorticity distribution ω_0, say a $NX \times NY$ matrix of Gaussian $N(0, 1)$ random variables. Alternatively, we can impose one or several Gaussian distributions on the domain, as individual vortices.

Another initialization which is more intuitive is to start with a velocity field (u_0, v_0). In our experiments, we borrow the velocity fields from two very common scenarios, one is the steady inviscid flow past an arbitrary obstacle, and the other is the velocity shear condition between two adjacent but opposite flows. Having computed the velocity field, we can derive the corresponding vorticity via differentiation in Fourier space, $\hat{\omega_0} = ik_y\hat{u_0} - ik_x\hat{v_0}$.

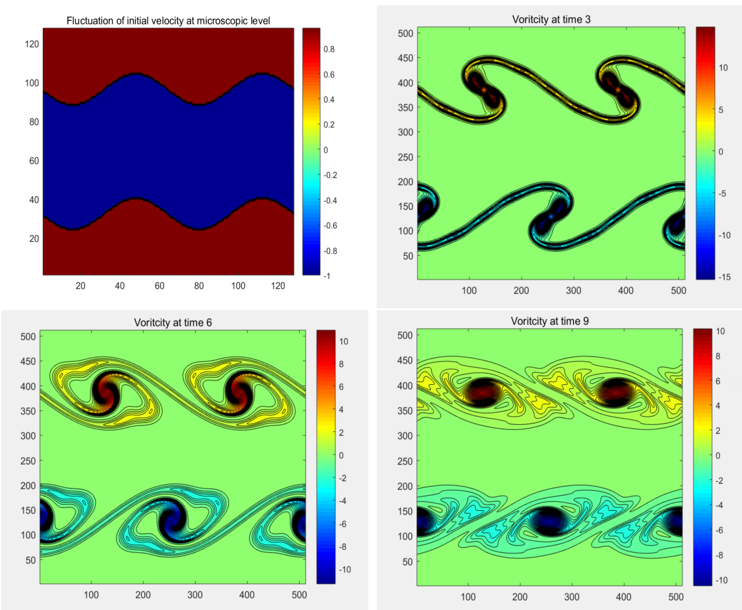

Figure 1: Upper left: the initial velocity distribution. The red region moves right while the blue region moves left. The sinusoidal waves represent the microscopic fluctuations of at the boundary of the two opposite flows. The following three snapshots represent the evolution of the intricate patterns arising from the vortices, for which the Kelvin-Helmholtz instability is well-known.

To obtain the steady inviscid velocity field, we solve the stream function via $\Delta\psi = 0$ which is a simple Laplace equation subject to the boundary condition that $\psi \equiv 0$ inside the domain of the obstacles. First, we set up obstacle matrix D, a $0, 1$-matrix where 1 denotes the location of the obstacle elements. Next, we initialize the stream function matrix ψ_0 as a constant slope tilted toward the bottom, so that there will be a horizontal velocity $-\psi_y = u$ across the domain. Then, we repeatedly average the stream function matrix over adjacent entries (the entries on the boundary are held fixed), to approximate the solution of the Laplace equation. The initial velocity field (u_0, v_0) can then be derived as the curl of stream function matrix (differentiation is carried out by FFT, ik, $IFFT$). Note that since the stream function matrix is tilted, it is not periodic $\psi(x, 1) \neq \psi(x, NY)$), so direct application of FFT will result in highly oscillatory wavenumbers (Fourier coefficients for functions with jump discontinuity are alternating) and consequently highly oscillatory and counter-intuitive velocity. Our solution is to make a mirror copy of ψ's matrix so that the double-matrix becomes symmetric.

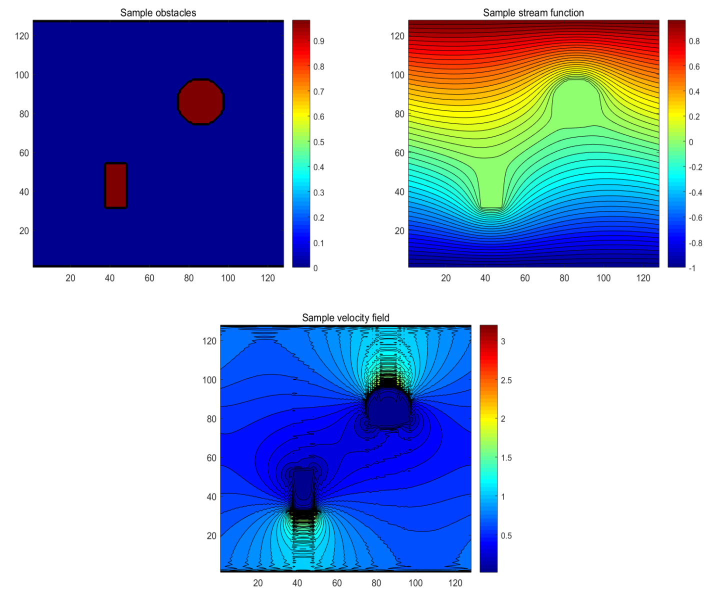

Figure 2: Left: Position of obstacles. Right: Solution of stream function. Bottom: Magnitude of velocity field derived from stream function

3.2 Approximation of viscous flow past obstacle

Given a subdomain D which serves as obstacle for the flow, the goal of simulation is to force the velocity to vanish on the boundary ∂D. It corresponds to the "no-slip" condition of viscous flow, meaning that the flow is dragged to become arbitrarily small when arbitrarily close to the boundary and that there is a thin layer of stagnation near the boundary. Effectively, we can model the obstacle by the condition $\vec{u} \equiv \vec{0}$ or $\psi \equiv$ constant inside D.

3.2.1 Damping method

Our damping method is similar to the classical penalization method, which adds to the Navier-Stokes equation a penalty term $-\lambda \mathbf{1}_D \vec{u}$, symbolizing the amount of non-zero velocity inside the obstacle's domain D. Yet, numerically this method is indirect, involving at each time step, additional computation of velocity and convolution with boundary matrix, which are further complicated by the need of dealiasing. Our testing also indicates that the penalization method, when equipped with non-negligible penalty rate, is prone to instability.

Hence, we propose to perform damping directly on the stream function. The procedure is as follows: At each time step during simulation, compute ψ_t from $\hat{\omega}_t$ and update it by $\psi_t \leftarrow (1 - \lambda \mathbf{1}_D)\psi_t$, where λ is the rate of damping.

3.2.2 Pushing method

The pushing method is similar to damping. First, during the preprocessing step, we compute ψ_0, the stream function of the steady inviscid flow past the given obstacle D. Then, at each time step during simulation, we update ψ_t by $\psi_t \leftarrow (1 - \lambda)\psi_t + \lambda\psi_0$. Intuitively, the stream function is being pushed toward a flow that respects the obstacle.

In both method, when choosing an optimal λ we need to balance between sharpening the obstacle's influence on the flow and preventing the flow to prematurely converge to a steady flow (in the pushing method) or dissipate away (in the damping method).

4 Result

4.1 No Force Field

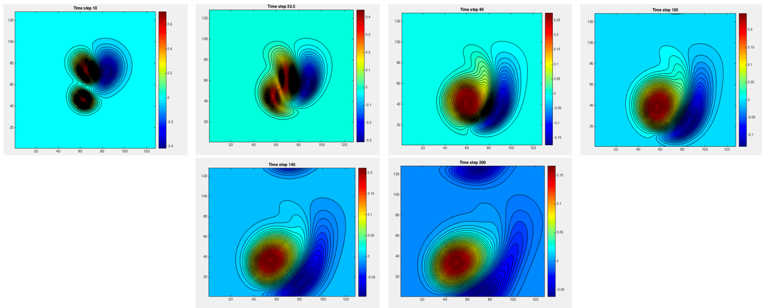

Figure 3: Simulation with No Force Field and Three Vortex

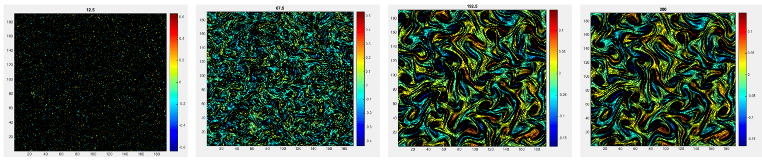

Figure 4: Simulation with No Force Field and Uniform Vortex Distribution

4.2 Force Field-Circular Obstacle

We generate a circular force field in the space to simulate the flow dynamics over an obstacle through pushing and damping method elaborated in the previous section.

4.2.1 Damping Method

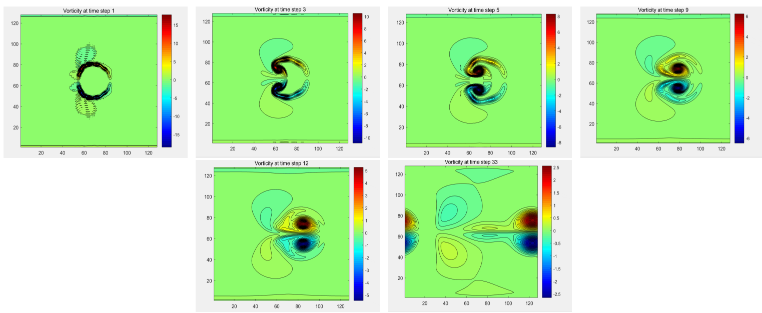

Figure 5: Control Group with Smooth Circular Force Field Using Damping Method

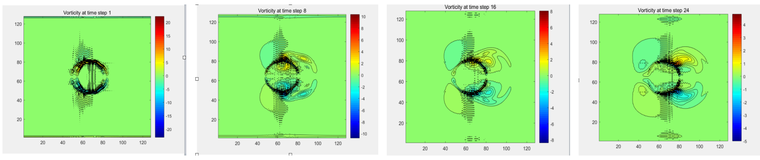

Figure 6: Simulation With Smooth Circular Force Field Using Damping Method

4.2.2 Experiment Finding Using Pushing Method

Implementing the pushing method presented in the previous section, we further simulated the evolution of vorticity under various damping coefficients and different Reynolds numbers. Below are the MATLAB simulation and result.

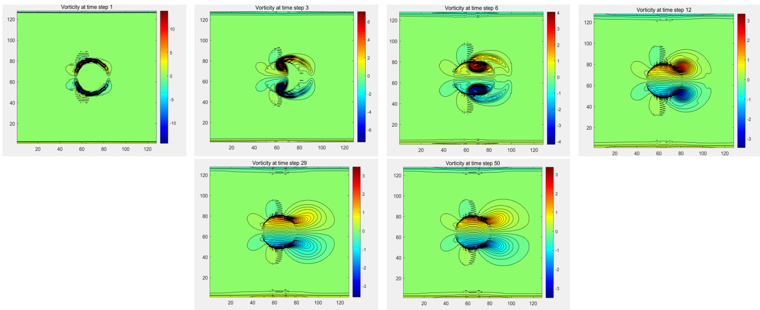

Figure 7: Smooth Force Field with Damping Coefficient $\lambda = 0.0001$

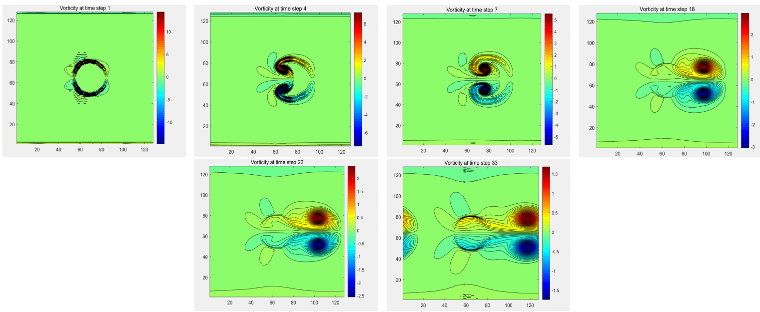

Figure 8: Smooth Force Field with Damping Coefficient $\lambda = 0.00001$

4.2.3 Evolution of Vorticity Under Different Magnitude of Reynolds Number

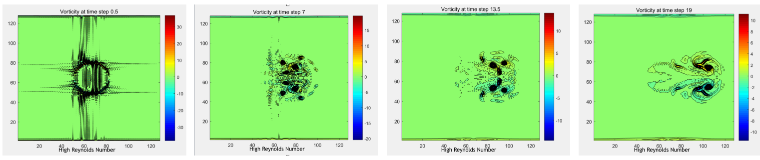

Figure 9: Evolution of Vorticity Under High Reynolds Number: The inertia force has a stronger effect on the stability of flow than viscous force has, thus generating more turbulence in the flow field

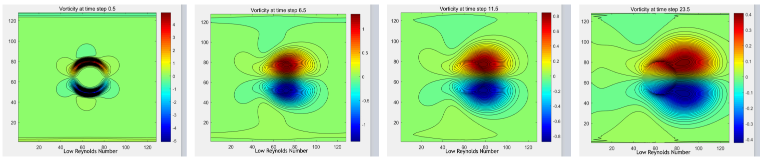

Figure 10: Evolution of Vorticity Under Low Reynolds Number: As viscous force being the leading force in the flow field, evolution of vorticity would decay in a stable way

5 Conclusion

The problem solved here arose from the formation of a local storm in Manhattan. To simplify the problem, our model is set to solve a 2D viscous and incompressible flow through a smooth sphere using spectral method. We solved the Navier-Stokes Equation for spectral method.

The stream function is used to link the vorticity and velocity. For the initial conditions, we used two different methods. First, we used Gaussian random for the distribution of vorticity, with both one vortex, and multiple vortices. Then using Kelvin-Helmholtz instability. We created a shear velocity field as initial condition. To variate the vorticity field, we change the Reynold's number. For time-stepping, Crank-Nicholson method is used here and the numerical solution uses discrete Fourier transformation. However, with every round of the Fast Fourier Transformation, the linear terms get dropped with from the velocity field. So when extracting the components, the linear terms are added back to recover the complete velocity field.

There are two variables for our model: formation of an obstacle and change of vorticity. By variating these two, we create a change of the force field. The variation of vorticity is discussed before. Two methods are used to approximate the obstacle. Damping method with different damping coefficient and pushing method to push toward the solution of steady inviscid flow through the obstacle.

Using the above methods, we successfully simulated the models for different initial conditions as described before. In addition to that, we also simulated the model with no force field and three vortex and simulation with no force field and uniform vortex distribution. Simulations are also made with damping coefficients $\lambda = 0.0001$ and $\lambda = 0.00001$. Simulations are also made to show

how the Reynold's Number affects the vorticity field.

Back to our original problem of the formation of storm, our model is the simplest model with the use of spectral method over 2-D dynamic flows. The spectral method has a broad application on different types of flows with various shapes of obstacles. For the purpose of furthering our study of the formation of storms, we can propose the following research topics too:
i) 2D compressible flow passing through a circular obstacle
- since compressibility affects the wind and the forcefield created due to an obstacle
ii) 2D viscous and incompressible flow passing through two circular obstacles
- since a local storm is more likely to happen between two buildings
iii) 2D viscous and incompressible flow passing through a block obstacle
- circular obstacles are used here for simplification of the problem, while the buildings are more likely to be blocks

6 Bibliography

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