

# Introduction to Statistics: an integrated textbook and workbook using R

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# Introduction

Welcome to statistics!

If you want, you can also download this book as a PDF or EPUB file. Be aware that the print versions are missing some of the richer formatting of the online version. Besides, the recommended way to work through this material is to download the R notebook file (`.Rmd`) at the top of each chapter and work through it in RStudio.

## History and goals

In 2015, a group of interdisciplinary faculty at Westminster College (Salt Lake City, UT) started a process that led to the creation of a new Data Science program. Preparatory to creating a more rigorous introductory statistics course using the statistical software R, I wrote a series of 22 modules that filled a gap in the R training literature. Most R training at the time was focused either on learning to program using R as a computer language, or using R to do sophisticated statistical analysis. We needed our students to use R as a tool for elementary statistical methods and we needed the learning curve to be as gentle as possible. I decided early on that to make the modules more useful, they needed to be structured more like an interactive textbook rather than just a series of lab exercises, and so I spent the summer of 2016 writing a free, open-source, self-contained, and nearly fully-featured introductory statistics textbook. The first sections of the newly-created DATA 220 were offered in Fall, 2016, using the materials I created.

Since then, I have been revising and updating the modules a little every semester. At some point, however, it became clear that some big changes needed to happen:

- The modules were more or less aligned with the OpenIntro book *Introduction to Statistics with Randomization and Simulation* (ISRS) by David Diez, Christopher Barr, and Mine Çetinkaya-Rundel. That book has now been supplanted by *Introduction to Modern Statistics* (IMS) by

Mine Çetinkaya-Rundel and Johanna Hardin, also published through the OpenIntro project.

- The initial materials were written mostly using a mix of base R tools, some **tidyverse** tools, and the amazing resources of the **mosaic** package. I wanted to convert everything to be more aligned with **tidyverse** packages now that they are mature, well-supported, and becoming a *de facto* standard for doing data analysis in R.
- The initial choice of data sets that served as examples and exercises for students was guided by convenience. As I had only a short amount of time to write an entire textbook from scratch, I tended to grab the first data sets I could find that met the conditions needed for the statistical principles I was trying to illustrate. It has become clear in the last few years that the material will be more engaging with more interesting data sets. Ideally, we should use at least some data sets that speak to issues of social justice.
- Making statistics more inclusive requires us to confront some ugly chapters in the development of the subject. Statistical principles are often named after people. (These are supposedly the people who “discovered” the principle, but keep in mind Stigler’s Law of Eponymy which states that no scientific discovery is truly named after its original discoverer. In a neat bit of self-referential irony, Stephen Stigler was not the first person to make this observation.) The beliefs of some of these people were problematic. For example, Francis Galton (famous for the concept of “regression to the mean”), Karl Pearson (of the Pearson correlation coefficient), and Ronald Fisher (famous for many things, including the P-value) were all deeply involved in the eugenics movement of the late 19th and early 20th century. The previous modules almost never referenced this important historical background and context. Additionally, it’s important to discuss ethics, whether that be issues of data provenance, data manipulation, choice of analytic techniques, framing conclusions, and many other topics.

The efforts of my revisions are here online. I’ve tried to address all the concerns mentioned above:

- The chapter are arranged to align somewhat with *Introduction to Modern Statistics* (IMS). There isn’t quite a one-to-one correspondence, but teachers who want to use the chapters of my book to supplement instruction from IMS, or vice versa, should be able to do so pretty easily. One of the appendices of this book [ADD LINK] is a concordance that shows how the books’ chapters match up, along with some notes that explain when one book does more or less than the other.
- The book is now completely aligned with the **tidyverse** and other packages that are designed to integrate into the **tidyverse**. All plotting is done with **ggplot2** and all data manipulation is done with **dplyr** and **forcats**. [OTHERS?] Tables are created using **tabyl** from the **janitor** package. Inference is taught using the cool tools in the **infer** package.

- I have made an effort to find more interesting data sets. It's tremendously difficult to find data that is both fascinating on its merits and also meets the pedagogical requirements of an introductory statistics course. I would like to use even more data that addresses social justice issues. There's some in the book now, and I plan to incorporate even more in the future as I come across data sets that are suitable.
- When statistical tools are introduced, I have tried to give a little historical context about their development if I can. I've also tried to frame every step of the inferential process as a decision-making process that requires not only analytical expertise, but also solid ethical grounding. Again, there's a lot more I could do here, and my goal is to continue to develop more such discussion as I can in future revisions.

Now, instead of a bunch of separate module files, all the material is gathered in one place as chapters of a book. In each chapter (starting with Chapter 2), students can download the chapter as an R notebook file, open it in RStudio, and work through the material.

## Philosophy and pedagogy

To understand my statistics teaching philosophy, it's worth telling you a little about my background in statistics.

At the risk of undermining my own credibility, I'd like to tell you about the first statistics class I took. In the mid-2000s, I was working on my Ph.D. at the University of California, San Diego, studying geometric topology. To make a little extra money and get some teaching experience under my belt, I started teaching night and summer classes at Miramar College, a local community college in the San Diego Community College District. I had been there for several semesters, mostly teaching pre-calculus, calculus, and other lower-division math classes. One day, I got a call from my department chair with my assignment for the upcoming semester. I was scheduled to teach intro stats. I was about to respond, "Oh, I've never taken a stats class before." But remembering this was the way I earned money to be able to live in expensive San Diego County, I said, "Sounds great. By the way, do you happen to have an extra copy of the textbook we'll be using?"

Yes, the first statistics class I took was the one I taught. Not ideal, I know.

I was lucky to start teaching with *Intro Stats* by De Veaux, Velleman, and Bock, a book that was incredibly well-written and included a lot of resources for teachers like me. (I learned quickly that I wasn't the only math professor in the world who got thrown into teaching statistics classes with little to no training.) I got my full-time appointment at Westminster College in 2008 and continued to teach intro stats classes for many years to follow. As I mentioned

earlier, we started the Data Science program at Westminster College in 2016 and moved everything from our earlier hodgepodge of calculators, spreadsheets, and SPSS, over to R.

Eventually, I got interested in Bayesian statistics and read everything I could get my hands on. I became convinced that Bayesian statistics is the “right” way to do statistical analysis. I started teaching special topics courses in Bayesian Data Analysis and working with students on research projects that involved Bayesian methods. **If it were up to me, every introductory statistics class in the world would be taught using Bayesian methods.** I know that sounds like a strong statement. (And I put it in boldface, so it looks even stronger.) But I truly believe that in an alternate universe where Fisher and his disciples didn’t “win” the stats wars of the 20th century (and perhaps one in which computing power got a little more advanced a little earlier in the development of statistics), we would all be Bayesians. Bayesian thinking is far more intuitive and more closely aligned with our intuitions about probabilities and uncertainty.

Unfortunately, our current universe timeline didn’t play out that way. So we are left with frequentism. It’s not that I necessarily object to frequentist tools. All tools are just tools, after all. However, the standard form of frequentist inference, with its null hypothesis significance testing, P-values, and confidence intervals, can be confusing. It’s bad enough that professional researchers struggle with them. We teach undergraduate students in introductory classes.

Okay, so we are stuck not in the world we want, but the world we’ve got. At my institution and most others, intro stats is a service course that trains far more people who are outside the fields of mathematics and statistics. In that world, students will go on to careers where they interact with research that reports p-values and confidence intervals.

So what’s the best we can do for our students, given that limitation? We need to be laser-focused on teaching the frequentist logic of inference the best we can. I want student to see P-values in papers and know how to interpret those P-values correctly. I want students to understand what a confidence intervals tells them—and even more importantly, what it does not tell them. I want students to respect the severe limitations inherent in tests of significance. If we’re going to train frequentists, the least we can do is help them become good frequentists.

One source of inspiration for good statistical pedagogy comes from the Guidelines for Assessment and Instruction in Statistics Education (GAISE), a set of recommendations made by experienced stats educators and endorsed by the American Statistical Association. Their college guidelines are as follows:

1. Teach statistical thinking.
  - Teach statistics as an investigative process of problem-solving and decision-making.
  - Give students experience with multivariable thinking.



2. Focus on conceptual understanding.
3. Integrate real data with a context and purpose.
4. Foster active learning.
5. Use technology to explore concepts and analyze data.
6. Use assessments to improve and evaluate student learning.

In every element of this book, I've tried to follow these guidelines:

1. The first part of the book is an extensive guide for exploratory data analysis. The rest of the book is about inference in the context of specific research questions that are answered using statistical tools. While multi-variable thinking is a little harder to do in a intro stats class, I take the opportunity whenever possible to use graphs to explore more variables than we can handle with intro stats inferential techniques. I point out the the simple analyses taught in this class are only the first step in more comprehensive analyses that incorporate more information and control for confounders. I emphasize that students can continue their statistical growth by enrolling in more advanced stats classes.
2. I often tell students that if they forget everything else from their stats class, the one thing I want them to be able to do is interpret a P-value correctly. It's not intuitive, so it takes an entire semester to set up the idea of a sampling distribution and explain over and over again how the P-value relates to it. In this book, I try to reinforce the logic of inference until the students know it almost instinctively. A huge pedagogical advantage is derived by using randomization and simulation to keep students from getting lost in the clouds of theoretical probability distributions. But they also need to know about the latter too. Every hypothesis test is presented both ways, a task made easy when using the **infer** package.
3. This is the thing I struggle with the most. Finding good data is hard. Over the years, I've found a few data sets I really like, but my goal is to continue to revise the book to incorporate more interesting data, especially data that serves to highlight issues of social justice.
4. Back when I wrote the first set of modules that eventually became this book, the goal was to create assignments that merged content with activities so that students would be engaged in active learning. When these chapters are used in the classroom, students can collaborate with each other and with their professor. They learn by doing.
5. Unlike most books out there, this book does not try to be agnostic about technology. This book is about doing statistics in R.
6. This one I'll leave in the capable hands of the professors who use these materials. The chapter assignments should be completed and submitted, and that is one form of assessment. But I also believe in augmenting this material with other forms of assessment that may include supplemental assignments, open-ended data exploration, quizzes and tests, projects, etc.

## Course structure

As explained above, this book is meant to be a workbook that students complete as they're reading.

At Westminster College, we host RStudio Workbench on a server that is connected to our single sign-on (SSO) systems so that students can access RStudio through a browser using their campus online usernames and passwords. If you have the ability to convince your IT folks to get such a server up and running, it's highly worth it. Rather than spending the first day of class troubleshooting while students try to install software on their machines, you can just have them log in and get started right away. Campus admins install packages and tweak settings to make sure all students have a standardized interface and consistent experience.

If you don't have that luxury, you will need to have students download and install both R and RStudio. The installation processes for both pieces of software are very easy and straightforward for the majority of students. The book chapters here assume that the necessary packages are installed already, so if your students are running R on their own machines, they will need to use `install.packages` at the beginning of some of the chapters for any new packages that are introduced. (They are mentioned at the beginning of each chapter with instructions for installing them.)

Chapter 1 is fully online and introduces R and RStudio very gently using only commands at the Console. By the end of Chapter 1, they will have created a project called `intro_stats` in RStudio that should be used all semester to organize their work. There is a reminder at the beginning of all subsequent chapter to make sure they are in that project before starting to do any work. (Generally, there is no reason they will exit the project, but some students get curious and click on stuff.)

In Chapter 2, students are taught to click a link to download an R Notebook file (`.Rmd`). I have found that students struggle initially to get this file to the right place. If students are using RStudio Workbench online, they will need to use the "Upload" button in the Files tab in RStudio to get the file from their Downloads folder (or wherever they tell their machine to put downloaded files from the internet) into RStudio. If students are using R on their own machines, they will need to move the file from their Downloads folder into their project directory. There are some students who have never had to move files around on their computers, so this is a task that might require some guidance from classmates, TAs, or the professor. The location of the project directory and the downloaded files can vary from one machine to the next. They will have to use something like File Explorer for Windows or the Finder for MacOS, so there isn't a single set of instructions that will get all students' files successfully in the right place. Once the file is in the correct location, students can just click on it to open it in RStudio and start reading. Chapter 2 is all about using R Notebooks: markdown syntax, R code chunks, and inline code.

By Chapter 3, a rhythm is established that students will start to get used to:

- Open the book online and open RStudio.
- Install any packages in RStudio that are new to that chapter. (Not necessary for those using RStudio Workbench in a browser.)
- Check to make sure they're in the `intro_stats` project.
- Click the link online to download the R Notebook file.
- Move the R Notebook file from the Downloads folder to the project directory.
- Open up the R Notebook file.
- Restart R and Run All Chunks.
- Start reading and working.

Chapters 3 and 4 focus on exploratory data analysis for categorical and numerical data, respectively.

Chapter 5 is a primer on data manipulation using `dplyr`.

[FINISH ONCE ALL CHAPTER ARE LAID OUT]

## Onward and upward

I hope you enjoy the textbook. You can provide feedback two ways:

1. The preferred method is to file an issue on the Github page: [https://github.com/VectorPosse/intro\\_stats/issues](https://github.com/VectorPosse/intro_stats/issues)
2. Alternatively, send me an email: [sraleigh@westminstercollege.edu](mailto:sraleigh@westminstercollege.edu)



# Chapter 1

## Introduction to R

**Functions introduced in this chapter:**

```
<-, c, sum, mean, library, ?, ??, View, head, tail, str, NROW, NCOL, summary,  
$
```

### 1.1 Introduction

Welcome to R! This chapter will walk you through everything you need to know to get started using R.

As you go through this chapter (and all future chapters), please read slowly and carefully, and pay attention to detail. Many steps depend on the correct execution of all previous steps, so reading quickly and casually might come back to bite you later.

### 1.2 What is R?

R is a programming language specifically designed for doing statistics. Don't be intimidated by the word "programming" though. The goal of this course is not to make you a computer programmer. To use R to do statistics, you don't need know anything about programming at all. Every chapter throughout the whole course will give you examples of the commands you need to use. All you have to do is use those example commands as templates and make the necessary changes to adapt them to the data you're trying to analyze.

The greatest thing about R is that it is free and open source. This means that you can download it and use it for free, and also that you can inspect and modify

the source code for all R functions. This kind of transparency does not exist in commercial software. The net result is a robust, secure, widely-used language with literally tens of thousands of contributions from R users all over the world.

R has also become a standard tool for statistical analysis, from academia to industry to government. Although some commercial packages are still widely used, many practitioners are switching to R due to its cost (free!) and relative ease of use. After this course, you will be able to list some R experience on your résumé and your future employer will value this. It might even help get you a job!

### 1.3 RStudio

RStudio is an “Integrated Development Environment,” or IDE for short. An IDE is a tool for working with a programming language that is fancier than just a simple text editor. Most IDEs give you shortcuts, menus, debugging facilities, syntax highlighting, and other things to make your life as easy as possible.

Open RStudio so we can explore some of the areas you’ll be using in the future.

On the left side of your screen, you should see a big pane called the “Console”. There will be some startup text there, and below that, you should see a “command prompt”: the symbol “>” followed by a blinking cursor. (If the cursor is not blinking, that means that the focus is in another pane. Click anywhere in the Console and the cursor should start blinking again.)

A command prompt can be one of the more intimidating things about starting to use R. It’s just sitting there waiting for you to do something. Unlike other programs where you run commands from menus, R requires you to know what you need to type to make it work.

We’ll return to the Console in a moment.

Next, look at the upper-right corner of the screen. There are at least three tabs in this pane starting with “Environment”, “History”, and “Connections”. The “Environment” (also called the “Global Environment”) keeps track of things you define while working with R. There’s nothing to see there yet because we haven’t defined anything! The “History” tab will likewise be empty; again, we haven’t done anything yet. We won’t use the “Connections” tab in this course. (Depending on the version of RStudio you are using and its configuration, you may see additional tabs, but we won’t need them for this course.)

Now look at the lower-right corner of the screen. There are likely five tabs here: “Files”, “Plots”, “Packages”, “Help”, and “Viewer”. The “Files” tab will eventually contain the files you upload or create. “Plots” will show you the result of commands that produce graphs and charts. “Packages” will be explained later. “Help” is precisely what it sounds like; this will be a very useful place for

you to get to know. We will never use the “Viewer” tab, so don’t worry about it.

## 1.4 Try something!

So let’s do something in R! Go back to the Console and at the command prompt (the “>” symbol with the blinking cursor), type

```
1+1
```

and hit Enter.

Congratulations! You just ran your first command in R. It’s all downhill from here. R really is nothing more than a glorified calculator.

Okay, let’s do something slightly more sophisticated. It’s important to note that R is case-sensitive, which means that lowercase letters and uppercase letters are treated differently. Type the following, making sure you use a lowercase `c`, and hit Enter:

```
x <- c(1, 3, 4, 7, 9)
```

You have just created a “vector”. When we use the letter `c` and enclose a list of things in parentheses, we tell R to “combine” those elements. So, a vector is just a collection of data. The little arrow `<-` says to take what’s on the right and assign it to the symbol on the left. The vector `x` is now saved in memory. As long as you don’t terminate your current R session, this vector is available to you.

Check out the “Environment” pane now. You should see the vector `x` that you just created, along with some information about it. Next to `x`, it says `num`, which means your vector has numerical data. Then it says `[1:5]` which indicates that there are five elements in the vector `x`.

At the command prompt in the Console, type

```
x
```

and hit Enter. Yup, `x` is there. R knows what it is. You may be wondering about the `[1]` that appears at the beginning of the line. To see what that means, try typing this (and hit Enter—at some point here I’m going to stop reminding you to hit Enter after everything you type):

```
y <- letters
```

R is clever, so the alphabet is built in under the name `letters`.

Type

```
y
```

Now can you see what the `[1]` meant above? Assuming the letters spilled onto more than one line of the Console, you should see a number in brackets at the beginning of each line telling you the numerical position of the first entry in each new line.

Since we've done a few things, check out the "Global Environment" in the upper-right corner. You should see the two objects we've defined thus far, `x` and `y`. Now click on the "History" tab. Here you have all the commands you have run so far. This can be handy if you need to go back and re-run an earlier command, or if you want to modify an earlier command and it's easier to edit it slightly than type it all over again. To get an older command back into the Console, either double-click on it, or select it and click the "To Console" button at the top of the pane.

When we want to re-use an old command, it has usually not been that long since we last used it. In this case, there is an even more handy trick. Click in the Console so that the cursor is blinking at the blank command prompt. Now hit the up arrow on your keyboard. Do it again. Now hit the down arrow once or twice. This is a great way to access the most recently used commands from your command history.

Let's do something with `x`. Type

```
sum(x)
```

I bet you figured out what just happened.

Now try

```
mean(x)
```

What if we wanted to save the mean of those five numbers for use later? We can assign the result to another variable! Type the following and observe the effect in the Environment.

```
m <- mean(x)
```

It makes no difference what letter or combination of letters we use to name our variables. For example,



```
mean_x <- mean(x)
```

just saves the mean to a differently named variable. In general, variable names can be any combination of characters that are letters, numbers, underscore symbols (`_`), and dots (`.`). (In this course, we will prefer underscores over dots.) You cannot use spaces or any other special character in the names of variables.<sup>1</sup> You should avoid variable names that are the same words as predefined R functions; for example, we should not type `mean <- mean(x)`.

## 1.5 Load packages

Packages are collections of commands, functions, and sometimes data that people all over the world write and maintain. These packages extend the capabilities of R and add useful tools. For example, we would like to use the `palmerpenguins` package because it includes an interesting data set on penguins.

If you have installed R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type `install.packages("palmerpenguins")` if you've never used the `palmerpenguins` package before. If you are using RStudio Workbench through a browser, you may not be able to install packages because you may not have admin privileges. If you need a package that is not installed, contact the person who administers your server.

The data set is called `penguins`. Let's see what happens when we try to access this data set without loading the package that contains it. Try typing this:

```
penguins
```

You should have received an error. That makes sense because R doesn't know anything about a data set called `penguins`.

Now—assuming you have the `palmerpenguins` package installed—type this at the command prompt:

```
library(palmerpenguins)
```

It didn't look like anything happened. However, in the background, all the stuff in the `palmerpenguins` package became available to use.

Let's test that claim. Hit the up arrow twice and get back to where you see this at the Console (or you can manually re-type it, but that's no fun!):

---

<sup>1</sup>The official spec says that a valid variable name “consists of letters, numbers and the dot or underline characters and starts with a letter or the dot not followed by a number.”

```
penguins
```

Now R knows about the `penguins` data, so the last command printed some of it to the Console.

Go look at the “Packages” tab in the pane in the lower-right corner of the screen. Scroll down a little until you get to the “P”s. You should be able to find the `palmerpenguins` package. You’ll also notice a check mark by it, indicating that this package is loaded into your current R session.

You must use the `library` command in every new R session in which you want to use a package.<sup>2</sup> If you terminate your R session, R forgets about the package. If you are ever in a situation where you are trying to use a command and you know you’re typing it correctly, but you’re still getting an error, check to see if the package containing that command has been loaded with `library`. (Many R commands are “base R” commands, meaning they come with R and no special package is required to access them. The set of `letters` you used above is one such example.)

## 1.6 Getting help

There are four important ways to get help with R. The first is the obvious “Help” tab in the lower-right pane on your screen. Click on that tab now. In the search bar at the right, type `penguins` and hit Enter. Take a few minutes to read the help file.

Help files are only as good as their authors. Fortunately, most package developers are conscientious enough to write decent help files. But don’t be surprised if the help file doesn’t quite tell you what you want to know. And for highly technical R functions, sometimes the help files are downright inscrutable. Try looking at the help file for the `grep` function. Can you honestly say you have any idea what this command does or how you might use it? Over time, as you become more knowledgeable about how R works, these help files get less mysterious.

The second way of getting help is from the Console. Go to the Console and type

```
?letters
```

The question mark tells R you need help with the R command `letters`. This will bring up the help file in the same Help pane you were looking at before.

---

<sup>2</sup>If you have installed R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you’ll want to know that `install.packages` only has to be run once, the first time you want to install a package. If you’re using RStudio Workbench, you don’t even need to type that because your server admin will have already done it for you.

Sometimes, you don't know exactly what the name of the command is. For example, suppose we misremembered the name and thought it was `letter` instead of `letters`. Try typing this:

```
?letter
```

You should have received an error because there is no command called `letter`. Try this instead:

```
??letter
```

and scroll down a bit in the Help pane. Two question marks tell R not to be too picky about the spelling. This will bring up a whole bunch of possibilities in the Help pane, representing R's best guess as to what you might be searching for. (In this case, it's not easy to find. You'd have to know that the help file for `letters` appeared on a help page called `base::Constants`.)

The fourth way to get help—and often the most useful way—is to use your best friend Google. You don't want to just search for “R”. (That's the downside of using a single letter of the alphabet for the name of a programming language.) However, if you type “R \_\_\_\_\_” where you fill in the blank with the topic of interest, Google usually does a pretty good job sending you to relevant pages. Within the first few hits, in fact, you'll often see an online copy of the same help file you see in R. Frequently, the next few hits lead to StackOverflow where very knowledgeable people post very helpful responses to common questions.

Use Google to find out how to take the square root of a number in R. Test out your newly-discovered function on a few numbers to make sure it works.

## 1.7 Understanding the data

Let's go back to the penguins data contained in the `penguins` data set from the `palmerpenguins` package.

The first thing we do to understand a data set is to read the help file on it. (We've already done this for the `penguins` data.) Of course, this only works for data files that come with R or with a package that can be loaded into R. If you are using R to analyze your own data, presumably you don't need a help file. And if you're analyzing data from another source, you'll have to go to that source to find out about the data.

When you read the help file for `penguins`, you may have noticed that it described the “Format” as being “A tibble with 344 rows and 8 variables.” What is a “tibble”?

The word “tibble” is an R-specific term that describes data organized in a specific way. A more common term is “data frame” (or sometimes “data table”). The idea is that in a data frame, the rows and the columns have very specific interpretations.

Each row of a data frame represents a single object or observation. So in the `penguins` data, each row represents a penguin. If you have survey data, each row will usually represent a single person. But an “object” can be anything about which we collect data. State-level data might have 50 rows and each row represents an entire state.

Each column of a data frame represents a *variable*, which is a property, attribute, or measurement made about the objects in the data. For example, the help file mentions that various pieces of information are recorded about each penguin, like species, bill length, flipper length, body mass, sex, and so on. These are examples of variables. In a survey, for example, the variables will likely be the responses to individual questions.

We will use the terms tibble and data frame interchangeably in this course. They are not quite synonyms: tibbles are R-specific implementations of data frames, the latter being a more general term that applies in all statistical contexts. Nevertheless, there are no situations (at least not encountered in this course) where it makes any difference if a data set is called a tibble or a data frame.

We can also look at the data frame in “spreadsheet” form. Type

```
View(penguins)
```

(Be sure you’re using an upper-case “V” in `View`.) A new pane should open up in the upper-left corner of the screen. In that pane, the `penguins` data appears in a grid format, like a spreadsheet. The observations (individual penguins) are the rows and the variables (attributes and measurements about the penguins) are the columns. This will also let you sort each column by clicking on the arrows next to the variable name across the top.

Sometimes, we just need a little peek at the data. Try this to print just a few rows of data to the Console:

```
head(penguins)
```

We can customize this by specifying the number of rows to print. (Don’t forget about the up arrow trick!)

```
head(penguins, n = 10)
```

The `tail` command does something similar.

```
tail(penguins)
```

When we’re working with HTML documents like this one, it’s usually not necessary to use `View`, `head`, or `tail` because the HTML format will print the data frame a lot more neatly than it did in the Console. You do not need to type the following code; just look below it for the table that appears.

```
penguins
```

```
## # A tibble: 344 x 8
##   species island   bill_length_mm bill_depth_mm flipper_length_mm body_mass_g
##   <fct>   <fct>         <dbl>         <dbl>         <int>         <int>
## 1 Adelie  Torgersen         39.1          18.7          181          3750
## 2 Adelie  Torgersen         39.5          17.4          186          3800
## 3 Adelie  Torgersen         40.3           18          195          3250
## 4 Adelie  Torgersen          NA           NA           NA           NA
## 5 Adelie  Torgersen         36.7          19.3          193          3450
## 6 Adelie  Torgersen         39.3          20.6          190          3650
## 7 Adelie  Torgersen         38.9          17.8          181          3625
## 8 Adelie  Torgersen         39.2          19.6          195          4675
## 9 Adelie  Torgersen         34.1          18.1          193          3475
## 10 Adelie Torgersen         42           20.2          190          4250
## # ... with 334 more rows, and 2 more variables: sex <fct>, year <int>
```

You can scroll through the rows by using the numbers at the bottom or the “Next” button. You can scroll through the variables by clicking the little black arrow pointed to the right in the upper-right corner. The only thing you can’t do here that you can do with `View` is sort the columns.

We want to understand the “structure” of our data. For this, we use the `str` command. Try it:

```
str(penguins)
```

This tells us several important things. First it says that we are looking at a tibble with 344 observations of 8 variables. We can isolate those pieces of information separately as well, if needed:

```
NROW(penguins)
```

```
NCOL(penguins)
```

These give you the number of rows and columns, respectively.

The `str` command also tells us about each of the variables in our data set. We'll talk about these later.

We need to be able to summarize variables in the data set. The `summary` command is one way to do it:

```
summary(penguins)
```

You may not recognize terms like “Median” or “1st Qu.” or “3rd Qu.” yet. Nevertheless, you can see why this summary could come in handy.

## 1.8 Understanding the variables

When we want to look at only one variable at a time, we use the dollar sign to grab it. Try this:

```
penguins$body_mass_g
```

This will list the entire `body_mass_g` column, in other words, the body masses (in grams) of all the penguins in this particular study. If we only want to see the first few, we can use `head` like before.

```
head(penguins$body_mass_g)
```

If we want the structure of the variable `body_mass_g`, we do this:

```
str(penguins$body_mass_g)
```

Notice the letters `int` at the beginning of the line. That stands for “integer” which is another word for whole number. In other words, the penguins' body masses all appear in this data set as whole numbers. There are other data types you'll see in the future:

- **num:** This is for general numerical data (which can be integers as well as having decimal parts).
- **chr:** This means “character”, used for character strings, which can be any sequence of letters or numbers. For example, if the researcher recorded some notes for each penguin, these notes would be recorded in a character variable.
- **factor:** This is for categorical data, which is data that groups observations together into categories. For example, `species` is categorical. These are generally recorded like character strings, but factor variables have more structure because they take on a limited number of possible values corresponding to a generally small number of categories. We'll learn a lot more about factor variables in future chapters.

There are other data types, but the ones above are by far the most common that you'll encounter on a regular basis.

If we want to summarize only the variable `body_mass_g`, we can do this:

```
summary(penguins$body_mass_g)
```

While executing the commands above, you may have noticed entries listed as `NA`. These are “missing” values. It is worth paying attention to missing values and thinking carefully about why they might be missing. For now, just make a mental note that `NA` is the code R uses for data that is missing. (This would be the same as a blank cell in a spreadsheet.)

## 1.9 Projects

Using files in R requires you to be organized. R uses what's called a “working directory” to find the files it needs. Therefore, you can't just put files any old place and expect R to be able to find them.

One way of ensuring that files are all located where R can find them is to organize your work into projects. Look in the far upper-right corner of the RStudio screen. You should see some text that says **Project: (None)**. This means we are not currently in a project. We're going to create a new project in preparation for the next chapter on using R Markdown.

Open the drop-down menu here and select **New Project**. When the dialog box opens, select **New Directory**, then **New Project**.

You'll need to give your project a name. In general, this should be a descriptive name—one that could still remind you in several years what the project was about. The only thing to remember is that project names and file names should not have any spaces in them. In fact, you should avoid other kinds of special characters as well, like commas, number signs, etc. Stick to letters and numerals and you should be just fine. If you want a multi-word project name or file name, I recommend using underscores. R will allow you to name projects with spaces and modern operating systems are set up to handle file names with spaces, but there are certain things that either don't work at all or require awkward workarounds when file names have spaces. In this case, let's type `intro_stats` for the “Directory name”. Leave everything else alone and click **Create Project**.

You will see the screen refresh and R will restart.

You will see a new file called `intro_stats.Rproj` in the Files pane, but **you should never touch that file**. It's just for RStudio to keep track of your project details.

If everything works the way it should, creating a new project will create a new folder, put you in that folder, and automatically make it your working directory.

Any additional files you need for your project should be placed in this directory. In all future chapters, the first thing you will do is download the chapter file from the book website and place it here in your project folder. If you have installed R and RStudio on your own machine, you'll need to navigate your system to find the downloaded file and move or copy it to your project working directory. (This is done most easily using File Explorer in Windows and the Finder in MacOS.) If you are using RStudio Workbench through a web browser, you'll need to upload it to your project folder using the "Upload" button in the Files tab.

## 1.10 Conclusion

It is often said that there is a steep learning curve when learning R. This is true to some extent. R is harder to use at first than other types of software. Nevertheless, in this course, we will work hard to ease you over that first hurdle and get you moving relatively quickly. Don't get frustrated and don't give up! Learning R is worth the effort you put in. Eventually, you'll grow to appreciate the power and flexibility of R for accomplishing a huge variety of statistical tasks.

Onward and upward!



## Chapter 2

# Using R Markdown

2.0

### Functions introduced in this chapter

No R functions are introduced here, but R Markdown syntax is explained.

## 2.1 Introduction

This chapter will teach you how to use R Markdown to create quality documents that incorporate text and R code seamlessly.

First, though, let's make sure you are set up in your project in RStudio.

### 2.1.1 Are you in your project?

If you followed the directions at the end of the last chapter, you should have created a project called `intro_stats`. Let's make sure you're in that project.

**Look at the upper right corner of the RStudio screen. Does it say `intro_stats`? If so, congratulations! You are in your project.**

If you're not in the `intro_stats` project, click on whatever it does say in the upper right corner (probably `Project: (None)`). You can click “Open Project” but it's likely that the `intro_stats` project appears in the drop-down menu in your list of recently accessed projects. So click on the project `intro_stats`.

### 2.1.2 Install new packages

If you are using RStudio Workbench, you do not need to install any packages. (Any packages you need should already be installed by the server administrators.)

If you are using R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type the following commands at the Console:

```
install.packages("rmarkdown")  
install.packages("tidyverse")
```

### 2.1.3 Download the R notebook file

You need to download this chapter as an R Notebook (`.Rmd`) file. Please click the following link to do so:

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/02-using\\_r\\_markdown.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/02-using_r_markdown.Rmd)

The file is now likely sitting in a Downloads folder on your machine (or wherever you have set up for web files to download). If you have installed R and RStudio on your own machine, you will need to move the file from your Downloads folder into the `intro_stats` project directory you created at the end of the last chapter. (Again, if you haven't created the `intro_stats` project, please go back to Chapter 1 and follow the directions for doing that.) Moving files around is most easily done using File Explorer in Windows or the Finder in MacOS. If you are logged into RStudio Workbench instead, go to the Files tab and click the "Upload" button. From there, leave the first box alone ("Target directory"). Click the "Choose File" button and navigate to the folder on your machine containing the file `02_using-r-markdown.Rmd`. Select that file and click "OK" to upload the file. Then you will be able to open the file in RStudio simply by clicking on it.

If you are reading this text online in the browser, be aware that there are several instructions below that won't make any sense because you're not looking at the plain text file with all the code in it. Much of the material in this book can be read and enjoyed online, but the real learning comes from downloading the chapter files (starting with Chapter 2—this one) and working through them in RStudio.

## 2.2 What is R Markdown?

The first question should really be, "What is Markdown?"

Markdown is a way of using plain text with simple characters to indicate formatting choices in a document. For example, in a Markdown file, one can make headers by using number signs (or hashtags as the kids are calling them these days<sup>1</sup>). The notebook file itself is just a plain text file. To see the formatting, the file has to be converted to HTML, which is the format used for web pages. (This process is described below.)

R Markdown is a special version of Markdown that also allows you to include R code alongside the text. Here’s an example of a “code chunk”:

```
1 + 1
```

```
## [1] 2
```

Click the little dark green, right-facing arrow in the upper-right corner of the code chunk. (The icon I’m referring to is next to a faint gear icon and a lighter green icon with a downward-facing arrow.) When you “run” the code chunk like this, R produces output it. We’ll say more about code chunks later in this document.

This document—with text and code chunks together—is called an R Notebook file.

## 2.3 Previewing a document

There is a button in the toolbar right above the text that says “Preview”. Go ahead and push it. See what happens.

Once the pretty output is generated, take a few moments to look back and forth between it and the original R Notebook text file (the plain text in RStudio). You can see some tricks that we won’t need much (embedding web links, making lists, etc.) and some tricks that we will use in every chapter (like R code chunks).

At first, you’ll want to work back and forth between the R Notebook file and the HTML file to get used to how the formatting in the plain text file get translated to output in the HTML file. After a while, you will look at the HTML file less often and work mostly in the R Notebook file, only previewing when you are finished and ready to produce your final draft.

## 2.4 Literate programming

R Markdown is one way to implement a “literate programming” paradigm. The concept of literate programming was famously described by Donald Knuth, an

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<sup>1</sup>Also called “pound signs” or “octothorpes”. This is also an example of formatting a footnote!

eminent computer scientist. The idea is that computer programs should not appear in a sterile file that's full of hard-to-read, abstruse lines of computer code. Instead, functional computer code should appear interspersed with writing that explains the code.

## 2.5 Reproducible research

One huge benefit of organizing your work into R Notebooks is that it makes your work *reproducible*. This means that anyone with access to your data and your R Notebook file should be able to re-create the exact same analysis you did.

This is a far cry from what generally happens in research. For example, if I do all my work in Microsoft Excel, I make a series of choices in how I format and analyze my data and all those choices take the form of menu commands that I point and click with my mouse. There is no record of the exact sequence of clicks that took me from point A to B all the way to Z. All I have to show for my work is the “clean” spreadsheet and anything I’ve written down or communicated about my results. If there were any errors along the way, they would be very hard to track down.<sup>2</sup>

Reproducibility should be a minimum prerequisite for all statistical analysis. Sadly, that is not the case in most of the research world. We are training you to be better.

## 2.6 Structure of an R Notebook

Let’s start from the top. Look at the very beginning of the plain R Notebook file. (If you’re in RStudio, you are looking at the R Notebook file. If you are looking at the pretty HTML file, you’ll need to go back to RStudio.) The section at the very top of the file that starts and ends with three hyphens is called the YAML header. (Google it if you really care why.) The title of the document appears already, but you’ll need to substitute your name and today’s date in the obvious places. **Scroll up and do that now.**

You’ve made changes to the document, so you’ll need to push the “Preview” button again. Once that’s done, look at the resulting HTML document. The YAML header has been converted into a nicely formatted document header with the new information you’ve provided.

Next, there is some weird looking code with instructions not to touch it. I recommend heeding that advice. This code will allow you to answer questions

---

<sup>2</sup>If you think these errors are trivial, Google “Reinhart and Rogoff Excel error” to read about the catastrophic consequences of seemingly trivial Excel mistakes.

and have your responses appear in pretty blue boxes. In the body of the chapter, such answer boxes will be marked with tags `::: {.answer}` and `:::`. Let's try it:

Replace this text here with something else. Then preview the document and see how it appears in the HTML file.

**Be careful not to delete the two lines starting with the three colons (:::) that surround your text! If you mess this up, the rest of the document's formatting will get screwed up.**

To be clear, the colorful answer boxes are not part of the standard R Markdown tool set. That's why we had to define them manually near the top of the file. Note that the weird code itself does not show up in the HTML file. It works in the background to define the blue boxes that show up in the HTML file.

We also have section headers throughout, which in the R Notebook file look like:

## Section header

The hashtags are Markdown code for formatting headers. Additional hashtags will create subsections:

### Not quite as big

We could actually use a single number sign, but `#` makes a header as big as the title, which is too big. Therefore, we will prefer `##` for section headers and `###` for subsections.

**You do need to make sure that there is a blank line before and after each section header.** To see why, look at the HTML document at this spot: `##` Is this a new section? Do you see the problem?

Put a blank line before and after the line above that says "Is this a new section?" Preview one more time and make sure that the line now shows up as a proper section header.

## 2.7 Other formatting tricks

You can make text *italic* or **bold** by using asterisks. (Don't forget to look at the HTML to see the result.)

You can make bullet-point lists. These can be made with hyphens, but you'll need to start after a blank line, then put the hyphens at the beginning of each new line, followed by a space, as follows:

- First item
- Second item

If you want sub-items, indent at least two spaces and use a minus sign followed by a space.

- Item
  - Sub-item
  - Sub-item
- Item
- Item

Or you can make ordered lists. Just use numbers and R Markdown will do all the work for you. Sub-items work the same way as above. (Again, make sure you're starting after a blank line and that there is a space after the periods and hyphens.)

1. First Item
  - Sub-item
  - Sub-item
2. Second Item
3. Third Item

We can make horizontal rules. There are lots of ways of doing this, but I prefer a bunch of asterisks in a row.

---

There are many more formatting tricks available. For a good resource on all R Markdown stuff, click on this link for a “cheat sheet”. And note in the previous sentence the syntax for including hyperlinks in your document.<sup>3</sup>

## 2.8 R code chunks

The most powerful feature of R Markdown is the ability to do data analysis right inside the document. This is accomplished by including R code chunks.

---

<sup>3</sup>You can also access cheat sheets through the main Help menu in RStudio.

An R code chunk doesn't just show you the R code in your output file; it also runs that code and generates output that appears right below the code chunk.

An R code chunk starts with three “backticks” followed by the letter `r` enclosed in braces, and it ends with three more backticks. (The backtick is usually in the upper-left corner of your keyboard, next to the number 1 and sharing a key with the tilde ~.)

In RStudio, click the little dark green, right-facing arrow in the upper-right corner of the code chunk below, just as you did earlier.

```
# Here's some sample R code
test <- c(1, 2, 3, 4)
sum(test)
```

```
## [1] 10
```

After pushing the dark green arrow, you should notice that the output of the R code appeared like magic. If you preview the HTML output, you should see the same output appear. If you hover your mouse over the dark green arrow, you should see the words “Run Current Chunk”. We'll call this the Run button for short.

**We need to address something here that always confuses people new to R and R Markdown.** A number sign (aka “hashtag”) in an R Notebook gives us headers for sections and subsections. In R, however, a number sign indicates a “comment” line. In the R code above, the line `# Here's some sample R code` is not executed as R code. But you can clearly see that the two lines following were executed as R code. So be careful! Number signs inside and outside R code chunks behave very differently.

Typically, the first code chunk that appears in our document will load any packages we need. We will be using a package called **tidyverse** (which is really a collection of lots of different packages) throughout the course. We load it now. Click on the Run button (the dark green, right-facing arrow) in the code chunk below.

```
library(tidyverse)
```

```
## -- Attaching packages ----- tidyverse 1.3.1 --
```

```
## v ggplot2 3.3.6    v purrr   0.3.4
## v tibble  3.1.7    v dplyr   1.0.9
## v tidyr   1.2.0    v stringr 1.4.0
## v readr   2.1.2    v forcats 0.5.1
```

```
## -- Conflicts ----- tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

The output here consists of a bunch of information generated when trying to load the package. These are not errors, even though one section is labeled “Conflicts”. Usually, errors appear with the word “Error”, so it’s typically clear when something just didn’t work. Also note that once you’ve loaded a package, you don’t need to load it again until you restart your R session. For example, if you go back and try to run the code chunk above one more time, the output will disappear. That’s because `tidyverse` is already loaded, so the second “run” doesn’t actually generate output anymore.

Okay, let’s do something interesting now. We’ll revisit the `penguins` data set we introduced in the previous chapter. Remember, though, that this data set also lives in a package that needs to be loaded. Run the code chunk below to load the `palmerpenguins` package:

```
library(palmerpenguins)
```

Let’s see what happens when we try to run multiple commands in one code chunk:

```
head(penguins)
```

```
## # A tibble: 6 x 8
##   species island bill_length_mm bill_depth_mm flipper_length_~ body_mass_g sex
##   <fct>   <fct>         <dbl>         <dbl>         <int>         <int> <fct>
## 1 Adelie  Torge~           39.1           18.7           181           3750 male
## 2 Adelie  Torge~           39.5           17.4           186           3800 fema~
## 3 Adelie  Torge~           40.3           18            195           3250 fema~
## 4 Adelie  Torge~           NA            NA            NA            NA <NA>
## 5 Adelie  Torge~           36.7           19.3           193           3450 fema~
## 6 Adelie  Torge~           39.3           20.6           190           3650 male
## # ... with 1 more variable: year <int>
```

```
tail(penguins)
```

```
## # A tibble: 6 x 8
##   species island bill_length_mm bill_depth_mm flipper_length_~ body_mass_g sex
##   <fct>   <fct>         <dbl>         <dbl>         <int>         <int> <fct>
## 1 Chinst~ Dream           45.7           17            195           3650 fema~
## 2 Chinst~ Dream           55.8           19.8           207           4000 male
## 3 Chinst~ Dream           43.5           18.1           202           3400 fema~
```



```
## 4 Chinst~ Dream          49.6          18.2          193          3775 male
## 5 Chinst~ Dream          50.8           19          210          4100 male
## 6 Chinst~ Dream          50.2          18.7          198          3775 fema~
## # ... with 1 more variable: year <int>
```

```
str(penguins)
```

```
## tibble [344 x 8] (S3: tbl_df/tbl/data.frame)
## $ species      : Factor w/ 3 levels "Adelie","Chinstrap",...: 1 1 1 1 1 1 1 1 1 1 ...
## $ island       : Factor w/ 3 levels "Biscoe","Dream",...: 3 3 3 3 3 3 3 3 3 3 ...
## $ bill_length_mm : num [1:344] 39.1 39.5 40.3 NA 36.7 39.3 38.9 39.2 34.1 42 ...
## $ bill_depth_mm : num [1:344] 18.7 17.4 18 NA 19.3 20.6 17.8 19.6 18.1 20.2 ...
## $ flipper_length_mm: int [1:344] 181 186 195 NA 193 190 181 195 193 190 ...
## $ body_mass_g    : int [1:344] 3750 3800 3250 NA 3450 3650 3625 4675 3475 4250 ...
## $ sex           : Factor w/ 2 levels "female","male": 2 1 1 NA 1 2 1 2 NA NA ...
## $ year          : int [1:344] 2007 2007 2007 2007 2007 2007 2007 2007 2007 2007 ...
```

If you're looking at this in RStudio, it's a bit of a mess. RStudio did its best to give you what you asked for, but there are three separate commands here. The first two (`head` and `tail`) print some of the data, so the first two boxes of output are tables showing you the head and the tail of the data. The next one (`str`) normally just prints some information to the Console. So RStudio gave you an R Console box with the output of this command.

If you look at the HTML file, you can see the situation isn't as bad. Each command and its corresponding output appear nicely separated there.

Nevertheless, it will be good practice and a good habit to get into to put multiple output-generating commands in their own R code chunks. Run the following code chunks and compare the output to the mess you saw above:

```
head(penguins)
```

```
## # A tibble: 6 x 8
##   species island bill_length_mm bill_depth_mm flipper_length~ body_mass_g sex
##   <fct>   <fct>         <dbl>         <dbl>         <int>         <int> <fct>
## 1 Adelie Torge~         39.1           18.7           181           3750 male
## 2 Adelie Torge~         39.5           17.4           186           3800 fema~
## 3 Adelie Torge~         40.3            18           195           3250 fema~
## 4 Adelie Torge~          NA            NA            NA            NA <NA>
## 5 Adelie Torge~         36.7           19.3           193           3450 fema~
## 6 Adelie Torge~         39.3           20.6           190           3650 male
## # ... with 1 more variable: year <int>
```

```
tail(penguins)
```

```
## # A tibble: 6 x 8
##   species island bill_length_mm bill_depth_mm flipper_length_~ body_mass_g sex
##   <fct>   <fct>         <dbl>         <dbl>         <int>         <int> <fct>
## 1 Chinstrap Dream         45.7           17           195          3650 fema~
## 2 Chinstrap Dream         55.8           19.8         207          4000 male
## 3 Chinstrap Dream         43.5           18.1         202          3400 fema~
## 4 Chinstrap Dream         49.6           18.2         193          3775 male
## 5 Chinstrap Dream         50.8           19           210          4100 male
## 6 Chinstrap Dream         50.2           18.7         198          3775 fema~
## # ... with 1 more variable: year <int>
```

```
str(penguins)
```

```
## tibble [344 x 8] (S3: tbl_df/tbl/data.frame)
## $ species      : Factor w/ 3 levels "Adelie","Chinstrap",...: 1 1 1 1 1 1 1 1 1 1
## $ island       : Factor w/ 3 levels "Biscoe","Dream",...: 3 3 3 3 3 3 3 3 3 3
## $ bill_length_mm : num [1:344] 39.1 39.5 40.3 NA 36.7 39.3 38.9 39.2 34.1 42 ...
## $ bill_depth_mm : num [1:344] 18.7 17.4 18 NA 19.3 20.6 17.8 19.6 18.1 20.2 ...
## $ flipper_length_mm: int [1:344] 181 186 195 NA 193 190 181 195 193 190 ...
## $ body_mass_g    : int [1:344] 3750 3800 3250 NA 3450 3650 3625 4675 3475 4250 .
## $ sex           : Factor w/ 2 levels "female","male": 2 1 1 NA 1 2 1 2 NA NA ..
## $ year          : int [1:344] 2007 2007 2007 2007 2007 2007 2007 2007 2007 2007
```

This won't look any different in the HTML file, but it sure looks a lot cleaner in RStudio.

What about the two lines of the first code chunk we ran above?

```
test <- c(1, 2, 3, 4)
sum(test)
```

```
## [1] 10
```

Should these two lines be separated into two code chunks? If you run it, you'll see only one piece of output. That's because the line `test <- c(1, 2, 3, 4)` works invisibly in the background. The vector `test` gets assigned, but no output is produced. Try it and see (push the Run button):

```
test <- c(1, 2, 3, 4)
```

So while there's no harm in separating these lines and putting them in their own chunks, it's not strictly necessary. You really only need to separate lines when they produce output. (And even then, if you forget, RStudio will kindly give you multiple boxes of output.)

Suppose we define a new variable called `test2` in a code chunk. FOR PURPOSES OF THIS EXERCISE, DO NOT HIT THE RUN BUTTON YET! But do go look at the HTML file.

```
test2 <- c("a", "b", "c")
test2
```

```
## [1] "a" "b" "c"
```

The first line defines `test2` invisibly. The second line asks R to print the value of `test2`, but in the HTML file we see no output. That's because we have not run the code chunk yet. DON'T HIT THE RUN BUTTON YET!

Okay, now go to the Console in RStudio (in the lower left corner of the screen). Try typing `test2`. You should get an “Error: object ‘test2’ not found.”

Why does this happen? The Global Environment doesn't know about it yet. Look in the upper right corner of the screen, under the “Environment” tab. You should see `test`, but not `test2`.

Okay, NOW GO BACK AND CLICK THE RUN BUTTON IN THE LAST CHUNK ABOVE. The output appears in RStudio below the code chunk and the Global Environment has been updated.

The take home message is this:

### **Be sure to run all your code chunks in RStudio!**

In RStudio, look in the toolbar above this document, toward the right. You should see the word “Run” with a little drop-down menu next to it. Click on that drop-down menu and select “Run All”. Do you see what happened? All the code chunks ran again, and that means that anything in the Global Environment will now be updated to reflect the definitions made in the R Notebook.

It's a good idea to “Run All” when you first open a new R Notebook. This will ensure that all your code chunks have their output below them (meaning you don't have to go through and click the Run button manually for each chunk, one at a time) and the Global Environment will accurately reflect the variables you are using.

You can “Run All” from time to time, but it's easier just to “Run All” once at the beginning, and then Run individual R code chunks manually as you create them.

Now go back to the Environment tab and find the icon with the little broom on it. Click it. You will get a popup warning you that you about to “remove

all objects from the environment”. Click “Yes”. Now the Global Environment is empty. Go back to the “Run” menu and select “Run All”. All the objects you defined in the R Notebook file are back.

Clearing out your environment can be useful from time to time. Maybe you’ve been working on a chapter for a while and you’ve tried a bunch of stuff that didn’t work, or you went back and changed a bunch of code. Eventually, all that junk accumulates in your Global Environment and it can mess up your R Notebook. For example, let’s define a variable called `my_variable`.

```
my_variable <- 42
```

Then, let’s do some calculation with `my_variable`.

```
my_variable * 2
```

```
## [1] 84
```

Perhaps later you decide you don’t really need `my_variable`. Put a hashtag in front of the code `my_variable <- 42` to comment it out so that it will no longer run, but don’t touch the next code chunk where you multiply it by 2. Now try running the code chunk with `my_variable * 2` again. Note that `my_variable` is still sitting in your Global Environment, so you don’t get any error messages. R can still see and access `my_variable`.

Now go to the “Run” menu and select “Restart R and Run All Chunks”. This clears the Global Environment and runs all the R code starting from the top of the R Notebook. This time you will get an error message: `object 'my_variable' not found`. You’ve tried to calculate with a variable called `my_variable` that doesn’t exist anymore. (The line in which it was defined has been commented out.)

It’s best to make sure all your code chunks will run when loaded from a clean R session. The “Restart R and Run All Chunks” option is an easy way to both clear your environment and re-run all code chunks. You can do this as often as you want, but you will definitely want to do this one last time when you are done. **At the end of the chapter, when you are ready to prepare the final draft, please select “Restart R and Run All Chunks”. Make sure everything still works!**

To get rid of the error above, uncomment the line `my_variable <- 42` by removing the hashtag you added earlier.

## 2.9 Inline R commands

You don’t need a standalone R code chunk to do computations. One neat feature is the ability to use R to calculate things right in the middle of your text.

Here's an example. Suppose we wanted to compute the mean body mass (in grams) for the penguins in the `penguins` data set. We could do this:

```
mean(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 4201.754
```

(The `na.rm = TRUE` part is necessary because two of the penguins are missing body mass data. More on missing data in future chapters.)

But we can also do this inline by using backticks and putting the letter `r` inside the first backtick. Go to the HTML document to see how the following sentence appears:

The mean body mass for penguins in the `penguins` data set is 4201.754386 grams.

You can (and should) check to make sure your inline R code is working by checking the HTML output, but you don't necessarily need to go to the HTML file to find out. In RStudio, click so that the cursor is somewhere in the middle of the inline code chunk in the paragraph above. Now type Ctrl-Enter or Cmd-Enter (PC or Mac respectively). A little box should pop up that shows you the answer!

Notice that in addition to the inline R command that calculated the mean, I also enclosed `penguins` in backticks to make it stand out in the output. I'll continue to do that for all computer commands and R functions. But to be clear, putting a word in backticks is just a formatting trick. If you want inline R code, you also need the letter `r` followed by a space inside the backticks.

## 2.10 Copying and pasting

In future chapters, you will be shown how to run statistical analyses using R. Each chapter will give extensive explanations of the statistical concepts and demonstrations of the necessary R code. Afterwards, there will be one or more exercises that ask you to apply your new-found knowledge to run similar analyses on your own with different data.

The idea is that you should be able to copy and paste the R code from the previously worked examples. **But you must be thoughtful about how you do this.** The code cannot just be copied and pasted blindly. It must be modified so that it applies to the exercises with new data. This requires that you understand what the code is doing. You cannot effectively modify the code if you don't know which parts to modify.

There will also be exercises in which you are asked to provide your own explanations and interpretations of your analyses. These should **not** be copied

and pasted from any previous work. These exercises are designed to help you understand the statistical concepts, so they must be in your own words, using your own understanding.

In order to be successful in these chapters, you must do the following:

**1. Read every part of the chapter carefully!**

- It will be tempting to skim over the paragraphs quickly and just jump from code chunk to code chunk. This will be highly detrimental to your ability to gain the necessary understanding—not just to complete the chapter, but to succeed in statistics overall.

**2. Copy and paste thoughtfully!**

- Not every piece of code from the early part of the chapter will necessarily apply to the later exercises. And the code that does apply will need to be modified (sometimes quite heavily) to be able to run new analyses. Your job is to understand how the code works so that you can make changes to it without breaking things. If you don't understand a piece of code, don't copy and paste it until you've read and re-read the earlier exposition that explains how the code works.

One final note about copying and pasting. Sometimes, people will try to copy and paste code from the HTML output file. This is a bad idea. The HTML document uses special characters to make the output look pretty, but these characters don't actually work as plain text in an R Notebook. The same applies to things copied and pasted from a Word document or another website. If you need to copy and paste code, be sure to find the plain text R Notebook file (the one with the .Rmd extension here in RStudio) and copy and paste from that.

## 2.11 Conclusion

That's it! There wasn't too much you were asked to do for this assignment that will actually show up in the HTML output. (Make sure you did do the three things that were asked of you however: one was adding your name and the date to the YAML header, one was typing something in the blue answer box, and the last was to make a section header appear properly.) As you gain confidence and as we move into more serious stats material, you will be asked to do a lot more.

### 2.11.1 Preparing and submitting your assignment

If you look in your project folder, you should see three files:

```
intro_stats.Rproj  
02-using_r_markdown.Rmd  
02-using_r_markdown.nb.html
```

The first file (with extension `.Rproj`) you were instructed never to touch.

The next file (with extension `.Rmd`) is your R Notebook file. It's the file you're looking at right now. It is really nothing more than a plain text file, although when you open it in RStudio, some magic allows you to see the output from the code chunks you run.

Finally, you have a file with extension `.nb.html`. That is the pretty output file generated when you hit the “Preview” button. (If you happen to see other files in your project folder, you should ignore those and not mess with them.) This is the “final product” of your work.

There are several steps that you should follow at the end of each of every chapter.

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.





## Chapter 3

# Categorical data

2.0

### Functions introduced in this chapter

`glimpse`, `table`, `tabyl`, `adorn_pct_formatting`, `ggplot`, `geom_bar`,  
`adorn_percentages`, `mutate`, `as_factor`, `labs`, `tibble`, `geom_col`

### 3.1 Introduction

In this chapter, we'll learn about categorical data and how to summarize it using tables and graphs.

#### 3.1.1 Install new packages

If you are using RStudio Workbench, you do not need to install any packages. (Any packages you need should already be installed by the server administrators.)

If you are using R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type the following command at the Console:

```
install.packages("janitor")
```

### 3.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/03-categorical\\_data.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/03-categorical_data.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 3.1.3 Restart R and run all chunks

In RStudio, in the toolbar above this document, find the “Run” drop-down menu and select “Restart R and Run All Chunks.”

This does two important things:

1. R will restart. This will clear out the Global Environment and provide a fresh session for this new assignment. None of the clutter from previous chapters will be there to mess up your work in this chapter.
2. All the code chunks in this document will run so that you can see the output as you scroll past it. This saves you some effort in having to click the little green “Run” button in each code chunk as you come across it. (Also, if you forget to run one, that could cause errors later on, so this way, all the variables you need will be in the Global Environment for when they're needed later.) You will still need to click the green arrow for new code chunks that you create, of course.

At the end of the assignment, you will “Restart R and Run All Chunks” once again to make sure that everything works smoothly and there are no lingering errors.

### 3.1.4 Load packages

We load the `tidyverse` package since it also loads the `ggplot2` package that we'll use throughout the course to make graphs. It also loads several other packages, for example, one called `dplyr` to give us a command called `mutate`, and another called `forcats` to give us `as_factor`. (These will all be explained later.) The `janitor` package gives us the `tabyl` command for creating nice tables. Finally, We load the `palmerpenguins` package to work with the penguin data.

```
library(tidyverse)
library(janitor)

##
## Attaching package: 'janitor'

## The following objects are masked from 'package:stats':
##
##      chisq.test, fisher.test

library(palmerpenguins)
```

## 3.2 Categorical data

Data comes in different types depending on what is being measured. When people think of “data”, they often imagine *numerical data*, consisting of numbers. But there are other kinds of data as well.

In this chapter, we focus on *categorical data* that groups observations into categories.

For example, if we record the species of a penguin, that is not a number. It’s a word that classifies that penguin into one of a finite number of types. Whenever you see words in a data set, there’s a good chance that you’re looking at categorical data.

Even “numbers” can sometimes represent categorical data. For example, suppose in a survey there is a Yes/No question. Instead of seeing the words “Yes” or “No”, though, you might see a data set with ones and zeros, where 1 = Yes and 0 = No. The presence of numbers does not automatically make that data numerical. In fact, the data is categorical. Yes and No are categories that sort the survey respondents into two groups based on their responses to a certain question.

What about ZIP codes? They are recorded as numbers, and unlike the Yes/No example above, those numbers aren’t just substitutes for words. Nevertheless, ZIP codes are categorical. They sort addresses into a finite number of groups based on geographic proximity.

Another way to think of it is this: can the numerical values of ZIP codes be treated as numbers in any meaningful way? Can you take a sum or an average of ZIP codes? Sure, technically a computer can add up or average a set of ZIP codes, but would the result be a meaningful number? Since the answer is “no” we cannot think of ZIP codes as numbers, even though they are recorded that way.

**Exercise 1** Think of another type of data that would be recorded using numbers but should be thought of as categorical data.

Please write up your answer here.

### 3.3 Factor variables

R uses the term “factor variable” to refer to a categorical variable. Look at the structure of the `penguins` data below.

```
str(penguins)
```

```
## tibble [344 x 8] (S3: tbl_df/tbl/data.frame)
## $ species      : Factor w/ 3 levels "Adelie","Chinstrap",...: 1 1 1 1 1 1 1 1 1 1
## $ island       : Factor w/ 3 levels "Biscoe","Dream",...: 3 3 3 3 3 3 3 3 3 3
## $ bill_length_mm : num [1:344] 39.1 39.5 40.3 NA 36.7 39.3 38.9 39.2 34.1 42 ...
## $ bill_depth_mm : num [1:344] 18.7 17.4 18 NA 19.3 20.6 17.8 19.6 18.1 20.2 ...
## $ flipper_length_mm: int [1:344] 181 186 195 NA 193 190 181 195 193 190 ...
## $ body_mass_g    : int [1:344] 3750 3800 3250 NA 3450 3650 3625 4675 3475 4250 .
## $ sex           : Factor w/ 2 levels "female","male": 2 1 1 NA 1 2 1 2 NA NA ..
## $ year          : int [1:344] 2007 2007 2007 2007 2007 2007 2007 2007 2007 2007
```

The categorical variables `species`, `island`, and `sex` are coded correctly as factor variables.

The `tidyverse` package offers a function called `glimpse` that effectively does the same thing as `str`. We’ll use `glimpse` throughout the rest of the course.

```
glimpse(penguins)
```

```
## Rows: 344
## Columns: 8
## $ species      <fct> Adelie, Adelie, Adelie, Adelie, Adelie, Adelie, Adel~
## $ island       <fct> Torgersen, Torgersen, Torgersen, Torgersen, Torgersen~
## $ bill_length_mm <dbl> 39.1, 39.5, 40.3, NA, 36.7, 39.3, 38.9, 39.2, 34.1, ~
## $ bill_depth_mm <dbl> 18.7, 17.4, 18.0, NA, 19.3, 20.6, 17.8, 19.6, 18.1, ~
## $ flipper_length_mm <int> 181, 186, 195, NA, 193, 190, 181, 195, 193, 190, 186~
## $ body_mass_g   <int> 3750, 3800, 3250, NA, 3450, 3650, 3625, 4675, 3475, ~
## $ sex          <fct> male, female, female, NA, female, male, female, male~
## $ year         <int> 2007, 2007, 2007, 2007, 2007, 2007, 2007, 2007, 2007~
```

**Exercise 2** Look at the output of `str` versus `glimpse` above. Write down any advantages or disadvantages you see using one versus the other. (You may also want to check the help file for the two commands to see if they offer any clues as to why you might use one over the other.)

Please write up your answer here.

---

Your data set may already come with its variables coded correctly as factor variables, but often they are not. As described above, numbers are often used to represent categories, so R may think that those variables represent numerical data. Later, we'll see an example of this and learn how to handle categorical variables that are not coded as factor variables in R.

## 3.4 Summarizing one categorical variable

If you need to summarize a single categorical variable, a *frequency table* usually suffices. This is simply a table that counts up all the instances of each category. The word “frequency” is synonymous here with the word “count”.

We can use the `table` command:

```
table(penguins$species)
```

```
##
##      Adelie Chinstrap   Gentoo
##      152         68      124
```

Recall that the dollar sign means to grab the variable `species` from the tibble `penguins`.

You can also generate a *relative frequency table* which is a table that uses proportions or percentages instead of counts.

**NOTE:** For purposes of this course, we're going to be very careful about the terms *proportion* and *percentage*. For us, a proportion will always be a number between 0 and 1 whereas a percentage will be between 0 and 100. Calculating a percentage is the same as multiplying a proportion by 100.

The `table` command stops being convenient if you want proportions instead of counts. Instead, we will use the `tabyl` command from the `janitor` package that was loaded near the top of the chapter. The syntax for this command is a little different. The tibble goes first, followed by a comma, followed by the variable you want to summarize:

```
tabyl(penguins, species)
```

```
##   species    n  percent
##   Adelie  152 0.4418605
## Chinstrap   68 0.1976744
##   Gentoo  124 0.3604651
```

Now you get both counts and proportions. Note that in the output above, it's a little misleading to call the last column “percent”. These are actually proportions, and we would have to multiply by 100 to get percentages.

It's usually nice to have the column totals. We can achieve that by using an `adorn` function to get them as follows:

```
tabyl(penguins, species) %>%
  adorn_totals()
```

```
##   species    n  percent
##   Adelie  152 0.4418605
## Chinstrap   68 0.1976744
##   Gentoo  124 0.3604651
##      Total  344 1.0000000
```

We'll always include the totals at the bottom.

If you really want percentages, we can use a different `adorn` function:

```
tabyl(penguins, species) %>%
  adorn_pct_formatting()
```

```
##   species    n percent
##   Adelie  152   44.2%
## Chinstrap   68   19.8%
##   Gentoo  124   36.0%
```

Again, we'll also include `adorn_totals` so that we get the column totals.

```
tabyl(penguins, species) %>%
  adorn_totals() %>%
  adorn_pct_formatting()
```

```
##   species    n percent
##   Adelie  152   44.2%
## Chinstrap   68   19.8%
##   Gentoo  124   36.0%
##      Total  344  100.0%
```

The syntax above looks a little confusing with the unusual `%>%` symbols everywhere. You will learn more about that weird set of symbols in a later chapter. For now, you can just copy and paste this code and make any necessary changes to the tibble and/or variables names as needed.

**Exercise 3(a)** Use the `tabyl` command as above to create a frequency table for the sex of the penguins. Include the column totals at the bottom. (You will also get a relative frequency table for free.)

```
# Add code here to create a frequency table for sex
```

**Exercise 3(b)** In the table for sex that you just created, what does the row labeled `<NA>` mean?

Please write up your answer here.

**Exercise 3(c)** Now create a relative frequency table for sex that reports percentages and not proportions (still including the column totals at the bottom).

```
# Add code here that reports percentages instead of proportions
```

**Exercise 3(d)** In the previous tables, what is the difference between `percent` and `valid_percent`? Why are there two different sets of percentages being computed?

Please write up your answer here.

## 3.5 Graphing one categorical variable

When asked, “What type of graph should I use when graphing a single categorical variable?” the simple answer is “None.” If you do need to summarize a categorical variable, a frequency table usually suffices.

If you really, really want a graph, the standard type is a bar chart. But before we can create one, we need to start learning about the very important tool we will use throughout the course for graphing. It’s called `ggplot` and it’s part of a package called `ggplot2`.<sup>1</sup>

We don’t have to load the `ggplot2` package explicitly because it got loaded alongside a number of other packages when we called `library(tidyverse)` early on in the chapter.

---

<sup>1</sup>Why the “2”? It’s a long story. Google it if you’re interested in the history of the development of the `ggplot2` package.

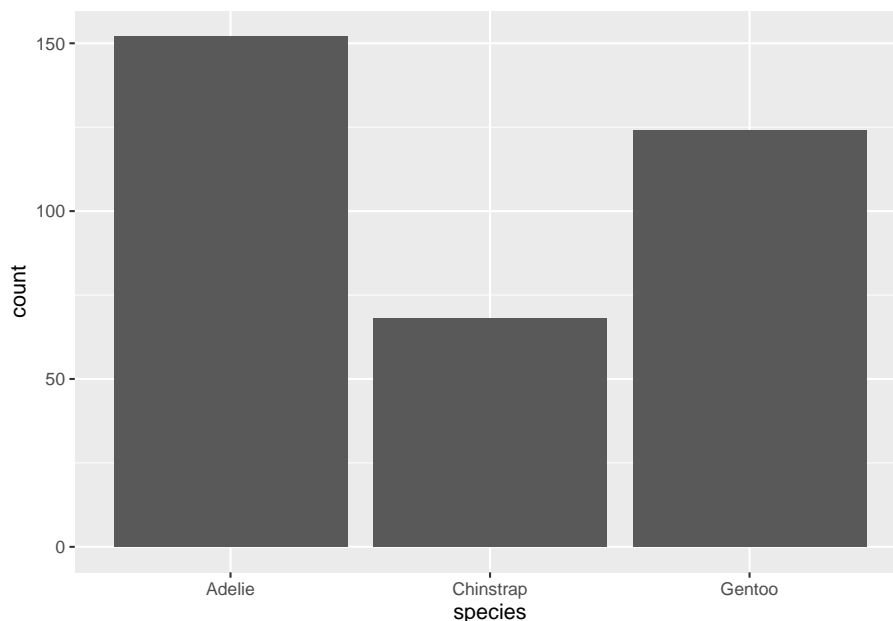
### 3.5.1 ggplot

The `ggplot` command is an all-purpose graphing utility. It uses a graphing philosophy derived from a book called *The Grammar of Graphics* by Leland Wilkinson. The basic idea is that each variable you want to plot should correspond to some element or “aesthetic” component of the graph. The obvious places for data to go are along the y-axis or x-axis, but other aesthetics are important too; graphs often use color, shape, or size to illustrate different aspects of data. Once these aesthetics have been defined, we will add “layers” to the graph. These are objects like dots, boxes, lines, or bars that dictate the type of graph we want to see.

In an introductory course, we won’t get too fancy with these graphs. But be aware that there’s a whole field of data visualization that studies clear and interesting ways to understand data graphically.

It will be easier to explain the `ggplot` syntax in the context of specific graph types, so let’s create a bar chart for species.

```
ggplot(penguins, aes(x = species)) +  
  geom_bar()
```



We’ll walk through this syntax step by step.

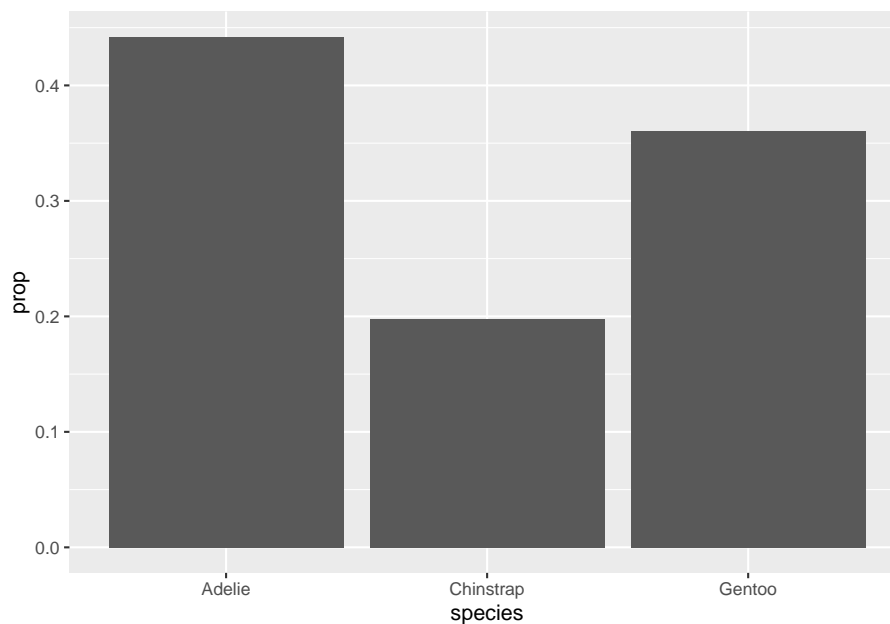
- The first argument of the `ggplot` command is the name of the tibble, in this case, `penguins`.



- Next we define the aesthetics using `aes` and parentheses. Inside the parentheses, we assign any variables we want to plot to aesthetics of the graph. For this analysis, we are only interested in the variable `species` and for a bar chart, the categorical variable typically goes on the x-axis. That's why it says `x = species` inside the `aes` argument.
- Finally, `ggplot` needs to know what kind of graph we want. Graph types are called “geoms” in the `ggplot` world, and `geom_bar()` tells `ggplot` to add a “bar chart layer”. Adding a layer is accomplished by literally typing a plus sign.

This can be modified somewhat to give proportions (relative frequencies) on the y-axis instead of counts. Unfortunately, the `ggplot` syntax is not very transparent here. My recommendation is to copy and paste the code below if you need to make a relative frequency bar chart in the future, making the necessary changes to the tibble and variable names, of course.

```
ggplot(penguins, aes(x = species, y = ..prop.., group = 1)) +  
  geom_bar()
```



These bar charts are the graphical analogues of a frequency table and a relative frequency table, respectively.

**Exercise 4** In a sentence or two at most, describe the distribution of species in this data set.

Please write up your answer here.

---

What about pie charts? Just. Don't.

Seriously. Pie charts suck.<sup>2</sup>

### 3.6 Summarizing two categorical variables

A table summarizing two categorical variables is called a *contingency table* (or pivot table, or cross-tabulation, or probably several other terms as well).

For example, we might pose the following question: is the distribution of sex among penguins in our data more or less balanced across the three species?

When we work with two variables, typically we think of one variable as *response* and the other as *predictor*. The response variable is usually the variable of main interest. A predictor variable is another attribute that might predict or explain more about the response variable.

For example, our question is concerned with the sex distribution of penguins. We could create a relative frequency table of sex alone to see if male and female penguins are balanced in the data. In fact, you did that very thing above and saw that, indeed, there were roughly equal numbers of male and female penguins. But is that still true when we divide up the data into the three groups representing the separate species?

Two variables are called *associated* when there is a relationship between them. For example, if sex and species were associated, then the distribution of sex would change depending on the species. Maybe one species of penguin had more females and another had fewer females. Our prediction of the sex distribution would change based on the value of the predictor variable **species**.

On the other hand, two variables that are not associated are called *independent*. Independent variables are not related. If the sex distribution were the same across all species, then knowledge of the species would not change our predictions about the sex of a penguin. It wouldn't matter because there was no relationship between sex and species.

Most research questions that involve two or more variables are fundamentally questions of whether a response variable is associated with one or more predictor variables, or whether they are independent.

Let's check the contingency table. The `tabyl` command will place the first variable listed across the rows and the second one listed down the columns.

---

<sup>2</sup><https://medium.com/the-mission/to-pie-charts-3b1f57bcb34a>

Since we always include column totals, we want the predictor variable to be the column variable so we can see how the predictor groups are distributed in the data. **Always list the response variable first.**

```
tabyl(penguins, sex, species) %>%
  adorn_totals()
```

```
##      sex Adelie Chinstrap Gentoo
## female      73         34      58
##  male      73         34      61
##   <NA>       6          0       5
##  Total    152        68     124
```

Each column is a group, and our question is whether the distribution of sexes in each column is similar.

The last row of totals is called the *marginal distribution* (because it sits in the “margin” of the contingency table). It is equivalent to a frequency table for species.

**3.6.0.0.1 Exercise 5** Counts can be misleading. For example, there are 73 female Adelie penguins, but only 34 female Chinstrap penguins. Does that mean that Adelie penguins are more likely to be female than Chinstrap penguins? Why or why not?

Please write up your answer here.

---

A more fair way to compare across columns is to create relative frequencies. We can do this with a slightly different **adorn** command. The following code says that we want to compute column proportions (yes, I know the command is called **adorn\_percentages**, but these are proportions):

```
tabyl(penguins, sex, species) %>%
  adorn_totals() %>%
  adorn_percentages("col")
```

```
##      sex      Adelie Chinstrap      Gentoo
## female 0.48026316      0.5 0.46774194
##  male  0.48026316      0.5 0.49193548
##   <NA> 0.03947368      0.0 0.04032258
##  Total 1.00000000      1.0 1.00000000
```

If we actually want percentages, we need one more line of code. This command—`adorn_pct_formatting`—is the same as we used before with frequency tables.

```
tabyl(penguins, sex, species) %>%
  adorn_totals() %>%
  adorn_percentages("col") %>%
  adorn_pct_formatting()
```

```
##      sex Adelie Chinstrap Gentoo
## female 48.0%    50.0%  46.8%
##  male  48.0%    50.0%  49.2%
##   <NA>  3.9%     0.0%   4.0%
##   Total 100.0%   100.0% 100.0%
```

Now we can see that each column adds up to 100%. In other words, each species is now on equal footing, and only the distribution of sexes within each group matters.

**3.6.0.0.2 Exercise 6(a)** What percentage of Adelie penguins are male? What percentage of Chinstrap penguins are male? What percentage of Gentoo penguins are male?

Please write up your answer here.

**3.6.0.0.3 Exercise 6(b)** Does sex appear to be associated with species for the penguins in this data set? Or are these variables independent?

Please write up your answer here.

---

The islands of Antarctica on which the penguins were observed and measured are recorded in the variable called `island`. Is the distribution of the three species of penguin the same (or similar) on the three islands?

**3.6.0.0.4 Exercise 7(a)** Choosing which variables play the roles of response and predictor can be tricky. For the question above, with `species` and `island`, which is response and which is predictor?

One way to think about this is to ask the following two questions and see which one is closer to the question asked:

- Given information about the species, are you interested in which island the penguin lives on? If so, `species` is a predictor and `island` is response. (You are using `species` to predict `island`.)

- Given information about the island, are you interested in the species of the penguin? If so, **island** is a predictor and **species** is response. (You are using **island** to predict **species**.)

Please write up your answer here.

**3.6.0.0.5 Exercise 7(b)** Create a contingency table with percentages. List **species** first, followed by **island**. (Hey, that's hint in case you need to go back and change your answer to part (a).)

```
# Add code here to create a contingency table with percentages.
```

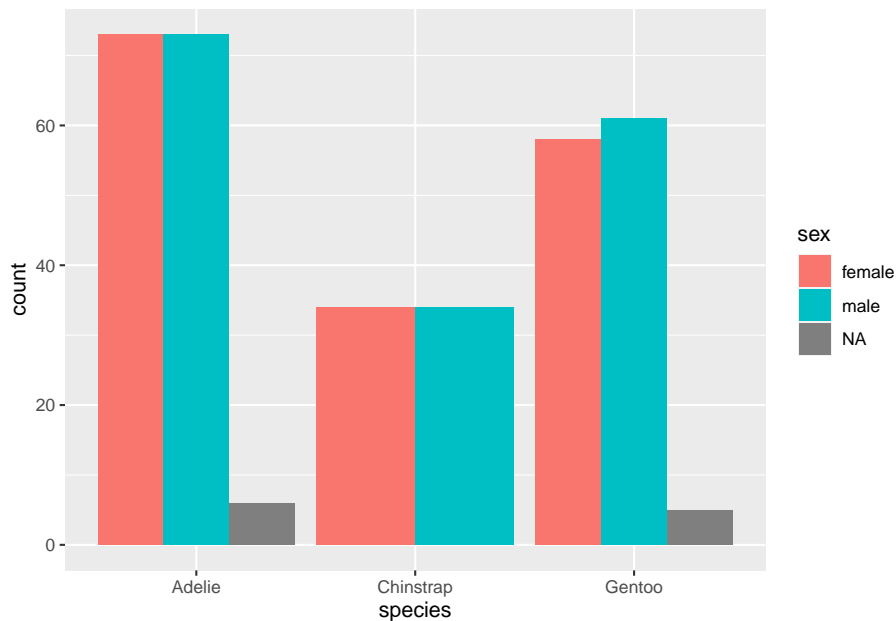
**3.6.0.0.6 Exercise 7(c)** Finally, comment on the association or independence of the two variables.

Please write up your answer here.

## 3.7 Graphing two categorical variables

A somewhat effective way to display two categorical variables is with a side-by-side bar chart. Here is the **ggplot** code for the relationship between **sex** and **species**.

```
ggplot(penguins, aes(fill = sex, x = species)) +  
  geom_bar(position = "dodge")
```

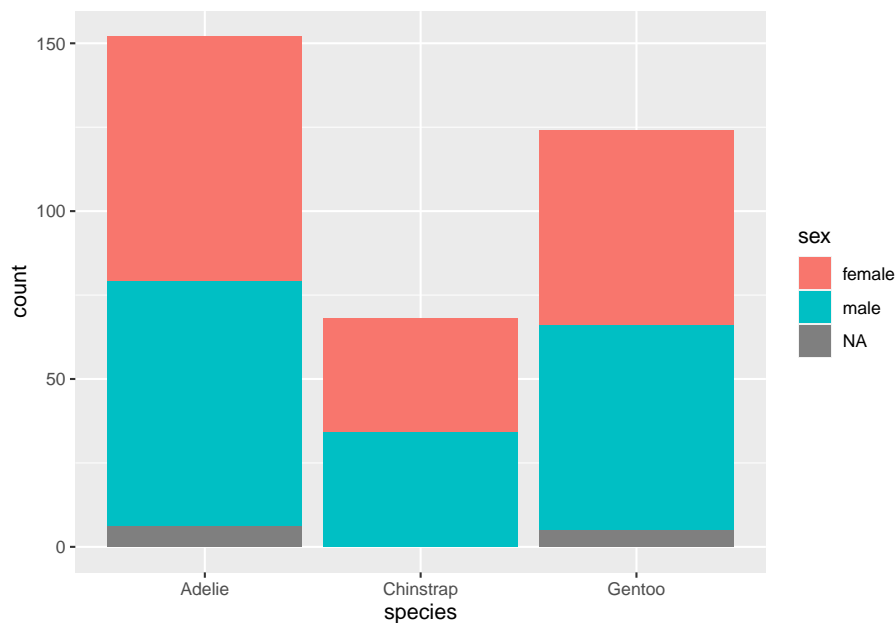


This is somewhat different from the first `ggplot` example you saw above, so let's take a moment to go through it.

- The first argument is the data frame `penguins`; no mystery there.
- The second aesthetic `x = species` also makes a lot of sense. As `species` is our predictor variable—we're using species to group the penguins, and then within each species, we're interested in the sex distribution—`species` goes on the x-axis.
- However, `sex` does not go on the y-axis! (This is a very common mistake for novices.) The y-axis of a bar chart is always a count or a proportion/percentage, so no variable should ever go on the y-axis of a bar chart. In that case, how does `sex` enter the picture? Through the use of color! The aesthetic `fill = sex` says to use the `sex` variable to shade or “fill” the bars with different colors. You'll also notice that `ggplot` makes a legend automatically with the colors so you can see which color corresponds to which value (in this case, “female”, “male”, or “NA” for the missing data).

Another unusual feature is the argument `position = "dodge"` in the `geom_bar` layer. Let's see what happens if we remove it.

```
ggplot(penguins, aes(fill = sex, x = species)) +  
  geom_bar()
```

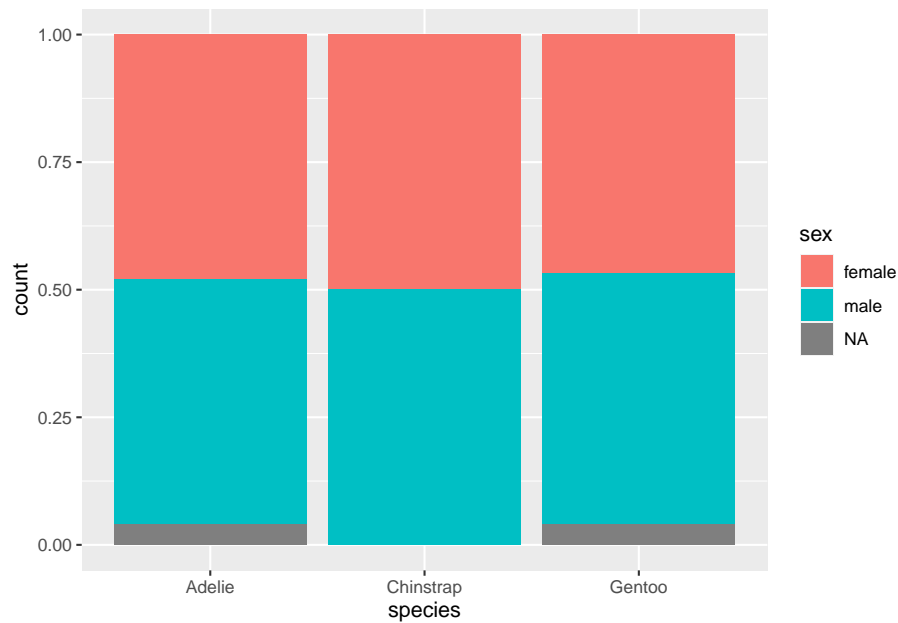


We get a stacked bar chart! This is another popular way of displaying two categorical variables, but we don't tend to prefer it. Notice how difficult it is to compare the number of females across species; since there is no common baseline for the red segments of each bar, it is harder to determine which ones are bigger or smaller. (In this case, it's fairly clear, but there are plenty of data sets for which the counts might be a lot closer.)

So let's agree to use side-by-side bar charts. There is still one aspect of the side-by-side bar chart that is misleading, though. For example, the red bar for Adelie penguins is bigger than the red bar for Gentoo penguins. Does this mean Adelie penguins are more likely to be female?

This is the same issue we identified in an exercise above. To fix this problem, a better option here would be to use relative frequencies (i.e., proportions/percentages within each group) instead of counts on the y-axis. This is analogous to using proportions/percentages in a contingency table. Unfortunately, it is rather difficult to do this with `ggplot`. A compromise is available: by using `position = fill`, you can create a stacked bar chart that scales every group to 100%. Making comparisons across groups can still be hard, as explained above for any kind of stacked bar chart, but it works okay if there are only two categories in the response variable (as is almost the case with `sex` here, although the missing data distorts things a little at the bottom).

```
ggplot(penguins, aes(fill = sex, x = species)) +  
  geom_bar(position = "fill")
```



This graph does correctly show that the sexes are pretty much equally balances across all three species.

**Exercise 8(a)** Using `species` and `island`, create a side-by-side bar chart. Be careful, though, to change the sample code above to make sure `species` is now the response variable (using the `fill` aesthetic) and that `island` is the explanatory variable (using `x`). (Hey, that's another hint to go back and look at the previous exercise and make sure you got part (a) right!)

```
# Add code here to make a side-by-side bar chart.
```

**Exercise 8(b)** Comment on the association or independence of the two variables.

Please write up your answer here.

### 3.8 Recoding factor variables

As mentioned earlier, there are situations where a categorical variable is not recorded in R as a factor variable. Let's look at the `year` variable:



```
glimpse(penguins$year)
```

```
## int [1:344] 2007 2007 2007 2007 2007 2007 2007 2007 2007 2007 2007 ...
```

These appear as integers. Yes, years are whole numbers, but why might this variable be treated as categorical data and not numerical data?

**Exercise 9(a)** Use the `tabyl` command to create a frequency table for `year`.

```
# Add code here to make a frequency table for year.
```

**Exercise 9(b)** Why is `year` better thought of as categorical data and not numerical data (at least for this data set—we’re not claiming years should always be treated as categorical)?

Please write up your answer here.

---

While the `tabyl` command seemed to work just fine with the `year` data in integer format, there are other commands that will not work so well. For example, `ggplot` often fails to do the right thing when a categorical variable is coded as a number. Therefore, we need a way to change numerically coded variables to factors.

The code below uses a command called `mutate` that takes an old variable and creates a new variable. (You’ll learn more about this command in a later chapter. For now, you can just copy and paste this code if you need it again.) The name of the new variable can be anything we want; we’ll just call it `year_fct`. Then the real work is being done by the `as_factor` command that converts the numeric `year` variable into a factor variable.

Observe the effect below:

```
penguins <- penguins %>%
  mutate(year_fct = as_factor(year))
glimpse(penguins)
```

```
## Rows: 344
## Columns: 9
## $ species      <fct> Adelie, Adelie, Adelie, Adelie, Adelie, Adelie, Adel~
## $ island       <fct> Torgersen, Torgersen, Torgersen, Torgersen, Torgerse~
## $ bill_length_mm <dbl> 39.1, 39.5, 40.3, NA, 36.7, 39.3, 38.9, 39.2, 34.1, ~
```

```
## $ bill_depth_mm      <dbl> 18.7, 17.4, 18.0, NA, 19.3, 20.6, 17.8, 19.6, 18.1, ~
## $ flipper_length_mm <int> 181, 186, 195, NA, 193, 190, 181, 195, 193, 190, 186~
## $ body_mass_g       <int> 3750, 3800, 3250, NA, 3450, 3650, 3625, 4675, 3475, ~
## $ sex               <fct> male, female, female, NA, female, male, female, male~
## $ year              <int> 2007, 2007, 2007, 2007, 2007, 2007, 2007, 2007, 2007~
## $ year_fct          <fct> 2007, 2007, 2007, 2007, 2007, 2007, 2007, 2007, 2007~
```

**Exercise 10(a)** Make a contingency table of the species measured in each year using counts. Use the `species` variable first, followed by the new factor variable `year_fct`. (Think about why that order makes sense. **We will always list the response variable first so that the categories of interest will be the rows and the groups will be the columns.**)

```
# Add code here to make a contingency table for species and year with counts.
```

**Exercise 10(b)** Make a contingency table of the species measured in each year using column percentages (*not* proportions). (Again, be sure to use the new factor variable `year_fct`, not the old variable `year`.)

```
# Add code here to make a contingency table for species and year with percentages.
```

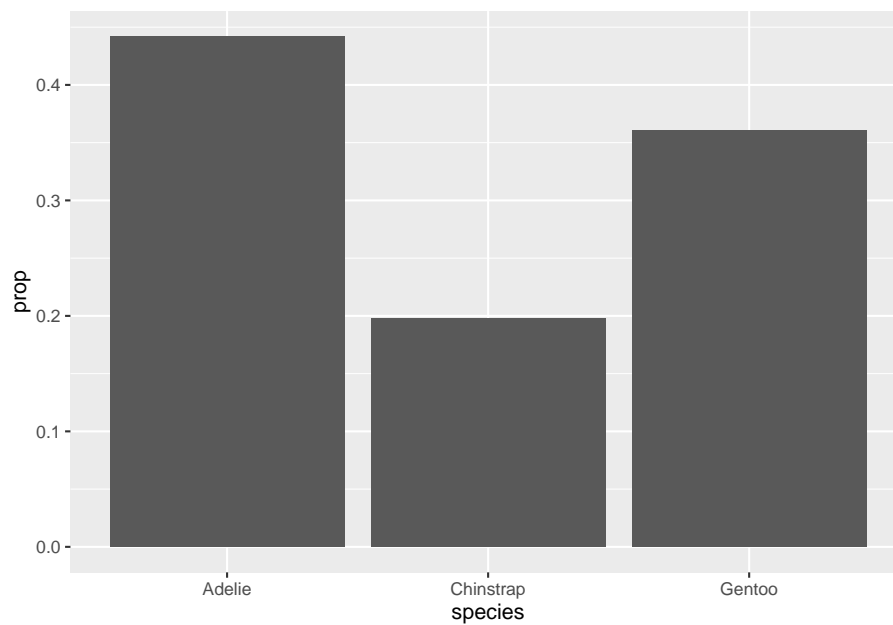
**Exercise 10(c)** How similar or dissimilar are the distributions of species across the three years of the study?

Please write up your answer here.

### 3.9 Publication-ready graphics

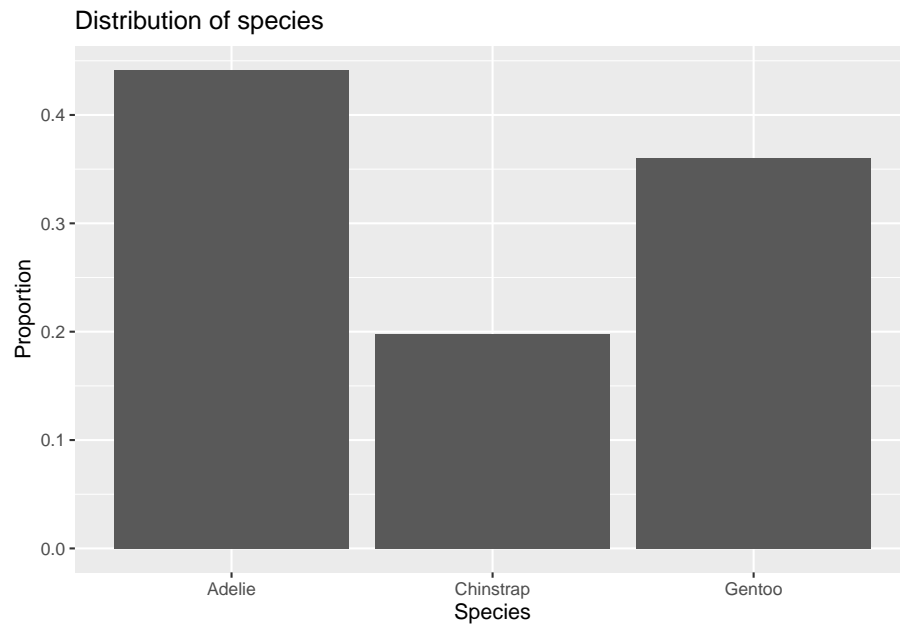
Let's go back to the first relative frequency bar chart from this chapter.

```
ggplot(penguins, aes(x = species, y = ..prop.., group = 1)) +
  geom_bar()
```



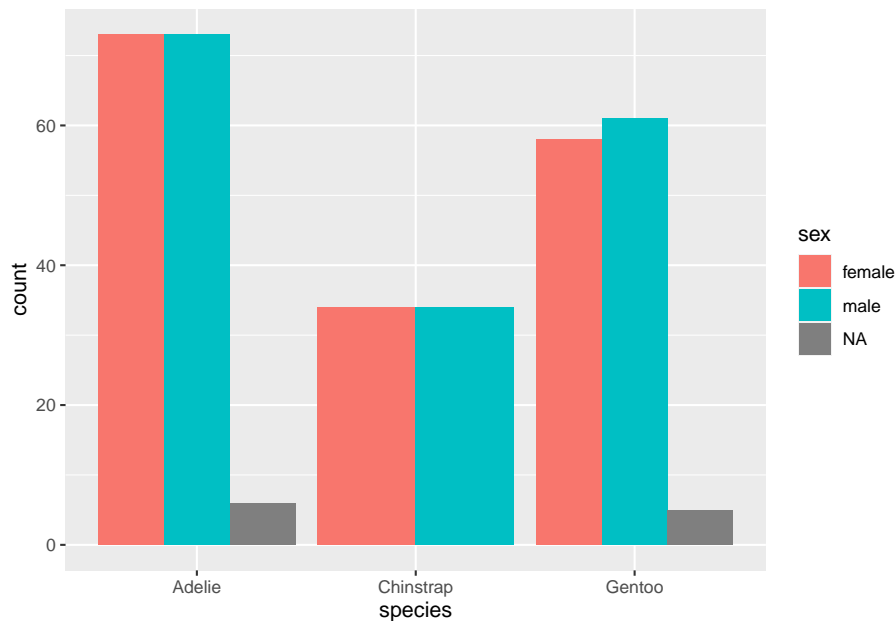
The variable name `species` is already informative, but the y-axis is labeled with “prop”. Also note that this graph could use a title. We can do all this with `labs` (for labels). Observe:

```
ggplot(penguins, aes(x = species, y = ..prop.., group = 1)) +  
  geom_bar() +  
  labs(title = "Distribution of species",  
        y = "Proportion",  
        x = "Species")
```



**Exercise 11** Modify the following side-by-side bar chart by adding a title and labels for both the fill variable and the x-axis variable. (Hint: you can use `fill = sex` inside the `labs` command just like you used `title`, `y`, and `x`.)

```
# Modify the following side-by-side bar chart by adding a title and
# labels for both the x-axis and the fill variable.
ggplot(penguins, aes(fill = sex, x = species)) +
  geom_bar(position = "dodge")
```



### 3.10 Plotting summary data

Everything we did above was summarizing *raw data*; that is, the data consisted of all the observations for each individual penguin. Often, though, when you find data out in the wild, that data will be summarized into a table already and you may not have access to the raw data.

For example, let's suppose that you found some data online, but it looked like this:

species	count
Adelie	152
Chinstrap	68
Gentoo	124

This raises two questions:

1. How would you get this data into R?
2. How would you plot the data?

To answer the first question, we show you how to create your own tibble. Here is the syntax:

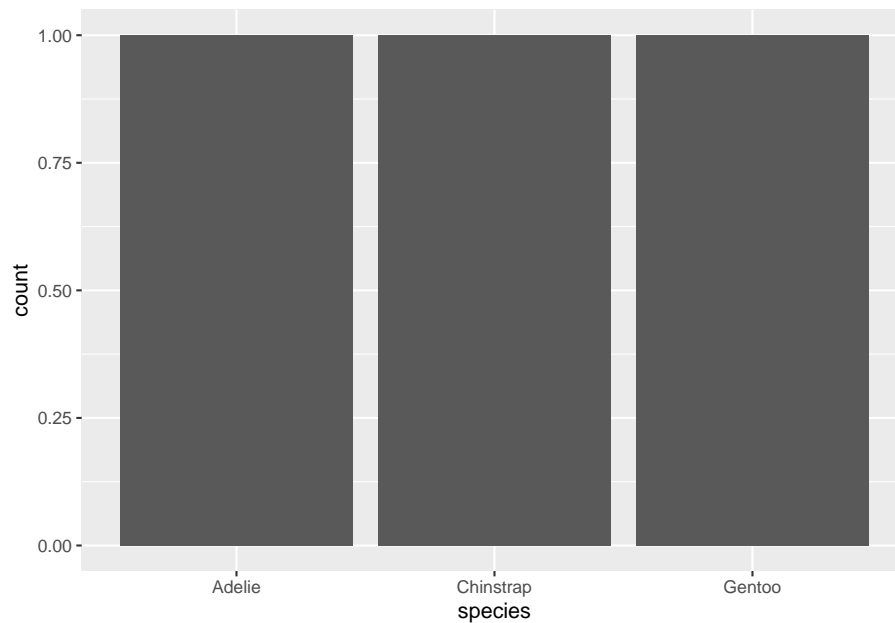
```
penguin_species_table <- tibble(  
  species = c("Adelie", "Chinstrap", "Gentoo"),  
  count = c(152, 68, 124)  
)  
penguin_species_table
```

```
## # A tibble: 3 x 2  
##   species   count  
##   <chr>     <dbl>  
## 1 Adelie     152  
## 2 Chinstrap   68  
## 3 Gentoo    124
```

Basically, the `tibble` command creates a new tibble. Then each column of data must be entered manually as a “vector” using the `c` to group all the data values together for each column. Be careful about the placement of quotation marks, commas, and parentheses.

Once we have our summary data, we want to make a bar chart. But this won’t work:

```
ggplot(penguin_species_table, aes(x = species)) +  
  geom_bar()
```



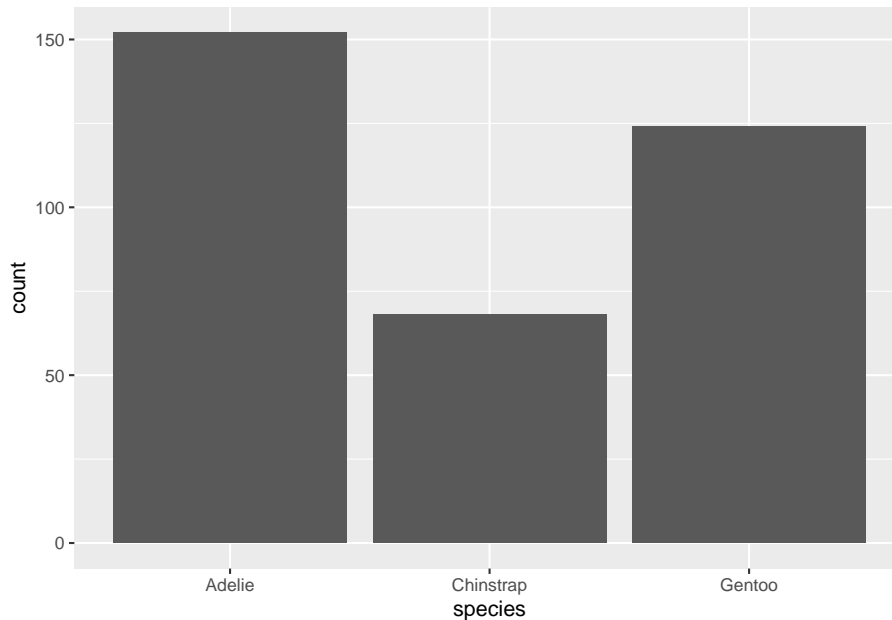
**Exercise 12** Explain what went wrong with the previous command? Why does `ggplot` think that each species has count 1?

Please write up your answer here.

---

Instead, we need to use `geom_col`. This works a lot like `geom_bar` except that it also requires a `y` value in its aesthetics to force the command to look for the counts in some other variable in the data.

```
ggplot(penguin_species_table, aes(x = species, y = count)) +  
  geom_col()
```



**Exercise 13(a)** Use the `tabyl` command to create a frequency table for `island`.

```
# Add code here to create a frequency table for island
```

**Exercise 13(b)** Use the `tibble` command to create a new tibble manually that contains the frequency data for the `island` variable. It should have two columns, one called `island` and the other called `count`. Name it `penguin_island_table`.

```
# Add code here to create a tibble with frequency data for island
```

**Exercise 13(c)** Use `ggplot` with `geom_col` to create a bar chart for island.

```
# Add code here to create a bar chart for island
```

## 3.11 Conclusion

You can summarize a single categorical variable using a frequency table. For only one categorical variable, a graph is usually overkill, but if you really want a graph, the bar chart is the best option. Both raw counts and proportions/percentages can be useful.

We use contingency tables to summarize two categorical variables. Unless groups are of equal size, raw counts can be incredibly misleading here. You should include proportions/percentages to be able to compare the distributions across groups. If the proportions/percentages are roughly the same, the variables are more likely to be independent, whereas if the proportions/percentages are different, there may be an association between the variables. For graphing, the best choice is usually a side-by-side bar chart. A stacked bar chart will also work, especially if using relative frequencies on the y-axis, but it can be hard to compare across groups when the response variable has three or more categories.

Sometimes we come across categorical data that is recorded using numbers. Many R commands will not work properly if they expect factors and receive numbers, so we use the `mutate` command to create a new variable along with `as_factor` to convert the numbers to categories.

Sometimes we come across summary data instead of raw data. We can then manually create tibbles with that summary data and use `geom_col` instead of `geom_bar` to graph it.

### 3.11.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1–2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.



If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.



## Chapter 4

# Numerical data

2.0

### Functions introduced in this chapter

`mean`, `sd`, `var`, `median`, `sort`, `IQR`, `quantile`, `summary`, `min`, `max`, `geom_histogram`,  
`geom_point`, `geom_boxplot`, `facet_grid`

## 4.1 Introduction

In this chapter, we'll learn about numerical data and how to summarize it through summary statistics and graphs.

### 4.1.1 Install new packages

There are no new packages used in this chapter.

### 4.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/04-numerical\\_data.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/04-numerical_data.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 4.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

### 4.1.4 Load packages

We load the `tidyverse` package to get `ggplot2` and the `palmerpenguins` package to work with the penguin data.

```
library(tidyverse)
library(palmerpenguins)
```

## 4.2 A note about mathematical notation

From time to time, we will use mathematical notation that can’t be typed directly on the keyboard. For example, let’s suppose we want to typeset the quadratic formula, which involves a complicated fraction as well as a square root symbol.

When such notation appears, it will be surrounded by double dollar signs as follows:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The R Notebook will interpret this special mathematical notation and render it on the screen as well as in the HTML document.<sup>1</sup> If the nicely formatted formula does not appear on your screen, place your cursor anywhere inside the math formula and hit Ctrl-Enter or Cmd-Enter (PC or Mac respectively).

Sometimes, we want such math to appear inline. We can do this with single dollar signs. For example, the distance formula is  $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ , a fact you may have learned a long time ago.

This will *not* render visually in the R Notebook, but it will show up in the HTML file. If you want to check that it worked properly without having to preview the HTML, you can either hover your cursor over the math formula and wait a second, or you can place your cursor anywhere inside the math formula and hit Ctrl-Enter or Cmd-Enter (PC or Mac respectively) to see a pop-up window previewing the mathematical content properly formatted.

You will be shown examples of any mathematical notation you need to use in any given chapter, so feel free to copy/paste/modify any math notation you need.

---

<sup>1</sup>This notation is part of a mathematical document preparation system called LaTeX, pronounced “Lay-tek” (not like the rubbery substance).

## 4.3 Statistics

The word “statistics” has several meanings. On one hand, it’s an entire field of study, as in the subject of this course. More specifically, though, a “statistic” is any kind of numerical summary of data. While there are many ways to summarize data, they mostly fall into two main flavors: measures of *center* and measures of *spread*. Measures of center try to estimate some kind of average, middle, or common value in data. Measures of spread try to estimate something like the width, range, variability, or uncertainty of data.

There are two pairs of measurements that we will learn about in this chapter: the mean/standard deviation, and the median/IQR.

### 4.3.1 Mean and standard deviation

The first pair of the summary statistics we’ll discuss consists of the mean and the standard deviation.

The *mean* of a variable  $y$ —denoted  $\bar{y}$  and pronounced “y bar”—is calculated by summing all the values of the variable, and dividing by the total number of observations. In formula form, this is

$$\bar{y} = \frac{\sum y}{n}.$$

This is a measure of center since it estimates the “middle” of a set of numbers. It is calculated in R using the `mean` command.

Throughout this chapter, we will be using the `penguins` data set. (If you need a reminder, look at the help file for `penguins` using one of the methods discussed in Chapter 2.)

If we want to calculate the mean body mass of our penguins (in grams), we type the following:

```
mean(penguins$body_mass_g)
```

```
## [1] NA
```

Unfortunately, this didn’t give us an answer. As you may recall from previous chapters, this is because we are missing several values of body mass in this data. We need an extra piece of code to tell R to ignore that missing data and give us the mean of the valid data.

```
mean(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 4201.754
```

(The term `na.rm` stands for “NA remove”).

We never leave such numbers without interpretation. In a full, contextually meaningful sentence, we might say, “The mean body mass of this group of penguins is approximately 4200 grams.”

Notice that we mentioned the penguins, placing this number in context, and we mentioned the units of measurement, grams. (Otherwise, what would this number mean? 4200 pounds? Okay, probably not, but you should always mention the units of measurement.) Also notice that we rounded the final value. A gram is a very small unit of measurement, so there is no need to report this value to many decimal places.

If we use inline code, we can say, “The mean body mass of this group of penguins is 4201.754386 grams.” There are ways of rounding this number as well, but it’s a bit of a hassle to do so in inline code.

The corresponding measure of spread is the *standard deviation*. Usually this is called  $s$  and is calculated using a much more complicated formula:

$$s = \sqrt{\frac{\sum (y - \bar{y})^2}{n - 1}}.$$

This is a measure of spread because the  $(y - \bar{y})$  term measures the how far away each data point is from the mean.

In R, this is calculated with the `sd` command. Again, we’ll need to add `na.rm = TRUE`.

```
sd(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 801.9545
```

“The standard deviation of this group of penguins is about 801 grams.”

Or using inline code:

“The standard deviation of this group of penguins is 801.9545357 grams.”

The mean and the standard deviation should always be reported together. One without the other is incomplete and potentially misleading.

Another related measurement is the *variance*, but this is nothing more than the standard deviation squared:

$$s^2 = \frac{\sum (y - \bar{y})^2}{n - 1}.$$

(Compare this formula to the one for the standard deviation. Nothing has changed except for the removal of the square root.) We rarely use the variance in an introductory stats class because it's not as interpretable as the standard deviation. The main reason for this is units. If the data units are grams, then both the mean and the standard deviation are also reported in grams. The variance has units of “grams squared”, but what does that even mean? If you need to calculate the variance in R, the command is `var`.

```
var(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 643131.1
```

You can check and see that the number above really is just 801.9545357 squared. Regarding the inline code in the previous sentence, remember, in the R Notebook, you can click inside the inline code and hit Ctrl-Enter or Cmd-Enter. In the HTML document, the number will be calculated and will magically appear.

### 4.3.2 Median and IQR

Another choice for measuring the center and spread of a data set is the median and the IQR.

The median is just the middle value if the list of values is ordered. In R, it is calculated using the `median` command.

```
median(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 4050
```

The median body mass of these penguins is 4050 grams.

The median value depends on whether there are an even or odd number of data points. If there are an odd number, there is a middle value in the list. Convince yourself this is true; for example, look at the numbers 1 through 7.

```
1:7
```

```
## [1] 1 2 3 4 5 6 7
```

The number 4 is in the middle of the list, with three numbers to either side.

However, if there are an even number of data points, there is no number right in the middle:

```
1:8
```

```
## [1] 1 2 3 4 5 6 7 8
```

The “midpoint” of this list would lie between 4 and 5. If this is the case, we calculate the median by taking the mean of the two numbers straddling the middle. In the case of 1 through 8 above, the median would be 4.5.

Let’s print out the entire `body_mass_g` variable, all 342 valid values (not including the missing values, of course). If we’re clever about it, we can see them in order using the `sort` command.

```
sort(penguins$body_mass_g)
```

```
## [1] 2700 2850 2850 2900 2900 2900 2900 2925 2975 3000 3000 3050 3050 3050 3050
## [16] 3075 3100 3150 3150 3150 3150 3175 3175 3200 3200 3200 3200 3200 3250 3250
## [31] 3250 3250 3250 3275 3300 3300 3300 3300 3300 3300 3325 3325 3325 3325 3325
## [46] 3350 3350 3350 3350 3350 3400 3400 3400 3400 3400 3400 3400 3400 3425 3425
## [61] 3450 3450 3450 3450 3450 3450 3450 3450 3475 3475 3475 3500 3500 3500 3500
## [76] 3500 3500 3500 3525 3525 3550 3550 3550 3550 3550 3550 3550 3550 3575
## [91] 3600 3600 3600 3600 3600 3600 3600 3625 3650 3650 3650 3650 3650 3650 3675
## [106] 3675 3700 3700 3700 3700 3700 3700 3700 3700 3700 3700 3700 3725 3725 3725
## [121] 3750 3750 3750 3750 3750 3775 3775 3775 3775 3800 3800 3800 3800 3800 3800
## [136] 3800 3800 3800 3800 3800 3800 3825 3850 3875 3900 3900 3900 3900 3900 3900
## [151] 3900 3900 3900 3900 3950 3950 3950 3950 3950 3950 3950 3950 3950 3975
## [166] 4000 4000 4000 4000 4000 4050 4050 4050 4050 4050 4050 4075 4100 4100 4100
## [181] 4100 4100 4150 4150 4150 4150 4150 4150 4200 4200 4200 4200 4200 4250 4250
## [196] 4250 4250 4250 4275 4300 4300 4300 4300 4300 4300 4300 4300 4350 4350 4375
## [211] 4400 4400 4400 4400 4400 4400 4400 4400 4450 4450 4450 4450 4450 4475 4500
## [226] 4500 4500 4550 4550 4575 4600 4600 4600 4600 4600 4625 4625 4650 4650 4650
## [241] 4650 4650 4675 4700 4700 4700 4700 4700 4700 4725 4725 4725 4750 4750 4750
## [256] 4750 4750 4775 4800 4800 4800 4850 4850 4850 4850 4875 4875 4875 4900 4900
## [271] 4925 4925 4950 4950 4975 5000 5000 5000 5000 5000 5000 5050 5050 5050 5100
## [286] 5100 5100 5150 5150 5200 5200 5200 5200 5250 5250 5250 5300 5300 5300 5300
## [301] 5350 5350 5350 5400 5400 5400 5400 5400 5450 5500 5500 5500 5500 5500 5550
## [316] 5550 5550 5550 5550 5550 5600 5600 5650 5650 5650 5700 5700 5700 5700 5700
## [331] 5750 5800 5800 5850 5850 5850 5950 5950 6000 6000 6050 6300
```

**Exercise 1** If there are 342 penguins in this data set with body mass data, between which two values in the list above would the median lie? In other



words, between what two positions in the list will the median be found? Verify that the median you find from this list is the same as the one we calculated with the `median` command above.

Please write up your answer here.

---

Calculating the *interquartile range*—or *IQR*—requires first the calculation of the first and third quartiles, denoted  $Q1$  and  $Q3$ . If the median is the 50% mark in the sorted data, the first and third quartiles are the 25% and the 75% marks, respectively. One way to compute these by hand is to calculate the median of the lower and upper halves of the data separately. Then again, it's hard to know how to split the data set into halves if there are an odd number of observations. There are many different methods for computing percentiles in general, but you don't need to worry too much about the particular implementation in R. Once you have  $Q1$  and  $Q3$ , the IQR is just

$$IQR = Q3 - Q1$$

In R, you can get the IQR by using—are you ready for this?—the `IQR` command.

```
IQR(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 1200
```

The IQR for this group of penguins is 1200 grams.

The IQR is a measure of spread because the distance between  $Q1$  and  $Q3$  measures the span of the “middle 50%” of the data.

A general function for computing any percentile in R is the `quantile` function. For example, since  $Q1$  is the 25th percentile, you can compute it as follows:

```
Q1 <- quantile(penguins$body_mass_g, 0.25, na.rm = TRUE)
Q1
```

```
## 25%
## 3550
```

The 25% label is cute, but somewhat unnecessary, and it will mess up a later command, so let's get rid of it:

```
Q1 <- unname(Q1)
Q1
```

```
## [1] 3550
```

**Exercise 2(a)** Now you compute Q3.

```
# Add code here to compute, store, and print out Q3
```

**Exercise 2(b)** Reassign Q3 using the `unname` command as we did above to strip the unnecessary label.

```
# Add code here that uses the unname command
```

**Exercise 2(c)** Finally, check that the IQR calculated above matches the value you get from subtracting Q3 minus Q1.

```
# Add code here to compute Q3 - Q1.
```

---

The median and the IQR should always be reported together.

Also, don't mix and match. For example, it doesn't really make sense to report the mean and the IQR. Nor should you report the median and the standard deviation. They go together in pairs: either the mean and the standard deviation together, or the median and the IQR together.

### 4.3.3 Robust statistics

Some statistics are more sensitive than others to features of the data. For example, outliers are data points that are far away from the bulk of the data. The mean and especially the standard deviation can change a lot when outliers are present. Also, skewness in the data frequently pulls the mean too far in the direction of the skew while simultaneously inflating the standard deviation. (We'll learn more about skewed data later in this chapter.)

On the other hand, the median and IQR are “robust”, meaning that they do not change much (or at all) in the presence of outliers and they tend to be good summaries even for skewed data.

**Exercise 3** Explain why the median and IQR are robust. In other words, why does an outlier have little or no influence on the median and IQR?

Please write up your answer here.

---

#### 4.3.4 Five-number summary

A *five-number summary* is the minimum, Q1, median, Q3, and maximum of a set of numbers.

The `summary` command in R gives you the five-number summary, and throws in the mean for good measure. (Note that it does not require `na.rm = TRUE`!)

```
summary(penguins$body_mass_g)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.     NA's
##      2700   3550   4050   4202   4750   6300         2
```

You can, of course, isolate the various pieces of this. You already know most of the commands below. (These individual commands all do require `na.rm = TRUE`.)

```
min(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 2700
```

```
median(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 4050
```

```
max(penguins$body_mass_g, na.rm = TRUE)
```

```
## [1] 6300
```

Remember the `quantile` function from earlier, where we computed Q1? We're going to use it in a new way. Instead of what we did earlier,

```
quantile(penguins$body_mass_g, 0.25, na.rm = TRUE),
```

what about this instead?

```
quantile(penguins$body_mass_g, na.rm = TRUE)
```

```
##   0%  25%  50%  75% 100%
## 2700 3550 4050 4750 6300
```

**Exercise 4** What is the difference between the way `quantile` was used in a previous exercise versus the way it was used here? How did that change the output?

Please write up your answer here.

---

Also, don't forget about the trick for using R commands inline. If you need to mention a statistic in the middle of a sentence, there is no need to break the sentence and display a code chunk. Be sure you're looking at the R notebook file (not the HTML file) to note that the numbers in the next sentence are not manually entered, but are calculated on the fly:

There are 344 penguins in this data set and their median body mass is 4050 grams.

**Exercise 5** Type a full, contextually meaningful sentence using inline R code (as above, but changing the commands) reporting the minimum and maximum body mass (in grams) in our data set.

Please write up your answer here.

## 4.4 Graphing one numerical variable

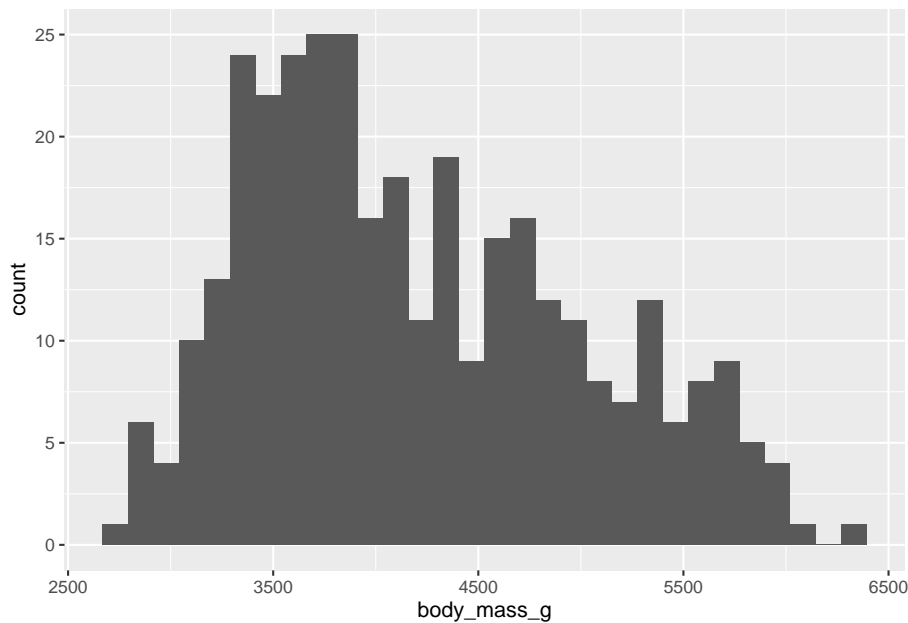
From the `penguins` data, let's consider again the body mass in grams. This is clearly a numerical variable.

The single most useful display of a single numerical variable is a histogram. Here is the `ggplot` command to do that:

```
ggplot(penguins, aes(x = body_mass_g)) +
  geom_histogram()
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```



#### 4.4.1 The shape of data

The way histograms work is to create “bins”, which are ranges of numbers along the x-axis. R goes through the data and counts how many observations fall into each bin. In that way, a histogram is somewhat like a bar chart. However, a bar chart uses bars to represent distinct, discrete categories, whereas a histogram uses bars that are all next to each other to represent values along a continuous numerical range. Histograms are meant to give you—at a quick glance—a sense of the “shape” of the data.

What do we mean by “shape”? Generally, we look for three things:

1. Modes

- Modes are peaks in the data. These are places where data tends to cluster, representing common values of the numerical variable. In the **penguin** data, there appears to be a big mode between about 3500 and 4000 grams. When data has one clear mode, we call the data *unimodal*. But data can also be *bimodal*, or more generally, *multimodal*. This often happens when the data contains multiple groups that are different from each other. In this case, we know there are three species of penguin in the data, so if those species are drastically different in their body mass, we might be looking at multimodal data. We’ll explore this question more later in the chapter. For now, it’s hard to say what’s going on because the above histogram has

a lot of spiky bars popping up all over. It's not completely obvious how many modes there might be.

## 2. Symmetry

- If there is one mode, we can also ask if the data is spread evenly to the left and right of that mode. If so, we call the data *symmetric*. No data is perfectly symmetric, but we are looking for overall balance between the areas to the left and right of the mode. When data is not symmetric, we call it *skewed*. Assuming that there is one big mode around 3500 or 4000, the body mass data above is skewed. There is clearly more data above the mode than below the mode. The right side of the histogram stretches out further to the right of the mode than to the left. Therefore, the body mass data is *right-skewed*. There is a longer “tail” to the right. If it were the opposite, it would be *left-skewed*. It is common for beginning students to confuse these two terms. Be aware that we are not concerned about where the mode is. We want to know which side has more data spread into a longer tail. That is the direction of the skewness.

## 3. Outliers.

- Outliers are data points that are far from the bulk of the data. The body mass data above appears to have no outliers. We are looking for a large gap between the main “mass” of data and any lingering data points far away from that mass. There is no such large gap in the histogram above.

**Whenever you are asked about the “shape” of a numerical variable, be sure to comment on (1) modes, (2) symmetry, and (3) outliers.**

Generally, the default binning for `ggplot` histograms is not great. This is by design. The creator of the `ggplot2` package, Hadley Wickham, said the following:

“In `ggplot2`, a very simple heuristic is used for the default number of bins: it uses 30, regardless of the data. This is perverse, and ignores all of the research on selecting good bin sizes automatically, but sends a clear message to the user that he or she needs to think about, and experiment with, the bin width. This message is reinforced with a warning that reminds the user to manually adjust the bin width.”

Indeed, if you look at the output from the graphing command above, you can see that `ggplot` informs you that you should pick a better value for the binwidth. You can also see that the bins aren't ideal. They are too narrow, which means that arbitrary differences between bins show up as “random” spikes all over the graph. These spikes can confuse the issue of how many modes appear in the data.

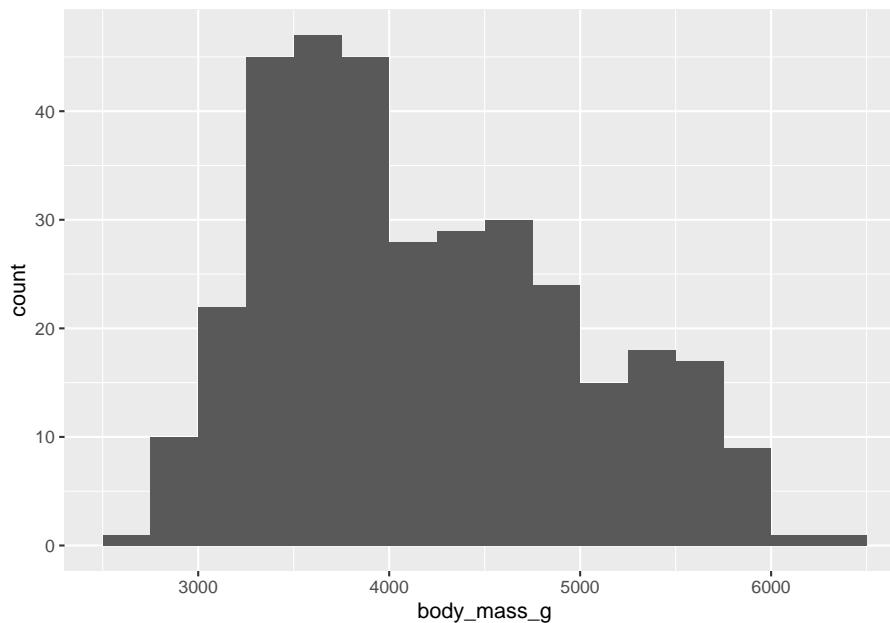
Instead, we should aim to use bins that show the overall shape of the data and smooth it out a bit. Look back at the scale of the x-axis to assess how wide each bar should be. There's no one correct answer. In this case, the bins ought to be a little wider. Since our x-axis goes from about 2500 to 6500, maybe we should try a binwidth of 250. And if 250 doesn't look good, nothing prevents us from trying a different number.

It's also easier to interpret the histogram when the bins' edges line up with numbers that are easy to see in the plot. Use **boundary** to determine where you want the bin boundaries to fall. For example, if we set the boundary to 3500, that means that one bar will start with its left edge at 3500. This is convenient because there is a tick mark labeled there on the x-axis. The boundary number is pretty arbitrary; once one boundary is set, it determines where all the other bins will line up. With a binwidth of 250, we'd get the same graph if the boundary were set to 3000 or 3250 or 5750, or even 0. Any other multiple of 250 would give the same graph.

We use **binwidth** and **boundary** inside the parentheses of the **geom\_histogram** to modify these parameters.

```
ggplot(penguins, aes(x = body_mass_g)) +  
  geom_histogram(binwidth = 250, boundary = 3500)
```

```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```

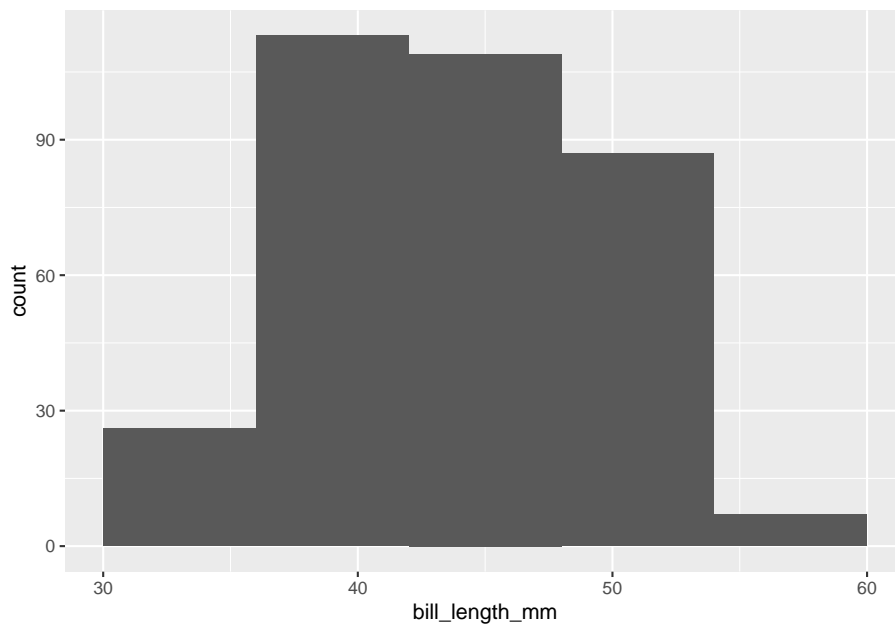


Even with the smoother look, it appears that there are multiple modes, maybe three? Do these correspond to the three species of penguin? Stay tuned.

**Exercise 6(a)** Here is a histogram of the penguin bill lengths (measured in millimeters):

```
ggplot(penguins, aes(x = bill_length_mm)) +  
  geom_histogram(binwidth = 6, boundary = 30)
```

```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```



Write a short paragraph describing the shape of the distribution of penguin bill lengths, focusing on the three key shape features (modes, symmetry, and outliers).

Please write up your answer here.

**Exercise 6(b)** The last question was a trick question!

Change the binwidth (no need to change the boundary) to something smaller to see more clearly the bimodal nature of the distribution.



```
# Add code here that changes the binwidth of the last histogram to see  
# the bimodal nature of the distribution.
```

**Exercise 7(a)** Make a histogram of the variable `flipper_length_mm`. Start with a histogram where you don't modify the binwidth or boundary.

```
# Add code here to create a histogram of flipper length
```

**Exercise 7(b)** By examining the scale on the x-axis above, repeat the command, but this time change the binwidth and the boundary until you are satisfied that the bins are neither too wide nor too narrow.

```
# Add code here to modify the histogram of flipper length,  
# adding binwidth and boundary
```

**Exercise 7(c)** Write a short paragraph describing the shape of the distribution of penguin flipper lengths, focusing on the three key shape features (modes, symmetry, and outliers).

Please write up your answer here.

#### 4.4.2 Less useful plot types

There are several other graph types that one might see for a single numerical variable: e.g., dotplots, stem-and-leaf plots, boxplots, etc. I'm not a big fan of dotplots or stem-and-leaf plots as they are just messier versions of histograms. I do like boxplots, but they are typically less informative than histograms. Boxplots are much better for comparing groups, and we'll see them later in the chapter.

### 4.5 Graphing two numerical variables

The proper graph for two numerical variables is a scatterplot. We graph the response variable on the y-axis and the predictor variable on the x-axis.

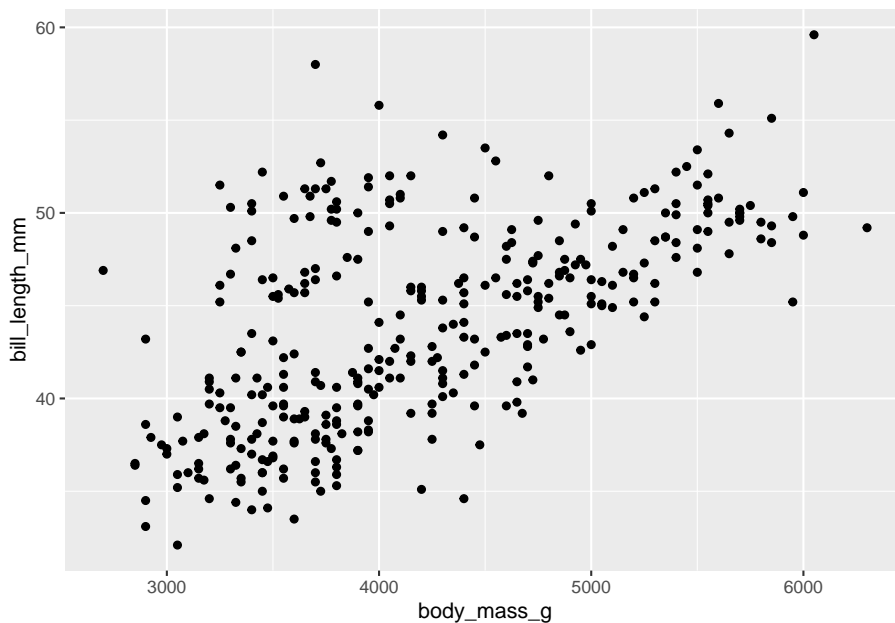
Let's consider a possible association between bill length and body mass. For this question, there is not really a strong preference for which variable serves as response and which variable serves as predictor. We'll consider bill length as the response variable and body mass as the predictor.

Since we are plotting two variables, we have two aesthetics, one on the y-axis (the response variable) and one on the x-axis (the predictor variable). Since

scatterplots use points to plot each data value, the correct layer to add is `geom_point()`.

```
ggplot(penguins, aes(y = bill_length_mm, x = body_mass_g)) +  
  geom_point()
```

```
## Warning: Removed 2 rows containing missing values (geom_point).
```



We are looking for evidence of a relationship between the two variables. This will manifest as a pattern in the data. We are interested in answering the following questions:

1. Linearity

- Is the association linear? In other words, do the data points lie roughly in a straight line pattern? The scatterplot above is a bit “cloudy” but generally moves from lower left to upper right in a straight (not curved pattern). It’s not a completely random scatter of dots.

2. Direction

- If the pattern is linear, it is a *positive* relationship or a *negative* one? Positive means that the line moves from lower left to upper right. Negative

means it moves from upper left to lower right. If you recall the direction of slopes from high school algebra class, a positive association corresponds to a line with a positive slope, and similarly for a negative association. In the data above, lower values of body mass correspond to lower bill lengths, and higher values of body mass correspond to higher bill lengths. So this is a positive association.

### 3. Strength

- If there is a pattern, how tight is the pattern? Do the data points stay close to a straight line, or are they pretty spread out and only generally moving in one direction. A strong relationship is one that is tightly packed around a line or curve. The relationship above is not strong. We might use terms like “weak”, “moderately weak”, or “moderate”, but definitely not strong.

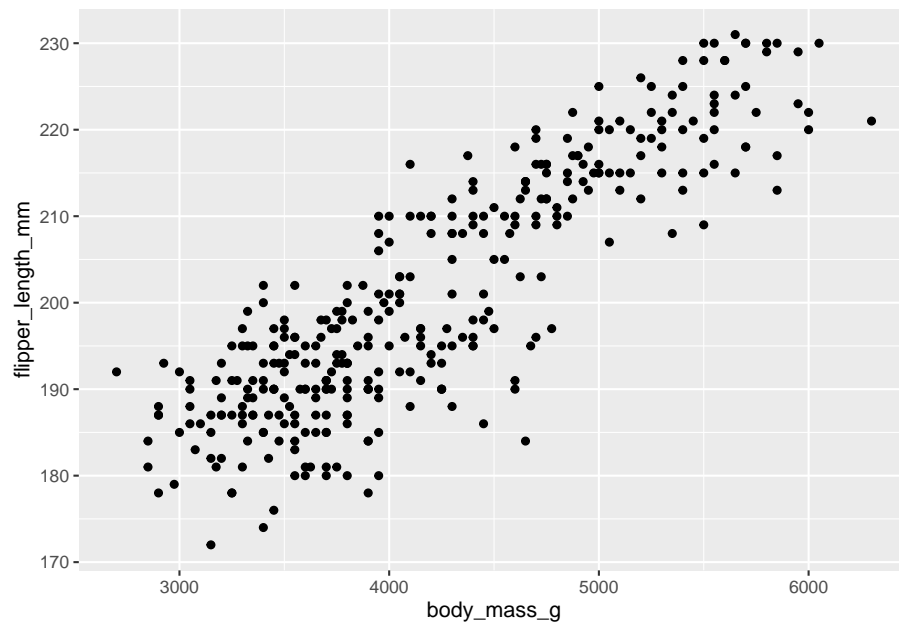
### 4. Outliers

- Are there outliers? These will be points that are isolated and relatively far from the bulk of the data. There are a few points above that are borderline, but none is a particularly strong outlier, especially give how spread out the rest of the data is.

**Exercise 8** Here is a scatterplot of

```
ggplot(penguins, aes(y = flipper_length_mm, x = body_mass_g)) +  
  geom_point()
```

```
## Warning: Removed 2 rows containing missing values (geom_point).
```



Write a short paragraph describing the association of penguin flipper lengths and body mass, focusing on the four key features (linearity, direction, strength, and outliers).

Please write up your answer here.

## 4.6 Graphing grouped numerical data

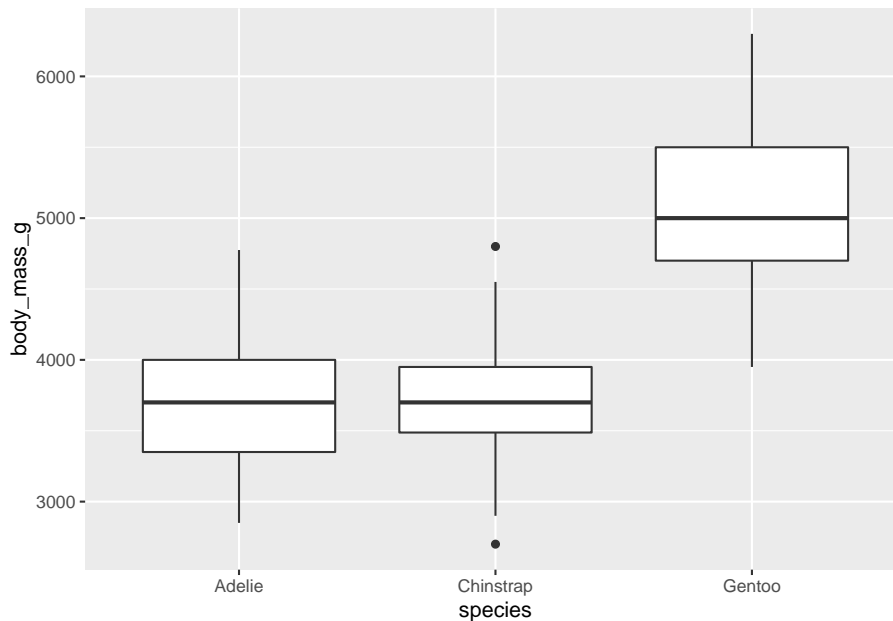
Suppose you want to analyze one numerical variable and one categorical variable. Usually, the idea here is that the categorical variable divides up the data into groups and you are interested in understanding the numerical variable for each group separately. Another way to say this is that your numerical variable is response and your categorical variable is predictor. (It is also possible for a categorical variable to be response and a numerical variable to be predictor. This is common in so-called “classification” problems. We will not cover this possibility in this course, but it is covered in more advanced courses.)

This turns out to be exactly what we need in the penguins data. Throughout the above exercises, there was a concern that the penguin measurements are fundamentally different among three different species of penguin.

Graphically, there are two good options here. The first is a side-by-side boxplot.

```
ggplot(penguins, aes(y = body_mass_g, x = species)) +  
  geom_boxplot()
```

```
## Warning: Removed 2 rows containing non-finite values (stat_boxplot).
```



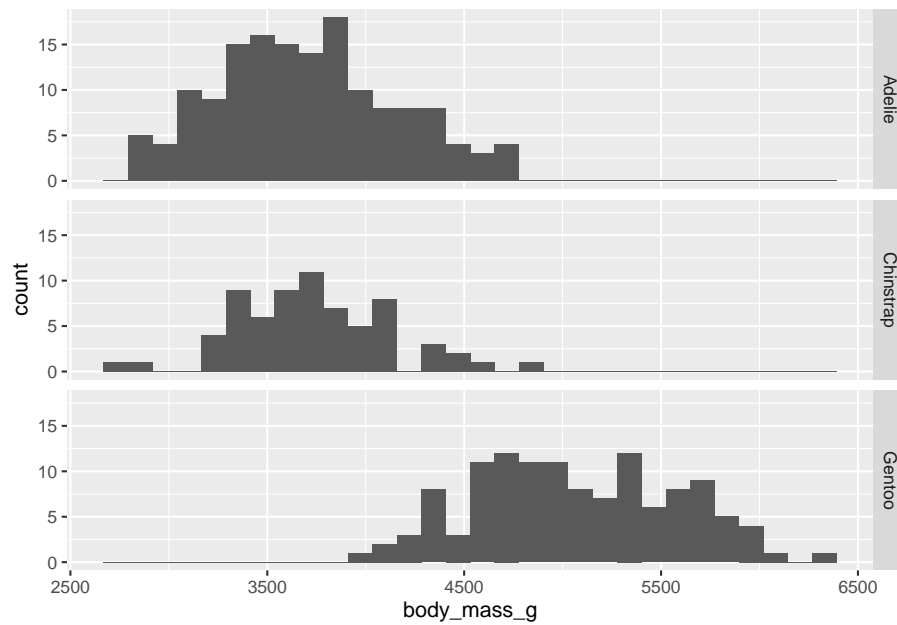
Notice the placement of the variables. The y-axis is `body_mass_g`, the numerical variable. The x-axis variable is `species`; the groups are placed along the x-axis. This is consistent with other graph types that place the response variable on the y-axis and the predictor variable on the x-axis.

The other possible graph is a stacked histogram. This uses a feature called “faceting” that creates a different plot for each group. The syntax is a little unusual.

```
ggplot(penguins, aes(x = body_mass_g)) +
  geom_histogram() +
  facet_grid(species ~ .)
```

```
## `stat_bin()` using `bins = 30`. Pick better value with `binwidth`.
```

```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```

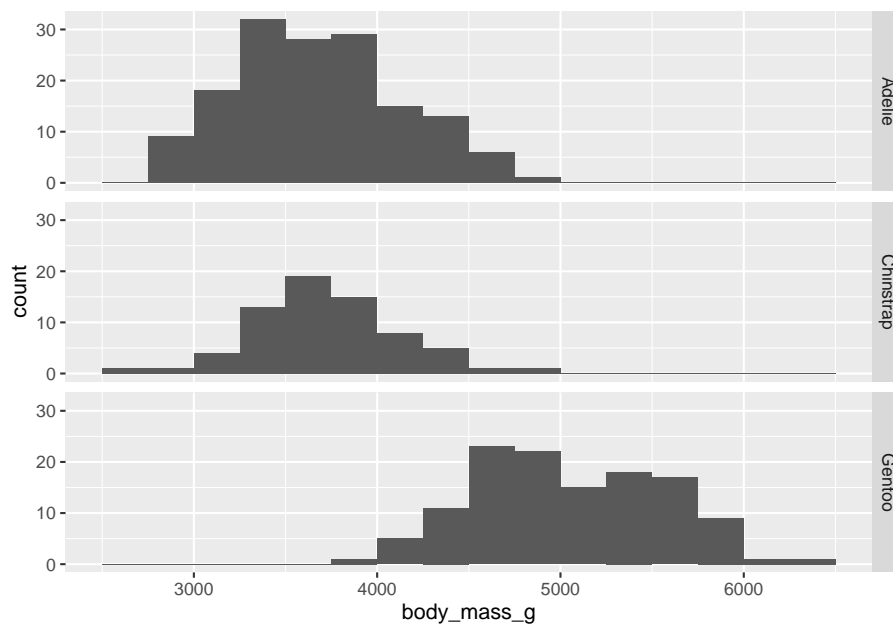


The argument `species ~ .` in the `facet_grid` function means, “Put each species on a different row.” We’ll explore this notation a little later.

As always, the default bins suck, so let’s change them.

```
ggplot(penguins, aes(x = body_mass_g)) +
  geom_histogram(binwidth = 250, boundary = 3500) +
  facet_grid(species ~ .)
```

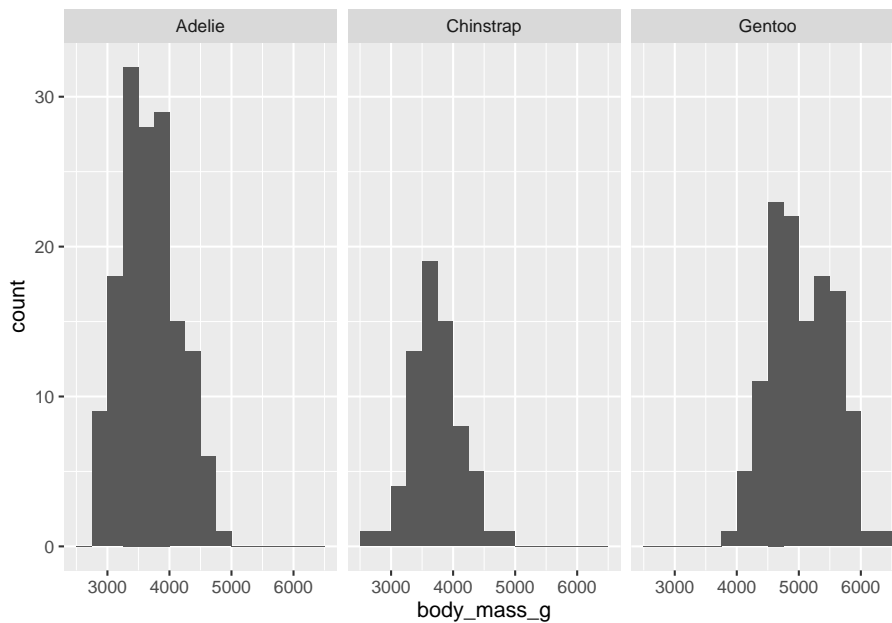
```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```



Consider the following subtle change in notation:

```
ggplot(penguins, aes(x = body_mass_g)) +  
  geom_histogram(binwidth = 250, boundary = 3500) +  
  facet_grid(. ~ species)
```

```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```



**Exercise 9(a)** Explain why that last graph (which might be called a side-by-side histogram) is less effective than the earlier stacked histogram. (Hint: what stays lined up when the histograms are stacked vertically rather than horizontally?)

Please write up your answer here.

**Exercise 9(b)** Can you figure out what's going on with the weird syntax of `species ~ .` vs `. ~ species`? Explain it in your own words.

Please write up your answer here.

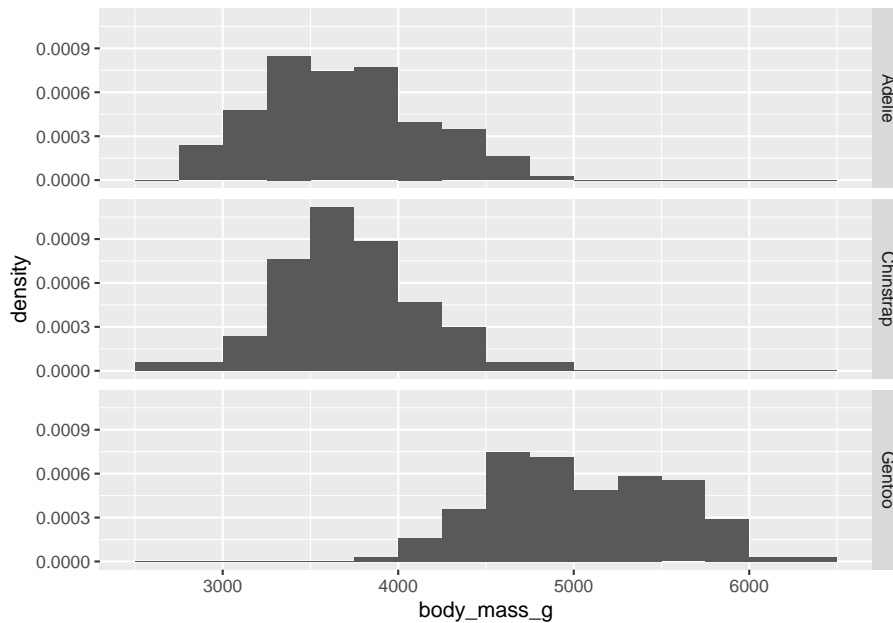
---

The other thing that kind of sucks is the fact that the y-axis is showing counts. That makes it harder to see the distribution of body mass among Chinstrap penguins, for example, as there are fewer of them in the data set. It would be nice to scale these using percentages.

```
ggplot(penguins, aes(x = body_mass_g)) +
  geom_histogram(aes(y = ..density..),
                 binwidth = 250, boundary = 3500) +
  facet_grid(species ~ .)
```



```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```



Due to some technical issues in `ggplot2`, these are not strictly proportions. (If you were to add up the heights of all the bars, they would not add up to 100%.) Nevertheless, the graph is still useful because it does scale the groups to put them on equal footing. In other words, it treats each group as if they all had the same sample size.

**Exercise 10** Choose a numerical variable that's not body mass and a categorical variable that's not species from the `penguins` data set. Make both a side-by-side boxplot and a stacked histogram. Discuss the resulting graphs. Comment on the association (or independence) of the two variables. If there is an association, be sure to focus on the four key features (linearity, direction, strength, and outliers).

```
# Add code here to create a side-by-side boxplot.
```

```
# Add code here to create a stacked histogram.
```

Please write up your answer here.

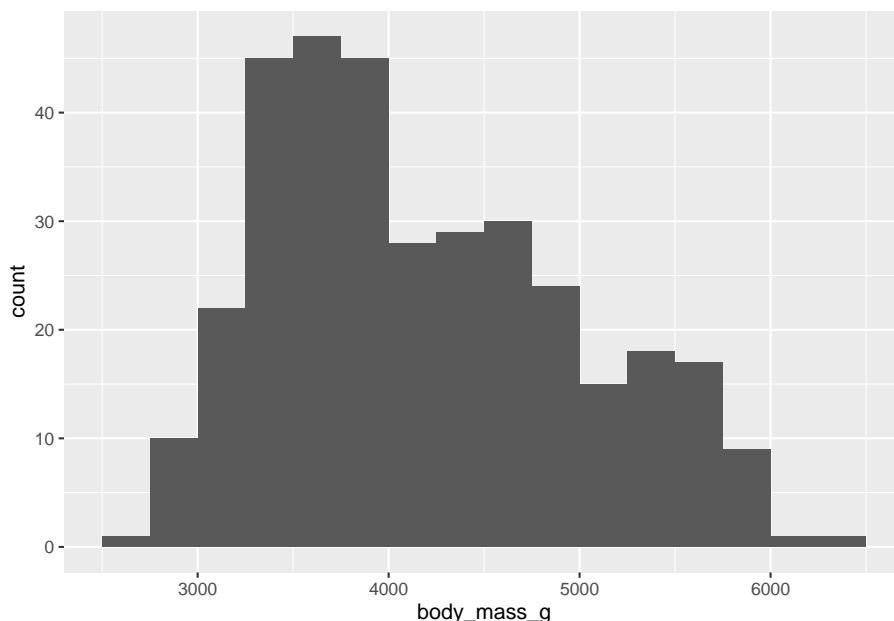
## 4.7 Publication-ready graphics

The great thing about `ggplot2` graphics is that they are already quite pretty. To take them from exploratory data analysis to the next level, there are a few things we can do to tidy them up.

Let's go back to the first histogram from this chapter.

```
ggplot(penguins, aes(x = body_mass_g)) +  
  geom_histogram(binwidth = 250, boundary = 3500)
```

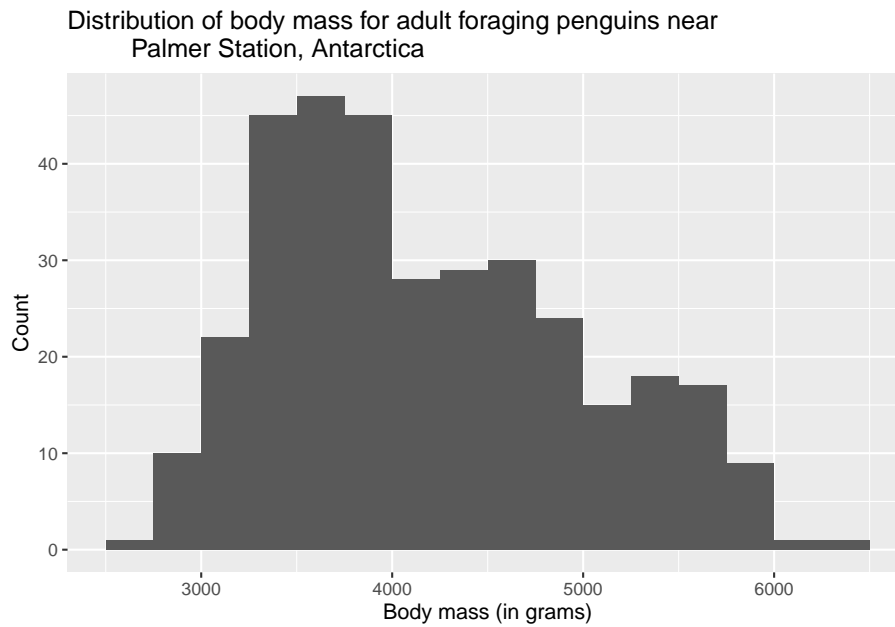
```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```



The variable names of this data set are already pretty informative, but we can do a little better with `labs` (for labels). Observe:

```
ggplot(penguins, aes(x = body_mass_g)) +  
  geom_histogram(binwidth = 250, boundary = 3500) +  
  labs(title = "Distribution of body mass for adult foraging penguins near  
         Palmer Station, Antarctica",  
        x = "Body mass (in grams)",  
        y = "Count")
```

```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```

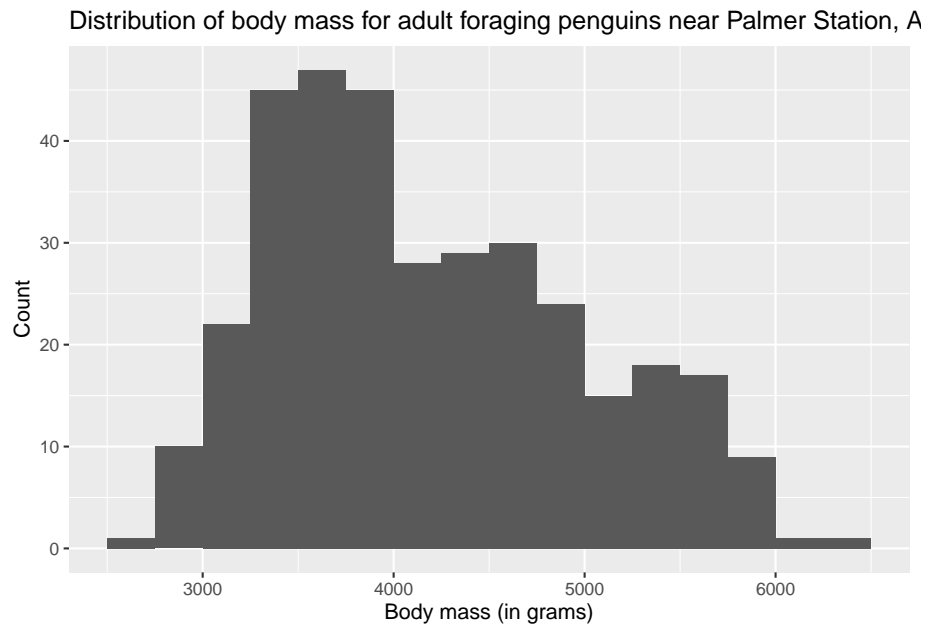


You can also see that we took the opportunity to mention the units of measurement (grams) for our variable in the x-axis label. This is good practice.

A quick note about formatting in R code chunks. Notice that I put different parts of the last `ggplot` command on their own separate lines. The command would still work if I did this:

```
ggplot(penguins, aes(x = body_mass_g)) + geom_histogram(binwidth = 250, boundary = 3500) + labs(t
```

```
## Warning: Removed 2 rows containing non-finite values (stat_bin).
```

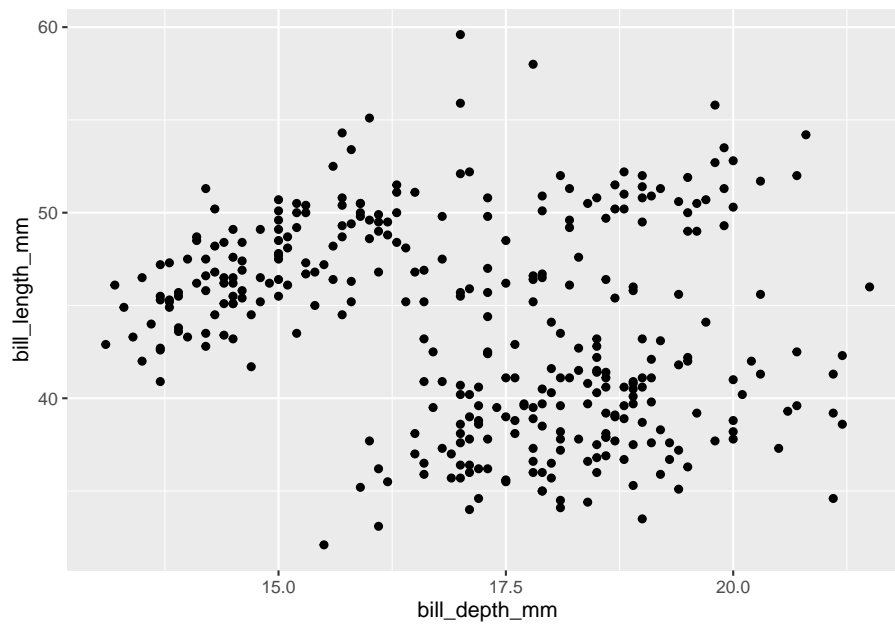


But it's much harder to read. If you find that your code is “wrapping” to the next line, find some spots like commas or plus signs to break the code. Be sure to break the line after the comma or plus sign.

**Exercise 11** Modify the following scatterplot by adding a title and labels for both the y-axis and x-axis.

```
# Modify the following scatterplot by adding a title and  
# labels for both the y-axis and x-axis.  
ggplot(penguins, aes(y = bill_length_mm, x = bill_depth_mm)) +  
  geom_point()
```

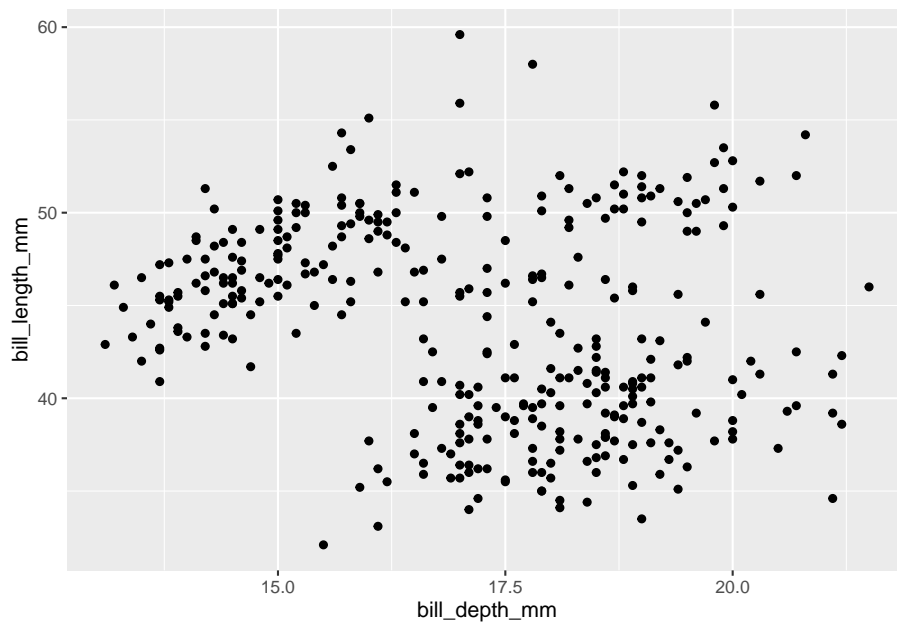
```
## Warning: Removed 2 rows containing missing values (geom_point).
```



**Exercise 12** The previous scatterplot looked a little funny due to some odd groupings that we suspect (as usual) might be due to multiple species being measures. Add a new aesthetic (so, inside the parentheses following `aes`) to the following code to assign `color = species`. Comment on what you see.

```
# Modify the code below to add color = species
ggplot(penguins, aes(y = bill_length_mm, x = bill_depth_mm)) +
  geom_point()
```

```
## Warning: Removed 2 rows containing missing values (geom_point).
```



Please write up your answer here.

---

Every part of the graph can be customized, from the color scheme to the tick marks on the axes, to the major and minor grid lines that appear on the background. We won't go into all that, but you can look at the `ggplot2` documentation online and search Google for examples if you want to dig in and figure out how to do some of that stuff. However, the default options are often (but not always) the best, so be careful that your messing around doesn't inadvertently make the graph less clear or less appealing.

## 4.8 Conclusion

Summary statistics are simple numbers that describe and summarize data sets. Measures of center tell us where the “middle” of our numerical data lies, and measures of spread tell us how spread out our numerical data is. These measures should always be reported in pairs, for example the mean/standard deviation, or the median/IQR.

The `ggplot2` package with its `ggplot` command is a very versatile tool for creating nice graphs relatively easily. For a single numerical variable, the standard graph type is a histogram. For two numerical variables, use a scatterplot. For a numerical response with a categorical predictor, use either a side-by-side boxplot or a stacked histogram.

### 4.8.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.





## Chapter 5

# Manipulating data

2.0

### Functions introduced in this chapter

`read_csv`, `select`, `rename`, `rm`, `filter`, `slice`, `arrange`, `mutate`, `all.equal`,  
`ifelse`, `transmute`, `summarise`, `group_by`, `%>%`, `count`

### 5.1 Introduction

This tutorial will import some data from the web and then explore it using the amazing `dplyr` package, a package which is quickly becoming the *de facto* standard among R users for manipulating data. It's part of the **tidyverse** that we've already used in several chapters.

#### 5.1.1 Install new packages

There are no new packages used in this chapter.

#### 5.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/05-manipulating\\_data.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/05-manipulating_data.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 5.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

### 5.1.4 Load packages

We load the `tidyverse` package as usual, but this time it is to give us access to the `dplyr` package, which is loaded alongside our other `tidyverse` packages like `ggplot2`. The `tidyverse` also has a package called `readr` that will allow us to import data from an external source (in this case, a web site).

```
library(tidyverse)
```

## 5.2 Importing CSV data

For most of the chapters, we use data sets that are either included in base R or included in a package that can be loaded into R. But it is useful to see how to get a data set from outside the R ecosystem. This depends a lot on the format of the data file, but a common format is a “comma-separated values” file, or CSV file. If you have a data set that is not formatted as a CSV file, it is usually pretty easy to open it in something like Google Spreadsheets or Microsoft Excel and then re-save it as a CSV file.

The file we’ll import is a random sample from all the commercial domestic flights that departed from Houston, Texas, in 2011.

We use the `read_csv` command to import a CSV file. In this case, we’re grabbing the file from a web page where the file is hosted. If you have a file on your computer, you can also put the file into your project directory and import it from there. Put the URL (for a web page) or the filename (for a file in your project directory) in quotes inside the `read_csv` command. We also need to assign the output to a tibble, so we’ve called it `hf` for “Houston flights”.

```
hf <- read_csv("https://vectorposse.github.io/intro_stats/data/hf.csv")
```

```
## Rows: 22758 Columns: 21
## -- Column specification -----
## Delimiter: ","
## chr (5): UniqueCarrier, TailNum, Origin, Dest, CancellationCode
```

```
## dbl (16): Year, Month, DayofMonth, DayOfWeek, DepTime, ArrTime, FlightNum, A...
##
## i Use `spec()` to retrieve the full column specification for this data.
## i Specify the column types or set `show_col_types = FALSE` to quiet this message.
```

```
hf
```

```
## # A tibble: 22,758 x 21
##   Year Month DayofMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>    <dbl>    <dbl>   <dbl>   <dbl> <chr>          <dbl>
## 1  2011     1      12        3    1419    1515 AA           428
## 2  2011     1      17        1    1530    1634 AA           428
## 3  2011     1      24        1    1356    1513 AA           428
## 4  2011     1       9        7     714     829 AA           460
## 5  2011     1      18        2     721     827 AA           460
## 6  2011     1      22        6     717     829 AA           460
## 7  2011     1      11        2    1953    2051 AA           533
## 8  2011     1      14        5    2119    2229 AA           533
## 9  2011     1      26        3    2009    2103 AA           533
## 10 2011     1      14        5    1629    1734 AA          1121
## # ... with 22,748 more rows, and 13 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>
```

```
glimpse(hf)
```

```
## Rows: 22,758
## Columns: 21
## $ Year          <dbl> 2011, 2011, 2011, 2011, 2011, 2011, 2011, 2011, 2011, 2011~
## $ Month         <dbl> 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1~
## $ DayofMonth    <dbl> 12, 17, 24, 9, 18, 22, 11, 14, 26, 14, 18, 20, 3, 12~
## $ DayOfWeek     <dbl> 3, 1, 1, 7, 2, 6, 2, 5, 3, 5, 2, 4, 1, 3, 6, 4, 1, 3~
## $ DepTime       <dbl> 1419, 1530, 1356, 714, 721, 717, 1953, 2119, 2009, 1~
## $ ArrTime       <dbl> 1515, 1634, 1513, 829, 827, 829, 2051, 2229, 2103, 1~
## $ UniqueCarrier <chr> "AA", "AA", "AA", "AA", "AA", "AA", "AA", "AA", "AA", "AA"~
## $ FlightNum     <dbl> 428, 428, 428, 460, 460, 460, 533, 533, 533, 1121, 1~
## $ TailNum       <chr> "N577AA", "N518AA", "N531AA", "N586AA", "N558AA", "N~
## $ ActualElapsedTime <dbl> 56, 64, 77, 75, 66, 72, 58, 70, 54, 65, 135, 144, 64~
## $ AirTime       <dbl> 41, 48, 43, 51, 46, 47, 44, 45, 39, 47, 114, 111, 46~
## $ ArrDelay      <dbl> 5, 84, 3, -6, -8, -6, -29, 69, -17, -11, 39, -1, -2,~
## $ DepDelay      <dbl> 19, 90, -4, -6, 1, -3, -12, 74, 4, -1, 44, -5, -1, 1~
## $ Origin        <chr> "IAH", "IAH", "IAH", "IAH", "IAH", "IAH", "IAH", "IA~
## $ Dest          <chr> "DFW", "DFW", "DFW", "DFW", "DFW", "DFW", "DFW", "DF~
```

```
## $ Distance      <dbl> 224, 224, 224, 224, 224, 224, 224, 224, 224, 224, 96~
## $ TaxiIn        <dbl> 4, 8, 6, 11, 7, 18, 3, 5, 9, 8, 7, 20, 5, 8, 8, 7, 4~
## $ TaxiOut       <dbl> 11, 8, 28, 13, 13, 7, 11, 20, 6, 10, 14, 13, 13, 10, ~
## $ Cancelled     <dbl> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0~
## $ CancellationCode <chr> NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, NA, ~
## $ Diverted      <dbl> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0~
```

The one disadvantage of a file imported from the internet or your computer is that it does not come with a help file. (Only packages in R have help files.) Hopefully you have access to some kind of information about the data you’re importing. In this case, we get lucky because the full Houston flights data set happens to be available in a package called `hflights`.

**Exercise 1** Go to the help tab in RStudio and search for `hflights`. Of the several options that appear, click the one from the `hflights` package (listed as `hflights::hflights`). Review the help file so you know what all the variables mean. Report below how many cases are in the original `hflights` data. What fraction of the original data has been sampled in the CSV file we imported above?

Please write up your answer here.

## 5.3 Introduction to dplyr

The `dplyr` package (pronounced “dee-ply-er”) contains tools for manipulating the rows and columns of tibbles. The key to using `dplyr` is to familiarize yourself with the “key verbs”:

- `select` (and `rename`)
- `filter` (and `slice`)
- `arrange`
- `mutate` (and `transmute`)
- `summarise` (with `group_by`)

We’ll consider these one by one. We won’t have time to cover every aspect of these functions. More information appears in the help files, as well as this very helpful “cheat sheet”: <https://raw.githubusercontent.com/rstudio/cheatsheets/main/data-transformation.pdf>

## 5.4 select

The `select` verb is very easy. It just selects some subset of variables (the columns of your data set).

The `select` command from the `dplyr` package illustrates one of the common issues R users face. Because the word “select” is pretty common, and selecting things is a common task, there are multiple packages that have a function called `select`. Depending on the order in which packages were loaded, R might get confused as to which version of `select` you want and try to apply the wrong one. One way to get the correct version is to specify the package in the syntax. Instead of typing `select`, we can type `dplyr::select` to ensure we get the version from the `dplyr` package. We’ll do this in all future uses of the `select` function. (The other functions in this chapter don’t cause us trouble because we don’t use any other packages whose functions conflict like this.)

Suppose all we wanted to see was the carrier, origin, and destination. We would type

```
hf_select <- dplyr::select(hf, UniqueCarrier, Origin, Dest)
hf_select
```

```
## # A tibble: 22,758 x 3
##   UniqueCarrier Origin Dest
##   <chr>          <chr> <chr>
## 1 AA           IAH   DFW
## 2 AA           IAH   DFW
## 3 AA           IAH   DFW
## 4 AA           IAH   DFW
## 5 AA           IAH   DFW
## 6 AA           IAH   DFW
## 7 AA           IAH   DFW
## 8 AA           IAH   DFW
## 9 AA           IAH   DFW
## 10 AA          IAH   DFW
## # ... with 22,748 more rows
```

A brief but important aside here: there is nothing special about the variable name `hf_select`. I could have typed

```
beef_gravy <- dplyr::select(hf, UniqueCarrier, Origin, Dest)
```

and it would work just as well. Generally speaking, though, you want to give variables a name that reflects the intent of your analysis.

Also, **it is important to assign the result to a new variable**. If I had typed

```
hf <- dplyr::select(hf, UniqueCarrier, Origin, Dest)
```

this would have overwritten the original tibble `hf` with this new version with only three variables. I want to preserve `hf` because I want to do other things with the entire data set later. The take-home message here is this: **Major**

**modifications to your data should generally be given a new variable name.** There are caveats here, though. Every time you create a new variable, you also fill up more memory with your creation. If you check your Global Environment, you'll see that both `hf` and `hf_select` are sitting in there. We'll have more to say about this in a moment.

Okay, back to the `select` function. The first argument of `select` is the tibble. After that, just list all the names of the variables you want to select.

If you don't like the names of the variables, you can change them as part of the select process.

```
hf_select <- dplyr::select(hf,
                           carrier = UniqueCarrier,
                           origin = Origin,
                           dest = Dest)
hf_select
```

```
## # A tibble: 22,758 x 3
##   carrier origin dest
##   <chr>   <chr> <chr>
## 1 AA      IAH    DFW
## 2 AA      IAH    DFW
## 3 AA      IAH    DFW
## 4 AA      IAH    DFW
## 5 AA      IAH    DFW
## 6 AA      IAH    DFW
## 7 AA      IAH    DFW
## 8 AA      IAH    DFW
## 9 AA      IAH    DFW
## 10 AA     IAH    DFW
## # ... with 22,748 more rows
```

(Note here that I am overwriting `hf_select` which had been defined slightly differently before. However, these two versions of `hf_select` are basically the same object, so no need to keep two copies here.)

There are a few notational shortcuts. For example, see what the following do.

```
hf_select2 <- dplyr::select(hf, DayOfWeek:UniqueCarrier)
hf_select2
```

```
## # A tibble: 22,758 x 4
##   DayOfWeek DepTime ArrTime UniqueCarrier
##   <dbl>   <dbl>   <dbl> <chr>
## 1       3     1419     1515 AA
```

```
## 2      1    1530    1634 AA
## 3      1    1356    1513 AA
## 4      7     714     829 AA
## 5      2     721     827 AA
## 6      6     717     829 AA
## 7      2    1953    2051 AA
## 8      5    2119    2229 AA
## 9      3    2009    2103 AA
## 10     5    1629    1734 AA
## # ... with 22,748 more rows
```

```
hf_select3 <- dplyr::select(hf, starts_with("Taxi"))
hf_select3
```

```
## # A tibble: 22,758 x 2
##   TaxiIn TaxiOut
##   <dbl>   <dbl>
## 1      4      11
## 2      8       8
## 3      6      28
## 4     11      13
## 5      7      13
## 6     18       7
## 7      3      11
## 8      5      20
## 9      9       6
## 10     8      10
## # ... with 22,748 more rows
```

**Exercise 2** What is contained in the new tibbles `hf_select2` and `hf_select3`? In other words, what does the colon (`:`) appear to do and what does `starts_with` appear to do in the `select` function?

Please write up your answer here.

---

The cheat sheet shows a lot more of these “helper functions” if you’re interested.

The other command that’s related to `select` is `rename`. The only difference is that `select` will throw away any columns you don’t select (which is what you want and expect, typically), whereas `rename` will keep all the columns, but rename those you designate.

**Exercise 3** Putting a minus sign in front of a variable name in the `select` command will remove the variable. Create a tibble called `hf_select4` that removes `Year`, `DayofMonth`, `DayOfWeek`, `FlightNum`, and `Diverted`. (Be careful with the unusual—and inconsistent!—capitalization in those variable names.) In the second part of the code chunk below, type `hf_select4` so that the tibble prints to the screen (just like in all the above examples).

```
# Add code here to define hf_select4.  
# Add code here to print hf_select4.
```

## 5.5 The `rm` command

Recall that earlier we mentioned the pros and cons of creating a new tibble every time we make a change. On one hand, making a new tibble instead of overwriting the original one will keep the original one available so that we can run different commands on it. On the other hand, making a new tibble does eat up a lot of memory.

One way to get rid of an object once we are done with it is the `rm` command, where `rm` is short for “remove”. When you run the code chunk below, you’ll see that all the tibbles we created with `select` will disappear from your Global Environment.

```
rm(hf_select, hf_select2, hf_select3)
```

If you need one these tibbles back later, you can always go back and re-run the code chunk that defined it.

We’ll use `rm` at the end of some of the following sections so that we don’t use up too much memory.

**Exercise 4** Remove `hf_select4` (that you created in Exercise 3) from the Global Environment.

```
# Add code here to remove hf_select4.
```

## 5.6 `filter`

The `filter` verb works a lot like `select`, but for rows instead of columns.

For example, let’s say we only want to see Delta flights. We use `filter`:



```
hf_filter <- filter(hf, UniqueCarrier == "DL")
hf_filter
```

```
## # A tibble: 265 x 21
##   Year Month DayofMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>      <dbl>      <dbl>   <dbl>   <dbl> <chr>          <dbl>
## 1  2011     1         4          2    1834    2134 DL             54
## 2  2011     1         5          3    1606    1903 DL             8
## 3  2011     1         5          3     543     834 DL          1248
## 4  2011     1         7          5    1603    1902 DL             8
## 5  2011     1         7          5    1245    1539 DL          1204
## 6  2011     1         7          5     933    1225 DL          1590
## 7  2011     1         8          6     921    1210 DL          1590
## 8  2011     1        12          3      NA      NA DL          1590
## 9  2011     1        13          4     928    1224 DL          1590
## 10 2011     1        13          4     656     947 DL          1900
## # ... with 255 more rows, and 13 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>
```

In the printout of the tibble above, if you can't see the `UniqueCarrier` column, click the black arrow on the right to scroll through the columns until you can see it. You can click "Next" at the bottom to scroll through the rows.

**Exercise 5** How many rows did we get in the `hf_filter` tibble? What do you notice about the `UniqueCarrier` of all those rows?

Please write up your answer here.

---

Just like `select`, the first argument of `filter` is the name of the tibble. Following that, you must specify some condition. Only rows meeting that condition will be included in the output.

One thing that is unusual here is the double equal sign (`UniqueCarrier == "DL"`). This won't be a mystery to people with programming experience, but it tends to be a sticking point for the rest of us. A single equals sign represents assignment. If I type `x = 3`, what I mean is, "Take the letter `x` and assign it the value 3." In R, we would also write `x <- 3` to mean the same thing. The first line of the code chunk below assigns `x` to be 3. Therefore, the following line that just says `x` creates the output "3".

```
x = 3
x
```

```
## [1] 3
```

On the other hand, `x == 3` means something completely different. This is a logical statement that is either true or false. Either `x` is 3, in which case we get `TRUE` or `x` is not 3, and we get `FALSE`.

```
x == 3
```

```
## [1] TRUE
```

(It's true because we just assigned `x` to be 3 in the previous code chunk!)

In the above `filter` command, we are saying, "Give me the rows where the value of `UniqueCarrier` is "DL", or, in other words, where the statement `UniqueCarrier == "DL"` is true.

As another example, suppose we wanted to find out all flights that leave before 6:00 a.m.

```
hf_filter2 <- filter(hf, DepTime < 600)
hf_filter2
```

```
## # A tibble: 230 x 21
##   Year Month DayofMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>    <dbl>    <dbl>    <dbl>    <dbl> <chr>          <dbl>
## 1  2011     1        20         4      556      912 AA             1994
## 2  2011     1        21         5      555      822 CO             446
## 3  2011     1        18         2      555      831 CO             446
## 4  2011     1        16         7      556      722 CO             199
## 5  2011     1         5         3      558     1009 CO              89
## 6  2011     1         1         6      558     1006 CO              89
## 7  2011     1         5         3      543      834 DL            1248
## 8  2011     1         3         1      555      749 US             270
## 9  2011     1         6         4      556      801 US             270
## 10 2011     1        13         4      552      713 US             270
## # ... with 220 more rows, and 13 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>
```

**Exercise 6** Look at the help file for `hflights` again. Why do we have to use the number 600 in the command above? (Read the description of the `DepTime` variable.)

Please write up your answer here.

If we need two or more conditions, we use `&` for “and” and `|` for “or”. The following will give us only the Delta flights that departed before 6:00 a.m.

```
hf_filter3 <- filter(hf, UniqueCarrier == "DL" & DepTime < 600)
hf_filter3
```

```
## # A tibble: 30 x 21
##   Year Month DayOfMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>      <dbl>      <dbl>   <dbl>   <dbl> <chr>          <dbl>
## 1  2011     1         5         3     543     834 DL           1248
## 2  2011     1        16         7     542     834 DL           1248
## 3  2011     1        19         3     538     844 DL           1248
## 4  2011     1        22         6     540     850 DL           1248
## 5  2011     1        26         3     540     851 DL           1248
## 6  2011     2        12         6     538     823 DL           1248
## 7  2011     2        15         2     539     840 DL           1248
## 8  2011     2        16         3     540     829 DL           1248
## 9  2011     2        21         1     552     856 DL           1248
## 10 2011     3         2         3     557     902 DL           2375
## # ... with 20 more rows, and 13 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>
```

Again, check the cheat sheet for more complicated condition-checking if needed.

**Exercise 7(a)** The symbol `!=` means “not equal to” in R. Use the `filter` command to create a tibble called `hf_filter4` that finds all flights *except* those flying into Salt Lake City (“SLC”). As before, print the output to the screen.

```
# Add code here to define hf_filter4.
# Add code here to print hf_filter4.
```

**Exercise 7(b)** Based on the output of the previous part, how many flights were there flying into SLC? (In other words, how many rows were removed from the original `hf` tibble to produce `hf_filter4`?)

Please write up your answer here.

**Exercise 8** Use the `rm` command to remove all the extra tibbles you created in this section with `filter`.

```
# Add code here to remove all filtered tibbles.
```

---

The `slice` command is related, but fairly useless in practice. It will allow you to extract rows by position. So `slice(hf, 1:10)` will give you the first 10 rows. As a general rule, the information available in a tibble should never depend on the order in which the rows appear. Therefore, no function you run should make any assumptions about the ordering of your data. The only reason one might want to think about the order of data is for convenience in presenting that data visually for someone to inspect. And that brings us to...

## 5.7 arrange

This just re-orders the rows, sorting on the values of one or more specified columns. As I mentioned before, in most data analyses you work with summaries of the data that do not depend on the order of the rows, so this is not quite as interesting as some of the other verbs. In fact, since the re-ordering is usually for the visual benefit of the reader, there is often no need to store the output in a new variable. We'll just print the output to the screen.

```
arrange(hf, ActualElapsedTime)
```

```
## # A tibble: 22,758 x 21
##   Year Month DayofMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>    <dbl>    <dbl>    <dbl>    <dbl> <chr>          <dbl>
## 1  2011    10         5         3    1656    1731 WN            2493
## 2  2011     4        13         3    1207    1243 WN            2025
## 3  2011     7        19         2    1043    1119 CO            1583
## 4  2011     2        22         2    1426    1503 WN            1773
## 5  2011     3        19         6    1629    1706 WN            3805
## 6  2011     5        31         2    1937    2014 WN             819
## 7  2011     7        16         6    1632    1709 WN             912
## 8  2011     8        22         1    1708    1745 WN            1754
## 9  2011     9        30         5    1955    2032 WN            1959
## 10 2011     9         1         4    1735    1812 WN            1754
## # ... with 22,748 more rows, and 13 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>
```

Scroll over to the `ActualElapsedTime` variable in the output above (using the black right arrow) to see that these are now sorted in ascending order.

**Exercise 9** How long is the shortest actual elapsed time? Why is this flight so short? (Hint: look at the destination.) Which airline flies that route? You may have to use your best friend Google to look up airport and airline codes.

Please write up your answer here.

---

If you want descending order, do this:

```
arrange(hf, desc(ActualElapsedTime))
```

```
## # A tibble: 22,758 x 21
##   Year Month DayofMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>    <dbl>    <dbl>   <dbl>   <dbl> <chr>          <dbl>
## 1  2011     2         4         5     941    1428 CO             1
## 2  2011    11         8         2     937    1417 CO             1
## 3  2011    11        11         5     930    1408 CO             1
## 4  2011    12        30         5     936    1413 CO             1
## 5  2011    12         8         4     935    1410 CO             1
## 6  2011    10        17         1     938    1311 CO             1
## 7  2011     6        27         1     936    1308 CO             1
## 8  2011     3        24         4     926    1256 CO             1
## 9  2011    12        27         2     935    1405 CO             1
##10  2011     3         9         3     933    1402 CO             1
## # ... with 22,748 more rows, and 13 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>
```

**Exercise 10** How long is the longest actual elapsed time? Why is this flight so long? Which airline flies that route? Again, you may have to use your best friend Google to look up airport and airline codes.

Please write up your answer here.

**Exercise 11(a)** You can sort by multiple columns. The first column listed will be the first in the sort order, and then within each level of that first variable, the next column will be sorted, etc. Print a tibble that sorts first by destination (`Dest`) and then by arrival time (`ArrTime`), both in the default ascending order.

```
# Add code here to sort hf first by Dest and then by ArrTime.
```

**Exercise 11(b)** Based on the output of the previous part, what is the first airport code alphabetically and to what city does it correspond? (Use Google if you need to link the airport code to a city name.) At what time did the earliest flight to that city arrive?

Please write up your answer here.

## 5.8 mutate

Frequently, we want to create new variables that combine information from one or more existing variables. We use `mutate` for this. For example, suppose we wanted to find the total time of the flight. We might do this by adding up the minutes from several variables: `TaxiOut`, `AirTime`, and `TaxiIn`, and assigning that sum to a new variable called `total`. Scroll all the way to the right in the output below (using the black right arrow) to see the new `total` variable.

```
hf_mutate <- mutate(hf, total = TaxiOut + AirTime + TaxiIn)
hf_mutate

## # A tibble: 22,758 x 22
##   Year Month DayofMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>      <dbl>      <dbl> <dbl>   <dbl> <chr>          <dbl>
## 1  2011     1         12          3  1419   1515 AA             428
## 2  2011     1         17          1  1530   1634 AA             428
## 3  2011     1         24          1  1356   1513 AA             428
## 4  2011     1          9          7   714    829 AA             460
## 5  2011     1         18          2   721    827 AA             460
## 6  2011     1         22          6   717    829 AA             460
## 7  2011     1         11          2  1953   2051 AA             533
## 8  2011     1         14          5  2119   2229 AA             533
## 9  2011     1         26          3  2009   2103 AA             533
## 10 2011     1         14          5  1629   1734 AA            1121
## # ... with 22,748 more rows, and 14 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>, total <dbl>
```

As it turns out, that was wasted effort because that variable already exists in `ActualElapsedTime`. The `all.equal` command below checks that both specified columns contain the exact same values.

```
all.equal(hf_mutate$total, hf$ActualElapsedTime)
```

```
## [1] TRUE
```

Perhaps we want a variable that just classifies a flight as arriving late or not. Scroll all the way to the right in the output below to see the new `late` variable.

```
hf_mutate2 <- mutate(hf, late = (ArrDelay > 0))
hf_mutate2
```

```
## # A tibble: 22,758 x 22
##   Year Month DayOfMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>      <dbl>      <dbl> <dbl>   <dbl> <chr>          <dbl>
## 1  2011     1         12          3  1419   1515 AA             428
## 2  2011     1         17          1  1530   1634 AA             428
## 3  2011     1         24          1  1356   1513 AA             428
## 4  2011     1          9          7   714    829 AA             460
## 5  2011     1         18          2   721    827 AA             460
## 6  2011     1         22          6   717    829 AA             460
## 7  2011     1         11          2  1953   2051 AA             533
## 8  2011     1         14          5  2119   2229 AA             533
## 9  2011     1         26          3  2009   2103 AA             533
## 10 2011     1         14          5  1629   1734 AA            1121
## # ... with 22,748 more rows, and 14 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>, late <lgl>
```

This one is a little tricky. Keep in mind that `ArrDelay > 0` is a logical condition that is either true or false, so that truth value is what is recorded in the `late` variable. If the arrival delay is a positive number of minutes, the flight is considered “late”, and if the arrival delay is zero or negative, it’s not late.

If we would rather see more descriptive words than `TRUE` or `FALSE`, we have to do something even more tricky. Look at the `late` variable in the output below.

```
hf_mutate3 <- mutate(hf,
                      late = as_factor(ifelse(ArrDelay > 0,
                                                "Late", "On time")))
hf_mutate3
```

```
## # A tibble: 22,758 x 22
##   Year Month DayOfMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
```

```
##      <dbl> <dbl>      <dbl>      <dbl>      <dbl>      <dbl> <chr>      <dbl>
## 1  2011      1        12          3      1419      1515 AA        428
## 2  2011      1        17          1      1530      1634 AA        428
## 3  2011      1        24          1     1356      1513 AA        428
## 4  2011      1         9          7       714      829 AA        460
## 5  2011      1        18          2       721      827 AA        460
## 6  2011      1        22          6       717      829 AA        460
## 7  2011      1        11          2     1953     2051 AA        533
## 8  2011      1        14          5     2119     2229 AA        533
## 9  2011      1        26          3     2009     2103 AA        533
## 10 2011      1        14          5     1629     1734 AA       1121
## # ... with 22,748 more rows, and 14 more variables: TailNum <chr>,
## #   ActualElapsedTime <dbl>, AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>,
## #   Origin <chr>, Dest <chr>, Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>,
## #   Cancelled <dbl>, CancellationCode <chr>, Diverted <dbl>, late <fct>
```

The `as_factor` command tells R that `late` should be a categorical variable. Without it, the variable would be a “character” variable, meaning a list of character strings. It won’t matter for us here, but in any future analysis, you want categorical data to be treated as such by R.

The main focus here is on the `ifelse` construction. The `ifelse` function takes a condition as its first argument. If the condition is true, it returns the value in the second slot, and if it’s false (the “else” part of if/else), it returns the value in the third slot. In other words, if `ArrDelay > 0`, this means the flight is late, so the new `late` variable should say “Late”; whereas, if `ArrDelay` is not greater than zero (so either zero or possibly negative if the flight arrived early), then the new variable should say “On Time”.

Having said that, I would generally recommend that you leave these kinds of variables as logical types. It’s much easier to summarize such variables in R, namely because R treats `TRUE` as 1 and `FALSE` as 0, allowing us to do things like this:

```
mean(hf_mutate2$late, na.rm = TRUE)
```

```
## [1] 0.4761522
```

This gives us the proportion of late flights.

Note that we needed `na.rm` as you’ve seen in previous chapter. For example, look at the 93rd row of the tibble:

```
slice(hf_mutate2, 93)
```

```
## # A tibble: 1 x 22
```



```
##   Year Month DayofMonth DayOfWeek DepTime ArrTime UniqueCarrier FlightNum
##   <dbl> <dbl>      <dbl>      <dbl>  <dbl>  <dbl> <chr>          <dbl>
## 1  2011     1         27         4      NA      NA CO             258
## # ... with 14 more variables: TailNum <chr>, ActualElapsedTime <dbl>,
## #   AirTime <dbl>, ArrDelay <dbl>, DepDelay <dbl>, Origin <chr>, Dest <chr>,
## #   Distance <dbl>, TaxiIn <dbl>, TaxiOut <dbl>, Cancelled <dbl>,
## #   CancellationCode <chr>, Diverted <dbl>, late <lgl>
```

Notice that all the times are missing. There are a bunch of rows like this. Since there is not always an arrival delay listed, the `ArrDelay` variable doesn't always have a value, and if `ArrDelay` is `NA`, the `late` variable will be too. So if we try to calculate the mean with just the `mean` command, this happens:

```
mean(hf_mutate2$late)
```

```
## [1] NA
```

**Exercise 12** Why does taking the mean of a bunch of zeros and ones give us the proportion of ones? (Think about the formula for the mean. What happens when we take the sum of all the zeros and ones, and what happens when we divide by the total?)

Please write up your answer here.

**Exercise 13** Create a new tibble called `hf_mutate4` that uses the `mutate` command to create a new variable called `dist_k` which measures the flight distance in kilometers instead of miles. (Hint: to get from miles to kilometers, multiply the distance by 1.60934.) Print the output to the screen.

```
# Add code here to define hf_mutate4.
# Add code here to print hf_mutate4.
```

---

A related verb is `transmute`. The only difference between `mutate` and `transmute` is that `mutate` creates the new column(s) and keeps all the old ones too, whereas `transmute` will throw away all the columns except the newly created ones. This is not something that you generally want to do, but there are exceptions. For example, if I was preparing a report and I needed only to talk about flights being late or not, it would do no harm (and would save some memory) to throw away everything except the `late` variable.

Before moving on to the next section, we'll clean up the extra tibbles lying around. You'll need to manually click the run button in the next code chunk since you have defined `hf_mutate4`.

```
rm(hf_mutate, hf_mutate2, hf_mutate3, hf_mutate4)
```

```
## Warning in rm(hf_mutate, hf_mutate2, hf_mutate3, hf_mutate4): object
## 'hf_mutate4' not found
```

## 5.9 summarise (with group\_by)

First, before you mention that `summarise` is spelled wrong...well, the author of the `dplyr` package is named Hadley Wickham (same author as the `ggplot2` package) and he is from New Zealand. So that's the way he spells it. He was nice enough to include the `summarize` function as an alias if you need to use it 'cause this is 'Murica!

The `summarise` function, by itself, is kind of boring, and doesn't do anything that couldn't be done more easily with base R functions.

```
summarise(hf, mean(Distance))
```

```
## # A tibble: 1 x 1
##   `mean(Distance)`
##             <dbl>
## 1             791.
```

```
mean(hf$Distance)
```

```
## [1] 790.5861
```

Where `summarise` shines is in combination with `group_by`. For example, let's suppose that we want to see average flight distances, but broken down by airline. We can do the following:

```
hf_summ_grouped <- group_by(hf, UniqueCarrier)
hf_summ <- summarise(hf_summ_grouped, mean(Distance))
hf_summ
```

```
## # A tibble: 15 x 2
##   UniqueCarrier `mean(Distance)`
##   <chr>             <dbl>
## 1 AA              470.
## 2 AS             1874
## 3 B6             1428
```

```
## 4 CO 1097.  
## 5 DL 723.  
## 6 EV 788.  
## 7 F9 883.  
## 8 FL 686.  
## 9 MQ 701.  
## 10 OO 823.  
## 11 UA 1204.  
## 12 US 982.  
## 13 WN 613.  
## 14 XE 590.  
## 15 YV 982.
```

### 5.9.1 Piping

This is a good spot to introduce a time-saving and helpful device called “piping”, denoted by the symbol `%>%`. We’ve seen this weird combination of symbols in past chapters, but we haven’t really explained what they do.

Piping always looks more complicated than it really is. The technical definition is that

```
x %>% f(y)
```

is equivalent to

```
f(x, y).
```

As a simple example, we could add two numbers like this:

```
sum(2, 3)
```

```
## [1] 5
```

Or using the pipe, we could do it like this:

```
2 %>% sum(3)
```

```
## [1] 5
```

All this is really saying is that the pipe takes the thing on its left, and plugs it into the first slot of the function on its right. So why do we care?

Let’s revisit the combination `group_by/summarise` example above. There are two ways to do this without pipes, and both are a little ugly. One way is above, where you have to keep reassigning the output to new variables (in the case above, to `hf_summ_grouped` and then `hf_summ`). The other way is to nest the functions:

```
summarise(group_by(hf, UniqueCarrier), mean(Distance))
```

```
## # A tibble: 15 x 2
##   UniqueCarrier `mean(Distance)`
##   <chr>          <dbl>
## 1 AA              470.
## 2 AS             1874
## 3 B6             1428
## 4 CO             1097.
## 5 DL              723.
## 6 EV              788.
## 7 F9              883
## 8 FL              686.
## 9 MQ              701.
## 10 OO             823.
## 11 UA            1204.
## 12 US              982.
## 13 WN              613.
## 14 XE              590.
## 15 YV              982.
```

This requires a lot of brain power to parse. In part, this is because the function is inside-out: first you group `hf` by `UniqueCarrier`, and then the result of that is summarized. Here's how the pipe fixes it:

```
hf %>%
  group_by(UniqueCarrier) %>%
  summarise(mean(Distance))
```

```
## # A tibble: 15 x 2
##   UniqueCarrier `mean(Distance)`
##   <chr>          <dbl>
## 1 AA              470.
## 2 AS             1874
## 3 B6             1428
## 4 CO             1097.
## 5 DL              723.
## 6 EV              788.
## 7 F9              883
## 8 FL              686.
## 9 MQ              701.
## 10 OO             823.
## 11 UA            1204.
## 12 US              982.
```

```
## 13 WN                613.
## 14 XE                590.
## 15 YV                982.
```

Look at the `group_by` line. The `group_by` function should take two arguments, the tibble, and then the grouping variable. It appears to have only one argument. But look at the previous line. The pipe says to insert whatever is on its left (`hf`) into the first slot of the function on its right (`group_by`). So the net effect is still to evaluate the function `group_by(hf, UniqueCarrier)`.

Now look at the `summarise` line. Again, `summarise` is a function of two inputs, but all we see is the part that finds the mean. The pipe at the end of the previous line tells the `summarise` function to insert the stuff already computed (the grouped tibble returned by `group_by(hf, UniqueCarrier)`) into the first slot of the `summarise` function.

Piping takes a little getting used to, but once you're good at it, you'll never go back. It's just makes more sense semantically. When I read the above set of commands, I see a set of instructions in chronological order:

- Start with the tibble `hf`.
- Next, group by the carrier.
- Next, summarize each group using the mean distance.

Now we can assign the result of all that to the new variable `hf_summ`:

```
hf_summ <- hf %>%
  group_by(UniqueCarrier) %>%
  summarise(mean(Distance))
hf_summ

## # A tibble: 15 x 2
##   UniqueCarrier `mean(Distance)`
##   <chr>          <dbl>
## 1 AA            470.
## 2 AS           1874
## 3 B6           1428
## 4 CO           1097.
## 5 DL            723.
## 6 EV            788.
## 7 F9            883
## 8 FL            686.
## 9 MQ            701.
## 10 OO           823.
## 11 UA          1204.
```

```
## 12 US          982.
## 13 WN          613.
## 14 XE          590.
## 15 YV          982.
```

Some people even take this one step further. The result of all the above is assigned to a new variable `hf_summ` that currently appears as the first command (`hf_summ <- ...`) But you could write this as

```
hf %>%
  group_by(UniqueCarrier) %>%
  summarise(mean(Distance)) -> hf_summ
```

Now it says the following:

- Start with the tibble `hf`.
- Next, group by the carrier.
- Next, summarize each group using the mean distance.
- *Finally*, assign the result to a new variable called `hf_summ`.

In other words, the arrow operator for assignment works both directions!

Let's try some counting. This one is common enough that `dplyr` doesn't even make us use `group_by` and `summarise`. We can just use the command `count`. What if we wanted to know how many flights correspond to each carrier?

```
hf_summ2 <- hf %>%
  count(UniqueCarrier)
hf_summ2
```

```
## # A tibble: 15 x 2
##   UniqueCarrier     n
##   <chr>         <int>
## 1 AA             325
## 2 AS              37
## 3 B6              70
## 4 CO            7004
## 5 DL             265
## 6 EV             221
## 7 F9              84
## 8 FL             214
## 9 MQ             465
## 10 OO            1607
## 11 UA             208
```

```
## 12 US          409
## 13 WN          4535
## 14 XE          7306
## 15 YV           8
```

Also note that we can give summary columns a new name if we wish. In `hf_summ`, we didn't give the new column an explicit name, so it showed up in our tibble as a column called `mean(Distance)`. If we want to change it, we can do this:

```
hf_summ <- hf %>%
  group_by(UniqueCarrier) %>%
  summarise(mean_dist = mean(Distance))
hf_summ
```

```
## # A tibble: 15 x 2
##   UniqueCarrier mean_dist
##   <chr>          <dbl>
## 1 AA             470.
## 2 AS            1874
## 3 B6            1428
## 4 CO            1097.
## 5 DL             723.
## 6 EV             788.
## 7 F9             883
## 8 FL             686.
## 9 MQ             701.
## 10 OO            823.
## 11 UA           1204.
## 12 US            982.
## 13 WN            613.
## 14 XE            590.
## 15 YV            982.
```

Look at the earlier version of `hf_summ` and compare it to the one above. Make sure you see that the name of the second column changed.

The new count column of `hf_summ2` is just called `n`. That's okay, but if we insist on giving it a more user-friendly name, we can do so as follows:

```
hf_summ2 <- hf %>%
  count(UniqueCarrier, name = "total_count")
hf_summ2
```

```
## # A tibble: 15 x 2
##   UniqueCarrier total_count
```

```
##      <chr>                <int>
##  1 AA                    325
##  2 AS                     37
##  3 B6                     70
##  4 CO                    7004
##  5 DL                     265
##  6 EV                     221
##  7 F9                      84
##  8 FL                     214
##  9 MQ                     465
## 10 OO                    1607
## 11 UA                     208
## 12 US                     409
## 13 WN                    4535
## 14 XE                    7306
## 15 YV                      8
```

This is a little different because it requires us to use a **name** argument and put the new name in quotes.

**Exercise 14(a)** Create a tibble called `hf_summ3` that lists the total count of flights for each day of the week. Be sure to use the pipe as above. Print the output to the screen. (You don't need to give the count column a new name.)

```
# Add code here to define hf_summ3.
# Add code here to print hf_summ3.
```

**Exercise 14(b)** According to the output in the previous part, what day of the week had the fewest flights? (Assume 1 = Monday.)

Please write up your answer here.

---

The tibbles created in this section are all just a few rows each. They don't take up much memory, so we don't really need to remove them. You can if you want, but it's not necessary.

## 5.10 Putting it all together

Often we need more than one of these verbs. In many data analyses, we need to do a sequence of operations to get at the answer we seek. This is most easily accomplished using a more complicated sequence of pipes.



Here's an example of multi-step piping. Let's say that we only care about Delta flights, and even then, we only want to know about the month of the flight and the departure delay. From there, we wish to group by month so we can find the maximum departure delay by month. Here is a solution, piping hot and ready to go. [groan]

```
hf_grand_finale <- hf %>%
  filter(UniqueCarrier == "DL") %>%
  dplyr::select(Month, DepDelay) %>%
  group_by(Month) %>%
  summarise(max_delay = max(DepDelay, na.rm = TRUE))
hf_grand_finale
```

```
## # A tibble: 12 x 2
##   Month max_delay
##   <dbl>     <dbl>
## 1     1         26
## 2     2        460
## 3     3        202
## 4     4         23
## 5     5        127
## 6     6        184
## 7     7        360
## 8     8         48
## 9     9        292
## 10    10         90
## 11    11         10
## 12    12         14
```

Go through each line of code carefully and translate it into English:

- We define a variable called `hf_grand_finale` that starts with the original `hf` data.
- We `filter` this data so that only Delta flights will be analyzed.
- We `select` the variables `Month` and `DepDelay`, throwing away all other variables that are not of interest to us. (Don't forget to use the `dplyr::select` syntax to make sure we get the right function!)
- We `group_by` month so that the results will be displayed by month.
- We `summarise` each month by listing the maximum value of `DepDelay` that appears within each month.
- We print the result to the screen.

Notice in the `summarise` line, we again took advantage of `dplyr`'s ability to rename any variable along the way, assigning our computation to the new variable `max_delay`. Also note the need for `na.rm = TRUE` so that the `max` command ignores any missing values.

A minor simplification results from the realization that `summarise` must throw away any variables it doesn't need. (Think about why for a second: what would `summarise` do with, say, `ArrTime` if we've only asked it to calculate the maximum value of `DepDelay` for each month?) So you could write this instead, removing the `select` clause:

```
hf_grand_finale <- hf %>%
  filter(UniqueCarrier == "DL") %>%
  group_by(Month) %>%
  summarise(max_delay = max(DepDelay, na.rm = TRUE))
hf_grand_finale
```

```
## # A tibble: 12 x 2
##   Month max_delay
##   <dbl>   <dbl>
## 1     1         26
## 2     2        460
## 3     3        202
## 4     4         23
## 5     5        127
## 6     6        184
## 7     7        360
## 8     8         48
## 9     9        292
## 10    10         90
## 11    11         10
## 12    12         14
```

Check that you get the same result. With *massive* data sets, it's possible that the selection and sequence of these verbs matter, but you don't see an appreciable difference here, even with 22758 rows. (There are ways of benchmarking performance in R, but that is a more advanced topic.)

**Exercise 15** Summarize in your own words what information is contained in the `hf_grand_finale` tibble. In other words, what are the numbers in the `max_delay` column telling us? Be specific!

Please write up your answer here.

The remaining exercises are probably the most challenging you've seen so far in the course. Take each slowly. Read the instructions carefully. Go back through the chapter and identify which “verb” needs to be used for each part of the task. Examine the sample code in those sections carefully to make sure you get the syntax right. Create a sequence of “pipes” to do each task, one-by-one. Copy and paste pieces of code from earlier and make minor changes to adapt the code to the given task.

**Exercise 16** Create a tibble that counts the flights to LAX grouped by day of the week. (Hint: you need to `filter` to get flights to LAX. Then you'll need to `count` using `DayOfWeek` as a grouping variable. Because you're using `count`, you don't need `group_by` or `summarise`.) Print the output to the screen.

```
# Add code here to count the flights to LAX
# grouped by day of the week.
# Print the output to the screen.
```

**Exercise 17** Create a tibble that finds the median distance flight for each airline. Sort the resulting tibble from highest distance to lowest. (Hint: You'll need to `group_by` carrier and `summarise` using the `median` function. Finally, you'll need to `arrange` the result according to the median distance variable that you just created.) Print the output to the screen.

```
# Add code here to find the median distance by airline.
# Print the output to the screen.
```

## 5.11 Conclusion

Raw data often doesn't come in the right form for us to run our analyses. The `dplyr` verbs are powerful tools for manipulating tibbles until they are in the right form.

### 5.11.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.



## Chapter 6

# Correlation

2.0

### Functions introduced in this chapter

`cor`

## 6.1 Introduction

In this chapter, we will learn about the concept of correlation, which is a way of measuring a linear relationship between two numerical variables.

### 6.1.1 Install new packages

If you are using RStudio Workbench, you do not need to install any packages. (Any packages you need should already be installed by the server administrators.)

If you are using R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type the following command at the Console:

```
install.packages("faraway")
```

### 6.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/06-correlation.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/06-correlation.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 6.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

### 6.1.4 Load packages

We load the now-standard `tidyverse` package. We also include the `faraway` package to access data about Chicago in the 1970s.

```
library(tidyverse)
library(faraway)
```

## 6.2 Redlining in Chicago

The data set we will use throughout this chapter is from Chicago in the 1970s studying the practice of “redlining”.

**Exercise 1** Do an internet search for “redlining”.

Consult at least two or three sources. Then, in your own words (not copied and pasted from any of the websites you consulted), explain what “redlining” means.

Please write up your answer here.

---

The `chredlin` data set appears in the `faraway` package accompanying a book by Julian Faraway (*Practical Regression and Anova using R*, 2002.) Faraway explains:

“In a study of insurance availability in Chicago, the U.S. Commission on Civil Rights attempted to examine charges by several community organizations that insurance companies were redlining their neighborhoods, i.e. canceling policies or refusing to insure or renew. First the Illinois Department of Insurance provided the number of cancellations, non-renewals, new policies, and renewals of homeowners and residential fire insurance policies by ZIP code for the months of December 1977 through February 1978. The companies that provided this information account for more than 70% of the homeowners insurance policies written in the City of Chicago. The department also supplied the number of FAIR plan policies written and renewed in Chicago by zip code for the months of December 1977 through May 1978. Since most FAIR plan policyholders secure such coverage only after they have been rejected by the voluntary market, rather than as a result of a preference for that type of insurance, the distribution of FAIR plan policies is another measure of insurance availability in the voluntary market.”

In other words, the degree to which residents obtained FAIR policies can be seen as an indirect measure of redlining. This participation in an “involuntary” market is thought to be largely driven by rejection of coverage under more traditional insurance plans.

### 6.2.1 Exploratory data analysis

Before we learn about correlation, let’s get to know our data a little better.

Type `?chredlin` at the Console to read the help file. While it’s not very informative about how the data was collected, it does have crucial information about the way the data is structured.

Here is the data set:

```
chredlin
```

```
##      race fire theft  age involact income side
## 60626 10.0  6.2   29 60.4     0.0 11.744    n
## 60640 22.2  9.5   44 76.5     0.1  9.323    n
## 60613 19.6 10.5   36 73.5     1.2  9.948    n
## 60657 17.3  7.7   37 66.9     0.5 10.656    n
## 60614 24.5  8.6   53 81.4     0.7  9.730    n
## 60610 54.0 34.1   68 52.6     0.3  8.231    n
## 60611  4.9 11.0   75 42.6     0.0 21.480    n
## 60625  7.1  6.9   18 78.5     0.0 11.104    n
## 60618  5.3  7.3   31 90.1     0.4 10.694    n
```

##	60647	21.5	15.1	25	89.8	1.1	9.631	n
##	60622	43.1	29.1	34	82.7	1.9	7.995	n
##	60631	1.1	2.2	14	40.2	0.0	13.722	n
##	60646	1.0	5.7	11	27.9	0.0	16.250	n
##	60656	1.7	2.0	11	7.7	0.0	13.686	n
##	60630	1.6	2.5	22	63.8	0.0	12.405	n
##	60634	1.5	3.0	17	51.2	0.0	12.198	n
##	60641	1.8	5.4	27	85.1	0.0	11.600	n
##	60635	1.0	2.2	9	44.4	0.0	12.765	n
##	60639	2.5	7.2	29	84.2	0.2	11.084	n
##	60651	13.4	15.1	30	89.8	0.8	10.510	n
##	60644	59.8	16.5	40	72.7	0.8	9.784	n
##	60624	94.4	18.4	32	72.9	1.8	7.342	n
##	60612	86.2	36.2	41	63.1	1.8	6.565	n
##	60607	50.2	39.7	147	83.0	0.9	7.459	n
##	60623	74.2	18.5	22	78.3	1.9	8.014	s
##	60608	55.5	23.3	29	79.0	1.5	8.177	s
##	60616	62.3	12.2	46	48.0	0.6	8.212	s
##	60632	4.4	5.6	23	71.5	0.3	11.230	s
##	60609	46.2	21.8	4	73.1	1.3	8.330	s
##	60653	99.7	21.6	31	65.0	0.9	5.583	s
##	60615	73.5	9.0	39	75.4	0.4	8.564	s
##	60638	10.7	3.6	15	20.8	0.0	12.102	s
##	60629	1.5	5.0	32	61.8	0.0	11.876	s
##	60636	48.8	28.6	27	78.1	1.4	9.742	s
##	60621	98.9	17.4	32	68.6	2.2	7.520	s
##	60637	90.6	11.3	34	73.4	0.8	7.388	s
##	60652	1.4	3.4	17	2.0	0.0	13.842	s
##	60620	71.2	11.9	46	57.0	0.9	11.040	s
##	60619	94.1	10.5	42	55.9	0.9	10.332	s
##	60649	66.1	10.7	43	67.5	0.4	10.908	s
##	60617	36.4	10.8	34	58.0	0.9	11.156	s
##	60655	1.0	4.8	19	15.2	0.0	13.323	s
##	60643	42.5	10.4	25	40.8	0.5	12.960	s
##	60628	35.1	15.6	28	57.8	1.0	11.260	s
##	60627	47.4	7.0	3	11.4	0.2	10.080	s
##	60633	34.0	7.1	23	49.2	0.3	11.428	s
##	60645	3.1	4.9	27	46.6	0.0	13.731	n

**Exercise 2** What do each of the rows of this data set represent? You'll need to refer to the help file. (They are *not* individual people.)

Please write up your answer here.



**Exercise 3** The `race` variable is numeric. Why? What do these numbers represent? (Again, refer to the help file.)

Please write up your answer here.

---

The `glimpse` command gives a concise overview of all the variables present.

```
glimpse(chredlin)
```

```
## Rows: 47
## Columns: 7
## $ race    <dbl> 10.0, 22.2, 19.6, 17.3, 24.5, 54.0, 4.9, 7.1, 5.3, 21.5, 43.1~
## $ fire    <dbl> 6.2, 9.5, 10.5, 7.7, 8.6, 34.1, 11.0, 6.9, 7.3, 15.1, 29.1, 2~
## $ theft   <dbl> 29, 44, 36, 37, 53, 68, 75, 18, 31, 25, 34, 14, 11, 11, 22, 1~
## $ age     <dbl> 60.4, 76.5, 73.5, 66.9, 81.4, 52.6, 42.6, 78.5, 90.1, 89.8, 8~
## $ involact <dbl> 0.0, 0.1, 1.2, 0.5, 0.7, 0.3, 0.0, 0.0, 0.4, 1.1, 1.9, 0.0, 0~
## $ income  <dbl> 11.744, 9.323, 9.948, 10.656, 9.730, 8.231, 21.480, 11.104, 1~
## $ side    <fct> n, n, n, n, n, n, n, n, n, n, n, n, n, n, n, n, n, n, n, n~
```

**Exercise 4(a)** Which variable listed above represents participation in the FAIR plan? How is it measured? (Again, refer to the help file.)

Please write up your answer here.

**Exercise 4(b)** Why is it important to analyze the number of plans *per 100 housing units* as opposed to the total number of plans across each ZIP code? (Hint: what happens if some ZIP codes are larger than others?)

Please write up your answer here.

---

We are interested in the association between `race` and `involact`. If redlining plays a role in driving people toward FAIR plan policies, we would expect there to be a relationship between the racial composition of a ZIP code and the number of FAIR plan policies obtained in that ZIP code.

**Exercise 5(a)** Since `race` is a numerical variable, what type of graph or chart is appropriate for visualizing it? (You may need to refer back to the “Numerical data” chapter.)

Please write up your answer here.

**Exercise 5(b)** Using `ggplot` code, create the type of graph you identified above. (Again, refer back to the “Numerical data” chapter for sample code if you’ve forgotten.) After creating the initial plot, be sure to go back and set the `binwidth` and `boundary` to sensible values.

```
# Add code here to create a plot of race
```

**Exercise 5(c)** Describe the shape of the `race` variable using the three key shape descriptors (modes, symmetry, and outliers).

Please write up your answer here.

**Exercise 5(d)** Create the same kind of graph as above, but for `involact`. (Again, go back and set the `binwidth` and `boundary` to sensible values.)

```
# Add code here to create a plot of race
```

**Exercise 5(e)** Describe the shape of the `involact` variable using the three key shape descriptors (modes, symmetry, and outliers).

Please write up your answer here.

**Exercise 5(f)** Since both `race` and `involact` are numerical variables, what type of graph or chart is appropriate for visualizing the relationship between them?

Please write up your answer here.

**Exercise 5(g)** For our research question, is `race` functioning as a predictor variable or as the response variable? What about `involact`? Why? Explain why it makes more sense to think of one of them as the predictor and the other as the response.

Please write up your answer here.

**Exercise 5(h)** Using `ggplot` code, create the type of graph you identified above. Be sure to put `involact` on the y-axis and `race` on the x-axis.

```
# Add code here to create a plot of involact against race
```

---

## 6.3 Correlation

The word *correlation* describes a linear relationship between two numerical variables. As long as certain conditions are met, we can calculate a statistic called the *correlation coefficient*, often denoted with a lowercase  $r$ .

There are several different ways to compute a statistic that measures correlation. The most common way, and the way we will learn in this chapter, is often attributed to an English mathematician named Karl Pearson. According to his Wikipedia page,

“Pearson was also a proponent of social Darwinism, eugenics and scientific racism.”

**Exercise 6** Do an internet search for each of the following terms:

- Social Darwinism
- Eugenics
- Scientific racism

Consult at least two or three sources for each term. Then, in your own words (not copied and pasted from any of the websites you consulted), explain what these terms mean.

Please write up your answer here.

---

While Pearson is often credited with its discovery, the so-called “Pearson correlation coefficient” was first developed by a French scientist, Auguste Bravais. Due to the misattribution of discovery, along with the desire to disassociate the useful tool of correlation from its problematic applications to racism and eugenics, we will just refer to it as the *correlation coefficient* (without a name attached).

The correlation coefficient,  $r$ , has some important properties.

- The correlation coefficient is a number between -1 and 1.
- A value close to 0 indicates little or no correlation.
- A value close to 1 indicates strong positive correlation.
- A value close to -1 indicates strong negative correlation.

In between 0 and 1 (or -1), we often use words like weak, moderately weak, moderate, and moderately strong. There are no exact cutoffs for when such

words apply. You must learn from experience how to judge scatterplots and  $r$  values to make such determinations.

A correlation is positive when low values of one variable are associated with low values of the other value. Similarly, high values of one variable are associated with high values of the other. For example, exercise is positively correlated with burning calories. Low exercise levels will burn a few calories; high exercise levels burn more calories, on average.

A correlation is negative when low values of one variable are associated with high values of the other value, and vice versa. For example, tooth brushing is negatively correlated with cavities. Less tooth brushing may result in more cavities; more tooth brushing is associated with fewer cavities, on average.

## 6.4 Conditions for correlation

Two variables are considered “associated” any time there is any type of relationship between them (i.e., they are not independent). However, in statistics, we reserve the word “correlation” for situations meeting more stringent conditions:

1. The two variables must be numerical.<sup>1</sup>
2. There is a somewhat linear relationship between the variables, as shown in a scatterplot.
3. There are no serious outliers.

For condition (2) above, keep in mind that real data in scatterplots very rarely lines up in a perfect straight line. Instead, you will see a “cloud” of dots. All we want to know is whether that cloud of dots mostly moves from one corner of the scatterplot to the other. Violations of this condition will usually be for one of two reasons:

- The dots are scattered completely randomly with no discernible pattern.
- The dots have a pattern or shape to them, but that shape is curved and not linear.

**Exercise 7** Check the three conditions for the relationship between **involact** and **race**. For conditions (2) and (3), you’ll need to check the scatterplot you created above. (You did create a scatterplot for one of the exercises above, right?)

Please write up your answer here.

---

<sup>1</sup>There are other ways of measuring association for variables that are not numerical, but these aren’t covered in this course.

- 1.
- 2.
- 3.

## 6.5 Calculating correlation

Since the conditions are met, We calculate the correlation coefficient using the `cor` command.

```
cor(chredlin$race, chredlin$involact)
```

```
## [1] 0.713754
```

The order of the variables doesn't matter; correlation is symmetric, so the `r` value is the same independent of the choice of response and predictor variables.

Since the correlation between `involact` and `race` is a positive number and slightly closer to 1 than 0, we might call this a “moderate” positive correlation. You can tell from the scatterplot above that the relationship is not a strong relationship. The words you choose should match the graphs you create and the statistics you calculate.

**Exercise 8(a)** Create a scatterplot of `income` against `race`. (Put `income` on the y-axis and `race` on the x-axis.)

```
# Add code here to create a scatterplot of income against race
```

**Exercise 8(b)** Check the three conditions for the relationship between `income` and `race`. Which condition is pretty seriously violated here?

Please write up your answer here.

- 1.
- 2.
- 3.

**Exercise 9(a)** Create a scatterplot of `theft` against `fire`. (Put `theft` on the y-axis and `fire` on the x-axis.)

```
# Add code here to create a scatterplot of theft against fire
```

**Exercise 9(b)** Check the three conditions for the relationship between `theft` and `fire`. Which condition is pretty seriously violated here?

- 1.
- 2.
- 3.

Please write up your answer here.

**Exercise 9(c)** Even though the conditions are not met, what if you calculated the correlation coefficient anyway? Try it.

*# Add code here to calculate the correlation coefficient between theft and fire*

**Exercise 9(d)** Suppose you hadn't looked at the scatterplot and you only saw the correlation coefficient you calculated in the previous part. What would your conclusion be about the relationship between `theft` and `fire`. Why would that conclusion be misleading?

Please write up your answer here.

The lesson learned here is that you should never try to interpret a correlation coefficient without looking at a plot of the data to assure that the conditions are met and that the result is a sensible thing to interpret.

## 6.6 Correlation is not causation

When two variables are correlated—indeed, associated in any way, not just in a linear relationship—that means that there is a relationship between them. However, that does not mean that one variable *causes* the other variable.

For example, we discovered above that there was a moderate correlation between the racial composition of a ZIP code and the new FAIR policies created in those ZIP codes. However, being part of a racial minority does not cause someone to seek out alternative forms of insurance, at least not directly. In this case, the racial composition of certain neighborhoods, though racist policies, affected the availability of certain forms of insurance for residents in those neighborhoods. And that, in turn, caused residents to seek other forms of insurance.

In the Chicago example, there is still likely a causal connection between one variable (`race`) and the other (`involact`), but it was indirect. In other cases, there is no causal connection at all. Here are a few of my favorite examples.

**Exercise 10** Ice cream sales are positively correlated with drowning deaths. Does eating ice cream cause you to drown? (Perhaps the myth about swimming within one hour of eating is really true!) Does drowning deaths cause ice cream sales to rise? (Perhaps people are so sad about all the drownings that they have to go out for ice cream to cheer themselves up?)

See if you can figure out the real reason why ice cream sales are positively correlated with drowning deaths.

Please write up your answer here.

---

In the Chicago example, the causal effect was indirect. In the example from the exercise above, there is no causation whatsoever between the two variables. Instead, the causal effect was generated by a third factor that caused both ice cream sales to go up, and also happened to cause drowning deaths to go up. (Or, equivalently stated, it caused ice cream sales to be low during certain times of the year and also caused the drowning deaths to be low as well.) Such a factor is called a *lurking variable*. When a correlation between two variables exists due solely to the intervention of a lurking variable, that correlation is called a *spurious correlation*. The correlation is real; a scatterplot of ice cream sales and drowning deaths would show a positive relationship. But the reasons for that correlation to exist have nothing to do with any kind of direct causal link between the two.

Here's another one:

**Exercise 11** Most studies involving children create a number of weird correlations. For example, the height of children is very strongly correlated to pretty much everything you can measure about scholastic aptitude. For example, vocabulary count (the number of words children can use fluently in a sentence) is strongly correlated to height. Are tall people just smarter than short people?

The answer is, of course, no. The correlation is spurious. So what's the lurking variable?

Please write up your answer here.

## 6.7 Observational studies versus experiments

So when is a statistical finding (like correlation, for example) evidence of a causal relationship? Before we can answer that question, we need a few more definitions.

A lot of data comes from “observational studies” where we simply observe or measure things as they are “in the wild,” so to speak. We don't interfere in any

way. We just write down what we see. Polls are usually observational in that we ask people questions and record their responses. We do not try to manipulate their responses in any way. We just ask the questions and observe the answers. Field studies are often observational. We go out in nature and write stuff down as we observe it.

Another way to gather data is an *experiment*. In an experiment, we introduce a manipulation or treatment to try to ascertain its effect. For example, if we're testing a new drug, we will likely give the drug to one group of patients and a *placebo* to the other.

**Exercise 12** Here's another internet rabbit hole for you. First, look up the definition of placebo. You do not need to write up your own version of that definition here; just familiarize yourself with the term if you're not already familiar with it. Next, find some websites about the *placebo effect* and read those.

Given what you have learned about the placebo effect, why is it important to have a placebo group in a drug trial? Why not just give one set of patients the drug and compare them to another group that takes no pill at all?

Please write up your answer here.

---

The goal of the experiment is to learn whether the *treatment* (in this example, the drug) is effective when compared to the *control* (in this example, the placebo).

Note that the word “effective” implies a causal claim. We want to know if the drug *causes* patients to get better.

Unlike an observational study, in which the relationship between variables can be caused by a lurking variable, in an experiment, we purposefully manipulate one of the variables and try to control all others. For example, we manipulate the drug variable (we purposefully give some people the drug and others the placebo). But we control the amount of the drug given and the schedule on which patients are required to take the pills.

There are lots of things we cannot control. For example, it would be very difficult to control the diet of every person in the experiment. Could diet play a role in whether a patient gets better? Sure, so how do we know diet is not a lurking variable? In the context of an experiment, lurking variables are often called “confounders” or “confounding variables”. (The two terms are basically synonymous.)

One way to mitigate the effect of confounders that we cannot directly control is to *randomize* the patients into the treatment and control groups. With random



selection, there will likely be people who have relatively healthy diets in both the control and treatment groups. If the drugs work, in theory they should still work better for the treatment group than for those taking the placebo. And likewise, patients with less healthy diets will generally be mixed up in both groups, and the drug should also work better for them.

The mantra of experimental design is, “Control as much as you can. Randomize to take care of the rest.”

There are lots of aspects of experimental design that we will not go into here (for example, blinding and blocking). But we will continue to mention the differences between observational studies and experiments in future chapters as we exercise caution in making causal claims.

## 6.8 Prediction versus explanation

Even when claims are not causal, we can use associations (and correlations more specifically) for purposes of *prediction*.

**Exercise 13** If I tell you that ice cream sales are high right now, can you make a reasonable prediction about the relative number of drowning deaths this month (high or low)? Why or why not?

Please write up your answer here.

---

So even when there is no direct causal link between two variables, if they are positively correlated, then large values of one variable are associated with large values of the other variable. So if I tell you one value is large, it is reasonable to predict that the other value will be large as well.

We use the language “predictor” variable and “response” variable to reinforce this idea.

In a properly designed and controlled experiment, we can use different language. In this case, we can *explain* the outcome using the treatment variable. If we’ve controlled for everything else, the only possible explanation for a difference between the treatment and control groups must be the treatment variable. If the patients get better on the drug (more so than those on the placebo) and we’ve controlled for every other possible confounding variable, the only possible explanation is that the drug works. The drug “explains” the difference in the response variable.

Be careful, as sometimes statisticians use the term “explanatory variable” to mean any kind of variable that predicts or explains. In this course, we will try to use the term “predictor variable” exclusively.

## 6.9 Conclusion

If we have two numerical variables that have a linear association between them (also assuming there are no serious outliers), we can compute the correlation coefficient that measures the strength and direction of that linear association.

Keep in mind that in an observational study, this correlation is a measure of association, but it does not signify that one variable causes the other. It's possible that one variable causes the other, but it's also possible that a third “lurking” variable is responsible for the association. Either way, the fact that a relationship exists means it is possible to use values of one variable to make reasonable predictions about the values of the other variable.

In a properly designed experiment, the manipulation of one variable while controlling for others (and randomizing to take care of other confounders) ensures that there is a causal link between the treatment variable and the response of interest. In this case, the treatment can “explain” the response, not just predict it.

### 6.9.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.

## Chapter 7

# Regression

2.0

### Functions introduced in this chapter

`geom_smooth`, `lm`, `tidy`, `augment`, `glance`

## 7.1 Introduction

In this chapter we will learn how to run a regression analysis. Regression provides a model for the linear relationship between two numerical variables.

### 7.1.1 Install new packages

If you are using RStudio Workbench, you do not need to install any packages. (Any packages you need should already be installed by the server administrators.)

If you are using R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type the following command at the Console:

```
install.packages("broom")
```

### 7.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/07-regression.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/07-regression.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 7.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

### 7.1.4 Load packages

We load the `tidyverse` package. The `faraway` package will give access to the Chicago redlining data introduced in the previous chapter and the `palmerpenguins` package gives us the `penguins` data. Finally, the `broom` package will provide tools for cleaning up the output of the regression analysis we perform.

```
library(tidyverse)
library(faraway)
library(palmerpenguins)
library(broom)
```

## 7.2 Regression

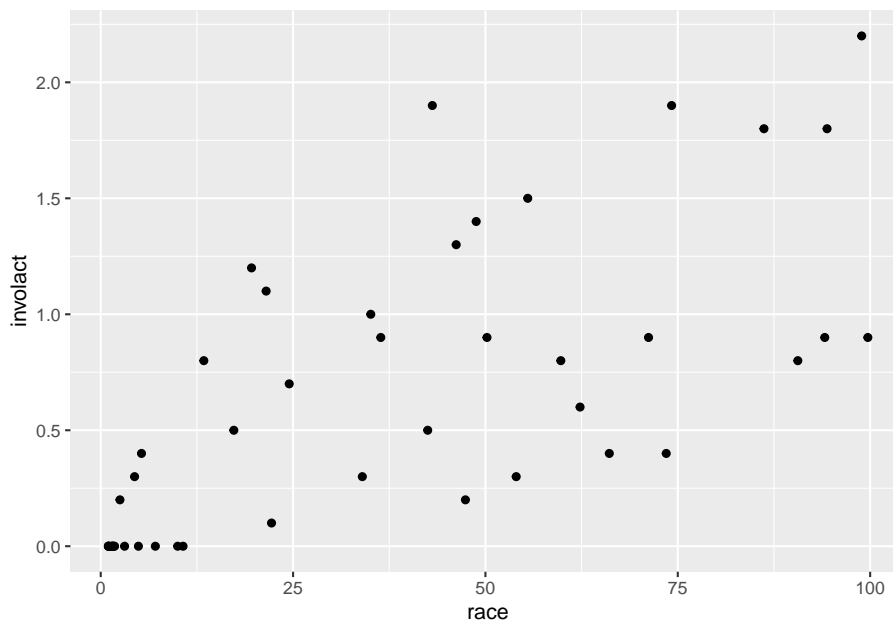
When we have a linear relationship between two numerical variables, we learned in the last chapter that we can compute the correlation coefficient. One serious limitation of the correlation coefficient is that it is only a single number, and therefore, it doesn't provide a whole lot of information about the nature of the linear relationship itself. It only gives clues as to the strength and direction of the association.

It will be helpful to model this linear relationship with an actual straight line. Such a line is called a *regression line*. It is also known as a *best-fit line* or *least-squares line* for reasons that we will get to later in the chapter.

The mathematics involved in figuring out what this line should be is more complicated than we cover in this book. Fortunately, R will do all the complicated calculations for us and we'll focus on understanding what they mean.

Recall the `chredlin` data set from the last chapter investigating the practice of redlining in Chicago in the 1970s. Let's review the scatterplot of `involact`, the number of FAIR policies per 100 housing units, against `race`, the racial composition of each ZIP code as a percentage of minority residents. (Recall that each row of the data represents an entire ZIP code.)

```
ggplot(chredlin, aes(y = involact, x = race)) +  
  geom_point()
```



**Exercise 1** Does the Chicago redlining data come from an observational study or an experiment? How do you know?

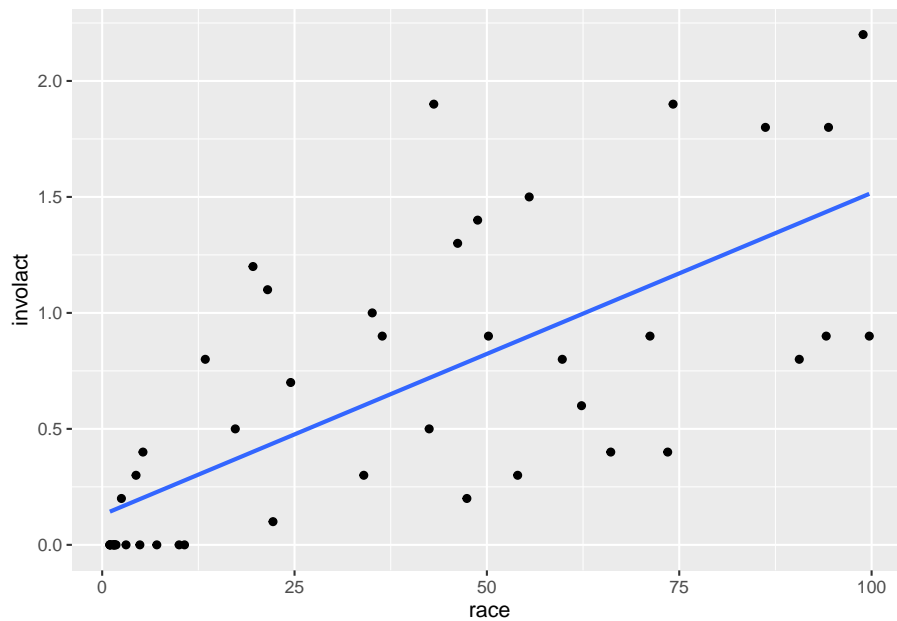
Please write up your answer here.

---

If certain conditions are met, we can graph a regression line; just add a `geom_smooth` layer to the scatterplot:

```
ggplot(chredlin, aes(y = involact, x = race)) +  
  geom_point() +  
  geom_smooth(method = lm, se = FALSE)
```

```
## `geom_smooth()` using formula 'y ~ x'
```



The `method = lm` argument is telling `ggplot` to use a “linear model”. The `se = FALSE` argument tells `ggplot` to draw just the line and nothing else. (What else might it try to draw? You are encouraged to go back to the code above and take out `se = FALSE` to see for yourself. However, we are not yet in a position to be able to explain the gray band that appears. We will return to this mystery in a future chapter.)

Of all possible lines, the blue line comes the closest to each point in the scatterplot. If we wiggled the line a little bit, it might get closer to a few points, but the net effect would be to make it further from other points. This is the mathematically optimal line of best fit.

### 7.3 Models

We used the word “model” when referring to the regression line above. What does that word mean in this context?

A model is something that represents something else, often on a smaller scale or in simplified form. A model is often an idealized form of something that may be quite messy or complex in reality. In statistics, a model is a representation of the way data is generated. For example, we may believe that as minority representation increases in a neighborhood, that neighborhood is more likely to be subject to racially discriminatory practices. We may even posit that the relationship is linear; i.e., for every percentage point increase in racial minorities, we expect some kind of proportional increase in racial discrimination, as

measured in this case by FAIR policies. We say that this is our hypothesis about the *data-generating process*: we suspect that the data we see results from a sociological process that uses the minority representation of a neighborhood to generate data about FAIR policies.

The assumption of a linear relationship between these two quantities is just that—an assumption. It is not necessarily “true”, whatever “true” might mean in this kind of question. It is a convenient device that makes a simplifying assumption in order to allow us to do something meaningful in a statistical analysis. If such a model—despite its simplifying caricature—helps us make meaningful predictions to study something important like racial discrimination, then the model is useful.

The first thing we acknowledge when working with a model is that the model does not generate the data in a rigid, deterministic way. If you look at the scatterplot above, even assuming the blue line represents a “correct” data-generating process, the data points don’t fall on the blue line. The blue line gives us only a sense of where the data might be, but there is additional space between the line and the points. These spaces are often referred to as *errors*. In statistics, the word “error” does not mean the same thing as “mistake”. Error is just the difference between an idealized model prediction and the real location of data. In the context of linear regression, we will use the term *residual* instead. After the model is done making a prediction, the residuals are “left over” to account for the difference between the model and the actual data.

The most important thing to remember about models is that they aren’t real. They are idealizations and simplifications. The degree to which we can trust models, then, comes down to certain assumptions we make about the data-generating process. These assumptions cannot be completely verified—after all, we will never know the exact data-generating process. But there are certain *conditions* we can check to know if the assumptions we make are reasonable.

**Exercise 2** Do an internet search for the phrase “statistical model” and/or “statistical modeling”. Read at least two or three sources. List below one important aspect of statistical modeling you find in your search that wasn’t mentioned in the paragraphs above. (Some of the sources you find may be a little technical. You should, for now, skip over the technical explanations. Try to find several sources that address the issue in non-technical ways. The additional information you mention below should be something non-technical that you understand.)

Please write up your answer here.

---

## 7.4 Checking conditions

We need to be careful here. Although we graphed the blue regression line above, we have not checked any conditions. Therefore, it is inappropriate to fit a regression line at this point. Once the line is seen, it cannot easily be “unseen”, and it’s crucial that you don’t trick your eyes into believing there is a linear relationship before checking the conditions that justify that belief.

The regression line we saw above makes no sense unless we know that regression is appropriate. The conditions for running a regression analysis include all the conditions you checked for a correlation analysis in the last chapter:

1. The two variables must be numerical.
2. There is a somewhat linear relationship between the variables, as shown in a scatterplot.
3. There are no serious outliers.

**Exercise 3** Check these three conditions for the regression between `involact` and `race` (using the scatterplot above for conditions (2) and (3).)

- 1.
- 2.
- 3.

---

However, there is an additional condition to check to ensure that our regression model is appropriate. It concerns the residuals, but as we haven’t computed anything yet, we have nothing to analyze. We’ll return to this condition later.

## 7.5 Calculating the regression line

What is the equation of the regression line? In your algebra class you learned that a line takes the form  $y = mx + b$  where  $m$  is the slope and  $b$  is the y-intercept. Statisticians write the equation in a slightly different form:

$$\hat{y} = b_0 + b_1x$$

The intercept is  $b_0$  and the slope is  $b_1$ . We use  $\hat{y}$  (pronounced “y hat”) instead of  $y$  because when we plug in values of  $x$ , we do not get back the exact values of  $y$  from the data. The line, after all, does not actually pass through most (if



any) actual data points. Instead, this equation gives us “predicted” values of  $y$  that lie on the regression line. These predicted  $y$  values are called  $\hat{y}$ .

To run a regression analysis and calculate the values of the intercept and slope, we use the `lm` command in R. (Again, `lm` stands for “linear model”.) This command requires us to specify a “formula” that tells R the relationship we want to model. It uses special syntax in a very specific order:

- The response variable,
- a “tilde” `~` (this key is usually in the upper-left corner of your keyboard, above the backtick),
- the predictor variable.

After a comma, we then specify the data set in which those variables live using `data =`. Here’s the whole command:

```
lm(involact ~ race, data = chredlin)

##
## Call:
## lm(formula = involact ~ race, data = chredlin)
##
## Coefficients:
## (Intercept)      race
##      0.12922      0.01388
```

**The response variable always goes before the tilde and the predictor variable always goes after.**

Let’s store that result for future use. The convention we’ll use in this book is to name things using the variables involved. For example,

```
involact_race_lm <- lm(involact ~ race, data = chredlin)
involact_race_lm

##
## Call:
## lm(formula = involact ~ race, data = chredlin)
##
## Coefficients:
## (Intercept)      race
##      0.12922      0.01388
```

The variable `involact_race_lm` now contains all the information we need about the linear regression model.

## 7.6 Interpreting the coefficients

Look at the output of the `lm` command above.

The intercept is 0.12922 and the slope is 0.01388. The number 0.12922 is labeled with `(Intercept)`, so that's pretty obvious. But how do we know the number 0.01388 corresponds to the slope? Process of elimination, I suppose. But there's another good reason too. The equation of the regression line can be written

$$\hat{y} = 0.12922 + 0.01388x$$

When we report the equation of the regression line, we typically use words instead of  $\hat{y}$  and  $x$  to make the equation more interpretable in the context of the problem. For example, for this data, we would write the equation as

$$\widehat{involact} = 0.12922 + 0.01388race$$

The slope is the *coefficient* of `race`, or the number attached to `race`. (The intercept is not attached to anything; it's just a constant term out front there.)

The slope  $b_1$  is always interpretable. This model predicts that one unit of increase in the x-direction corresponds to a change of 0.01388 units in the y-direction. Let's phrase it this way:

The model predicts that an increase of one percentage point in the composition of racial minorities corresponds to an increase of 0.01388 new FAIR policies per 100 housing units.

The intercept  $b_0$  is a different story. There is always a literal interpretation:

The model predicts that a ZIP code with 0% racial minorities will generate 0.12922 new FAIR policies.

In some cases (rarely), that interpretation might make sense. In most cases, though, it is physically impossible for the predictor variable to take a value of 0, or the value 0 is way outside the range of the data. Whenever we use a model to make a prediction outside of reasonable values, we call that *extrapolation*.

For the Chicago data, we likely don't have a case of extrapolation. While it is not literally true that any ZIP code has 0% racial minorities, we can see in the scatterplot that there are values very close to zero.

**Exercise 4** Use the `arrange` command from `dplyr` to sort the `chredlin` data frame by race (using the default ascending order). What is the value of `race` for the three ZIP codes with the smallest percentage of minority residents?

```
# Add code here to sort by race
```

Please write up your answer here.

---

Again, even though there are no ZIP codes with 0% racial minorities, there are a bunch that are close to zero, so the literal interpretation of the intercept is also likely a sensible one in this case.

**Exercise 5** Let's think through something else the intercept might be telling us in this case. The presumption is that FAIR policies are obtained mostly by folks who can't get insurance policies in other ways. Some of that is driven by racial discrimination, but maybe not all of it. What does the intercept have to say about the number of FAIR policies that are obtained *not* due to denial of coverage from racial discrimination?

Please write up your answer here.

## 7.7 Rescaling to make interpretations more meaningful

Let's revisit the interpretation of the slope:

The model predicts that an increase of one percentage point in the composition of racial minorities corresponds to an increase of 0.01388 new FAIR policies per 100 housing units.

This is a perfectly correct statement, but one percentage point change is not very much. It's hard to think about comparing two neighborhoods that differ by only one percent. This scale also makes the predicted change in the response variable hard to interpret. How many policies is 0.01388 per 100 housing units?

One way to make these kinds of statements more interpretable is to change the scale. What if we increase 10 percentage points instead of only 1 percentage point? In other words, what if we move 10 times as far along the x-axis. The response variable will also have to move 10 times as far. This is the new statement:

The model predicts that an increase of 10 percentage points in the composition of racial minorities corresponds to an increase of 0.1388 new FAIR policies per 100 housing units.

In this case, the decimal 0.1388 is maybe still not completely clear, but at least an increase of 10 percentage points is a meaningful difference between neighborhoods.

**Exercise 6** Since the last number is a *per capita* type measure, we can also rescale it. If the model predicts an increase in 0.1388 new FAIR policies per 100 households (corresponding to 10 percentage points increase in racial minorities), how many FAIR policies would that be in 1000 households?

Please write up your answer here.

## 7.8 The `tidy` command

Recall the output of the `lm` command:

```
involact_race_lm

##
## Call:
## lm(formula = involact ~ race, data = chredlin)
##
## Coefficients:
## (Intercept)      race
##    0.12922    0.01388
```

(We did not have to run `lm` again. We had this output stored in the variable `involact_race_lm`.)

That summary is fine, but what if we needed to reference the slope and intercept using inline code? Or what if we wanted to grab those numbers and use them in further calculations?

The problem is that the results of `lm` just print the output in an unstructured way. If we want structured input, we can use the `tidy` command from the `broom` package. This will take the results of `lm` and organize the output into a tibble.

```
tidy(involact_race_lm)
```

```
## # A tibble: 2 x 5
##   term      estimate std.error statistic    p.value
##   <chr>      <dbl>    <dbl>    <dbl>    <dbl>
## 1 (Intercept)  0.129    0.0966     1.34 0.188
## 2 race        0.0139   0.00203     6.84 0.0000000178
```

Let's store that tibble so we can refer to it in the future.

```
involact_race_tidy <- tidy(involact_race_lm)
involact_race_tidy
```

```
## # A tibble: 2 x 5
##   term      estimate std.error statistic    p.value
##   <chr>      <dbl>    <dbl>    <dbl>    <dbl>
## 1 (Intercept)  0.129    0.0966     1.34 0.188
## 2 race        0.0139   0.00203     6.84 0.0000000178
```

The intercept is stored in the `estimate` column, in the first row. The slope is stored in the same column, but in the second row. (There is a lot more information here to the right of the `estimate` column, but we will not know what these numbers mean until later in the course.)

We can grab the `estimate` column with the dollar sign as we've seen before:

```
involact_race_tidy$estimate
```

```
## [1] 0.12921803 0.01388235
```

This is a “vector” of two values, the intercept and the slope, respectively.

What if we want only one value at a time? We can grab individual elements of a vector using square brackets as follows:

```
involact_race_tidy$estimate[1]
```

```
## [1] 0.129218
```

```
involact_race_tidy$estimate[2]
```

```
## [1] 0.01388235
```

Here is the interpretation of the slope again, but this time, we'll use inline code:

The model predicts that an increase of 1 percentage points in the composition of racial minorities corresponds to an increase of 0.0138824 new FAIR policies per 100 housing units.

Click somewhere inside the backticks on the line above and hit Ctrl-Enter or Cmd-Enter (PC or Mac respectively). You should see the number 0.01388235 pop up. If you Preview the HTML version of the document, you will also see the number there (not the code).

What if we want to apply re-scaling to make this number more interpretable? The stuff inside the inline code chunk is just R code, so we can do any kind of calculation with it we want.

The model predicts that an increase of 10 percentage points in the composition of racial minorities corresponds to an increase of 0.1388235 new FAIR policies per 100 housing units.

Now the number will be 0.1388235, ten times as large.

**Exercise 7** Copy and paste the interpretation of the intercept from earlier, but replace the number 0.12922 with an inline code chunk that grabs that number from the `estimate` column of the `involact_race_tidy` tibble. (Remember that the intercept is the *first* element of that vector, not the second element like the slope.)

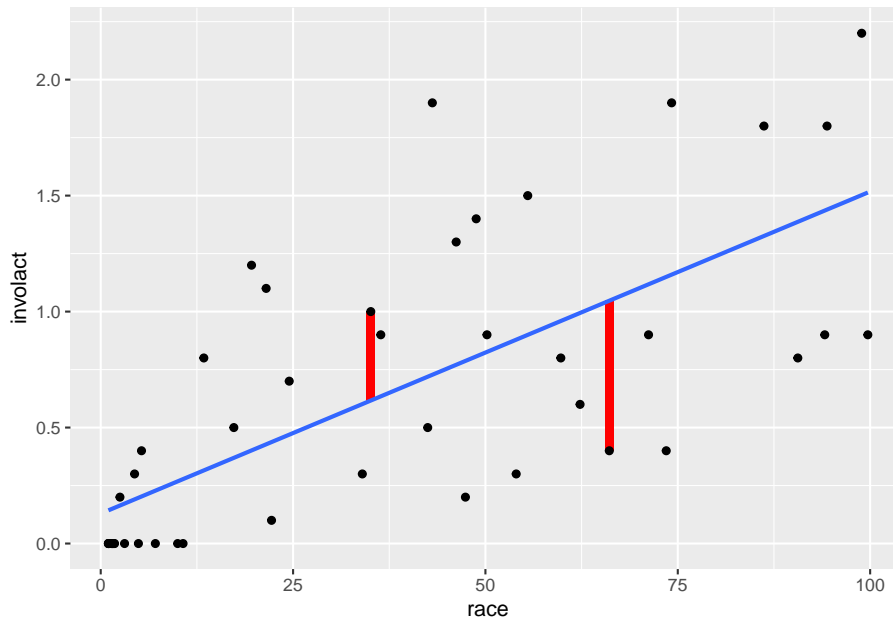
Please write up your answer here.

## 7.9 Residuals

Earlier, we promised to revisit the topic of residuals. Residuals are measured as the vertical distances from each data point to the regression line. We can see that visually below. (Don't worry about the complexity of the `ggplot` code used to create this picture. You will not need to create a plot like this on your own, so just focus on the graph that is created below.)

```
ggplot(chredlin, aes(y = involact, x = race)) +
  geom_segment(x = 35.1, xend = 35.1,
              y = 0.6164886, yend = 0.6164886 + 0.38351139,
              color = "red", size = 2) +
  geom_segment(x = 66.1, xend = 66.1,
              y = 1.0468415, yend = 1.0468415 - 0.64684154,
              color = "red", size = 2) +
  geom_point() +
  geom_smooth(method = lm, se = FALSE)
```

```
## `geom_smooth()` using formula 'y ~ x'
```



The graph above shows the regression line and two of the residuals as red line segments. (There is a residual for all 47 ZIP codes; only two are shown in this graph.) The one on the left corresponds to ZIP code with 35% racial minority. The regression line predicts that, if the model were true, such a ZIP code would have a value of `involact` of about 0.6. But the actual data for that ZIP code has an `involact` value of 1. The residual is the difference, about 0.4. In other words, the true data point is 0.4 units higher than the model prediction. This represents a *positive* residual; the actual data is 0.4 units *above* the line. Data points that lie below the regression line have *negative* residuals.

**Exercise 8** Look at the residual on the right. This corresponds to a ZIP code with about 66% racial minorities. First, estimate the value of `involact` that the model predicts for this ZIP code. (This is the y-value of the point on the regression line.) Next, report the actual `involact` value for this ZIP code. Finally, subtract these two numbers to get an approximate value for the residual. Should this residual be a positive number or a negative number?

You can just estimate with your eyeballs for now. You don't need to be super precise.

Please write up your answer here.

---

More formally, let's call the residual  $e$ . This is standard notation, as “e” stands for “error”. Again, though, it's not an error in the sense of a mistake. It's an error in the sense that the model is not perfectly accurate, so it doesn't predict the data points exactly. The degree to which the prediction misses is the “error” or “residual”. It is given by the following formula:

$$e = y - \hat{y}$$

**Exercise 9** There are two symbols on the right-hand side of the equation above,  $y$  and  $\hat{y}$ . Which one is the actual data value and which one is the predicted value (the one on the line)?

Please write up your answer here.

---

The residuals are used to determine the regression line. The correct regression line will be the one that results in the smallest residuals overall. How do we measure the overall set of residuals? We can't just calculate the average residual. Because the regression line should go through the middle of the data, the positive residuals will cancel out the negative residuals and the mean residual will just be zero. That's not very useful.

Instead, what we do is *square* the residuals. That makes all of them positive. Then we add together all the squared residuals and that sum is the thing we try to minimize. Well, we don't do that manually because it's hard, so we let the computer do that for us. Because the regression line minimizes the sum of the squared residuals, the regression line is often called the *least-squares* line.

Recall earlier when we mentioned that there was one additional condition to check in order for linear regression to make sense. This condition is that **there should not be any kind of pattern in the residuals**.

We know that some of the points are going to lie above the line (positive residuals) and some of the points will lie below the line (negative residuals). What we need is for the spread of the residuals to be pretty balanced across the length of the regression line and for the residuals not to form any kind of curved pattern.

To check this condition, we'll need to calculate the residuals first. To do so, we introduce a new function from the **broom** package. Whereas **tidy** serves up information about the intercept and the slope of the regression line, **augment** gives us extra information for each data point.

```
involact_race_aug <- augment(involact_race_lm)
involact_race_aug
```



```
## # A tibble: 47 x 9
##   .rownames involact race .fitted .resid .hat .sigma .cooksd .std.resid
##   <chr>      <dbl> <dbl> <dbl> <dbl> <dbl> <dbl> <dbl> <dbl>
## 1 60626      0    10    0.268 -0.268 0.0341 0.452 0.00651 -0.608
## 2 60640     0.1  22.2    0.437 -0.337 0.0246 0.451 0.00731 -0.761
## 3 60613     1.2  19.6    0.401  0.799 0.0261 0.437 0.0436    1.80
## 4 60657     0.5  17.3    0.369  0.131 0.0277 0.453 0.00124  0.295
## 5 60614     0.7  24.5    0.469  0.231 0.0235 0.453 0.00326  0.520
## 6 60610     0.3  54     0.879 -0.579 0.0287 0.445 0.0253  -1.31
## 7 60611     0    4.9    0.197 -0.197 0.0398 0.453 0.00417  -0.448
## 8 60625     0    7.1    0.228 -0.228 0.0372 0.453 0.00517  -0.517
## 9 60618     0.4   5.3    0.203  0.197 0.0393 0.453 0.00411  0.448
## 10 60647    1.1  21.5    0.428  0.672 0.0250 0.442 0.0295   1.52
## # ... with 37 more rows
```

The first three columns consist of the row names (the ZIP codes) followed by the actual data values we started with for `involact` and `race`. But now we’ve “augmented” the original data with some new stuff too. (We won’t learn about anything past the fifth column in this course, though.)

The fourth column—called `.fitted`—is  $\hat{y}$ , or the point on the line that corresponds to the given  $x$  value. Let’s check and make sure this is working as advertised.

The regression equation from above is

$$\widehat{involact} = 0.12922 + 0.01388race$$

Take, for example, the first row in the tibble above, the one corresponding to ZIP code 60626. The value of `race` is 10.0. Plug that value into the equation above:

$$\widehat{involact} = 0.12922 + 0.01388(10.0) = 0.268$$

The model predicts that a ZIP code with 10% racial minorities will have about 0.268 new FAIR policies per 100 housing units. The corresponding number in the `.fitted` column is 0.2680416, so that’s correct.

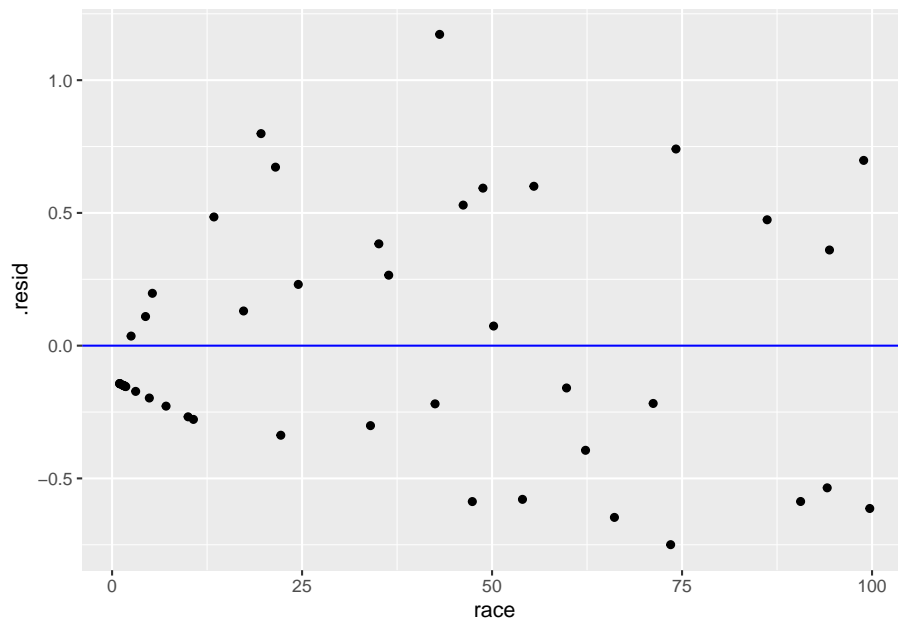
Now skip over to the fifth column of the `augment` output, the one that says `.resid`. If this is the residual  $e$ , then it should be  $y - \hat{y}$ . Since  $y$  is the actual value of `involact` and  $\hat{y}$  is the value predicted by the model, we should get for the first row of output

$$e = y - \hat{y} = 0.0 - 0.268 = -0.268$$

Yup, it works!

To check for patterns in the residuals, we'll create a *residual plot*. A residual plot graphs the residuals above each value along the x-axis. (In the command below, we also add a blue horizontal reference line so that it is clear which points have positive or negative residuals.)

```
ggplot(involact_race_aug, aes(y = .resid, x = race)) +
  geom_point() +
  geom_hline(yintercept = 0, color = "blue")
```



Pay close attention to the `ggplot` code. Notice that the tibble in the first slot is *not* `chredlin` as it was before. The residuals we need to plot are not stored in the raw `chredlin` data. We had to calculate the residuals using the `augment` command, and those residuals are then stored in a different place that we named `involact_race_aug`. In the latter tibble, the residuals themselves are stored in a variable called `.resid`. (Don't forget the dot in `.resid`.)

We are looking for systematic patterns in the residuals. A good residual plot should look like the most boring plot you've ever seen.

For the most part, the residual plot above looks pretty good. The one exception is the clustering near the left edge of the graph.

**Exercise 10** Refer back and forth between the original scatterplot created earlier (with the regression line) and the residual plot above. Can you explain why there is a line of data points with negative residuals along the left edge of the residual plot?

Please write up your answer here.

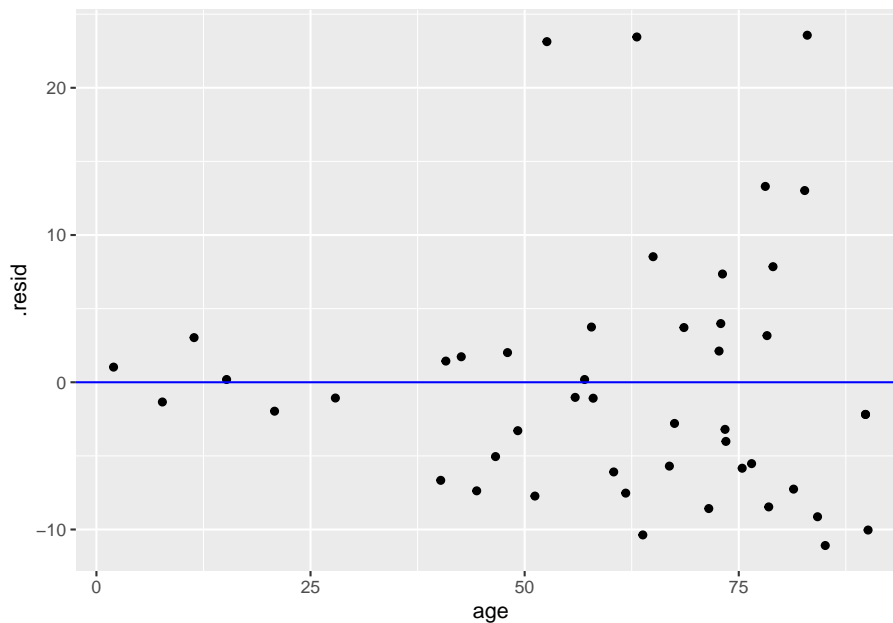
---

Residual patterns that are problematic often involve curved data (where the dots follow a curve around the horizontal reference line instead of spreading evenly around it) and *heteroscedasticity*, which is a fanning out pattern from left to right.

Other than the weird cluster of points at the left, the rest of the residual plot looks pretty good. Ignoring those ZIP codes with 0 FAIR policies, the rest of the residuals stretch, on average, about the same height above and below the line across the whole width of the plot. There is only one slightly large residual at about the 40% mark, but it's not extreme, and it doesn't look like a severe outlier in the original scatterplot.

What does a bad residual plot look like? The code below will run an ill-advised regression analysis on `fire`, the number of fires (per 100 housing units), against `age`, the percent of housing units built before 1939. The residual plot appears below.

```
fire_age_lm <- lm(fire ~ age, data = chredlin)
fire_age_aug <- augment(fire_age_lm)
ggplot(fire_age_aug, aes(y = .resid, x = age)) +
  geom_point() +
  geom_hline(yintercept = 0, color = "blue")
```



**Exercise 11** Using the vocabulary established above, explain why the residual plot above is bad.

Please write up your answer here.

---

Of course, we should never even get as far as running a regression analysis and making a residual plot if we perform exploratory data analysis as we're supposed to.

**Exercise 12(a)** If you were truly interested in investigating an association between the fire risk and the age of buildings in a ZIP code, the first thing you would do is create a scatterplot. Go ahead and do that below. Use `fire` as the response variable and `age` as the predictor.

*# Add code here to create a scatterplot of fire against age*

**Exercise 12(b)** From the scatterplot above, explain why you wouldn't even get as far as running a regression analysis. (Think of the conditions.)

Please write up your answer here.

---

To review, the conditions for a regression analysis are as follows (including the newest fourth condition):

1. The two variables must be numerical.
2. There is a somewhat linear relationship between the variables, as shown in a scatterplot.
3. There are no serious outliers.
4. **There is no pattern in the residuals.**

## 7.10 $R^2$

We've seen that the correlation coefficient  $r$  is of limited utility. In addition to being only a single statistic to summarize a linear association, the number doesn't have any kind of intrinsic meaning. It can only be judged by how close it is to 0 or 1 (or -1) in conjunction with a scatterplot to give you a sense of the strength of the correlation. **In particular, some people try to interpret  $r$  as some kind of percentage, but it's not.**

On the other hand, when we square the correlation coefficient, we *do* get an interpretable number. For some reason, instead of writing  $r^2$ , statisticians write  $R^2$ , with a capital R. (I can't find the historical reason why this is so.) In any event,  $R^2$  can be interpreted as a percentage! It represents the percent of variation in the y variable that can be explained by variation in the x variable.

Here we introduce the last of the **broom** functions: **glance**. Whereas **tidy** reports the intercept and slope, and **augment** reports values associated to each data point separately, the **glance** function gathers up summaries for the entire model. (Do not confuse **glance** with **glimpse**. The latter is a nicer version of **str** that just summarizes the variables in a tibble.)

```
involact_race_glance <- glance(involact_race_lm)
involact_race_glance

## # A tibble: 1 x 12
##   r.squared adj.r.squared sigma statistic      p.value    df logLik   AIC    BIC
##   <dbl>         <dbl> <dbl>      <dbl>      <dbl> <dbl> <dbl> <dbl> <dbl>
## 1     0.509         0.499 0.449      46.7 0.0000000178     1  -28.0  62.0  67.6
## # ... with 3 more variables: deviance <dbl>, df.residual <int>, nobs <int>
```

A more advanced statistics course might discuss the other model summaries present in the **glance** output. The  $R^2$  value is stored in the **r.squared** (inexplicably, now written with a lowercase r). Its value is 0.51. We will word it this way:

51% of the variability in FAIR policies can be accounted for by  
variability in racial composition.

Another way to think about this is to imagine all the factors that might go into the number of FAIR policies obtained in a ZIP code. That number varies across ZIP codes, with some ZIP codes having essentially 0 FAIR policies per 100 housing units, and others having quite a bit more, up to 2 or more per 100 housing units. What accounts for this discrepancy among ZIP codes? Is it the varying racial composition of those neighborhoods? To some degree, yes. We have seen that more racially diverse neighborhoods, on average, require more FAIR policies. But is race the only factor? Probably not. Income, for example, might play a role. People in low income neighborhoods may not be able to acquire traditional insurance due to its cost or their poor credit, etc. That also accounts for some of the variability among ZIP codes. Are there likely even more factors? Most assuredly. In fact, if 51% of the variability in FAIR policies can be accounted for by variability in racial composition. then 49% must be accounted for by other variables. These other variables may or may not be collected in our data, and we will never be able to determine all the factors that go into varying FAIR policy numbers.

$R^2$  is a measure of the fit of the model. High values of  $R^2$  mean that the line predicts the data values closely, whereas lower values of  $R^2$  mean that there is still a lot of variability left in the residuals (again, due to other factors that are not measured in the model).

**Exercise 13** Calculate the correlation coefficient `r` between `involact` and `race` using the `cor` command. (You might have to look back at the last chapter to remember the syntax.) Store that value as `r`.

In a separate code chunk, square that value using the command `r^2`. Verify that the square of the correlation coefficient is the same as the  $R^2$  value reported in the `glance` output above.

```
# Add code here to calculate the correlation coefficient
```

```
# Add code here to square the correlation coefficient
```

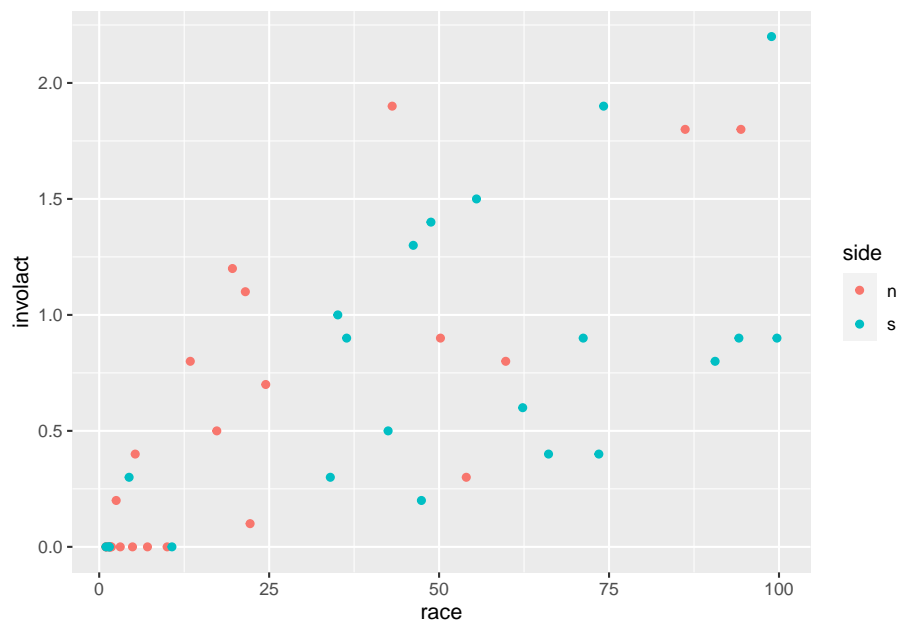
## 7.11 Multiple predictors

The discussion of  $R^2$  above highlights the fact that a single predictor will rarely account for all or even most of the variability in a response variable. Is there a way to take other predictors into account?

The answer is yes, and the statistical technique involved is called multiple regression. Multiple regression is a deep subject, worthy of entire courses. Suffice it to say here that more advanced stats courses go into the ways in which multiple predictors can be included in a regression.

One easy thing we can do is incorporate a categorical variable into a graph and see if that categorical variable might play a role in the regression analysis. For example, there is a variance called `side` in `chredlin` that indicates whether the ZIP code is on the north side (n) or south side(s) of Chicago. As described in an earlier chapter, we can use color to distinguish between the ZIP codes.

```
ggplot(chredlin, aes(y = involact, x = race, color = side)) +  
  geom_point()
```



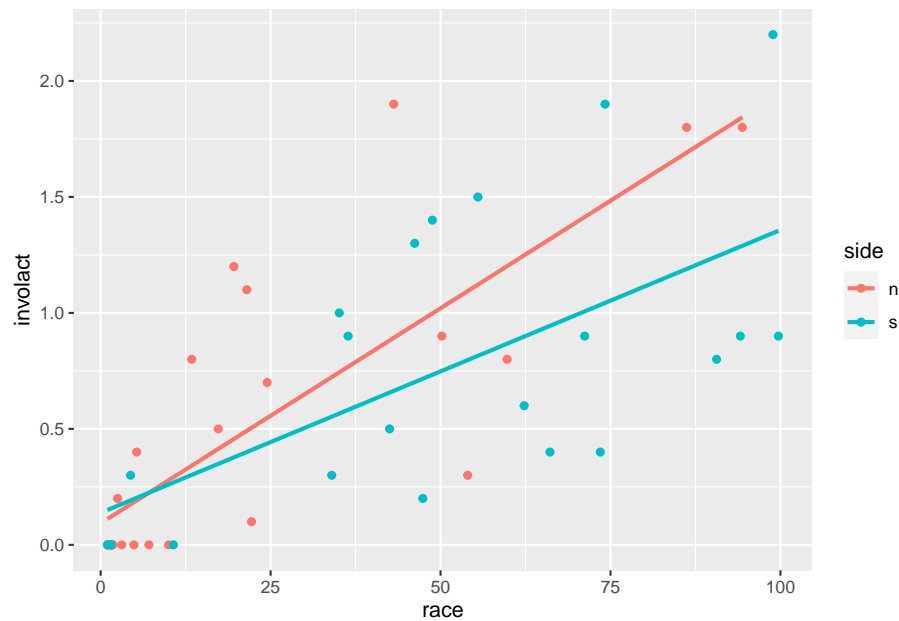
**Exercise 14** Do neighborhoods with higher percent racial minorities tend to be on the north or south side of Chicago?

Please write up your answer here.

Does this affect the regression? We haven't checked the conditions carefully for this new question, so we will exercise caution in coming to any definitive conclusions. But visually, there does appear to be a difference in the models generated for ZIP codes on the north versus south sides:

```
ggplot(chredlin, aes(y = involact, x = race, color = side)) +
  geom_point() +
  geom_smooth(method = lm, se = FALSE)
```

```
## `geom_smooth()` using formula 'y ~ x'
```



**Exercise 15** Although the slopes appear to be different, this is quite misleading. Focus on just the red dots. Which regression condition appears to be violated if we only consider the north side regression? How does that violation appear to affect the slope of the regression line?

Please write up your answer here.

## 7.12 Your turn

Let's revisit the `penguins` data. Imagine that it was much easier to measure body mass than it was to measure flipper length. (I'm not a penguin expert, so I don't know if that's true, but it seems plausible. Weighing a penguin can be done without human contact, for example.) Can we accurately predict flipper length from body mass? (This means that `flipper_length_mm` should be the response variable on the y-axis and `body_mass_g` should be the predictor variable on the x-axis.)

**Exercise 16(a)** Create a scatterplot of the data. Do *not* include a regression line yet. (In other words, there should be no `geom_smooth` in this plot.)

```
# Add code here to create a scatterplot of the data
```



**Exercise 16(b)** Use the scatterplot above to check the first three conditions of regression.

- 1.
- 2.
- 3.

**Exercise 16(c)** As we're reasonably satisfied that the first three conditions are met and regression is worth pursuing, run the `lm` command to perform the regression analysis. Assign the output to the name `fl_bm_lm`. Be sure to type the variable name `fl_bm_lm` on its own line so that the output is printed in this file.

Then use `tidy`, `augment`, and `glance` respectively on the output. Assign the output to the names `fl_bm_tidy`, `fl_bm_aug`, and `fl_bm_glance`. Again, in each code chunk, type the output variable name on its own line to ensure that it prints in this file.

```
# Add code here to generate and print regression output with lm
```

```
# Add code here to "tidy" and print the output from lm
```

```
# Add code here to "augment" and print the output from lm
```

```
# Add code here to "glance" at and print the output from lm
```

**Exercise 16(d)** Use the `augment` output from above to create a residual plot with a blue horizontal reference line.

```
# Add code here to create a residual plot
```

**Exercise 16(e)** Use the residual plot to check the fourth regression condition.

Please write up your answer here.

**Exercise 16(f)** With all the conditions met, plot the regression line on top of the scatterplot of the data. (Use `geom_smooth` with `method = lm` and `se = FALSE` as in the examples earlier.)

```
# Add code here to plot the regression line on the scatterplot
```

**Exercise 16(g)** Using the values of the intercept and slope from the `tidy` output, write the regression equation mathematically (enclosing your answer in double dollar signs as above), using contextually meaningful variable names.

*write – math – here*

**Exercise 16(h)** Interpret the slope in a full, contextually meaningful sentence.

Please write up your answer here.

**Exercise 16(i)** Give a literal interpretation of the intercept. Then comment on the appropriateness of that interpretation. (In other words, does the intercept make sense, or is it a case of extrapolation?)

Please write up your answer here.

**Exercise 16(j)** Use the equation of the regression line to predict the flipper length of a penguin with body mass 4200 grams. Show your work. Then put that prediction into a full, contextually meaningful sentence.

Please write up your answer here.

**Exercise 16(k)** Using the value of  $R^2$  from the `glance` output for the model of flipper length by body mass, write a full, contextually meaningful sentence interpreting that value.

Please write up your answer here.

**Exercise 16(l)** Add `color = species` to the `aes` portion of the `ggplot` command to look at the regression lines for the three different species separately. Comment on the slopes of those three regression lines.

```
# Add code here to plot regressions by species
```

Please write up your answer here.

## 7.13 Conclusion

Going beyond mere correlation, a regression analysis allows us to specify a linear model in the form of an equation. Assuming the conditions are met, this allows us to say more about the association. For example, the slope predicts how

the response changes when comparing two values of the predictor. In fact, we can use the regression line to make a prediction for any reasonable value of the predictor (being careful not to extrapolate). Because regression is only a model, these predictions will not be exactly correct. Real data comes with residuals, meaning deviations from the idealized predictions of the model. But if those residuals are relatively small then the  $R^2$  value will be large and the model does a good job making reasonably accurate predictions.

### 7.13.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.



## Chapter 8

# Introduction to randomization, Part 1

2.0

### Functions introduced in this chapter

`set.seed`, `rflip`, `do`

## 8.1 Introduction

In this module, we'll learn about randomization and simulation. When we want to understand how sampling works, it's helpful to simulate the process of drawing samples repeatedly from a population. In the days before computing, this was very difficult to do. Now, a few simple lines of computer code can generate thousands (even millions) of random samples, often in a matter of seconds or less.

### 8.1.1 Install new packages

If you are using RStudio Workbench, you do not need to install any packages. (Any packages you need should already be installed by the server administrators.)

If you are using R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type the following command at the Console:

```
install.packages("mosaic")
```

### 8.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/08-intro\\_to\\_randomization\\_1.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/08-intro_to_randomization_1.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 8.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

### 8.1.4 Load packages

We load the `tidyverse` package. The `mosaic` package contains some tools for making it easier to learn about randomization and simulation.

```
library(tidyverse)
library(mosaic)
```

```
## Registered S3 method overwritten by 'mosaic':
##   method                                from
##   fortify.SpatialPolygonsDataFrame ggplot2
```

```
##
## The 'mosaic' package masks several functions from core packages in order to add
## additional features. The original behavior of these functions should not be affected.
```

```
##
## Attaching package: 'mosaic'
```

```
## The following object is masked from 'package:Matrix':
##
##   mean
```

```
## The following objects are masked from 'package:faraway':
##
##   ilogit, logit
```

```
## The following objects are masked from 'package:dplyr':  
##  
##     count, do, tally  
  
## The following object is masked from 'package:purrr':  
##  
##     cross  
  
## The following object is masked from 'package:ggplot2':  
##  
##     stat  
  
## The following objects are masked from 'package:stats':  
##  
##     binom.test, cor, cor.test, cov, fivenum, IQR, median, prop.test,  
##     quantile, sd, t.test, var  
  
## The following objects are masked from 'package:base':  
##  
##     max, mean, min, prod, range, sample, sum
```

## 8.2 Sample and population

The goal of the next few chapters is to help you think about the process of sampling from a population. What do these terms mean?

A *population* is a group of objects we would like to study. If that sounds vague, that's because it is. A population can be a group of any size and of any type of thing in which we're interested. Often, populations refer to groups of people. For example, in an election, the population of interest is all voters. But if you're a biologist, you might study populations of other kinds of organisms. If you're an engineer, you might study populations of bolts on bridges. If you're in finance, you might study populations of loans.

Populations are usually inaccessible in their entirety. It is impossible to survey every voter in any reasonably sized election, for example. Therefore, to study them, we have to collect a *sample*. A sample is a subset of the population. We might conduct a poll of 2000 voters to try to learn about voting intentions for the entire population. Of course, for that to work, the sample has to be *representative* of its population. We'll have more to say about that in the future.

### 8.3 Flipping a coin

Before we talk about how samples are obtained from populations in the real world, we're going to perform some simulations.

One of the simplest acts to simulate is flipping a coin. We could get an actual coin and physically flip it over and over again, but that is time-consuming and annoying. It is much easier to flip a “virtual” coin inside the computer. One way to accomplish this in R is to use the `rflip` command from the `mosaic` package. To make sure we're flipping a fair coin, we'll say that we want a 50% chance of heads by including the parameter `prob = 0.5`.

One more bit of technical detail. Since there will be some randomness involved here, we will need to include an R command to ensure that we all get the same results every time this code runs. This is called “setting the seed”. Don't worry too much about what this is doing under the hood. The basic idea is that two people who start with the same seed will generate the same sequence of “random” numbers.

The seed 1234 in the chunk below is totally arbitrary. It could have been any number at all. (And, in fact, we'll use different numbers just for fun.) If you change the seed, you will get different output, so we all need to use the same seed. But the actual common value we all use for the seed is irrelevant.

Here is one coin flip with a 50% chance of coming up heads:

```
set.seed(1234)
rflip(1, prob = 0.5)

##
## Flipping 1 coin [ Prob(Heads) = 0.5 ] ...
##
## T
##
## Number of Heads: 0 [Proportion Heads: 0]
```

Here are ten coin flips, each with a 50% chance of coming up heads:

```
set.seed(1234)
rflip(10, prob = 0.5)

##
## Flipping 10 coins [ Prob(Heads) = 0.5 ] ...
##
## T H H H H T T H H
##
## Number of Heads: 7 [Proportion Heads: 0.7]
```



Just to confirm that this is a random process, let's flip ten coins again (but without setting the seed again):

```
rflip(10, prob = 0.5)

##
## Flipping 10 coins [ Prob(Heads) = 0.5 ] ...
##
## H H T H T H T T T T
##
## Number of Heads: 4 [Proportion Heads: 0.4]
```

If we return to the previous seed of 1234, we should obtain the same ten coin flips we did at first:

```
set.seed(1234)
rflip(10, prob = 0.5)

##
## Flipping 10 coins [ Prob(Heads) = 0.5 ] ...
##
## T H H H H T T H H
##
## Number of Heads: 7 [Proportion Heads: 0.7]
```

And just to see the effect of setting a different seed:

```
set.seed(9999)
rflip(10, prob = 0.5)

##
## Flipping 10 coins [ Prob(Heads) = 0.5 ] ...
##
## H H H T H H T H H H
##
## Number of Heads: 8 [Proportion Heads: 0.8]
```

**Exercise 1** In ten coin flips, how many would you generally expect to come up heads? Is that the actual number of heads you saw in the simulations above? Why aren't the simulations coming up with the expected number of heads each time?

Please write up your answer here.

## 8.4 Multiple simulations

Suppose now that you are not the only person flipping coins. Suppose a bunch of people in a room are all flipping coins. We'll start with ten coin flips per person, a task that could be reasonably done even without a computer.

You might observe three heads in ten flips. Fine, but what about everyone else in the room? What numbers of heads will they see?

The `do` command from `mosaic` is a way of doing something multiple times. Imagine there are twenty people in the room, each flipping a coin ten times, each time with a 50% probability of coming up heads. Observe:

```
set.seed(12345)
do(20) * rflip(10, prob = 0.5)
```

##		n	heads	tails	prop
##	1	10	2	8	0.2
##	2	10	5	5	0.5
##	3	10	5	5	0.5
##	4	10	4	6	0.4
##	5	10	4	6	0.4
##	6	10	7	3	0.7
##	7	10	6	4	0.6
##	8	10	5	5	0.5
##	9	10	7	3	0.7
##	10	10	7	3	0.7
##	11	10	6	4	0.6
##	12	10	7	3	0.7
##	13	10	7	3	0.7
##	14	10	6	4	0.6
##	15	10	7	3	0.7
##	16	10	6	4	0.6
##	17	10	7	3	0.7
##	18	10	3	7	0.3
##	19	10	4	6	0.4
##	20	10	7	3	0.7

The syntax could not be any simpler: `do(20) *` means, literally, “do twenty times.” In other words, this command is telling R to repeat an action twenty times, where the action is flipping a single coin ten times.

You'll notice that in place of a list of outcomes (H or T) of all the individual flips, we have instead a summary of the number of heads and tails each person sees. Each row represents a person, and the columns give information about

each person’s flips. (There are `n = 10` flips for each person, but then the number of heads/tails—and the corresponding “proportion” of heads—changes from person to person.)

Looking at the above rows and columns, we see that the output of our little coin-flipping experiment is actually stored in a data frame! Let’s give it a name and work with it.

```
set.seed(12345)
coin_flips_20_10 <- do(20) * rflip(10, prob = 0.5)
coin_flips_20_10
```

```
##      n heads tails prop
## 1  10      2      8 0.2
## 2  10      5      5 0.5
## 3  10      5      5 0.5
## 4  10      4      6 0.4
## 5  10      4      6 0.4
## 6  10      7      3 0.7
## 7  10      6      4 0.6
## 8  10      5      5 0.5
## 9  10      7      3 0.7
## 10 10      7      3 0.7
## 11 10      6      4 0.6
## 12 10      7      3 0.7
## 13 10      7      3 0.7
## 14 10      6      4 0.6
## 15 10      7      3 0.7
## 16 10      6      4 0.6
## 17 10      7      3 0.7
## 18 10      3      7 0.3
## 19 10      4      6 0.4
## 20 10      7      3 0.7
```

It is significant that we can store our outcomes this way. Because we have a data frame, we can apply all our data analysis tools (graphs, charts, tables, summary statistics, etc.) to the “data” generated from our set of simulations.

For example, what is the mean number of heads these twenty people observed?

```
mean(coin_flips_20_10$heads)
```

```
## [1] 5.6
```

**Exercise 2** The data frame `coin_flips_20_10` contains four variables: `n`, `heads`, `tails`, and `prop`. In the code chunk above, we calculated `mean(coin_flips_20_10$heads)` which gave us the mean count of heads for all people flipping coins. Instead of calculating the mean count of heads, change the variable from `heads` to `prop` to calculate the mean *proportion* of heads. Then explain why your answer makes sense in light of the mean count of heads calculated above.

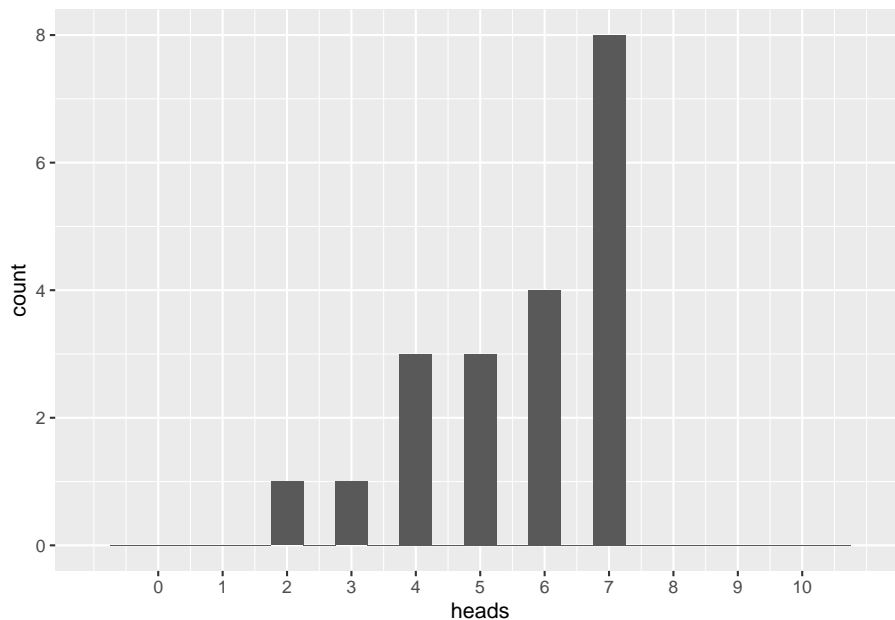
*# Add code here to calculate the mean proportion of heads.*

Please write up your answer here.

Let's look at a histogram of the number of heads we see in the simulated flips. (The fancy stuff in `scale_x_continuous` is just making sure that the x-axis goes from 0 to 10 and that the tick marks appear on each whole number.)

```
ggplot(coin_flips_20_10, aes(x = heads)) +
  geom_histogram(binwidth = 0.5) +
  scale_x_continuous(limits = c(-1, 11), breaks = seq(0, 10, 1))
```

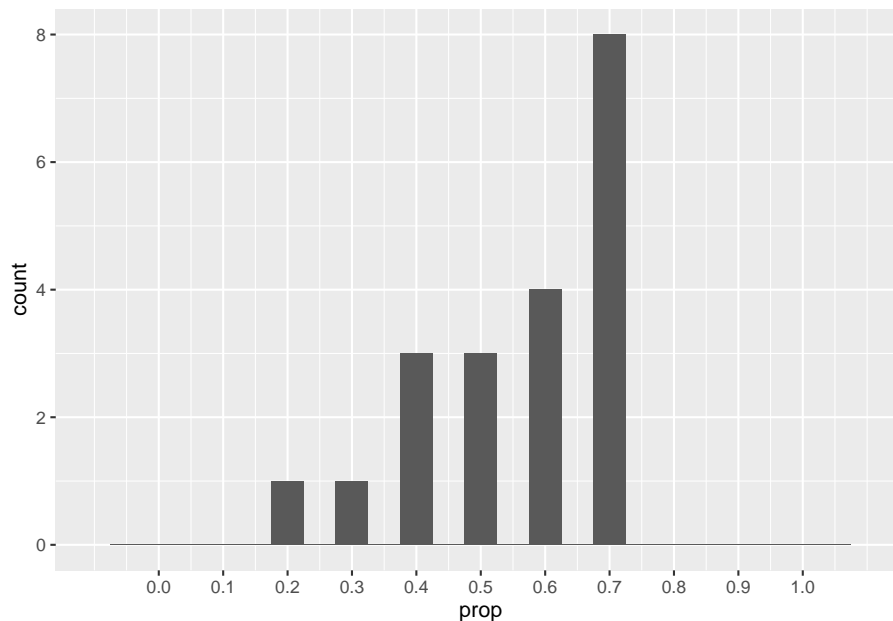
## Warning: Removed 2 rows containing missing values (geom\_bar).



Let's do the same thing, but now let's consider the *proportion* of heads.

```
ggplot(coin_flips_20_10, aes(x = prop)) +
  geom_histogram(binwidth = 0.05) +
  scale_x_continuous(limits = c(-0.1, 1.1), breaks = seq(0, 1, 0.1))
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



## 8.5 Bigger and better!

With only twenty people, it was possible that, for example, nobody would get all heads or all tails. Indeed, in `coin_flips_20_10` there were no people who got all heads or all tails. Also, there were more people with six and seven heads than with five heads, even though we “expected” the average to be five heads. There is nothing particularly significant about that; it happened by pure chance alone. Another run through the above commands would generate a somewhat different outcome. That’s what happens when things are random.

Instead, let’s imagine that we recruited way more people to flip coins with us. Let’s try it again with 2000 people:

```
set.seed(1234)
coin_flips_2000_10 <- do(2000) * rflip(10, prob = 0.5)
coin_flips_2000_10
```

##	n	heads	tails	prop
## 1	10	4	6	0.4
## 2	10	4	6	0.4
## 3	10	4	6	0.4
## 4	10	6	4	0.6
## 5	10	5	5	0.5
## 6	10	4	6	0.4
## 7	10	4	6	0.4
## 8	10	4	6	0.4
## 9	10	3	7	0.3
## 10	10	1	9	0.1
## 11	10	5	5	0.5
## 12	10	5	5	0.5
## 13	10	7	3	0.7
## 14	10	7	3	0.7
## 15	10	5	5	0.5
## 16	10	3	7	0.3
## 17	10	5	5	0.5
## 18	10	5	5	0.5
## 19	10	9	1	0.9
## 20	10	6	4	0.6
## 21	10	7	3	0.7
## 22	10	2	8	0.2
## 23	10	6	4	0.6
## 24	10	6	4	0.6
## 25	10	5	5	0.5
## 26	10	4	6	0.4
## 27	10	5	5	0.5
## 28	10	5	5	0.5
## 29	10	6	4	0.6
## 30	10	6	4	0.6
## 31	10	3	7	0.3
## 32	10	3	7	0.3
## 33	10	4	6	0.4
## 34	10	5	5	0.5
## 35	10	7	3	0.7
## 36	10	6	4	0.6
## 37	10	4	6	0.4
## 38	10	3	7	0.3
## 39	10	7	3	0.7
## 40	10	6	4	0.6
## 41	10	6	4	0.6
## 42	10	3	7	0.3
## 43	10	7	3	0.7
## 44	10	9	1	0.9
## 45	10	7	3	0.7

## 46	10	5	5	0.5
## 47	10	4	6	0.4
## 48	10	6	4	0.6
## 49	10	7	3	0.7
## 50	10	8	2	0.8
## 51	10	6	4	0.6
## 52	10	5	5	0.5
## 53	10	7	3	0.7
## 54	10	7	3	0.7
## 55	10	5	5	0.5
## 56	10	6	4	0.6
## 57	10	5	5	0.5
## 58	10	5	5	0.5
## 59	10	7	3	0.7
## 60	10	3	7	0.3
## 61	10	4	6	0.4
## 62	10	6	4	0.6
## 63	10	6	4	0.6
## 64	10	6	4	0.6
## 65	10	5	5	0.5
## 66	10	6	4	0.6
## 67	10	5	5	0.5
## 68	10	4	6	0.4
## 69	10	4	6	0.4
## 70	10	4	6	0.4
## 71	10	4	6	0.4
## 72	10	4	6	0.4
## 73	10	7	3	0.7
## 74	10	3	7	0.3
## 75	10	7	3	0.7
## 76	10	6	4	0.6
## 77	10	6	4	0.6
## 78	10	4	6	0.4
## 79	10	7	3	0.7
## 80	10	4	6	0.4
## 81	10	4	6	0.4
## 82	10	1	9	0.1
## 83	10	7	3	0.7
## 84	10	7	3	0.7
## 85	10	7	3	0.7
## 86	10	3	7	0.3
## 87	10	6	4	0.6
## 88	10	4	6	0.4
## 89	10	7	3	0.7
## 90	10	4	6	0.4
## 91	10	3	7	0.3

##	92	10	4	6	0.4
##	93	10	5	5	0.5
##	94	10	6	4	0.6
##	95	10	6	4	0.6
##	96	10	4	6	0.4
##	97	10	7	3	0.7
##	98	10	5	5	0.5
##	99	10	5	5	0.5
##	100	10	4	6	0.4
##	101	10	6	4	0.6
##	102	10	3	7	0.3
##	103	10	5	5	0.5
##	104	10	6	4	0.6
##	105	10	5	5	0.5
##	106	10	6	4	0.6
##	107	10	2	8	0.2
##	108	10	4	6	0.4
##	109	10	4	6	0.4
##	110	10	2	8	0.2
##	111	10	5	5	0.5
##	112	10	4	6	0.4
##	113	10	5	5	0.5
##	114	10	4	6	0.4
##	115	10	1	9	0.1
##	116	10	5	5	0.5
##	117	10	2	8	0.2
##	118	10	8	2	0.8
##	119	10	4	6	0.4
##	120	10	7	3	0.7
##	121	10	5	5	0.5
##	122	10	7	3	0.7
##	123	10	5	5	0.5
##	124	10	6	4	0.6
##	125	10	4	6	0.4
##	126	10	6	4	0.6
##	127	10	8	2	0.8
##	128	10	2	8	0.2
##	129	10	6	4	0.6
##	130	10	4	6	0.4
##	131	10	6	4	0.6
##	132	10	3	7	0.3
##	133	10	3	7	0.3
##	134	10	5	5	0.5
##	135	10	6	4	0.6
##	136	10	3	7	0.3
##	137	10	7	3	0.7



##	138	10	6	4	0.6
##	139	10	5	5	0.5
##	140	10	5	5	0.5
##	141	10	4	6	0.4
##	142	10	7	3	0.7
##	143	10	3	7	0.3
##	144	10	4	6	0.4
##	145	10	4	6	0.4
##	146	10	6	4	0.6
##	147	10	6	4	0.6
##	148	10	6	4	0.6
##	149	10	7	3	0.7
##	150	10	8	2	0.8
##	151	10	3	7	0.3
##	152	10	3	7	0.3
##	153	10	4	6	0.4
##	154	10	4	6	0.4
##	155	10	3	7	0.3
##	156	10	2	8	0.2
##	157	10	3	7	0.3
##	158	10	7	3	0.7
##	159	10	5	5	0.5
##	160	10	3	7	0.3
##	161	10	4	6	0.4
##	162	10	6	4	0.6
##	163	10	4	6	0.4
##	164	10	5	5	0.5
##	165	10	4	6	0.4
##	166	10	4	6	0.4
##	167	10	3	7	0.3
##	168	10	4	6	0.4
##	169	10	4	6	0.4
##	170	10	4	6	0.4
##	171	10	4	6	0.4
##	172	10	4	6	0.4
##	173	10	7	3	0.7
##	174	10	3	7	0.3
##	175	10	8	2	0.8
##	176	10	5	5	0.5
##	177	10	8	2	0.8
##	178	10	4	6	0.4
##	179	10	5	5	0.5
##	180	10	3	7	0.3
##	181	10	7	3	0.7
##	182	10	5	5	0.5
##	183	10	4	6	0.4

## 184	10	3	7	0.3
## 185	10	6	4	0.6
## 186	10	6	4	0.6
## 187	10	7	3	0.7
## 188	10	3	7	0.3
## 189	10	5	5	0.5
## 190	10	7	3	0.7
## 191	10	4	6	0.4
## 192	10	6	4	0.6
## 193	10	4	6	0.4
## 194	10	5	5	0.5
## 195	10	5	5	0.5
## 196	10	8	2	0.8
## 197	10	9	1	0.9
## 198	10	5	5	0.5
## 199	10	7	3	0.7
## 200	10	5	5	0.5
## 201	10	4	6	0.4
## 202	10	5	5	0.5
## 203	10	3	7	0.3
## 204	10	5	5	0.5
## 205	10	6	4	0.6
## 206	10	3	7	0.3
## 207	10	4	6	0.4
## 208	10	3	7	0.3
## 209	10	4	6	0.4
## 210	10	9	1	0.9
## 211	10	4	6	0.4
## 212	10	5	5	0.5
## 213	10	6	4	0.6
## 214	10	3	7	0.3
## 215	10	5	5	0.5
## 216	10	7	3	0.7
## 217	10	4	6	0.4
## 218	10	6	4	0.6
## 219	10	4	6	0.4
## 220	10	4	6	0.4
## 221	10	4	6	0.4
## 222	10	4	6	0.4
## 223	10	10	0	1.0
## 224	10	4	6	0.4
## 225	10	3	7	0.3
## 226	10	8	2	0.8
## 227	10	7	3	0.7
## 228	10	6	4	0.6
## 229	10	6	4	0.6

##	230	10	4	6	0.4
##	231	10	6	4	0.6
##	232	10	4	6	0.4
##	233	10	6	4	0.6
##	234	10	3	7	0.3
##	235	10	4	6	0.4
##	236	10	4	6	0.4
##	237	10	5	5	0.5
##	238	10	3	7	0.3
##	239	10	4	6	0.4
##	240	10	7	3	0.7
##	241	10	8	2	0.8
##	242	10	6	4	0.6
##	243	10	6	4	0.6
##	244	10	7	3	0.7
##	245	10	6	4	0.6
##	246	10	6	4	0.6
##	247	10	8	2	0.8
##	248	10	4	6	0.4
##	249	10	4	6	0.4
##	250	10	4	6	0.4
##	251	10	4	6	0.4
##	252	10	5	5	0.5
##	253	10	5	5	0.5
##	254	10	3	7	0.3
##	255	10	4	6	0.4
##	256	10	5	5	0.5
##	257	10	6	4	0.6
##	258	10	6	4	0.6
##	259	10	6	4	0.6
##	260	10	8	2	0.8
##	261	10	5	5	0.5
##	262	10	5	5	0.5
##	263	10	1	9	0.1
##	264	10	6	4	0.6
##	265	10	3	7	0.3
##	266	10	4	6	0.4
##	267	10	6	4	0.6
##	268	10	7	3	0.7
##	269	10	7	3	0.7
##	270	10	5	5	0.5
##	271	10	5	5	0.5
##	272	10	5	5	0.5
##	273	10	5	5	0.5
##	274	10	6	4	0.6
##	275	10	5	5	0.5

##	276	10	6	4	0.6
##	277	10	6	4	0.6
##	278	10	5	5	0.5
##	279	10	5	5	0.5
##	280	10	5	5	0.5
##	281	10	10	0	1.0
##	282	10	5	5	0.5
##	283	10	7	3	0.7
##	284	10	4	6	0.4
##	285	10	5	5	0.5
##	286	10	6	4	0.6
##	287	10	6	4	0.6
##	288	10	3	7	0.3
##	289	10	6	4	0.6
##	290	10	5	5	0.5
##	291	10	7	3	0.7
##	292	10	4	6	0.4
##	293	10	4	6	0.4
##	294	10	3	7	0.3
##	295	10	8	2	0.8
##	296	10	2	8	0.2
##	297	10	5	5	0.5
##	298	10	4	6	0.4
##	299	10	7	3	0.7
##	300	10	3	7	0.3
##	301	10	3	7	0.3
##	302	10	6	4	0.6
##	303	10	6	4	0.6
##	304	10	6	4	0.6
##	305	10	4	6	0.4
##	306	10	5	5	0.5
##	307	10	4	6	0.4
##	308	10	5	5	0.5
##	309	10	3	7	0.3
##	310	10	6	4	0.6
##	311	10	6	4	0.6
##	312	10	5	5	0.5
##	313	10	4	6	0.4
##	314	10	3	7	0.3
##	315	10	5	5	0.5
##	316	10	3	7	0.3
##	317	10	4	6	0.4
##	318	10	6	4	0.6
##	319	10	4	6	0.4
##	320	10	2	8	0.2
##	321	10	5	5	0.5

##	322	10	6	4	0.6
##	323	10	4	6	0.4
##	324	10	6	4	0.6
##	325	10	4	6	0.4
##	326	10	4	6	0.4
##	327	10	6	4	0.6
##	328	10	5	5	0.5
##	329	10	7	3	0.7
##	330	10	4	6	0.4
##	331	10	3	7	0.3
##	332	10	4	6	0.4
##	333	10	5	5	0.5
##	334	10	5	5	0.5
##	335	10	6	4	0.6
##	336	10	4	6	0.4
##	337	10	3	7	0.3
##	338	10	6	4	0.6
##	339	10	4	6	0.4
##	340	10	2	8	0.2
##	341	10	7	3	0.7
##	342	10	3	7	0.3
##	343	10	6	4	0.6
##	344	10	4	6	0.4
##	345	10	0	10	0.0
##	346	10	3	7	0.3
##	347	10	6	4	0.6
##	348	10	5	5	0.5
##	349	10	7	3	0.7
##	350	10	3	7	0.3
##	351	10	6	4	0.6
##	352	10	7	3	0.7
##	353	10	6	4	0.6
##	354	10	8	2	0.8
##	355	10	6	4	0.6
##	356	10	4	6	0.4
##	357	10	8	2	0.8
##	358	10	2	8	0.2
##	359	10	4	6	0.4
##	360	10	6	4	0.6
##	361	10	2	8	0.2
##	362	10	4	6	0.4
##	363	10	5	5	0.5
##	364	10	4	6	0.4
##	365	10	7	3	0.7
##	366	10	6	4	0.6
##	367	10	6	4	0.6

##	368	10	2	8	0.2
##	369	10	4	6	0.4
##	370	10	6	4	0.6
##	371	10	2	8	0.2
##	372	10	4	6	0.4
##	373	10	2	8	0.2
##	374	10	4	6	0.4
##	375	10	8	2	0.8
##	376	10	6	4	0.6
##	377	10	6	4	0.6
##	378	10	6	4	0.6
##	379	10	6	4	0.6
##	380	10	6	4	0.6
##	381	10	6	4	0.6
##	382	10	8	2	0.8
##	383	10	4	6	0.4
##	384	10	6	4	0.6
##	385	10	4	6	0.4
##	386	10	3	7	0.3
##	387	10	6	4	0.6
##	388	10	4	6	0.4
##	389	10	6	4	0.6
##	390	10	5	5	0.5
##	391	10	4	6	0.4
##	392	10	6	4	0.6
##	393	10	6	4	0.6
##	394	10	5	5	0.5
##	395	10	4	6	0.4
##	396	10	6	4	0.6
##	397	10	4	6	0.4
##	398	10	7	3	0.7
##	399	10	4	6	0.4
##	400	10	6	4	0.6
##	401	10	3	7	0.3
##	402	10	6	4	0.6
##	403	10	7	3	0.7
##	404	10	4	6	0.4
##	405	10	6	4	0.6
##	406	10	3	7	0.3
##	407	10	7	3	0.7
##	408	10	8	2	0.8
##	409	10	4	6	0.4
##	410	10	6	4	0.6
##	411	10	4	6	0.4
##	412	10	3	7	0.3
##	413	10	4	6	0.4

##	414	10	7	3	0.7
##	415	10	3	7	0.3
##	416	10	5	5	0.5
##	417	10	5	5	0.5
##	418	10	7	3	0.7
##	419	10	6	4	0.6
##	420	10	5	5	0.5
##	421	10	6	4	0.6
##	422	10	3	7	0.3
##	423	10	5	5	0.5
##	424	10	4	6	0.4
##	425	10	5	5	0.5
##	426	10	5	5	0.5
##	427	10	3	7	0.3
##	428	10	6	4	0.6
##	429	10	4	6	0.4
##	430	10	6	4	0.6
##	431	10	7	3	0.7
##	432	10	7	3	0.7
##	433	10	5	5	0.5
##	434	10	4	6	0.4
##	435	10	4	6	0.4
##	436	10	3	7	0.3
##	437	10	4	6	0.4
##	438	10	5	5	0.5
##	439	10	7	3	0.7
##	440	10	5	5	0.5
##	441	10	5	5	0.5
##	442	10	7	3	0.7
##	443	10	8	2	0.8
##	444	10	6	4	0.6
##	445	10	5	5	0.5
##	446	10	4	6	0.4
##	447	10	3	7	0.3
##	448	10	5	5	0.5
##	449	10	6	4	0.6
##	450	10	7	3	0.7
##	451	10	9	1	0.9
##	452	10	5	5	0.5
##	453	10	5	5	0.5
##	454	10	3	7	0.3
##	455	10	5	5	0.5
##	456	10	5	5	0.5
##	457	10	5	5	0.5
##	458	10	3	7	0.3
##	459	10	3	7	0.3

##	460	10	5	5	0.5
##	461	10	4	6	0.4
##	462	10	7	3	0.7
##	463	10	7	3	0.7
##	464	10	3	7	0.3
##	465	10	4	6	0.4
##	466	10	5	5	0.5
##	467	10	5	5	0.5
##	468	10	3	7	0.3
##	469	10	8	2	0.8
##	470	10	5	5	0.5
##	471	10	6	4	0.6
##	472	10	5	5	0.5
##	473	10	7	3	0.7
##	474	10	4	6	0.4
##	475	10	4	6	0.4
##	476	10	5	5	0.5
##	477	10	2	8	0.2
##	478	10	6	4	0.6
##	479	10	6	4	0.6
##	480	10	2	8	0.2
##	481	10	6	4	0.6
##	482	10	5	5	0.5
##	483	10	5	5	0.5
##	484	10	6	4	0.6
##	485	10	4	6	0.4
##	486	10	5	5	0.5
##	487	10	6	4	0.6
##	488	10	3	7	0.3
##	489	10	3	7	0.3
##	490	10	6	4	0.6
##	491	10	4	6	0.4
##	492	10	7	3	0.7
##	493	10	4	6	0.4
##	494	10	6	4	0.6
##	495	10	4	6	0.4
##	496	10	8	2	0.8
##	497	10	5	5	0.5
##	498	10	6	4	0.6
##	499	10	6	4	0.6
##	500	10	4	6	0.4
##	501	10	4	6	0.4
##	502	10	5	5	0.5
##	503	10	3	7	0.3
##	504	10	3	7	0.3
##	505	10	6	4	0.6



##	506	10	5	5	0.5
##	507	10	6	4	0.6
##	508	10	5	5	0.5
##	509	10	5	5	0.5
##	510	10	6	4	0.6
##	511	10	5	5	0.5
##	512	10	4	6	0.4
##	513	10	6	4	0.6
##	514	10	5	5	0.5
##	515	10	5	5	0.5
##	516	10	9	1	0.9
##	517	10	4	6	0.4
##	518	10	2	8	0.2
##	519	10	3	7	0.3
##	520	10	4	6	0.4
##	521	10	2	8	0.2
##	522	10	6	4	0.6
##	523	10	6	4	0.6
##	524	10	7	3	0.7
##	525	10	5	5	0.5
##	526	10	7	3	0.7
##	527	10	7	3	0.7
##	528	10	2	8	0.2
##	529	10	4	6	0.4
##	530	10	8	2	0.8
##	531	10	5	5	0.5
##	532	10	6	4	0.6
##	533	10	8	2	0.8
##	534	10	3	7	0.3
##	535	10	4	6	0.4
##	536	10	6	4	0.6
##	537	10	8	2	0.8
##	538	10	4	6	0.4
##	539	10	4	6	0.4
##	540	10	6	4	0.6
##	541	10	5	5	0.5
##	542	10	4	6	0.4
##	543	10	5	5	0.5
##	544	10	5	5	0.5
##	545	10	3	7	0.3
##	546	10	4	6	0.4
##	547	10	6	4	0.6
##	548	10	4	6	0.4
##	549	10	6	4	0.6
##	550	10	4	6	0.4
##	551	10	6	4	0.6

##	552	10	3	7	0.3
##	553	10	5	5	0.5
##	554	10	6	4	0.6
##	555	10	5	5	0.5
##	556	10	8	2	0.8
##	557	10	2	8	0.2
##	558	10	5	5	0.5
##	559	10	4	6	0.4
##	560	10	5	5	0.5
##	561	10	4	6	0.4
##	562	10	6	4	0.6
##	563	10	6	4	0.6
##	564	10	4	6	0.4
##	565	10	2	8	0.2
##	566	10	3	7	0.3
##	567	10	6	4	0.6
##	568	10	3	7	0.3
##	569	10	5	5	0.5
##	570	10	7	3	0.7
##	571	10	8	2	0.8
##	572	10	6	4	0.6
##	573	10	4	6	0.4
##	574	10	6	4	0.6
##	575	10	3	7	0.3
##	576	10	4	6	0.4
##	577	10	5	5	0.5
##	578	10	7	3	0.7
##	579	10	4	6	0.4
##	580	10	4	6	0.4
##	581	10	2	8	0.2
##	582	10	6	4	0.6
##	583	10	5	5	0.5
##	584	10	5	5	0.5
##	585	10	5	5	0.5
##	586	10	6	4	0.6
##	587	10	6	4	0.6
##	588	10	8	2	0.8
##	589	10	5	5	0.5
##	590	10	8	2	0.8
##	591	10	5	5	0.5
##	592	10	6	4	0.6
##	593	10	7	3	0.7
##	594	10	3	7	0.3
##	595	10	4	6	0.4
##	596	10	2	8	0.2
##	597	10	5	5	0.5

##	598	10	6	4	0.6
##	599	10	6	4	0.6
##	600	10	7	3	0.7
##	601	10	4	6	0.4
##	602	10	6	4	0.6
##	603	10	6	4	0.6
##	604	10	5	5	0.5
##	605	10	5	5	0.5
##	606	10	7	3	0.7
##	607	10	7	3	0.7
##	608	10	6	4	0.6
##	609	10	3	7	0.3
##	610	10	4	6	0.4
##	611	10	9	1	0.9
##	612	10	6	4	0.6
##	613	10	5	5	0.5
##	614	10	4	6	0.4
##	615	10	6	4	0.6
##	616	10	4	6	0.4
##	617	10	7	3	0.7
##	618	10	3	7	0.3
##	619	10	6	4	0.6
##	620	10	5	5	0.5
##	621	10	7	3	0.7
##	622	10	5	5	0.5
##	623	10	5	5	0.5
##	624	10	5	5	0.5
##	625	10	6	4	0.6
##	626	10	3	7	0.3
##	627	10	4	6	0.4
##	628	10	8	2	0.8
##	629	10	6	4	0.6
##	630	10	6	4	0.6
##	631	10	5	5	0.5
##	632	10	3	7	0.3
##	633	10	5	5	0.5
##	634	10	4	6	0.4
##	635	10	6	4	0.6
##	636	10	7	3	0.7
##	637	10	5	5	0.5
##	638	10	4	6	0.4
##	639	10	4	6	0.4
##	640	10	5	5	0.5
##	641	10	3	7	0.3
##	642	10	4	6	0.4
##	643	10	5	5	0.5

##	644	10	7	3	0.7
##	645	10	5	5	0.5
##	646	10	5	5	0.5
##	647	10	5	5	0.5
##	648	10	4	6	0.4
##	649	10	5	5	0.5
##	650	10	7	3	0.7
##	651	10	3	7	0.3
##	652	10	6	4	0.6
##	653	10	6	4	0.6
##	654	10	8	2	0.8
##	655	10	7	3	0.7
##	656	10	4	6	0.4
##	657	10	7	3	0.7
##	658	10	5	5	0.5
##	659	10	7	3	0.7
##	660	10	6	4	0.6
##	661	10	2	8	0.2
##	662	10	8	2	0.8
##	663	10	2	8	0.2
##	664	10	6	4	0.6
##	665	10	4	6	0.4
##	666	10	3	7	0.3
##	667	10	5	5	0.5
##	668	10	6	4	0.6
##	669	10	6	4	0.6
##	670	10	4	6	0.4
##	671	10	7	3	0.7
##	672	10	2	8	0.2
##	673	10	2	8	0.2
##	674	10	6	4	0.6
##	675	10	5	5	0.5
##	676	10	8	2	0.8
##	677	10	5	5	0.5
##	678	10	5	5	0.5
##	679	10	5	5	0.5
##	680	10	5	5	0.5
##	681	10	6	4	0.6
##	682	10	4	6	0.4
##	683	10	2	8	0.2
##	684	10	6	4	0.6
##	685	10	4	6	0.4
##	686	10	5	5	0.5
##	687	10	5	5	0.5
##	688	10	6	4	0.6
##	689	10	6	4	0.6

##	690	10	4	6	0.4
##	691	10	4	6	0.4
##	692	10	4	6	0.4
##	693	10	5	5	0.5
##	694	10	5	5	0.5
##	695	10	5	5	0.5
##	696	10	5	5	0.5
##	697	10	6	4	0.6
##	698	10	6	4	0.6
##	699	10	5	5	0.5
##	700	10	7	3	0.7
##	701	10	2	8	0.2
##	702	10	7	3	0.7
##	703	10	7	3	0.7
##	704	10	1	9	0.1
##	705	10	5	5	0.5
##	706	10	5	5	0.5
##	707	10	4	6	0.4
##	708	10	4	6	0.4
##	709	10	6	4	0.6
##	710	10	3	7	0.3
##	711	10	4	6	0.4
##	712	10	5	5	0.5
##	713	10	8	2	0.8
##	714	10	3	7	0.3
##	715	10	6	4	0.6
##	716	10	5	5	0.5
##	717	10	4	6	0.4
##	718	10	2	8	0.2
##	719	10	3	7	0.3
##	720	10	1	9	0.1
##	721	10	3	7	0.3
##	722	10	6	4	0.6
##	723	10	3	7	0.3
##	724	10	5	5	0.5
##	725	10	5	5	0.5
##	726	10	7	3	0.7
##	727	10	7	3	0.7
##	728	10	3	7	0.3
##	729	10	4	6	0.4
##	730	10	5	5	0.5
##	731	10	7	3	0.7
##	732	10	6	4	0.6
##	733	10	7	3	0.7
##	734	10	8	2	0.8
##	735	10	6	4	0.6

##	736	10	2	8	0.2
##	737	10	6	4	0.6
##	738	10	6	4	0.6
##	739	10	5	5	0.5
##	740	10	4	6	0.4
##	741	10	6	4	0.6
##	742	10	5	5	0.5
##	743	10	5	5	0.5
##	744	10	4	6	0.4
##	745	10	5	5	0.5
##	746	10	4	6	0.4
##	747	10	3	7	0.3
##	748	10	5	5	0.5
##	749	10	6	4	0.6
##	750	10	6	4	0.6
##	751	10	7	3	0.7
##	752	10	4	6	0.4
##	753	10	4	6	0.4
##	754	10	5	5	0.5
##	755	10	6	4	0.6
##	756	10	6	4	0.6
##	757	10	3	7	0.3
##	758	10	5	5	0.5
##	759	10	4	6	0.4
##	760	10	5	5	0.5
##	761	10	5	5	0.5
##	762	10	5	5	0.5
##	763	10	5	5	0.5
##	764	10	4	6	0.4
##	765	10	5	5	0.5
##	766	10	5	5	0.5
##	767	10	5	5	0.5
##	768	10	5	5	0.5
##	769	10	7	3	0.7
##	770	10	3	7	0.3
##	771	10	2	8	0.2
##	772	10	6	4	0.6
##	773	10	8	2	0.8
##	774	10	5	5	0.5
##	775	10	7	3	0.7
##	776	10	6	4	0.6
##	777	10	5	5	0.5
##	778	10	7	3	0.7
##	779	10	3	7	0.3
##	780	10	5	5	0.5
##	781	10	6	4	0.6

##	782	10	3	7	0.3
##	783	10	4	6	0.4
##	784	10	5	5	0.5
##	785	10	5	5	0.5
##	786	10	7	3	0.7
##	787	10	5	5	0.5
##	788	10	5	5	0.5
##	789	10	2	8	0.2
##	790	10	6	4	0.6
##	791	10	5	5	0.5
##	792	10	8	2	0.8
##	793	10	5	5	0.5
##	794	10	4	6	0.4
##	795	10	6	4	0.6
##	796	10	5	5	0.5
##	797	10	7	3	0.7
##	798	10	6	4	0.6
##	799	10	5	5	0.5
##	800	10	5	5	0.5
##	801	10	3	7	0.3
##	802	10	4	6	0.4
##	803	10	3	7	0.3
##	804	10	3	7	0.3
##	805	10	3	7	0.3
##	806	10	5	5	0.5
##	807	10	5	5	0.5
##	808	10	7	3	0.7
##	809	10	4	6	0.4
##	810	10	7	3	0.7
##	811	10	5	5	0.5
##	812	10	5	5	0.5
##	813	10	5	5	0.5
##	814	10	5	5	0.5
##	815	10	5	5	0.5
##	816	10	4	6	0.4
##	817	10	7	3	0.7
##	818	10	4	6	0.4
##	819	10	4	6	0.4
##	820	10	3	7	0.3
##	821	10	6	4	0.6
##	822	10	6	4	0.6
##	823	10	6	4	0.6
##	824	10	8	2	0.8
##	825	10	3	7	0.3
##	826	10	3	7	0.3
##	827	10	6	4	0.6

##	828	10	7	3	0.7
##	829	10	5	5	0.5
##	830	10	3	7	0.3
##	831	10	6	4	0.6
##	832	10	6	4	0.6
##	833	10	5	5	0.5
##	834	10	6	4	0.6
##	835	10	5	5	0.5
##	836	10	8	2	0.8
##	837	10	5	5	0.5
##	838	10	5	5	0.5
##	839	10	3	7	0.3
##	840	10	2	8	0.2
##	841	10	4	6	0.4
##	842	10	6	4	0.6
##	843	10	7	3	0.7
##	844	10	7	3	0.7
##	845	10	3	7	0.3
##	846	10	3	7	0.3
##	847	10	3	7	0.3
##	848	10	4	6	0.4
##	849	10	5	5	0.5
##	850	10	6	4	0.6
##	851	10	4	6	0.4
##	852	10	3	7	0.3
##	853	10	4	6	0.4
##	854	10	5	5	0.5
##	855	10	4	6	0.4
##	856	10	6	4	0.6
##	857	10	6	4	0.6
##	858	10	7	3	0.7
##	859	10	5	5	0.5
##	860	10	5	5	0.5
##	861	10	4	6	0.4
##	862	10	6	4	0.6
##	863	10	4	6	0.4
##	864	10	6	4	0.6
##	865	10	6	4	0.6
##	866	10	6	4	0.6
##	867	10	2	8	0.2
##	868	10	4	6	0.4
##	869	10	3	7	0.3
##	870	10	5	5	0.5
##	871	10	7	3	0.7
##	872	10	5	5	0.5
##	873	10	5	5	0.5



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##	875	10	6	4	0.6
##	876	10	7	3	0.7
##	877	10	4	6	0.4
##	878	10	3	7	0.3
##	879	10	5	5	0.5
##	880	10	7	3	0.7
##	881	10	6	4	0.6
##	882	10	7	3	0.7
##	883	10	8	2	0.8
##	884	10	6	4	0.6
##	885	10	3	7	0.3
##	886	10	6	4	0.6
##	887	10	4	6	0.4
##	888	10	4	6	0.4
##	889	10	5	5	0.5
##	890	10	5	5	0.5
##	891	10	7	3	0.7
##	892	10	5	5	0.5
##	893	10	7	3	0.7
##	894	10	5	5	0.5
##	895	10	6	4	0.6
##	896	10	3	7	0.3
##	897	10	6	4	0.6
##	898	10	4	6	0.4
##	899	10	4	6	0.4
##	900	10	2	8	0.2
##	901	10	7	3	0.7
##	902	10	7	3	0.7
##	903	10	6	4	0.6
##	904	10	7	3	0.7
##	905	10	4	6	0.4
##	906	10	3	7	0.3
##	907	10	3	7	0.3
##	908	10	3	7	0.3
##	909	10	6	4	0.6
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##	911	10	5	5	0.5
##	912	10	8	2	0.8
##	913	10	7	3	0.7
##	914	10	5	5	0.5
##	915	10	3	7	0.3
##	916	10	6	4	0.6
##	917	10	3	7	0.3
##	918	10	6	4	0.6
##	919	10	4	6	0.4

##	920	10	8	2	0.8
##	921	10	5	5	0.5
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##	923	10	2	8	0.2
##	924	10	6	4	0.6
##	925	10	3	7	0.3
##	926	10	5	5	0.5
##	927	10	4	6	0.4
##	928	10	3	7	0.3
##	929	10	6	4	0.6
##	930	10	5	5	0.5
##	931	10	5	5	0.5
##	932	10	4	6	0.4
##	933	10	4	6	0.4
##	934	10	4	6	0.4
##	935	10	7	3	0.7
##	936	10	3	7	0.3
##	937	10	2	8	0.2
##	938	10	5	5	0.5
##	939	10	3	7	0.3
##	940	10	6	4	0.6
##	941	10	5	5	0.5
##	942	10	6	4	0.6
##	943	10	5	5	0.5
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##	945	10	4	6	0.4
##	946	10	3	7	0.3
##	947	10	3	7	0.3
##	948	10	4	6	0.4
##	949	10	4	6	0.4
##	950	10	5	5	0.5
##	951	10	9	1	0.9
##	952	10	3	7	0.3
##	953	10	7	3	0.7
##	954	10	8	2	0.8
##	955	10	7	3	0.7
##	956	10	6	4	0.6
##	957	10	5	5	0.5
##	958	10	5	5	0.5
##	959	10	7	3	0.7
##	960	10	5	5	0.5
##	961	10	4	6	0.4
##	962	10	5	5	0.5
##	963	10	7	3	0.7
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##	965	10	4	6	0.4

##	966	10	5	5	0.5
##	967	10	8	2	0.8
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##	969	10	4	6	0.4
##	970	10	6	4	0.6
##	971	10	6	4	0.6
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##	975	10	6	4	0.6
##	976	10	4	6	0.4
##	977	10	4	6	0.4
##	978	10	5	5	0.5
##	979	10	8	2	0.8
##	980	10	5	5	0.5
##	981	10	6	4	0.6
##	982	10	5	5	0.5
##	983	10	4	6	0.4
##	984	10	3	7	0.3
##	985	10	7	3	0.7
##	986	10	6	4	0.6
##	987	10	4	6	0.4
##	988	10	4	6	0.4
##	989	10	4	6	0.4
##	990	10	5	5	0.5
##	991	10	7	3	0.7
##	992	10	2	8	0.2
##	993	10	4	6	0.4
##	994	10	5	5	0.5
##	995	10	5	5	0.5
##	996	10	4	6	0.4
##	997	10	7	3	0.7
##	998	10	4	6	0.4
##	999	10	4	6	0.4
##	1000	10	2	8	0.2
##	1001	10	8	2	0.8
##	1002	10	5	5	0.5
##	1003	10	4	6	0.4
##	1004	10	6	4	0.6
##	1005	10	5	5	0.5
##	1006	10	3	7	0.3
##	1007	10	7	3	0.7
##	1008	10	5	5	0.5
##	1009	10	6	4	0.6
##	1010	10	5	5	0.5
##	1011	10	6	4	0.6

##	1012	10	7	3	0.7
##	1013	10	4	6	0.4
##	1014	10	3	7	0.3
##	1015	10	7	3	0.7
##	1016	10	5	5	0.5
##	1017	10	7	3	0.7
##	1018	10	8	2	0.8
##	1019	10	5	5	0.5
##	1020	10	6	4	0.6
##	1021	10	4	6	0.4
##	1022	10	6	4	0.6
##	1023	10	7	3	0.7
##	1024	10	5	5	0.5
##	1025	10	6	4	0.6
##	1026	10	5	5	0.5
##	1027	10	4	6	0.4
##	1028	10	5	5	0.5
##	1029	10	6	4	0.6
##	1030	10	3	7	0.3
##	1031	10	4	6	0.4
##	1032	10	5	5	0.5
##	1033	10	3	7	0.3
##	1034	10	6	4	0.6
##	1035	10	5	5	0.5
##	1036	10	5	5	0.5
##	1037	10	4	6	0.4
##	1038	10	5	5	0.5
##	1039	10	4	6	0.4
##	1040	10	7	3	0.7
##	1041	10	5	5	0.5
##	1042	10	6	4	0.6
##	1043	10	4	6	0.4
##	1044	10	9	1	0.9
##	1045	10	4	6	0.4
##	1046	10	6	4	0.6
##	1047	10	6	4	0.6
##	1048	10	5	5	0.5
##	1049	10	3	7	0.3
##	1050	10	8	2	0.8
##	1051	10	4	6	0.4
##	1052	10	6	4	0.6
##	1053	10	6	4	0.6
##	1054	10	7	3	0.7
##	1055	10	5	5	0.5
##	1056	10	5	5	0.5
##	1057	10	6	4	0.6

##	1058	10	5	5	0.5
##	1059	10	7	3	0.7
##	1060	10	7	3	0.7
##	1061	10	3	7	0.3
##	1062	10	4	6	0.4
##	1063	10	8	2	0.8
##	1064	10	5	5	0.5
##	1065	10	7	3	0.7
##	1066	10	6	4	0.6
##	1067	10	6	4	0.6
##	1068	10	4	6	0.4
##	1069	10	6	4	0.6
##	1070	10	5	5	0.5
##	1071	10	6	4	0.6
##	1072	10	6	4	0.6
##	1073	10	4	6	0.4
##	1074	10	5	5	0.5
##	1075	10	4	6	0.4
##	1076	10	4	6	0.4
##	1077	10	5	5	0.5
##	1078	10	6	4	0.6
##	1079	10	6	4	0.6
##	1080	10	4	6	0.4
##	1081	10	7	3	0.7
##	1082	10	3	7	0.3
##	1083	10	3	7	0.3
##	1084	10	3	7	0.3
##	1085	10	2	8	0.2
##	1086	10	4	6	0.4
##	1087	10	4	6	0.4
##	1088	10	4	6	0.4
##	1089	10	9	1	0.9
##	1090	10	7	3	0.7
##	1091	10	8	2	0.8
##	1092	10	6	4	0.6
##	1093	10	4	6	0.4
##	1094	10	4	6	0.4
##	1095	10	5	5	0.5
##	1096	10	4	6	0.4
##	1097	10	7	3	0.7
##	1098	10	5	5	0.5
##	1099	10	8	2	0.8
##	1100	10	3	7	0.3
##	1101	10	3	7	0.3
##	1102	10	6	4	0.6
##	1103	10	7	3	0.7

## 1104 10	6	4	0.6
## 1105 10	5	5	0.5
## 1106 10	5	5	0.5
## 1107 10	6	4	0.6
## 1108 10	8	2	0.8
## 1109 10	5	5	0.5
## 1110 10	7	3	0.7
## 1111 10	7	3	0.7
## 1112 10	5	5	0.5
## 1113 10	3	7	0.3
## 1114 10	5	5	0.5
## 1115 10	4	6	0.4
## 1116 10	3	7	0.3
## 1117 10	5	5	0.5
## 1118 10	4	6	0.4
## 1119 10	4	6	0.4
## 1120 10	2	8	0.2
## 1121 10	7	3	0.7
## 1122 10	5	5	0.5
## 1123 10	8	2	0.8
## 1124 10	6	4	0.6
## 1125 10	5	5	0.5
## 1126 10	6	4	0.6
## 1127 10	5	5	0.5
## 1128 10	4	6	0.4
## 1129 10	5	5	0.5
## 1130 10	7	3	0.7
## 1131 10	5	5	0.5
## 1132 10	4	6	0.4
## 1133 10	4	6	0.4
## 1134 10	6	4	0.6
## 1135 10	5	5	0.5
## 1136 10	6	4	0.6
## 1137 10	5	5	0.5
## 1138 10	4	6	0.4
## 1139 10	3	7	0.3
## 1140 10	6	4	0.6
## 1141 10	6	4	0.6
## 1142 10	4	6	0.4
## 1143 10	4	6	0.4
## 1144 10	2	8	0.2
## 1145 10	2	8	0.2
## 1146 10	8	2	0.8
## 1147 10	5	5	0.5
## 1148 10	4	6	0.4
## 1149 10	4	6	0.4

## 1150 10	5	5	0.5
## 1151 10	5	5	0.5
## 1152 10	5	5	0.5
## 1153 10	6	4	0.6
## 1154 10	6	4	0.6
## 1155 10	7	3	0.7
## 1156 10	4	6	0.4
## 1157 10	3	7	0.3
## 1158 10	7	3	0.7
## 1159 10	4	6	0.4
## 1160 10	5	5	0.5
## 1161 10	5	5	0.5
## 1162 10	5	5	0.5
## 1163 10	7	3	0.7
## 1164 10	6	4	0.6
## 1165 10	5	5	0.5
## 1166 10	4	6	0.4
## 1167 10	7	3	0.7
## 1168 10	6	4	0.6
## 1169 10	7	3	0.7
## 1170 10	5	5	0.5
## 1171 10	6	4	0.6
## 1172 10	6	4	0.6
## 1173 10	7	3	0.7
## 1174 10	4	6	0.4
## 1175 10	7	3	0.7
## 1176 10	7	3	0.7
## 1177 10	3	7	0.3
## 1178 10	6	4	0.6
## 1179 10	5	5	0.5
## 1180 10	5	5	0.5
## 1181 10	5	5	0.5
## 1182 10	6	4	0.6
## 1183 10	2	8	0.2
## 1184 10	5	5	0.5
## 1185 10	2	8	0.2
## 1186 10	6	4	0.6
## 1187 10	6	4	0.6
## 1188 10	3	7	0.3
## 1189 10	4	6	0.4
## 1190 10	4	6	0.4
## 1191 10	4	6	0.4
## 1192 10	6	4	0.6
## 1193 10	7	3	0.7
## 1194 10	3	7	0.3
## 1195 10	3	7	0.3

##	1196	10	3	7	0.3
##	1197	10	4	6	0.4
##	1198	10	3	7	0.3
##	1199	10	1	9	0.1
##	1200	10	6	4	0.6
##	1201	10	7	3	0.7
##	1202	10	2	8	0.2
##	1203	10	4	6	0.4
##	1204	10	5	5	0.5
##	1205	10	6	4	0.6
##	1206	10	4	6	0.4
##	1207	10	4	6	0.4
##	1208	10	5	5	0.5
##	1209	10	6	4	0.6
##	1210	10	3	7	0.3
##	1211	10	2	8	0.2
##	1212	10	3	7	0.3
##	1213	10	3	7	0.3
##	1214	10	4	6	0.4
##	1215	10	5	5	0.5
##	1216	10	5	5	0.5
##	1217	10	6	4	0.6
##	1218	10	6	4	0.6
##	1219	10	4	6	0.4
##	1220	10	3	7	0.3
##	1221	10	5	5	0.5
##	1222	10	5	5	0.5
##	1223	10	4	6	0.4
##	1224	10	7	3	0.7
##	1225	10	5	5	0.5
##	1226	10	4	6	0.4
##	1227	10	5	5	0.5
##	1228	10	5	5	0.5
##	1229	10	3	7	0.3
##	1230	10	6	4	0.6
##	1231	10	5	5	0.5
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##	1233	10	5	5	0.5
##	1234	10	6	4	0.6
##	1235	10	4	6	0.4
##	1236	10	5	5	0.5
##	1237	10	4	6	0.4
##	1238	10	6	4	0.6
##	1239	10	6	4	0.6
##	1240	10	7	3	0.7
##	1241	10	8	2	0.8



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##	1243	10	6	4	0.6
##	1244	10	5	5	0.5
##	1245	10	4	6	0.4
##	1246	10	6	4	0.6
##	1247	10	4	6	0.4
##	1248	10	8	2	0.8
##	1249	10	2	8	0.2
##	1250	10	5	5	0.5
##	1251	10	4	6	0.4
##	1252	10	6	4	0.6
##	1253	10	6	4	0.6
##	1254	10	4	6	0.4
##	1255	10	2	8	0.2
##	1256	10	7	3	0.7
##	1257	10	5	5	0.5
##	1258	10	7	3	0.7
##	1259	10	5	5	0.5
##	1260	10	6	4	0.6
##	1261	10	6	4	0.6
##	1262	10	5	5	0.5
##	1263	10	6	4	0.6
##	1264	10	4	6	0.4
##	1265	10	7	3	0.7
##	1266	10	4	6	0.4
##	1267	10	3	7	0.3
##	1268	10	4	6	0.4
##	1269	10	5	5	0.5
##	1270	10	3	7	0.3
##	1271	10	5	5	0.5
##	1272	10	4	6	0.4
##	1273	10	7	3	0.7
##	1274	10	5	5	0.5
##	1275	10	4	6	0.4
##	1276	10	8	2	0.8
##	1277	10	5	5	0.5
##	1278	10	4	6	0.4
##	1279	10	3	7	0.3
##	1280	10	4	6	0.4
##	1281	10	5	5	0.5
##	1282	10	5	5	0.5
##	1283	10	4	6	0.4
##	1284	10	7	3	0.7
##	1285	10	4	6	0.4
##	1286	10	3	7	0.3
##	1287	10	4	6	0.4

##	1288	10	4	6	0.4
##	1289	10	5	5	0.5
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##	1291	10	7	3	0.7
##	1292	10	6	4	0.6
##	1293	10	5	5	0.5
##	1294	10	5	5	0.5
##	1295	10	7	3	0.7
##	1296	10	2	8	0.2
##	1297	10	4	6	0.4
##	1298	10	2	8	0.2
##	1299	10	4	6	0.4
##	1300	10	6	4	0.6
##	1301	10	4	6	0.4
##	1302	10	6	4	0.6
##	1303	10	5	5	0.5
##	1304	10	9	1	0.9
##	1305	10	5	5	0.5
##	1306	10	5	5	0.5
##	1307	10	5	5	0.5
##	1308	10	5	5	0.5
##	1309	10	6	4	0.6
##	1310	10	1	9	0.1
##	1311	10	6	4	0.6
##	1312	10	2	8	0.2
##	1313	10	6	4	0.6
##	1314	10	6	4	0.6
##	1315	10	7	3	0.7
##	1316	10	9	1	0.9
##	1317	10	5	5	0.5
##	1318	10	4	6	0.4
##	1319	10	6	4	0.6
##	1320	10	3	7	0.3
##	1321	10	4	6	0.4
##	1322	10	3	7	0.3
##	1323	10	6	4	0.6
##	1324	10	6	4	0.6
##	1325	10	6	4	0.6
##	1326	10	4	6	0.4
##	1327	10	6	4	0.6
##	1328	10	6	4	0.6
##	1329	10	5	5	0.5
##	1330	10	5	5	0.5
##	1331	10	3	7	0.3
##	1332	10	6	4	0.6
##	1333	10	2	8	0.2

##	1334	10	4	6	0.4
##	1335	10	8	2	0.8
##	1336	10	3	7	0.3
##	1337	10	4	6	0.4
##	1338	10	5	5	0.5
##	1339	10	4	6	0.4
##	1340	10	7	3	0.7
##	1341	10	3	7	0.3
##	1342	10	3	7	0.3
##	1343	10	7	3	0.7
##	1344	10	7	3	0.7
##	1345	10	4	6	0.4
##	1346	10	3	7	0.3
##	1347	10	7	3	0.7
##	1348	10	3	7	0.3
##	1349	10	4	6	0.4
##	1350	10	4	6	0.4
##	1351	10	7	3	0.7
##	1352	10	5	5	0.5
##	1353	10	6	4	0.6
##	1354	10	8	2	0.8
##	1355	10	3	7	0.3
##	1356	10	7	3	0.7
##	1357	10	4	6	0.4
##	1358	10	4	6	0.4
##	1359	10	4	6	0.4
##	1360	10	3	7	0.3
##	1361	10	4	6	0.4
##	1362	10	7	3	0.7
##	1363	10	7	3	0.7
##	1364	10	9	1	0.9
##	1365	10	5	5	0.5
##	1366	10	8	2	0.8
##	1367	10	5	5	0.5
##	1368	10	7	3	0.7
##	1369	10	3	7	0.3
##	1370	10	8	2	0.8
##	1371	10	9	1	0.9
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##	1373	10	6	4	0.6
##	1374	10	6	4	0.6
##	1375	10	8	2	0.8
##	1376	10	6	4	0.6
##	1377	10	3	7	0.3
##	1378	10	3	7	0.3
##	1379	10	5	5	0.5

## 1380	10	6	4	0.6
## 1381	10	4	6	0.4
## 1382	10	7	3	0.7
## 1383	10	8	2	0.8
## 1384	10	7	3	0.7
## 1385	10	5	5	0.5
## 1386	10	5	5	0.5
## 1387	10	6	4	0.6
## 1388	10	4	6	0.4
## 1389	10	6	4	0.6
## 1390	10	6	4	0.6
## 1391	10	6	4	0.6
## 1392	10	3	7	0.3
## 1393	10	5	5	0.5
## 1394	10	4	6	0.4
## 1395	10	2	8	0.2
## 1396	10	5	5	0.5
## 1397	10	4	6	0.4
## 1398	10	6	4	0.6
## 1399	10	3	7	0.3
## 1400	10	6	4	0.6
## 1401	10	6	4	0.6
## 1402	10	3	7	0.3
## 1403	10	4	6	0.4
## 1404	10	6	4	0.6
## 1405	10	5	5	0.5
## 1406	10	6	4	0.6
## 1407	10	6	4	0.6
## 1408	10	4	6	0.4
## 1409	10	4	6	0.4
## 1410	10	6	4	0.6
## 1411	10	4	6	0.4
## 1412	10	7	3	0.7
## 1413	10	5	5	0.5
## 1414	10	6	4	0.6
## 1415	10	5	5	0.5
## 1416	10	4	6	0.4
## 1417	10	7	3	0.7
## 1418	10	7	3	0.7
## 1419	10	6	4	0.6
## 1420	10	3	7	0.3
## 1421	10	6	4	0.6
## 1422	10	3	7	0.3
## 1423	10	6	4	0.6
## 1424	10	8	2	0.8
## 1425	10	5	5	0.5

##	1426	10	6	4	0.6
##	1427	10	3	7	0.3
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##	1461	10	5	5	0.5
##	1462	10	7	3	0.7
##	1463	10	3	7	0.3
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##	1467	10	4	6	0.4
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##	1470	10	6	4	0.6
##	1471	10	6	4	0.6

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##	1475	10	3	7	0.3
##	1476	10	6	4	0.6
##	1477	10	7	3	0.7
##	1478	10	6	4	0.6
##	1479	10	5	5	0.5
##	1480	10	9	1	0.9
##	1481	10	7	3	0.7
##	1482	10	6	4	0.6
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##	1497	10	5	5	0.5
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##	1499	10	5	5	0.5
##	1500	10	0	10	0.0
##	1501	10	4	6	0.4
##	1502	10	3	7	0.3
##	1503	10	6	4	0.6
##	1504	10	4	6	0.4
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##	1509	10	4	6	0.4
##	1510	10	6	4	0.6
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##	1512	10	4	6	0.4
##	1513	10	4	6	0.4
##	1514	10	3	7	0.3
##	1515	10	2	8	0.2
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##	1517	10	3	7	0.3

##	1518	10	8	2	0.8
##	1519	10	4	6	0.4
##	1520	10	6	4	0.6
##	1521	10	7	3	0.7
##	1522	10	5	5	0.5
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##	1526	10	6	4	0.6
##	1527	10	5	5	0.5
##	1528	10	6	4	0.6
##	1529	10	6	4	0.6
##	1530	10	7	3	0.7
##	1531	10	7	3	0.7
##	1532	10	3	7	0.3
##	1533	10	7	3	0.7
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##	1535	10	3	7	0.3
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##	1552	10	6	4	0.6
##	1553	10	2	8	0.2
##	1554	10	5	5	0.5
##	1555	10	5	5	0.5
##	1556	10	2	8	0.2
##	1557	10	7	3	0.7
##	1558	10	6	4	0.6
##	1559	10	4	6	0.4
##	1560	10	7	3	0.7
##	1561	10	7	3	0.7
##	1562	10	4	6	0.4
##	1563	10	4	6	0.4

##	1564	10	6	4	0.6
##	1565	10	4	6	0.4
##	1566	10	6	4	0.6
##	1567	10	4	6	0.4
##	1568	10	6	4	0.6
##	1569	10	6	4	0.6
##	1570	10	5	5	0.5
##	1571	10	6	4	0.6
##	1572	10	6	4	0.6
##	1573	10	4	6	0.4
##	1574	10	4	6	0.4
##	1575	10	6	4	0.6
##	1576	10	9	1	0.9
##	1577	10	4	6	0.4
##	1578	10	6	4	0.6
##	1579	10	4	6	0.4
##	1580	10	4	6	0.4
##	1581	10	5	5	0.5
##	1582	10	2	8	0.2
##	1583	10	6	4	0.6
##	1584	10	4	6	0.4
##	1585	10	8	2	0.8
##	1586	10	8	2	0.8
##	1587	10	4	6	0.4
##	1588	10	3	7	0.3
##	1589	10	6	4	0.6
##	1590	10	4	6	0.4
##	1591	10	4	6	0.4
##	1592	10	6	4	0.6
##	1593	10	4	6	0.4
##	1594	10	3	7	0.3
##	1595	10	4	6	0.4
##	1596	10	7	3	0.7
##	1597	10	5	5	0.5
##	1598	10	4	6	0.4
##	1599	10	8	2	0.8
##	1600	10	6	4	0.6
##	1601	10	7	3	0.7
##	1602	10	5	5	0.5
##	1603	10	5	5	0.5
##	1604	10	3	7	0.3
##	1605	10	5	5	0.5
##	1606	10	5	5	0.5
##	1607	10	4	6	0.4
##	1608	10	7	3	0.7
##	1609	10	4	6	0.4



## 1610	10	5	5	0.5
## 1611	10	6	4	0.6
## 1612	10	4	6	0.4
## 1613	10	6	4	0.6
## 1614	10	3	7	0.3
## 1615	10	7	3	0.7
## 1616	10	6	4	0.6
## 1617	10	5	5	0.5
## 1618	10	3	7	0.3
## 1619	10	6	4	0.6
## 1620	10	9	1	0.9
## 1621	10	6	4	0.6
## 1622	10	7	3	0.7
## 1623	10	8	2	0.8
## 1624	10	5	5	0.5
## 1625	10	4	6	0.4
## 1626	10	3	7	0.3
## 1627	10	3	7	0.3
## 1628	10	4	6	0.4
## 1629	10	8	2	0.8
## 1630	10	6	4	0.6
## 1631	10	5	5	0.5
## 1632	10	5	5	0.5
## 1633	10	5	5	0.5
## 1634	10	5	5	0.5
## 1635	10	4	6	0.4
## 1636	10	8	2	0.8
## 1637	10	6	4	0.6
## 1638	10	4	6	0.4
## 1639	10	6	4	0.6
## 1640	10	7	3	0.7
## 1641	10	4	6	0.4
## 1642	10	7	3	0.7
## 1643	10	5	5	0.5
## 1644	10	6	4	0.6
## 1645	10	3	7	0.3
## 1646	10	6	4	0.6
## 1647	10	4	6	0.4
## 1648	10	3	7	0.3
## 1649	10	4	6	0.4
## 1650	10	4	6	0.4
## 1651	10	6	4	0.6
## 1652	10	3	7	0.3
## 1653	10	6	4	0.6
## 1654	10	8	2	0.8
## 1655	10	4	6	0.4

## 1656	10	4	6	0.4
## 1657	10	5	5	0.5
## 1658	10	6	4	0.6
## 1659	10	3	7	0.3
## 1660	10	5	5	0.5
## 1661	10	5	5	0.5
## 1662	10	5	5	0.5
## 1663	10	3	7	0.3
## 1664	10	8	2	0.8
## 1665	10	5	5	0.5
## 1666	10	6	4	0.6
## 1667	10	5	5	0.5
## 1668	10	4	6	0.4
## 1669	10	7	3	0.7
## 1670	10	4	6	0.4
## 1671	10	5	5	0.5
## 1672	10	3	7	0.3
## 1673	10	3	7	0.3
## 1674	10	3	7	0.3
## 1675	10	6	4	0.6
## 1676	10	3	7	0.3
## 1677	10	6	4	0.6
## 1678	10	4	6	0.4
## 1679	10	8	2	0.8
## 1680	10	4	6	0.4
## 1681	10	6	4	0.6
## 1682	10	4	6	0.4
## 1683	10	6	4	0.6
## 1684	10	6	4	0.6
## 1685	10	4	6	0.4
## 1686	10	6	4	0.6
## 1687	10	7	3	0.7
## 1688	10	6	4	0.6
## 1689	10	5	5	0.5
## 1690	10	5	5	0.5
## 1691	10	6	4	0.6
## 1692	10	6	4	0.6
## 1693	10	7	3	0.7
## 1694	10	5	5	0.5
## 1695	10	6	4	0.6
## 1696	10	5	5	0.5
## 1697	10	5	5	0.5
## 1698	10	5	5	0.5
## 1699	10	3	7	0.3
## 1700	10	7	3	0.7
## 1701	10	6	4	0.6

##	1702	10	5	5	0.5
##	1703	10	4	6	0.4
##	1704	10	5	5	0.5
##	1705	10	8	2	0.8
##	1706	10	3	7	0.3
##	1707	10	7	3	0.7
##	1708	10	5	5	0.5
##	1709	10	4	6	0.4
##	1710	10	4	6	0.4
##	1711	10	6	4	0.6
##	1712	10	6	4	0.6
##	1713	10	6	4	0.6
##	1714	10	6	4	0.6
##	1715	10	5	5	0.5
##	1716	10	7	3	0.7
##	1717	10	3	7	0.3
##	1718	10	7	3	0.7
##	1719	10	4	6	0.4
##	1720	10	6	4	0.6
##	1721	10	5	5	0.5
##	1722	10	1	9	0.1
##	1723	10	6	4	0.6
##	1724	10	1	9	0.1
##	1725	10	5	5	0.5
##	1726	10	4	6	0.4
##	1727	10	5	5	0.5
##	1728	10	4	6	0.4
##	1729	10	5	5	0.5
##	1730	10	6	4	0.6
##	1731	10	6	4	0.6
##	1732	10	5	5	0.5
##	1733	10	5	5	0.5
##	1734	10	4	6	0.4
##	1735	10	5	5	0.5
##	1736	10	5	5	0.5
##	1737	10	3	7	0.3
##	1738	10	5	5	0.5
##	1739	10	5	5	0.5
##	1740	10	7	3	0.7
##	1741	10	4	6	0.4
##	1742	10	4	6	0.4
##	1743	10	5	5	0.5
##	1744	10	4	6	0.4
##	1745	10	2	8	0.2
##	1746	10	8	2	0.8
##	1747	10	5	5	0.5

## 1748	10	4	6	0.4
## 1749	10	6	4	0.6
## 1750	10	6	4	0.6
## 1751	10	7	3	0.7
## 1752	10	5	5	0.5
## 1753	10	4	6	0.4
## 1754	10	4	6	0.4
## 1755	10	5	5	0.5
## 1756	10	2	8	0.2
## 1757	10	7	3	0.7
## 1758	10	2	8	0.2
## 1759	10	4	6	0.4
## 1760	10	5	5	0.5
## 1761	10	6	4	0.6
## 1762	10	5	5	0.5
## 1763	10	3	7	0.3
## 1764	10	5	5	0.5
## 1765	10	8	2	0.8
## 1766	10	5	5	0.5
## 1767	10	6	4	0.6
## 1768	10	4	6	0.4
## 1769	10	7	3	0.7
## 1770	10	6	4	0.6
## 1771	10	5	5	0.5
## 1772	10	4	6	0.4
## 1773	10	5	5	0.5
## 1774	10	6	4	0.6
## 1775	10	6	4	0.6
## 1776	10	3	7	0.3
## 1777	10	3	7	0.3
## 1778	10	4	6	0.4
## 1779	10	3	7	0.3
## 1780	10	5	5	0.5
## 1781	10	6	4	0.6
## 1782	10	5	5	0.5
## 1783	10	5	5	0.5
## 1784	10	4	6	0.4
## 1785	10	3	7	0.3
## 1786	10	6	4	0.6
## 1787	10	5	5	0.5
## 1788	10	7	3	0.7
## 1789	10	2	8	0.2
## 1790	10	4	6	0.4
## 1791	10	5	5	0.5
## 1792	10	5	5	0.5
## 1793	10	5	5	0.5

##	1794	10	6	4	0.6
##	1795	10	7	3	0.7
##	1796	10	5	5	0.5
##	1797	10	6	4	0.6
##	1798	10	4	6	0.4
##	1799	10	5	5	0.5
##	1800	10	6	4	0.6
##	1801	10	6	4	0.6
##	1802	10	6	4	0.6
##	1803	10	2	8	0.2
##	1804	10	4	6	0.4
##	1805	10	5	5	0.5
##	1806	10	5	5	0.5
##	1807	10	7	3	0.7
##	1808	10	2	8	0.2
##	1809	10	5	5	0.5
##	1810	10	6	4	0.6
##	1811	10	5	5	0.5
##	1812	10	4	6	0.4
##	1813	10	5	5	0.5
##	1814	10	4	6	0.4
##	1815	10	4	6	0.4
##	1816	10	7	3	0.7
##	1817	10	7	3	0.7
##	1818	10	8	2	0.8
##	1819	10	3	7	0.3
##	1820	10	5	5	0.5
##	1821	10	4	6	0.4
##	1822	10	6	4	0.6
##	1823	10	6	4	0.6
##	1824	10	6	4	0.6
##	1825	10	5	5	0.5
##	1826	10	5	5	0.5
##	1827	10	5	5	0.5
##	1828	10	5	5	0.5
##	1829	10	7	3	0.7
##	1830	10	4	6	0.4
##	1831	10	4	6	0.4
##	1832	10	6	4	0.6
##	1833	10	4	6	0.4
##	1834	10	3	7	0.3
##	1835	10	5	5	0.5
##	1836	10	7	3	0.7
##	1837	10	6	4	0.6
##	1838	10	7	3	0.7
##	1839	10	4	6	0.4

##	1840	10	6	4	0.6
##	1841	10	6	4	0.6
##	1842	10	8	2	0.8
##	1843	10	4	6	0.4
##	1844	10	6	4	0.6
##	1845	10	3	7	0.3
##	1846	10	2	8	0.2
##	1847	10	4	6	0.4
##	1848	10	5	5	0.5
##	1849	10	3	7	0.3
##	1850	10	6	4	0.6
##	1851	10	5	5	0.5
##	1852	10	9	1	0.9
##	1853	10	1	9	0.1
##	1854	10	6	4	0.6
##	1855	10	7	3	0.7
##	1856	10	5	5	0.5
##	1857	10	9	1	0.9
##	1858	10	8	2	0.8
##	1859	10	6	4	0.6
##	1860	10	5	5	0.5
##	1861	10	4	6	0.4
##	1862	10	5	5	0.5
##	1863	10	4	6	0.4
##	1864	10	8	2	0.8
##	1865	10	4	6	0.4
##	1866	10	6	4	0.6
##	1867	10	3	7	0.3
##	1868	10	7	3	0.7
##	1869	10	5	5	0.5
##	1870	10	7	3	0.7
##	1871	10	7	3	0.7
##	1872	10	9	1	0.9
##	1873	10	4	6	0.4
##	1874	10	7	3	0.7
##	1875	10	6	4	0.6
##	1876	10	7	3	0.7
##	1877	10	7	3	0.7
##	1878	10	5	5	0.5
##	1879	10	6	4	0.6
##	1880	10	6	4	0.6
##	1881	10	4	6	0.4
##	1882	10	5	5	0.5
##	1883	10	5	5	0.5
##	1884	10	4	6	0.4
##	1885	10	5	5	0.5

##	1886	10	6	4	0.6
##	1887	10	5	5	0.5
##	1888	10	3	7	0.3
##	1889	10	6	4	0.6
##	1890	10	2	8	0.2
##	1891	10	4	6	0.4
##	1892	10	6	4	0.6
##	1893	10	4	6	0.4
##	1894	10	6	4	0.6
##	1895	10	4	6	0.4
##	1896	10	4	6	0.4
##	1897	10	4	6	0.4
##	1898	10	6	4	0.6
##	1899	10	5	5	0.5
##	1900	10	7	3	0.7
##	1901	10	4	6	0.4
##	1902	10	3	7	0.3
##	1903	10	6	4	0.6
##	1904	10	6	4	0.6
##	1905	10	2	8	0.2
##	1906	10	5	5	0.5
##	1907	10	3	7	0.3
##	1908	10	4	6	0.4
##	1909	10	5	5	0.5
##	1910	10	4	6	0.4
##	1911	10	5	5	0.5
##	1912	10	6	4	0.6
##	1913	10	8	2	0.8
##	1914	10	7	3	0.7
##	1915	10	3	7	0.3
##	1916	10	4	6	0.4
##	1917	10	4	6	0.4
##	1918	10	4	6	0.4
##	1919	10	4	6	0.4
##	1920	10	4	6	0.4
##	1921	10	4	6	0.4
##	1922	10	3	7	0.3
##	1923	10	5	5	0.5
##	1924	10	4	6	0.4
##	1925	10	8	2	0.8
##	1926	10	5	5	0.5
##	1927	10	5	5	0.5
##	1928	10	3	7	0.3
##	1929	10	6	4	0.6
##	1930	10	7	3	0.7
##	1931	10	4	6	0.4

## 1932	10	5	5	0.5
## 1933	10	4	6	0.4
## 1934	10	3	7	0.3
## 1935	10	6	4	0.6
## 1936	10	7	3	0.7
## 1937	10	5	5	0.5
## 1938	10	5	5	0.5
## 1939	10	5	5	0.5
## 1940	10	5	5	0.5
## 1941	10	3	7	0.3
## 1942	10	4	6	0.4
## 1943	10	3	7	0.3
## 1944	10	7	3	0.7
## 1945	10	4	6	0.4
## 1946	10	3	7	0.3
## 1947	10	4	6	0.4
## 1948	10	5	5	0.5
## 1949	10	6	4	0.6
## 1950	10	6	4	0.6
## 1951	10	4	6	0.4
## 1952	10	9	1	0.9
## 1953	10	5	5	0.5
## 1954	10	5	5	0.5
## 1955	10	5	5	0.5
## 1956	10	4	6	0.4
## 1957	10	3	7	0.3
## 1958	10	7	3	0.7
## 1959	10	6	4	0.6
## 1960	10	3	7	0.3
## 1961	10	4	6	0.4
## 1962	10	7	3	0.7
## 1963	10	7	3	0.7
## 1964	10	6	4	0.6
## 1965	10	6	4	0.6
## 1966	10	4	6	0.4
## 1967	10	7	3	0.7
## 1968	10	6	4	0.6
## 1969	10	5	5	0.5
## 1970	10	4	6	0.4
## 1971	10	4	6	0.4
## 1972	10	1	9	0.1
## 1973	10	7	3	0.7
## 1974	10	3	7	0.3
## 1975	10	4	6	0.4
## 1976	10	5	5	0.5
## 1977	10	4	6	0.4



```
## 1978 10      4      6 0.4
## 1979 10      3      7 0.3
## 1980 10      3      7 0.3
## 1981 10      4      6 0.4
## 1982 10      4      6 0.4
## 1983 10      5      5 0.5
## 1984 10      4      6 0.4
## 1985 10      2      8 0.2
## 1986 10      4      6 0.4
## 1987 10      4      6 0.4
## 1988 10      4      6 0.4
## 1989 10      5      5 0.5
## 1990 10      7      3 0.7
## 1991 10      3      7 0.3
## 1992 10      4      6 0.4
## 1993 10      6      4 0.6
## 1994 10      4      6 0.4
## 1995 10      7      3 0.7
## 1996 10      4      6 0.4
## 1997 10      6      4 0.6
## 1998 10      6      4 0.6
## 1999 10      3      7 0.3
## 2000 10      8      2 0.8
```

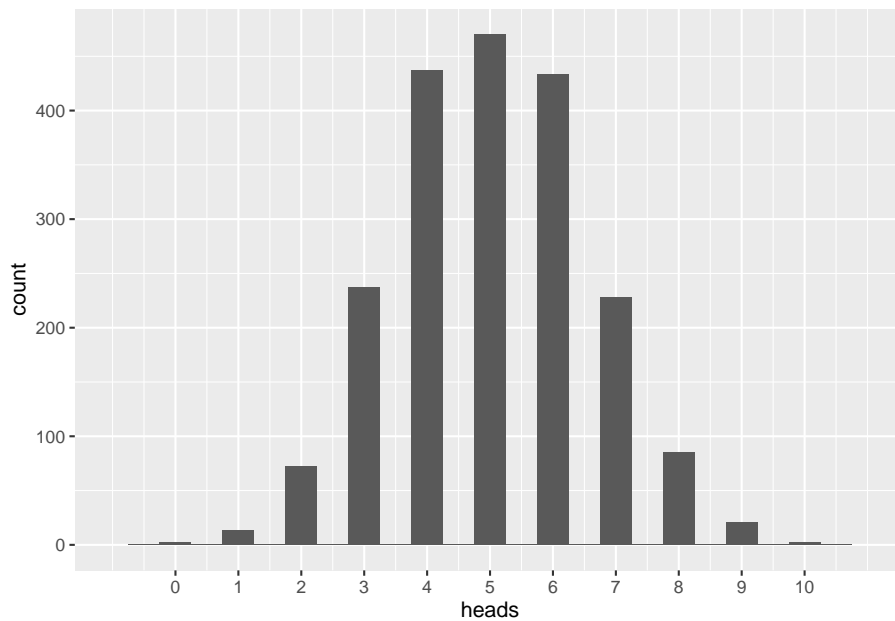
This is the same idea as before, but now there are 2000 rows in the data frame instead of 20.

```
mean(coin_flips_2000_10$heads)
```

```
## [1] 5.0245
```

```
ggplot(coin_flips_2000_10, aes(x = heads)) +
  geom_histogram(binwidth = 0.5) +
  scale_x_continuous(limits = c(-1, 11), breaks = seq(0, 10, 1))
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```

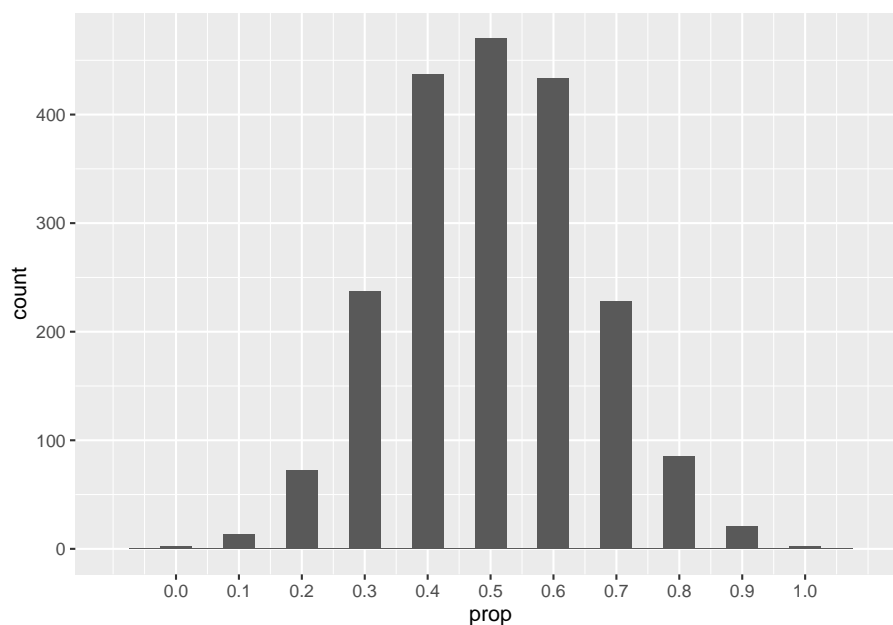


This is helpful. In contrast with the set of simulations with twenty people, the last histogram gives us something closer to what we expect. The mode is at five heads, and every possible number of heads is represented, with decreasing counts as one moves away from five. With 2000 people flipping coins, all possible outcomes—including rare ones—are better represented.

Here is the the same histogram, but this time with the proportion of heads instead of the count of heads:

```
ggplot(coin_flips_2000_10, aes(x = prop)) +
  geom_histogram(binwidth = 0.05) +
  scale_x_continuous(limits = c(-0.1, 1.1), breaks = seq(0, 1, 0.1))
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



**Exercise 3** Do you think the shape of the distribution would be appreciably different if we used 20,000 or even 200,000 people? Why or why not? (Normally, I would encourage you to test your theory by trying it in R. However, it takes a *long* time to simulate that many flips and I don't want you to tie up resources and memory. Think through this in your head.)

Please write up your answer here.

---

From now on, we will insist on using at least a thousand simulations—if not more—to make sure that we represent the full range of possible outcomes.<sup>1</sup>

## 8.6 More flips

Now let's increase the number of coin flips each person performs. We'll still use 2000 simulations (imagine 2000 people all flipping coins), but this time, each person will flip the coin 1000 times instead of only 10 times. The first code chunk below accounts for a substantial amount of the time it takes to run the code in this document.

---

<sup>1</sup>There is some theory behind choosing the number of times we need to simulate, but we're not going to get into all that.

```
set.seed(1234)
coin_flips_2000_1000 <- do(2000) * rflip(1000, prob = 0.5)
coin_flips_2000_1000
```

##		n	heads	tails	prop
## 1		1000	485	515	0.485
## 2		1000	515	485	0.515
## 3		1000	481	519	0.481
## 4		1000	508	492	0.508
## 5		1000	499	501	0.499
## 6		1000	516	484	0.516
## 7		1000	497	503	0.497
## 8		1000	497	503	0.497
## 9		1000	494	506	0.494
## 10		1000	528	472	0.528
## 11		1000	495	505	0.495
## 12		1000	483	517	0.483
## 13		1000	520	480	0.520
## 14		1000	528	472	0.528
## 15		1000	478	522	0.478
## 16		1000	516	484	0.516
## 17		1000	493	507	0.493
## 18		1000	524	476	0.524
## 19		1000	473	527	0.473
## 20		1000	516	484	0.516
## 21		1000	529	471	0.529
## 22		1000	516	484	0.516
## 23		1000	535	465	0.535
## 24		1000	491	509	0.491
## 25		1000	500	500	0.500
## 26		1000	497	503	0.497
## 27		1000	507	493	0.507
## 28		1000	515	485	0.515
## 29		1000	493	507	0.493
## 30		1000	482	518	0.482
## 31		1000	485	515	0.485
## 32		1000	493	507	0.493
## 33		1000	498	502	0.498
## 34		1000	490	510	0.490
## 35		1000	485	515	0.485
## 36		1000	495	505	0.495
## 37		1000	488	512	0.488
## 38		1000	496	504	0.496
## 39		1000	491	509	0.491
## 40		1000	488	512	0.488

## 41	1000	488	512	0.488
## 42	1000	524	476	0.524
## 43	1000	500	500	0.500
## 44	1000	516	484	0.516
## 45	1000	514	486	0.514
## 46	1000	479	521	0.479
## 47	1000	488	512	0.488
## 48	1000	469	531	0.469
## 49	1000	515	485	0.515
## 50	1000	520	480	0.520
## 51	1000	486	514	0.486
## 52	1000	507	493	0.507
## 53	1000	509	491	0.509
## 54	1000	467	533	0.467
## 55	1000	467	533	0.467
## 56	1000	504	496	0.504
## 57	1000	483	517	0.483
## 58	1000	513	487	0.513
## 59	1000	518	482	0.518
## 60	1000	493	507	0.493
## 61	1000	516	484	0.516
## 62	1000	507	493	0.507
## 63	1000	509	491	0.509
## 64	1000	508	492	0.508
## 65	1000	511	489	0.511
## 66	1000	491	509	0.491
## 67	1000	524	476	0.524
## 68	1000	515	485	0.515
## 69	1000	524	476	0.524
## 70	1000	510	490	0.510
## 71	1000	482	518	0.482
## 72	1000	498	502	0.498
## 73	1000	507	493	0.507
## 74	1000	490	510	0.490
## 75	1000	501	499	0.501
## 76	1000	502	498	0.502
## 77	1000	520	480	0.520
## 78	1000	528	472	0.528
## 79	1000	504	496	0.504
## 80	1000	501	499	0.501
## 81	1000	507	493	0.507
## 82	1000	486	514	0.486
## 83	1000	500	500	0.500
## 84	1000	505	495	0.505
## 85	1000	494	506	0.494
## 86	1000	505	495	0.505

##	87	1000	512	488	0.512
##	88	1000	521	479	0.521
##	89	1000	497	503	0.497
##	90	1000	501	499	0.501
##	91	1000	489	511	0.489
##	92	1000	497	503	0.497
##	93	1000	500	500	0.500
##	94	1000	470	530	0.470
##	95	1000	511	489	0.511
##	96	1000	504	496	0.504
##	97	1000	460	540	0.460
##	98	1000	493	507	0.493
##	99	1000	477	523	0.477
##	100	1000	489	511	0.489
##	101	1000	511	489	0.511
##	102	1000	519	481	0.519
##	103	1000	491	509	0.491
##	104	1000	464	536	0.464
##	105	1000	493	507	0.493
##	106	1000	497	503	0.497
##	107	1000	515	485	0.515
##	108	1000	491	509	0.491
##	109	1000	472	528	0.472
##	110	1000	505	495	0.505
##	111	1000	503	497	0.503
##	112	1000	489	511	0.489
##	113	1000	530	470	0.530
##	114	1000	510	490	0.510
##	115	1000	521	479	0.521
##	116	1000	488	512	0.488
##	117	1000	453	547	0.453
##	118	1000	489	511	0.489
##	119	1000	486	514	0.486
##	120	1000	481	519	0.481
##	121	1000	495	505	0.495
##	122	1000	484	516	0.484
##	123	1000	534	466	0.534
##	124	1000	500	500	0.500
##	125	1000	497	503	0.497
##	126	1000	524	476	0.524
##	127	1000	494	506	0.494
##	128	1000	505	495	0.505
##	129	1000	479	521	0.479
##	130	1000	493	507	0.493
##	131	1000	488	512	0.488
##	132	1000	482	518	0.482

##	133	1000	519	481	0.519
##	134	1000	497	503	0.497
##	135	1000	531	469	0.531
##	136	1000	481	519	0.481
##	137	1000	510	490	0.510
##	138	1000	500	500	0.500
##	139	1000	476	524	0.476
##	140	1000	493	507	0.493
##	141	1000	490	510	0.490
##	142	1000	469	531	0.469
##	143	1000	484	516	0.484
##	144	1000	534	466	0.534
##	145	1000	491	509	0.491
##	146	1000	510	490	0.510
##	147	1000	507	493	0.507
##	148	1000	495	505	0.495
##	149	1000	526	474	0.526
##	150	1000	497	503	0.497
##	151	1000	510	490	0.510
##	152	1000	496	504	0.496
##	153	1000	470	530	0.470
##	154	1000	502	498	0.502
##	155	1000	485	515	0.485
##	156	1000	516	484	0.516
##	157	1000	513	487	0.513
##	158	1000	510	490	0.510
##	159	1000	484	516	0.484
##	160	1000	517	483	0.517
##	161	1000	512	488	0.512
##	162	1000	492	508	0.492
##	163	1000	513	487	0.513
##	164	1000	478	522	0.478
##	165	1000	503	497	0.503
##	166	1000	485	515	0.485
##	167	1000	489	511	0.489
##	168	1000	477	523	0.477
##	169	1000	508	492	0.508
##	170	1000	530	470	0.530
##	171	1000	476	524	0.476
##	172	1000	510	490	0.510
##	173	1000	475	525	0.475
##	174	1000	479	521	0.479
##	175	1000	497	503	0.497
##	176	1000	505	495	0.505
##	177	1000	506	494	0.506
##	178	1000	514	486	0.514

##	179	1000	511	489	0.511
##	180	1000	536	464	0.536
##	181	1000	487	513	0.487
##	182	1000	489	511	0.489
##	183	1000	487	513	0.487
##	184	1000	503	497	0.503
##	185	1000	493	507	0.493
##	186	1000	530	470	0.530
##	187	1000	496	504	0.496
##	188	1000	495	505	0.495
##	189	1000	481	519	0.481
##	190	1000	503	497	0.503
##	191	1000	482	518	0.482
##	192	1000	504	496	0.504
##	193	1000	513	487	0.513
##	194	1000	523	477	0.523
##	195	1000	512	488	0.512
##	196	1000	512	488	0.512
##	197	1000	508	492	0.508
##	198	1000	528	472	0.528
##	199	1000	498	502	0.498
##	200	1000	529	471	0.529
##	201	1000	516	484	0.516
##	202	1000	490	510	0.490
##	203	1000	498	502	0.498
##	204	1000	499	501	0.499
##	205	1000	502	498	0.502
##	206	1000	498	502	0.498
##	207	1000	503	497	0.503
##	208	1000	521	479	0.521
##	209	1000	509	491	0.509
##	210	1000	509	491	0.509
##	211	1000	492	508	0.492
##	212	1000	496	504	0.496
##	213	1000	516	484	0.516
##	214	1000	494	506	0.494
##	215	1000	487	513	0.487
##	216	1000	509	491	0.509
##	217	1000	487	513	0.487
##	218	1000	490	510	0.490
##	219	1000	520	480	0.520
##	220	1000	495	505	0.495
##	221	1000	500	500	0.500
##	222	1000	491	509	0.491
##	223	1000	511	489	0.511
##	224	1000	475	525	0.475



##	225	1000	515	485	0.515
##	226	1000	477	523	0.477
##	227	1000	501	499	0.501
##	228	1000	509	491	0.509
##	229	1000	490	510	0.490
##	230	1000	498	502	0.498
##	231	1000	494	506	0.494
##	232	1000	521	479	0.521
##	233	1000	477	523	0.477
##	234	1000	510	490	0.510
##	235	1000	517	483	0.517
##	236	1000	506	494	0.506
##	237	1000	477	523	0.477
##	238	1000	490	510	0.490
##	239	1000	524	476	0.524
##	240	1000	503	497	0.503
##	241	1000	514	486	0.514
##	242	1000	506	494	0.506
##	243	1000	482	518	0.482
##	244	1000	507	493	0.507
##	245	1000	504	496	0.504
##	246	1000	501	499	0.501
##	247	1000	482	518	0.482
##	248	1000	480	520	0.480
##	249	1000	511	489	0.511
##	250	1000	497	503	0.497
##	251	1000	471	529	0.471
##	252	1000	510	490	0.510
##	253	1000	523	477	0.523
##	254	1000	485	515	0.485
##	255	1000	505	495	0.505
##	256	1000	507	493	0.507
##	257	1000	473	527	0.473
##	258	1000	495	505	0.495
##	259	1000	465	535	0.465
##	260	1000	501	499	0.501
##	261	1000	460	540	0.460
##	262	1000	499	501	0.499
##	263	1000	524	476	0.524
##	264	1000	514	486	0.514
##	265	1000	503	497	0.503
##	266	1000	469	531	0.469
##	267	1000	496	504	0.496
##	268	1000	489	511	0.489
##	269	1000	507	493	0.507
##	270	1000	466	534	0.466

##	271	1000	482	518	0.482
##	272	1000	520	480	0.520
##	273	1000	513	487	0.513
##	274	1000	492	508	0.492
##	275	1000	486	514	0.486
##	276	1000	498	502	0.498
##	277	1000	507	493	0.507
##	278	1000	494	506	0.494
##	279	1000	499	501	0.499
##	280	1000	498	502	0.498
##	281	1000	459	541	0.459
##	282	1000	495	505	0.495
##	283	1000	498	502	0.498
##	284	1000	495	505	0.495
##	285	1000	488	512	0.488
##	286	1000	518	482	0.518
##	287	1000	502	498	0.502
##	288	1000	503	497	0.503
##	289	1000	476	524	0.476
##	290	1000	495	505	0.495
##	291	1000	495	505	0.495
##	292	1000	503	497	0.503
##	293	1000	482	518	0.482
##	294	1000	518	482	0.518
##	295	1000	514	486	0.514
##	296	1000	520	480	0.520
##	297	1000	498	502	0.498
##	298	1000	523	477	0.523
##	299	1000	516	484	0.516
##	300	1000	483	517	0.483
##	301	1000	504	496	0.504
##	302	1000	505	495	0.505
##	303	1000	502	498	0.502
##	304	1000	486	514	0.486
##	305	1000	540	460	0.540
##	306	1000	510	490	0.510
##	307	1000	507	493	0.507
##	308	1000	482	518	0.482
##	309	1000	509	491	0.509
##	310	1000	486	514	0.486
##	311	1000	474	526	0.474
##	312	1000	511	489	0.511
##	313	1000	484	516	0.484
##	314	1000	499	501	0.499
##	315	1000	496	504	0.496
##	316	1000	505	495	0.505

##	317	1000	487	513	0.487
##	318	1000	520	480	0.520
##	319	1000	483	517	0.483
##	320	1000	515	485	0.515
##	321	1000	513	487	0.513
##	322	1000	509	491	0.509
##	323	1000	520	480	0.520
##	324	1000	509	491	0.509
##	325	1000	480	520	0.480
##	326	1000	524	476	0.524
##	327	1000	507	493	0.507
##	328	1000	509	491	0.509
##	329	1000	493	507	0.493
##	330	1000	464	536	0.464
##	331	1000	526	474	0.526
##	332	1000	513	487	0.513
##	333	1000	505	495	0.505
##	334	1000	509	491	0.509
##	335	1000	500	500	0.500
##	336	1000	499	501	0.499
##	337	1000	520	480	0.520
##	338	1000	491	509	0.491
##	339	1000	488	512	0.488
##	340	1000	483	517	0.483
##	341	1000	508	492	0.508
##	342	1000	474	526	0.474
##	343	1000	482	518	0.482
##	344	1000	485	515	0.485
##	345	1000	516	484	0.516
##	346	1000	511	489	0.511
##	347	1000	490	510	0.490
##	348	1000	519	481	0.519
##	349	1000	493	507	0.493
##	350	1000	508	492	0.508
##	351	1000	492	508	0.492
##	352	1000	500	500	0.500
##	353	1000	503	497	0.503
##	354	1000	478	522	0.478
##	355	1000	511	489	0.511
##	356	1000	495	505	0.495
##	357	1000	472	528	0.472
##	358	1000	468	532	0.468
##	359	1000	504	496	0.504
##	360	1000	478	522	0.478
##	361	1000	485	515	0.485
##	362	1000	503	497	0.503

##	363	1000	487	513	0.487
##	364	1000	482	518	0.482
##	365	1000	485	515	0.485
##	366	1000	507	493	0.507
##	367	1000	477	523	0.477
##	368	1000	504	496	0.504
##	369	1000	502	498	0.502
##	370	1000	492	508	0.492
##	371	1000	485	515	0.485
##	372	1000	491	509	0.491
##	373	1000	502	498	0.502
##	374	1000	483	517	0.483
##	375	1000	510	490	0.510
##	376	1000	508	492	0.508
##	377	1000	500	500	0.500
##	378	1000	501	499	0.501
##	379	1000	518	482	0.518
##	380	1000	528	472	0.528
##	381	1000	500	500	0.500
##	382	1000	486	514	0.486
##	383	1000	487	513	0.487
##	384	1000	511	489	0.511
##	385	1000	483	517	0.483
##	386	1000	485	515	0.485
##	387	1000	485	515	0.485
##	388	1000	520	480	0.520
##	389	1000	486	514	0.486
##	390	1000	492	508	0.492
##	391	1000	519	481	0.519
##	392	1000	478	522	0.478
##	393	1000	509	491	0.509
##	394	1000	494	506	0.494
##	395	1000	482	518	0.482
##	396	1000	490	510	0.490
##	397	1000	488	512	0.488
##	398	1000	538	462	0.538
##	399	1000	483	517	0.483
##	400	1000	515	485	0.515
##	401	1000	489	511	0.489
##	402	1000	511	489	0.511
##	403	1000	486	514	0.486
##	404	1000	501	499	0.501
##	405	1000	497	503	0.497
##	406	1000	515	485	0.515
##	407	1000	514	486	0.514
##	408	1000	504	496	0.504

##	409	1000	526	474	0.526
##	410	1000	481	519	0.481
##	411	1000	505	495	0.505
##	412	1000	504	496	0.504
##	413	1000	511	489	0.511
##	414	1000	510	490	0.510
##	415	1000	494	506	0.494
##	416	1000	515	485	0.515
##	417	1000	510	490	0.510
##	418	1000	488	512	0.488
##	419	1000	490	510	0.490
##	420	1000	506	494	0.506
##	421	1000	489	511	0.489
##	422	1000	514	486	0.514
##	423	1000	524	476	0.524
##	424	1000	492	508	0.492
##	425	1000	502	498	0.502
##	426	1000	519	481	0.519
##	427	1000	500	500	0.500
##	428	1000	516	484	0.516
##	429	1000	515	485	0.515
##	430	1000	496	504	0.496
##	431	1000	479	521	0.479
##	432	1000	481	519	0.481
##	433	1000	521	479	0.521
##	434	1000	485	515	0.485
##	435	1000	492	508	0.492
##	436	1000	507	493	0.507
##	437	1000	507	493	0.507
##	438	1000	497	503	0.497
##	439	1000	516	484	0.516
##	440	1000	491	509	0.491
##	441	1000	518	482	0.518
##	442	1000	490	510	0.490
##	443	1000	502	498	0.502
##	444	1000	521	479	0.521
##	445	1000	504	496	0.504
##	446	1000	495	505	0.495
##	447	1000	500	500	0.500
##	448	1000	513	487	0.513
##	449	1000	497	503	0.497
##	450	1000	488	512	0.488
##	451	1000	497	503	0.497
##	452	1000	532	468	0.532
##	453	1000	519	481	0.519
##	454	1000	487	513	0.487

##	455	1000	500	500	0.500
##	456	1000	509	491	0.509
##	457	1000	506	494	0.506
##	458	1000	508	492	0.508
##	459	1000	524	476	0.524
##	460	1000	520	480	0.520
##	461	1000	509	491	0.509
##	462	1000	551	449	0.551
##	463	1000	512	488	0.512
##	464	1000	497	503	0.497
##	465	1000	500	500	0.500
##	466	1000	493	507	0.493
##	467	1000	508	492	0.508
##	468	1000	514	486	0.514
##	469	1000	524	476	0.524
##	470	1000	508	492	0.508
##	471	1000	493	507	0.493
##	472	1000	513	487	0.513
##	473	1000	515	485	0.515
##	474	1000	494	506	0.494
##	475	1000	487	513	0.487
##	476	1000	464	536	0.464
##	477	1000	511	489	0.511
##	478	1000	484	516	0.484
##	479	1000	527	473	0.527
##	480	1000	485	515	0.485
##	481	1000	495	505	0.495
##	482	1000	515	485	0.515
##	483	1000	484	516	0.484
##	484	1000	464	536	0.464
##	485	1000	541	459	0.541
##	486	1000	512	488	0.512
##	487	1000	506	494	0.506
##	488	1000	500	500	0.500
##	489	1000	522	478	0.522
##	490	1000	507	493	0.507
##	491	1000	521	479	0.521
##	492	1000	511	489	0.511
##	493	1000	486	514	0.486
##	494	1000	501	499	0.501
##	495	1000	515	485	0.515
##	496	1000	473	527	0.473
##	497	1000	499	501	0.499
##	498	1000	515	485	0.515
##	499	1000	519	481	0.519
##	500	1000	488	512	0.488

##	501	1000	508	492	0.508
##	502	1000	484	516	0.484
##	503	1000	484	516	0.484
##	504	1000	502	498	0.502
##	505	1000	489	511	0.489
##	506	1000	495	505	0.495
##	507	1000	519	481	0.519
##	508	1000	521	479	0.521
##	509	1000	506	494	0.506
##	510	1000	515	485	0.515
##	511	1000	499	501	0.499
##	512	1000	514	486	0.514
##	513	1000	527	473	0.527
##	514	1000	504	496	0.504
##	515	1000	469	531	0.469
##	516	1000	489	511	0.489
##	517	1000	503	497	0.503
##	518	1000	531	469	0.531
##	519	1000	497	503	0.497
##	520	1000	499	501	0.499
##	521	1000	483	517	0.483
##	522	1000	501	499	0.501
##	523	1000	481	519	0.481
##	524	1000	516	484	0.516
##	525	1000	491	509	0.491
##	526	1000	486	514	0.486
##	527	1000	492	508	0.492
##	528	1000	498	502	0.498
##	529	1000	522	478	0.522
##	530	1000	487	513	0.487
##	531	1000	477	523	0.477
##	532	1000	501	499	0.501
##	533	1000	490	510	0.490
##	534	1000	487	513	0.487
##	535	1000	490	510	0.490
##	536	1000	484	516	0.484
##	537	1000	489	511	0.489
##	538	1000	502	498	0.502
##	539	1000	490	510	0.490
##	540	1000	493	507	0.493
##	541	1000	509	491	0.509
##	542	1000	523	477	0.523
##	543	1000	501	499	0.501
##	544	1000	482	518	0.482
##	545	1000	498	502	0.498
##	546	1000	481	519	0.481

##	547	1000	502	498	0.502
##	548	1000	499	501	0.499
##	549	1000	504	496	0.504
##	550	1000	487	513	0.487
##	551	1000	481	519	0.481
##	552	1000	483	517	0.483
##	553	1000	488	512	0.488
##	554	1000	491	509	0.491
##	555	1000	532	468	0.532
##	556	1000	509	491	0.509
##	557	1000	495	505	0.495
##	558	1000	493	507	0.493
##	559	1000	519	481	0.519
##	560	1000	475	525	0.475
##	561	1000	523	477	0.523
##	562	1000	474	526	0.474
##	563	1000	461	539	0.461
##	564	1000	479	521	0.479
##	565	1000	528	472	0.528
##	566	1000	502	498	0.502
##	567	1000	503	497	0.503
##	568	1000	501	499	0.501
##	569	1000	487	513	0.487
##	570	1000	504	496	0.504
##	571	1000	504	496	0.504
##	572	1000	509	491	0.509
##	573	1000	493	507	0.493
##	574	1000	498	502	0.498
##	575	1000	488	512	0.488
##	576	1000	514	486	0.514
##	577	1000	482	518	0.482
##	578	1000	483	517	0.483
##	579	1000	500	500	0.500
##	580	1000	485	515	0.485
##	581	1000	503	497	0.503
##	582	1000	476	524	0.476
##	583	1000	518	482	0.518
##	584	1000	502	498	0.502
##	585	1000	496	504	0.496
##	586	1000	501	499	0.501
##	587	1000	501	499	0.501
##	588	1000	520	480	0.520
##	589	1000	489	511	0.489
##	590	1000	499	501	0.499
##	591	1000	484	516	0.484
##	592	1000	504	496	0.504



##	593	1000	510	490	0.510
##	594	1000	499	501	0.499
##	595	1000	490	510	0.490
##	596	1000	503	497	0.503
##	597	1000	486	514	0.486
##	598	1000	489	511	0.489
##	599	1000	505	495	0.505
##	600	1000	493	507	0.493
##	601	1000	490	510	0.490
##	602	1000	482	518	0.482
##	603	1000	522	478	0.522
##	604	1000	525	475	0.525
##	605	1000	503	497	0.503
##	606	1000	471	529	0.471
##	607	1000	501	499	0.501
##	608	1000	504	496	0.504
##	609	1000	495	505	0.495
##	610	1000	504	496	0.504
##	611	1000	494	506	0.494
##	612	1000	530	470	0.530
##	613	1000	484	516	0.484
##	614	1000	489	511	0.489
##	615	1000	500	500	0.500
##	616	1000	508	492	0.508
##	617	1000	492	508	0.492
##	618	1000	478	522	0.478
##	619	1000	534	466	0.534
##	620	1000	489	511	0.489
##	621	1000	503	497	0.503
##	622	1000	504	496	0.504
##	623	1000	484	516	0.484
##	624	1000	494	506	0.494
##	625	1000	483	517	0.483
##	626	1000	509	491	0.509
##	627	1000	520	480	0.520
##	628	1000	489	511	0.489
##	629	1000	501	499	0.501
##	630	1000	500	500	0.500
##	631	1000	483	517	0.483
##	632	1000	514	486	0.514
##	633	1000	513	487	0.513
##	634	1000	499	501	0.499
##	635	1000	492	508	0.492
##	636	1000	464	536	0.464
##	637	1000	508	492	0.508
##	638	1000	506	494	0.506

##	639	1000	499	501	0.499
##	640	1000	500	500	0.500
##	641	1000	512	488	0.512
##	642	1000	491	509	0.491
##	643	1000	510	490	0.510
##	644	1000	487	513	0.487
##	645	1000	484	516	0.484
##	646	1000	475	525	0.475
##	647	1000	501	499	0.501
##	648	1000	478	522	0.478
##	649	1000	490	510	0.490
##	650	1000	493	507	0.493
##	651	1000	510	490	0.510
##	652	1000	493	507	0.493
##	653	1000	519	481	0.519
##	654	1000	542	458	0.542
##	655	1000	495	505	0.495
##	656	1000	527	473	0.527
##	657	1000	537	463	0.537
##	658	1000	509	491	0.509
##	659	1000	461	539	0.461
##	660	1000	502	498	0.502
##	661	1000	508	492	0.508
##	662	1000	496	504	0.496
##	663	1000	487	513	0.487
##	664	1000	510	490	0.510
##	665	1000	488	512	0.488
##	666	1000	517	483	0.517
##	667	1000	503	497	0.503
##	668	1000	456	544	0.456
##	669	1000	470	530	0.470
##	670	1000	475	525	0.475
##	671	1000	510	490	0.510
##	672	1000	492	508	0.492
##	673	1000	492	508	0.492
##	674	1000	506	494	0.506
##	675	1000	492	508	0.492
##	676	1000	485	515	0.485
##	677	1000	500	500	0.500
##	678	1000	499	501	0.499
##	679	1000	512	488	0.512
##	680	1000	490	510	0.490
##	681	1000	502	498	0.502
##	682	1000	489	511	0.489
##	683	1000	499	501	0.499
##	684	1000	493	507	0.493

##	685	1000	494	506	0.494
##	686	1000	515	485	0.515
##	687	1000	488	512	0.488
##	688	1000	487	513	0.487
##	689	1000	504	496	0.504
##	690	1000	504	496	0.504
##	691	1000	481	519	0.481
##	692	1000	487	513	0.487
##	693	1000	512	488	0.512
##	694	1000	512	488	0.512
##	695	1000	474	526	0.474
##	696	1000	498	502	0.498
##	697	1000	504	496	0.504
##	698	1000	510	490	0.510
##	699	1000	501	499	0.501
##	700	1000	517	483	0.517
##	701	1000	507	493	0.507
##	702	1000	478	522	0.478
##	703	1000	536	464	0.536
##	704	1000	484	516	0.484
##	705	1000	482	518	0.482
##	706	1000	485	515	0.485
##	707	1000	510	490	0.510
##	708	1000	487	513	0.487
##	709	1000	484	516	0.484
##	710	1000	504	496	0.504
##	711	1000	499	501	0.499
##	712	1000	507	493	0.507
##	713	1000	490	510	0.490
##	714	1000	511	489	0.511
##	715	1000	521	479	0.521
##	716	1000	507	493	0.507
##	717	1000	504	496	0.504
##	718	1000	489	511	0.489
##	719	1000	487	513	0.487
##	720	1000	502	498	0.502
##	721	1000	502	498	0.502
##	722	1000	491	509	0.491
##	723	1000	484	516	0.484
##	724	1000	500	500	0.500
##	725	1000	512	488	0.512
##	726	1000	491	509	0.491
##	727	1000	496	504	0.496
##	728	1000	485	515	0.485
##	729	1000	523	477	0.523
##	730	1000	515	485	0.515

##	731	1000	503	497	0.503
##	732	1000	509	491	0.509
##	733	1000	487	513	0.487
##	734	1000	508	492	0.508
##	735	1000	480	520	0.480
##	736	1000	499	501	0.499
##	737	1000	495	505	0.495
##	738	1000	502	498	0.502
##	739	1000	516	484	0.516
##	740	1000	493	507	0.493
##	741	1000	484	516	0.484
##	742	1000	475	525	0.475
##	743	1000	483	517	0.483
##	744	1000	508	492	0.508
##	745	1000	523	477	0.523
##	746	1000	502	498	0.502
##	747	1000	503	497	0.503
##	748	1000	519	481	0.519
##	749	1000	483	517	0.483
##	750	1000	484	516	0.484
##	751	1000	501	499	0.501
##	752	1000	494	506	0.494
##	753	1000	511	489	0.511
##	754	1000	507	493	0.507
##	755	1000	493	507	0.493
##	756	1000	501	499	0.501
##	757	1000	507	493	0.507
##	758	1000	507	493	0.507
##	759	1000	522	478	0.522
##	760	1000	475	525	0.475
##	761	1000	501	499	0.501
##	762	1000	478	522	0.478
##	763	1000	504	496	0.504
##	764	1000	506	494	0.506
##	765	1000	499	501	0.499
##	766	1000	492	508	0.492
##	767	1000	503	497	0.503
##	768	1000	501	499	0.501
##	769	1000	512	488	0.512
##	770	1000	491	509	0.491
##	771	1000	503	497	0.503
##	772	1000	484	516	0.484
##	773	1000	525	475	0.525
##	774	1000	527	473	0.527
##	775	1000	514	486	0.514
##	776	1000	507	493	0.507

##	777	1000	485	515	0.485
##	778	1000	482	518	0.482
##	779	1000	502	498	0.502
##	780	1000	492	508	0.492
##	781	1000	494	506	0.494
##	782	1000	501	499	0.501
##	783	1000	492	508	0.492
##	784	1000	502	498	0.502
##	785	1000	516	484	0.516
##	786	1000	505	495	0.505
##	787	1000	497	503	0.497
##	788	1000	492	508	0.492
##	789	1000	497	503	0.497
##	790	1000	511	489	0.511
##	791	1000	499	501	0.499
##	792	1000	507	493	0.507
##	793	1000	493	507	0.493
##	794	1000	491	509	0.491
##	795	1000	480	520	0.480
##	796	1000	512	488	0.512
##	797	1000	520	480	0.520
##	798	1000	482	518	0.482
##	799	1000	511	489	0.511
##	800	1000	517	483	0.517
##	801	1000	497	503	0.497
##	802	1000	513	487	0.513
##	803	1000	502	498	0.502
##	804	1000	521	479	0.521
##	805	1000	505	495	0.505
##	806	1000	479	521	0.479
##	807	1000	508	492	0.508
##	808	1000	516	484	0.516
##	809	1000	500	500	0.500
##	810	1000	517	483	0.517
##	811	1000	479	521	0.479
##	812	1000	493	507	0.493
##	813	1000	507	493	0.507
##	814	1000	519	481	0.519
##	815	1000	496	504	0.496
##	816	1000	497	503	0.497
##	817	1000	498	502	0.498
##	818	1000	500	500	0.500
##	819	1000	507	493	0.507
##	820	1000	527	473	0.527
##	821	1000	463	537	0.463
##	822	1000	506	494	0.506

##	823	1000	511	489	0.511
##	824	1000	523	477	0.523
##	825	1000	515	485	0.515
##	826	1000	527	473	0.527
##	827	1000	519	481	0.519
##	828	1000	490	510	0.490
##	829	1000	505	495	0.505
##	830	1000	511	489	0.511
##	831	1000	469	531	0.469
##	832	1000	492	508	0.492
##	833	1000	497	503	0.497
##	834	1000	523	477	0.523
##	835	1000	480	520	0.480
##	836	1000	493	507	0.493
##	837	1000	529	471	0.529
##	838	1000	523	477	0.523
##	839	1000	499	501	0.499
##	840	1000	523	477	0.523
##	841	1000	501	499	0.501
##	842	1000	505	495	0.505
##	843	1000	523	477	0.523
##	844	1000	504	496	0.504
##	845	1000	492	508	0.492
##	846	1000	470	530	0.470
##	847	1000	493	507	0.493
##	848	1000	511	489	0.511
##	849	1000	485	515	0.485
##	850	1000	510	490	0.510
##	851	1000	498	502	0.498
##	852	1000	506	494	0.506
##	853	1000	501	499	0.501
##	854	1000	519	481	0.519
##	855	1000	514	486	0.514
##	856	1000	489	511	0.489
##	857	1000	513	487	0.513
##	858	1000	533	467	0.533
##	859	1000	485	515	0.485
##	860	1000	499	501	0.499
##	861	1000	490	510	0.490
##	862	1000	508	492	0.508
##	863	1000	482	518	0.482
##	864	1000	496	504	0.496
##	865	1000	496	504	0.496
##	866	1000	525	475	0.525
##	867	1000	500	500	0.500
##	868	1000	480	520	0.480

##	869	1000	493	507	0.493
##	870	1000	500	500	0.500
##	871	1000	489	511	0.489
##	872	1000	503	497	0.503
##	873	1000	479	521	0.479
##	874	1000	500	500	0.500
##	875	1000	499	501	0.499
##	876	1000	502	498	0.502
##	877	1000	485	515	0.485
##	878	1000	515	485	0.515
##	879	1000	512	488	0.512
##	880	1000	509	491	0.509
##	881	1000	499	501	0.499
##	882	1000	477	523	0.477
##	883	1000	515	485	0.515
##	884	1000	490	510	0.490
##	885	1000	505	495	0.505
##	886	1000	499	501	0.499
##	887	1000	495	505	0.495
##	888	1000	527	473	0.527
##	889	1000	514	486	0.514
##	890	1000	513	487	0.513
##	891	1000	505	495	0.505
##	892	1000	504	496	0.504
##	893	1000	482	518	0.482
##	894	1000	499	501	0.499
##	895	1000	491	509	0.491
##	896	1000	474	526	0.474
##	897	1000	513	487	0.513
##	898	1000	492	508	0.492
##	899	1000	504	496	0.504
##	900	1000	511	489	0.511
##	901	1000	488	512	0.488
##	902	1000	534	466	0.534
##	903	1000	485	515	0.485
##	904	1000	471	529	0.471
##	905	1000	511	489	0.511
##	906	1000	502	498	0.502
##	907	1000	517	483	0.517
##	908	1000	520	480	0.520
##	909	1000	525	475	0.525
##	910	1000	517	483	0.517
##	911	1000	495	505	0.495
##	912	1000	497	503	0.497
##	913	1000	493	507	0.493
##	914	1000	496	504	0.496

##	915	1000	472	528	0.472
##	916	1000	503	497	0.503
##	917	1000	512	488	0.512
##	918	1000	488	512	0.488
##	919	1000	482	518	0.482
##	920	1000	496	504	0.496
##	921	1000	474	526	0.474
##	922	1000	502	498	0.502
##	923	1000	490	510	0.490
##	924	1000	516	484	0.516
##	925	1000	488	512	0.488
##	926	1000	489	511	0.489
##	927	1000	477	523	0.477
##	928	1000	511	489	0.511
##	929	1000	486	514	0.486
##	930	1000	482	518	0.482
##	931	1000	486	514	0.486
##	932	1000	506	494	0.506
##	933	1000	492	508	0.492
##	934	1000	482	518	0.482
##	935	1000	509	491	0.509
##	936	1000	511	489	0.511
##	937	1000	477	523	0.477
##	938	1000	507	493	0.507
##	939	1000	506	494	0.506
##	940	1000	497	503	0.497
##	941	1000	506	494	0.506
##	942	1000	495	505	0.495
##	943	1000	513	487	0.513
##	944	1000	511	489	0.511
##	945	1000	486	514	0.486
##	946	1000	486	514	0.486
##	947	1000	511	489	0.511
##	948	1000	492	508	0.492
##	949	1000	475	525	0.475
##	950	1000	490	510	0.490
##	951	1000	488	512	0.488
##	952	1000	493	507	0.493
##	953	1000	485	515	0.485
##	954	1000	509	491	0.509
##	955	1000	486	514	0.486
##	956	1000	504	496	0.504
##	957	1000	477	523	0.477
##	958	1000	512	488	0.512
##	959	1000	501	499	0.501
##	960	1000	487	513	0.487



##	961	1000	493	507	0.493
##	962	1000	492	508	0.492
##	963	1000	512	488	0.512
##	964	1000	505	495	0.505
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##	966	1000	494	506	0.494
##	967	1000	493	507	0.493
##	968	1000	502	498	0.502
##	969	1000	498	502	0.498
##	970	1000	498	502	0.498
##	971	1000	517	483	0.517
##	972	1000	525	475	0.525
##	973	1000	530	470	0.530
##	974	1000	503	497	0.503
##	975	1000	486	514	0.486
##	976	1000	525	475	0.525
##	977	1000	503	497	0.503
##	978	1000	493	507	0.493
##	979	1000	485	515	0.485
##	980	1000	485	515	0.485
##	981	1000	529	471	0.529
##	982	1000	508	492	0.508
##	983	1000	495	505	0.495
##	984	1000	488	512	0.488
##	985	1000	519	481	0.519
##	986	1000	515	485	0.515
##	987	1000	464	536	0.464
##	988	1000	524	476	0.524
##	989	1000	522	478	0.522
##	990	1000	520	480	0.520
##	991	1000	508	492	0.508
##	992	1000	512	488	0.512
##	993	1000	504	496	0.504
##	994	1000	481	519	0.481
##	995	1000	450	550	0.450
##	996	1000	500	500	0.500
##	997	1000	499	501	0.499
##	998	1000	487	513	0.487
##	999	1000	481	519	0.481
##	1000	1000	498	502	0.498
##	1001	1000	520	480	0.520
##	1002	1000	492	508	0.492
##	1003	1000	532	468	0.532
##	1004	1000	512	488	0.512
##	1005	1000	503	497	0.503
##	1006	1000	482	518	0.482

##	1007	1000	486	514	0.486
##	1008	1000	518	482	0.518
##	1009	1000	469	531	0.469
##	1010	1000	468	532	0.468
##	1011	1000	471	529	0.471
##	1012	1000	524	476	0.524
##	1013	1000	500	500	0.500
##	1014	1000	514	486	0.514
##	1015	1000	510	490	0.510
##	1016	1000	478	522	0.478
##	1017	1000	518	482	0.518
##	1018	1000	503	497	0.503
##	1019	1000	512	488	0.512
##	1020	1000	506	494	0.506
##	1021	1000	492	508	0.492
##	1022	1000	513	487	0.513
##	1023	1000	499	501	0.499
##	1024	1000	469	531	0.469
##	1025	1000	497	503	0.497
##	1026	1000	491	509	0.491
##	1027	1000	508	492	0.508
##	1028	1000	498	502	0.498
##	1029	1000	500	500	0.500
##	1030	1000	513	487	0.513
##	1031	1000	502	498	0.502
##	1032	1000	528	472	0.528
##	1033	1000	482	518	0.482
##	1034	1000	497	503	0.497
##	1035	1000	510	490	0.510
##	1036	1000	509	491	0.509
##	1037	1000	490	510	0.490
##	1038	1000	500	500	0.500
##	1039	1000	470	530	0.470
##	1040	1000	481	519	0.481
##	1041	1000	510	490	0.510
##	1042	1000	465	535	0.465
##	1043	1000	501	499	0.501
##	1044	1000	495	505	0.495
##	1045	1000	490	510	0.490
##	1046	1000	491	509	0.491
##	1047	1000	497	503	0.497
##	1048	1000	495	505	0.495
##	1049	1000	532	468	0.532
##	1050	1000	497	503	0.497
##	1051	1000	510	490	0.510
##	1052	1000	488	512	0.488

##	1053	1000	480	520	0.480
##	1054	1000	532	468	0.532
##	1055	1000	484	516	0.484
##	1056	1000	512	488	0.512
##	1057	1000	491	509	0.491
##	1058	1000	498	502	0.498
##	1059	1000	495	505	0.495
##	1060	1000	482	518	0.482
##	1061	1000	495	505	0.495
##	1062	1000	489	511	0.489
##	1063	1000	486	514	0.486
##	1064	1000	515	485	0.515
##	1065	1000	500	500	0.500
##	1066	1000	494	506	0.494
##	1067	1000	520	480	0.520
##	1068	1000	516	484	0.516
##	1069	1000	497	503	0.497
##	1070	1000	511	489	0.511
##	1071	1000	499	501	0.499
##	1072	1000	475	525	0.475
##	1073	1000	480	520	0.480
##	1074	1000	508	492	0.508
##	1075	1000	487	513	0.487
##	1076	1000	483	517	0.483
##	1077	1000	500	500	0.500
##	1078	1000	502	498	0.502
##	1079	1000	471	529	0.471
##	1080	1000	526	474	0.526
##	1081	1000	494	506	0.494
##	1082	1000	507	493	0.507
##	1083	1000	508	492	0.508
##	1084	1000	487	513	0.487
##	1085	1000	493	507	0.493
##	1086	1000	504	496	0.504
##	1087	1000	514	486	0.514
##	1088	1000	512	488	0.512
##	1089	1000	499	501	0.499
##	1090	1000	531	469	0.531
##	1091	1000	485	515	0.485
##	1092	1000	515	485	0.515
##	1093	1000	475	525	0.475
##	1094	1000	473	527	0.473
##	1095	1000	487	513	0.487
##	1096	1000	481	519	0.481
##	1097	1000	486	514	0.486
##	1098	1000	466	534	0.466

```
## 1099 1000 475 525 0.475
## 1100 1000 513 487 0.513
## 1101 1000 497 503 0.497
## 1102 1000 523 477 0.523
## 1103 1000 491 509 0.491
## 1104 1000 521 479 0.521
## 1105 1000 489 511 0.489
## 1106 1000 512 488 0.512
## 1107 1000 496 504 0.496
## 1108 1000 517 483 0.517
## 1109 1000 533 467 0.533
## 1110 1000 527 473 0.527
## 1111 1000 533 467 0.533
## 1112 1000 497 503 0.497
## 1113 1000 490 510 0.490
## 1114 1000 481 519 0.481
## 1115 1000 491 509 0.491
## 1116 1000 489 511 0.489
## 1117 1000 472 528 0.472
## 1118 1000 511 489 0.511
## 1119 1000 494 506 0.494
## 1120 1000 545 455 0.545
## 1121 1000 498 502 0.498
## 1122 1000 490 510 0.490
## 1123 1000 516 484 0.516
## 1124 1000 475 525 0.475
## 1125 1000 494 506 0.494
## 1126 1000 537 463 0.537
## 1127 1000 481 519 0.481
## 1128 1000 495 505 0.495
## 1129 1000 488 512 0.488
## 1130 1000 490 510 0.490
## 1131 1000 486 514 0.486
## 1132 1000 527 473 0.527
## 1133 1000 501 499 0.501
## 1134 1000 505 495 0.505
## 1135 1000 502 498 0.502
## 1136 1000 494 506 0.494
## 1137 1000 495 505 0.495
## 1138 1000 517 483 0.517
## 1139 1000 480 520 0.480
## 1140 1000 477 523 0.477
## 1141 1000 505 495 0.505
## 1142 1000 516 484 0.516
## 1143 1000 526 474 0.526
## 1144 1000 518 482 0.518
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##	1145	1000	495	505	0.495
##	1146	1000	511	489	0.511
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##	1149	1000	498	502	0.498
##	1150	1000	504	496	0.504
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##	1152	1000	487	513	0.487
##	1153	1000	504	496	0.504
##	1154	1000	496	504	0.496
##	1155	1000	512	488	0.512
##	1156	1000	477	523	0.477
##	1157	1000	514	486	0.514
##	1158	1000	511	489	0.511
##	1159	1000	475	525	0.475
##	1160	1000	464	536	0.464
##	1161	1000	448	552	0.448
##	1162	1000	526	474	0.526
##	1163	1000	538	462	0.538
##	1164	1000	499	501	0.499
##	1165	1000	487	513	0.487
##	1166	1000	509	491	0.509
##	1167	1000	501	499	0.501
##	1168	1000	481	519	0.481
##	1169	1000	509	491	0.509
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##	1173	1000	489	511	0.489
##	1174	1000	475	525	0.475
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##	1176	1000	473	527	0.473
##	1177	1000	513	487	0.513
##	1178	1000	517	483	0.517
##	1179	1000	497	503	0.497
##	1180	1000	469	531	0.469
##	1181	1000	520	480	0.520
##	1182	1000	457	543	0.457
##	1183	1000	532	468	0.532
##	1184	1000	500	500	0.500
##	1185	1000	514	486	0.514
##	1186	1000	522	478	0.522
##	1187	1000	517	483	0.517
##	1188	1000	518	482	0.518
##	1189	1000	503	497	0.503
##	1190	1000	506	494	0.506

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##	1192	1000	509	491	0.509
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##	1194	1000	511	489	0.511
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##	1196	1000	513	487	0.513
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##	1198	1000	512	488	0.512
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##	1202	1000	527	473	0.527
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##	1235	1000	533	467	0.533
##	1236	1000	487	513	0.487

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##	1242	1000	497	503	0.497
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##	1245	1000	512	488	0.512
##	1246	1000	526	474	0.526
##	1247	1000	487	513	0.487
##	1248	1000	508	492	0.508
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##	1255	1000	500	500	0.500
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##	1257	1000	529	471	0.529
##	1258	1000	518	482	0.518
##	1259	1000	510	490	0.510
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##	1261	1000	498	502	0.498
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##	1276	1000	480	520	0.480
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##	1278	1000	486	514	0.486
##	1279	1000	506	494	0.506
##	1280	1000	492	508	0.492
##	1281	1000	512	488	0.512
##	1282	1000	522	478	0.522

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##	1284	1000	494	506	0.494
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##	1288	1000	494	506	0.494
##	1289	1000	525	475	0.525
##	1290	1000	506	494	0.506
##	1291	1000	496	504	0.496
##	1292	1000	524	476	0.524
##	1293	1000	475	525	0.475
##	1294	1000	465	535	0.465
##	1295	1000	495	505	0.495
##	1296	1000	517	483	0.517
##	1297	1000	502	498	0.502
##	1298	1000	494	506	0.494
##	1299	1000	518	482	0.518
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##	1301	1000	513	487	0.513
##	1302	1000	522	478	0.522
##	1303	1000	494	506	0.494
##	1304	1000	499	501	0.499
##	1305	1000	493	507	0.493
##	1306	1000	535	465	0.535
##	1307	1000	495	505	0.495
##	1308	1000	507	493	0.507
##	1309	1000	509	491	0.509
##	1310	1000	500	500	0.500
##	1311	1000	480	520	0.480
##	1312	1000	524	476	0.524
##	1313	1000	489	511	0.489
##	1314	1000	504	496	0.504
##	1315	1000	516	484	0.516
##	1316	1000	521	479	0.521
##	1317	1000	532	468	0.532
##	1318	1000	518	482	0.518
##	1319	1000	500	500	0.500
##	1320	1000	502	498	0.502
##	1321	1000	491	509	0.491
##	1322	1000	529	471	0.529
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##	1324	1000	489	511	0.489
##	1325	1000	496	504	0.496
##	1326	1000	515	485	0.515
##	1327	1000	498	502	0.498
##	1328	1000	495	505	0.495



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##	1331	1000	515	485	0.515
##	1332	1000	491	509	0.491
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##	1336	1000	515	485	0.515
##	1337	1000	483	517	0.483
##	1338	1000	497	503	0.497
##	1339	1000	496	504	0.496
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##	1347	1000	504	496	0.504
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##	1349	1000	489	511	0.489
##	1350	1000	512	488	0.512
##	1351	1000	527	473	0.527
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##	1356	1000	491	509	0.491
##	1357	1000	504	496	0.504
##	1358	1000	502	498	0.502
##	1359	1000	471	529	0.471
##	1360	1000	492	508	0.492
##	1361	1000	488	512	0.488
##	1362	1000	494	506	0.494
##	1363	1000	531	469	0.531
##	1364	1000	473	527	0.473
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##	1368	1000	530	470	0.530
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##	1370	1000	517	483	0.517
##	1371	1000	526	474	0.526
##	1372	1000	515	485	0.515
##	1373	1000	488	512	0.488
##	1374	1000	455	545	0.455

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##	1376	1000	494	506	0.494
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##	1379	1000	472	528	0.472
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##	1381	1000	488	512	0.488
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##	1383	1000	520	480	0.520
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##	1387	1000	519	481	0.519
##	1388	1000	490	510	0.490
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##	1394	1000	500	500	0.500
##	1395	1000	519	481	0.519
##	1396	1000	493	507	0.493
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##	1398	1000	489	511	0.489
##	1399	1000	494	506	0.494
##	1400	1000	508	492	0.508
##	1401	1000	513	487	0.513
##	1402	1000	514	486	0.514
##	1403	1000	516	484	0.516
##	1404	1000	502	498	0.502
##	1405	1000	496	504	0.496
##	1406	1000	483	517	0.483
##	1407	1000	516	484	0.516
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##	1414	1000	463	537	0.463
##	1415	1000	521	479	0.521
##	1416	1000	483	517	0.483
##	1417	1000	469	531	0.469
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##	1419	1000	496	504	0.496
##	1420	1000	482	518	0.482

##	1421	1000	477	523	0.477
##	1422	1000	536	464	0.536
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##	1424	1000	505	495	0.505
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##	1426	1000	517	483	0.517
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##	1437	1000	506	494	0.506
##	1438	1000	492	508	0.492
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##	1463	1000	510	490	0.510
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##	1466	1000	475	525	0.475

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##	1470	1000	468	532	0.468
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##	1472	1000	486	514	0.486
##	1473	1000	508	492	0.508
##	1474	1000	517	483	0.517
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##	1483	1000	510	490	0.510
##	1484	1000	500	500	0.500
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##	1488	1000	486	514	0.486
##	1489	1000	467	533	0.467
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##	1494	1000	497	503	0.497
##	1495	1000	514	486	0.514
##	1496	1000	490	510	0.490
##	1497	1000	518	482	0.518
##	1498	1000	508	492	0.508
##	1499	1000	480	520	0.480
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##	1511	1000	496	504	0.496
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##	1516	1000	510	490	0.510
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##	1527	1000	528	472	0.528
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##	1560	1000	500	500	0.500
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##	1563	1000	483	517	0.483
##	1564	1000	517	483	0.517
##	1565	1000	509	491	0.509
##	1566	1000	510	490	0.510
##	1567	1000	488	512	0.488
##	1568	1000	491	509	0.491
##	1569	1000	526	474	0.526
##	1570	1000	484	516	0.484
##	1571	1000	494	506	0.494
##	1572	1000	498	502	0.498
##	1573	1000	481	519	0.481
##	1574	1000	520	480	0.520
##	1575	1000	504	496	0.504
##	1576	1000	512	488	0.512
##	1577	1000	510	490	0.510
##	1578	1000	503	497	0.503
##	1579	1000	501	499	0.501
##	1580	1000	495	505	0.495
##	1581	1000	497	503	0.497
##	1582	1000	533	467	0.533
##	1583	1000	521	479	0.521
##	1584	1000	492	508	0.492
##	1585	1000	496	504	0.496
##	1586	1000	484	516	0.484
##	1587	1000	487	513	0.487
##	1588	1000	495	505	0.495
##	1589	1000	476	524	0.476
##	1590	1000	483	517	0.483
##	1591	1000	520	480	0.520
##	1592	1000	502	498	0.502
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##	1595	1000	510	490	0.510
##	1596	1000	500	500	0.500
##	1597	1000	517	483	0.517
##	1598	1000	513	487	0.513
##	1599	1000	491	509	0.491
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##	1603	1000	493	507	0.493
##	1604	1000	485	515	0.485

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##	1606	1000	496	504	0.496
##	1607	1000	480	520	0.480
##	1608	1000	498	502	0.498
##	1609	1000	530	470	0.530
##	1610	1000	470	530	0.470
##	1611	1000	516	484	0.516
##	1612	1000	514	486	0.514
##	1613	1000	500	500	0.500
##	1614	1000	469	531	0.469
##	1615	1000	495	505	0.495
##	1616	1000	489	511	0.489
##	1617	1000	503	497	0.503
##	1618	1000	475	525	0.475
##	1619	1000	492	508	0.492
##	1620	1000	504	496	0.504
##	1621	1000	488	512	0.488
##	1622	1000	492	508	0.492
##	1623	1000	516	484	0.516
##	1624	1000	479	521	0.479
##	1625	1000	502	498	0.502
##	1626	1000	490	510	0.490
##	1627	1000	493	507	0.493
##	1628	1000	517	483	0.517
##	1629	1000	509	491	0.509
##	1630	1000	498	502	0.498
##	1631	1000	517	483	0.517
##	1632	1000	497	503	0.497
##	1633	1000	519	481	0.519
##	1634	1000	493	507	0.493
##	1635	1000	500	500	0.500
##	1636	1000	501	499	0.501
##	1637	1000	486	514	0.486
##	1638	1000	502	498	0.502
##	1639	1000	500	500	0.500
##	1640	1000	505	495	0.505
##	1641	1000	464	536	0.464
##	1642	1000	500	500	0.500
##	1643	1000	502	498	0.502
##	1644	1000	488	512	0.488
##	1645	1000	480	520	0.480
##	1646	1000	491	509	0.491
##	1647	1000	529	471	0.529
##	1648	1000	490	510	0.490
##	1649	1000	487	513	0.487
##	1650	1000	494	506	0.494

##	1651	1000	527	473	0.527
##	1652	1000	493	507	0.493
##	1653	1000	512	488	0.512
##	1654	1000	512	488	0.512
##	1655	1000	481	519	0.481
##	1656	1000	486	514	0.486
##	1657	1000	459	541	0.459
##	1658	1000	487	513	0.487
##	1659	1000	481	519	0.481
##	1660	1000	544	456	0.544
##	1661	1000	479	521	0.479
##	1662	1000	513	487	0.513
##	1663	1000	501	499	0.501
##	1664	1000	480	520	0.480
##	1665	1000	489	511	0.489
##	1666	1000	491	509	0.491
##	1667	1000	503	497	0.503
##	1668	1000	527	473	0.527
##	1669	1000	506	494	0.506
##	1670	1000	487	513	0.487
##	1671	1000	506	494	0.506
##	1672	1000	506	494	0.506
##	1673	1000	485	515	0.485
##	1674	1000	525	475	0.525
##	1675	1000	520	480	0.520
##	1676	1000	490	510	0.490
##	1677	1000	508	492	0.508
##	1678	1000	488	512	0.488
##	1679	1000	505	495	0.505
##	1680	1000	485	515	0.485
##	1681	1000	508	492	0.508
##	1682	1000	473	527	0.473
##	1683	1000	503	497	0.503
##	1684	1000	526	474	0.526
##	1685	1000	496	504	0.496
##	1686	1000	524	476	0.524
##	1687	1000	498	502	0.498
##	1688	1000	540	460	0.540
##	1689	1000	486	514	0.486
##	1690	1000	491	509	0.491
##	1691	1000	499	501	0.499
##	1692	1000	521	479	0.521
##	1693	1000	496	504	0.496
##	1694	1000	501	499	0.501
##	1695	1000	485	515	0.485
##	1696	1000	482	518	0.482



##	1697	1000	510	490	0.510
##	1698	1000	488	512	0.488
##	1699	1000	499	501	0.499
##	1700	1000	486	514	0.486
##	1701	1000	496	504	0.496
##	1702	1000	504	496	0.504
##	1703	1000	499	501	0.499
##	1704	1000	484	516	0.484
##	1705	1000	489	511	0.489
##	1706	1000	491	509	0.491
##	1707	1000	515	485	0.515
##	1708	1000	476	524	0.476
##	1709	1000	508	492	0.508
##	1710	1000	485	515	0.485
##	1711	1000	483	517	0.483
##	1712	1000	529	471	0.529
##	1713	1000	552	448	0.552
##	1714	1000	483	517	0.483
##	1715	1000	511	489	0.511
##	1716	1000	479	521	0.479
##	1717	1000	496	504	0.496
##	1718	1000	511	489	0.511
##	1719	1000	530	470	0.530
##	1720	1000	501	499	0.501
##	1721	1000	505	495	0.505
##	1722	1000	527	473	0.527
##	1723	1000	495	505	0.495
##	1724	1000	496	504	0.496
##	1725	1000	494	506	0.494
##	1726	1000	486	514	0.486
##	1727	1000	495	505	0.495
##	1728	1000	503	497	0.503
##	1729	1000	493	507	0.493
##	1730	1000	475	525	0.475
##	1731	1000	493	507	0.493
##	1732	1000	501	499	0.501
##	1733	1000	511	489	0.511
##	1734	1000	487	513	0.487
##	1735	1000	480	520	0.480
##	1736	1000	471	529	0.471
##	1737	1000	482	518	0.482
##	1738	1000	527	473	0.527
##	1739	1000	494	506	0.494
##	1740	1000	500	500	0.500
##	1741	1000	527	473	0.527
##	1742	1000	521	479	0.521

##	1743	1000	498	502	0.498
##	1744	1000	487	513	0.487
##	1745	1000	488	512	0.488
##	1746	1000	534	466	0.534
##	1747	1000	492	508	0.492
##	1748	1000	491	509	0.491
##	1749	1000	516	484	0.516
##	1750	1000	496	504	0.496
##	1751	1000	496	504	0.496
##	1752	1000	497	503	0.497
##	1753	1000	508	492	0.508
##	1754	1000	488	512	0.488
##	1755	1000	526	474	0.526
##	1756	1000	495	505	0.495
##	1757	1000	510	490	0.510
##	1758	1000	504	496	0.504
##	1759	1000	496	504	0.496
##	1760	1000	501	499	0.501
##	1761	1000	562	438	0.562
##	1762	1000	505	495	0.505
##	1763	1000	493	507	0.493
##	1764	1000	513	487	0.513
##	1765	1000	506	494	0.506
##	1766	1000	517	483	0.517
##	1767	1000	499	501	0.499
##	1768	1000	489	511	0.489
##	1769	1000	488	512	0.488
##	1770	1000	516	484	0.516
##	1771	1000	479	521	0.479
##	1772	1000	494	506	0.494
##	1773	1000	506	494	0.506
##	1774	1000	497	503	0.497
##	1775	1000	485	515	0.485
##	1776	1000	482	518	0.482
##	1777	1000	518	482	0.518
##	1778	1000	483	517	0.483
##	1779	1000	496	504	0.496
##	1780	1000	480	520	0.480
##	1781	1000	487	513	0.487
##	1782	1000	511	489	0.511
##	1783	1000	507	493	0.507
##	1784	1000	474	526	0.474
##	1785	1000	506	494	0.506
##	1786	1000	493	507	0.493
##	1787	1000	497	503	0.497
##	1788	1000	507	493	0.507

##	1789	1000	535	465	0.535
##	1790	1000	501	499	0.501
##	1791	1000	514	486	0.514
##	1792	1000	528	472	0.528
##	1793	1000	486	514	0.486
##	1794	1000	482	518	0.482
##	1795	1000	484	516	0.484
##	1796	1000	503	497	0.503
##	1797	1000	528	472	0.528
##	1798	1000	507	493	0.507
##	1799	1000	478	522	0.478
##	1800	1000	536	464	0.536
##	1801	1000	500	500	0.500
##	1802	1000	489	511	0.489
##	1803	1000	527	473	0.527
##	1804	1000	487	513	0.487
##	1805	1000	515	485	0.515
##	1806	1000	481	519	0.481
##	1807	1000	496	504	0.496
##	1808	1000	489	511	0.489
##	1809	1000	524	476	0.524
##	1810	1000	513	487	0.513
##	1811	1000	503	497	0.503
##	1812	1000	493	507	0.493
##	1813	1000	495	505	0.495
##	1814	1000	506	494	0.506
##	1815	1000	513	487	0.513
##	1816	1000	485	515	0.485
##	1817	1000	498	502	0.498
##	1818	1000	483	517	0.483
##	1819	1000	502	498	0.502
##	1820	1000	501	499	0.501
##	1821	1000	498	502	0.498
##	1822	1000	505	495	0.505
##	1823	1000	495	505	0.495
##	1824	1000	517	483	0.517
##	1825	1000	504	496	0.504
##	1826	1000	499	501	0.499
##	1827	1000	496	504	0.496
##	1828	1000	499	501	0.499
##	1829	1000	481	519	0.481
##	1830	1000	496	504	0.496
##	1831	1000	488	512	0.488
##	1832	1000	492	508	0.492
##	1833	1000	495	505	0.495
##	1834	1000	528	472	0.528

##	1835	1000	520	480	0.520
##	1836	1000	516	484	0.516
##	1837	1000	496	504	0.496
##	1838	1000	493	507	0.493
##	1839	1000	511	489	0.511
##	1840	1000	491	509	0.491
##	1841	1000	469	531	0.469
##	1842	1000	487	513	0.487
##	1843	1000	490	510	0.490
##	1844	1000	475	525	0.475
##	1845	1000	491	509	0.491
##	1846	1000	510	490	0.510
##	1847	1000	491	509	0.491
##	1848	1000	512	488	0.512
##	1849	1000	503	497	0.503
##	1850	1000	485	515	0.485
##	1851	1000	508	492	0.508
##	1852	1000	497	503	0.497
##	1853	1000	512	488	0.512
##	1854	1000	511	489	0.511
##	1855	1000	506	494	0.506
##	1856	1000	516	484	0.516
##	1857	1000	499	501	0.499
##	1858	1000	499	501	0.499
##	1859	1000	490	510	0.490
##	1860	1000	488	512	0.488
##	1861	1000	499	501	0.499
##	1862	1000	522	478	0.522
##	1863	1000	464	536	0.464
##	1864	1000	487	513	0.487
##	1865	1000	512	488	0.512
##	1866	1000	504	496	0.504
##	1867	1000	504	496	0.504
##	1868	1000	501	499	0.501
##	1869	1000	526	474	0.526
##	1870	1000	534	466	0.534
##	1871	1000	503	497	0.503
##	1872	1000	496	504	0.496
##	1873	1000	497	503	0.497
##	1874	1000	517	483	0.517
##	1875	1000	508	492	0.508
##	1876	1000	501	499	0.501
##	1877	1000	482	518	0.482
##	1878	1000	498	502	0.498
##	1879	1000	510	490	0.510
##	1880	1000	503	497	0.503

##	1881	1000	502	498	0.502
##	1882	1000	476	524	0.476
##	1883	1000	507	493	0.507
##	1884	1000	500	500	0.500
##	1885	1000	493	507	0.493
##	1886	1000	507	493	0.507
##	1887	1000	500	500	0.500
##	1888	1000	509	491	0.509
##	1889	1000	510	490	0.510
##	1890	1000	500	500	0.500
##	1891	1000	512	488	0.512
##	1892	1000	527	473	0.527
##	1893	1000	484	516	0.484
##	1894	1000	458	542	0.458
##	1895	1000	497	503	0.497
##	1896	1000	502	498	0.502
##	1897	1000	496	504	0.496
##	1898	1000	505	495	0.505
##	1899	1000	513	487	0.513
##	1900	1000	543	457	0.543
##	1901	1000	506	494	0.506
##	1902	1000	508	492	0.508
##	1903	1000	528	472	0.528
##	1904	1000	472	528	0.472
##	1905	1000	492	508	0.492
##	1906	1000	493	507	0.493
##	1907	1000	482	518	0.482
##	1908	1000	501	499	0.501
##	1909	1000	504	496	0.504
##	1910	1000	504	496	0.504
##	1911	1000	499	501	0.499
##	1912	1000	491	509	0.491
##	1913	1000	507	493	0.507
##	1914	1000	463	537	0.463
##	1915	1000	499	501	0.499
##	1916	1000	486	514	0.486
##	1917	1000	483	517	0.483
##	1918	1000	515	485	0.515
##	1919	1000	475	525	0.475
##	1920	1000	495	505	0.495
##	1921	1000	495	505	0.495
##	1922	1000	504	496	0.504
##	1923	1000	484	516	0.484
##	1924	1000	523	477	0.523
##	1925	1000	491	509	0.491
##	1926	1000	472	528	0.472

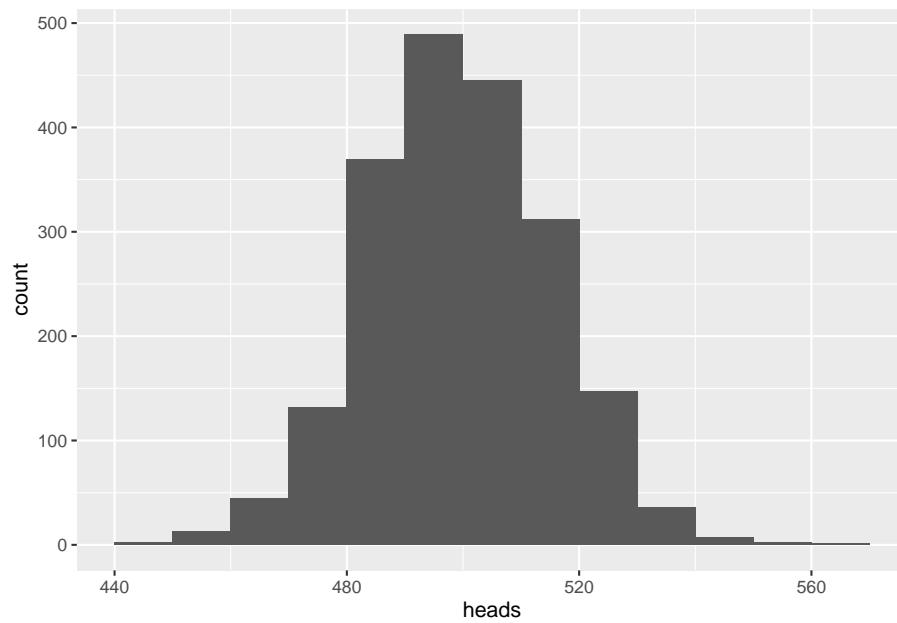
##	1927	1000	498	502	0.498
##	1928	1000	514	486	0.514
##	1929	1000	473	527	0.473
##	1930	1000	485	515	0.485
##	1931	1000	502	498	0.502
##	1932	1000	491	509	0.491
##	1933	1000	499	501	0.499
##	1934	1000	498	502	0.498
##	1935	1000	492	508	0.492
##	1936	1000	502	498	0.502
##	1937	1000	477	523	0.477
##	1938	1000	518	482	0.518
##	1939	1000	520	480	0.520
##	1940	1000	469	531	0.469
##	1941	1000	500	500	0.500
##	1942	1000	509	491	0.509
##	1943	1000	482	518	0.482
##	1944	1000	519	481	0.519
##	1945	1000	488	512	0.488
##	1946	1000	488	512	0.488
##	1947	1000	517	483	0.517
##	1948	1000	510	490	0.510
##	1949	1000	519	481	0.519
##	1950	1000	486	514	0.486
##	1951	1000	496	504	0.496
##	1952	1000	503	497	0.503
##	1953	1000	503	497	0.503
##	1954	1000	528	472	0.528
##	1955	1000	506	494	0.506
##	1956	1000	484	516	0.484
##	1957	1000	504	496	0.504
##	1958	1000	494	506	0.494
##	1959	1000	492	508	0.492
##	1960	1000	487	513	0.487
##	1961	1000	518	482	0.518
##	1962	1000	475	525	0.475
##	1963	1000	498	502	0.498
##	1964	1000	473	527	0.473
##	1965	1000	509	491	0.509
##	1966	1000	459	541	0.459
##	1967	1000	508	492	0.508
##	1968	1000	499	501	0.499
##	1969	1000	514	486	0.514
##	1970	1000	511	489	0.511
##	1971	1000	504	496	0.504
##	1972	1000	490	510	0.490

```
## 1973 1000 518 482 0.518
## 1974 1000 487 513 0.487
## 1975 1000 498 502 0.498
## 1976 1000 515 485 0.515
## 1977 1000 521 479 0.521
## 1978 1000 492 508 0.492
## 1979 1000 522 478 0.522
## 1980 1000 498 502 0.498
## 1981 1000 510 490 0.510
## 1982 1000 495 505 0.495
## 1983 1000 529 471 0.529
## 1984 1000 483 517 0.483
## 1985 1000 505 495 0.505
## 1986 1000 497 503 0.497
## 1987 1000 493 507 0.493
## 1988 1000 491 509 0.491
## 1989 1000 525 475 0.525
## 1990 1000 490 510 0.490
## 1991 1000 498 502 0.498
## 1992 1000 524 476 0.524
## 1993 1000 506 494 0.506
## 1994 1000 485 515 0.485
## 1995 1000 502 498 0.502
## 1996 1000 491 509 0.491
## 1997 1000 479 521 0.479
## 1998 1000 524 476 0.524
## 1999 1000 505 495 0.505
## 2000 1000 507 493 0.507
```

```
mean(coin_flips_2000_1000$heads)
```

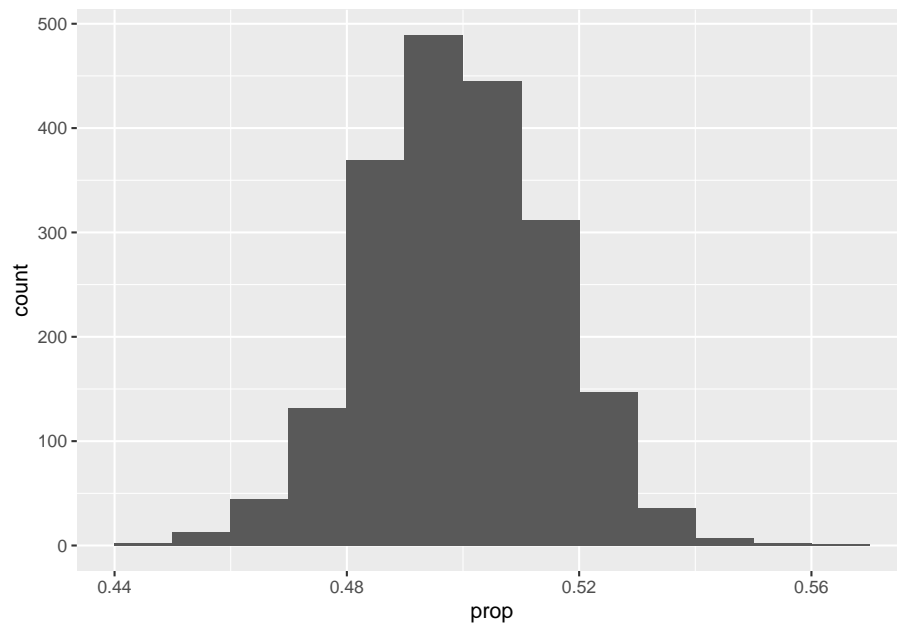
```
## [1] 499.9055
```

```
ggplot(coin_flips_2000_1000, aes(x = heads)) +
  geom_histogram(binwidth = 10, boundary = 500)
```



And now the same histogram, but with proportions:

```
ggplot(coin_flips_2000_1000, aes(x = prop)) +  
  geom_histogram(binwidth = 0.01, boundary = 0.5)
```





**Exercise 4** Comment on the histogram above. Describe its shape using the vocabulary of the three important features (modes, symmetry, outliers). Why do you think it's shaped like this?

Please write up your answer here.

**Exercise 5** Given the amount of randomness involved (each person is tossing coins which randomly come up heads or tails), why do we see so much structure and orderliness in the histograms?

Please write up your answer here.

## 8.7 But who cares about coin flips?

It's fair to ask why we go to all this trouble to talk about coin flips. The most pressing research questions of our day do not involve people sitting around and flipping coins, either physically or virtually.

But now substitute “heads” and “tails” with “cancer” and “no cancer”. Or “guilty” and “not guilty”. Or “shot” and “not shot”. The fact is that many important issues are measured as variables with two possible outcomes. There is some underlying “probability” of seeing one outcome over the other. (It doesn't have to be 50% like the coin.) Statistical methods—including simulation—can say a lot about what we “expect” to see if these outcomes are truly random. More importantly, when we see outcomes that *aren't* consistent with our simulations, we may wonder if there is some underlying mechanism that may be not so random after all. It may not look like it on first blush, but this idea is at the core of the scientific method.

For example, let's suppose that 85% of U.S. adults support some form of background checks for gun buyers.<sup>2</sup> Now, imagine we went out and surveyed a random group of people and asked them a simple yes/no question about their support for background checks. What might we see?

Let's simulate. Imagine flipping a coin, but instead of coming up heads 50% of the time, suppose it were possible for the coin to come up heads 85% of the time.<sup>3</sup> A sequence of heads and tails with this weird coin would be much like randomly surveying people and asking them about background checks.

We can make a “virtual” weird coin with the `rflip` command by specifying how often we want heads to come up.

<sup>2</sup>This is likely close to the truth. See this article: <https://iop.harvard.edu/get-involved/harvard-political-review/vast-majority-americans-support-universal-background-checks>

<sup>3</sup>The idea of a “weighted” coin that can do this comes up all the time in probability and statistics courses, but it seems that it's not likely one could actually manufacture a coin that came up heads more or less than 50% of the time when flipped. See this paper for more details: <http://www.stat.columbia.edu/~gelman/research/published/diceRev2.pdf>

```
set.seed(1234)
rflip(1, prob = 0.85)
```

```
##
## Flipping 1 coin [ Prob(Heads) = 0.85 ] ...
##
## H
##
## Number of Heads: 1 [Proportion Heads: 1]
```

If we flip our weird coin a bunch of times, we can see that our coin is not fair. Indeed, it appears to come up heads way more often than not:

```
set.seed(1234)
rflip(100, prob = 0.85)
```

```
##
## Flipping 100 coins [ Prob(Heads) = 0.85 ] ...
##
## H H H H T H H H H H H H H T H H H H H H H H H H H T H H H H H H H H
## H H T H H H H H H H H H H H H H H H H H H T H H H H H H H H H H T
## H H H H H H H H T H H H H T H H H T H T H H H H H H H H
##
## Number of Heads: 90 [Proportion Heads: 0.9]
```

The results from the above code can be thought of as a survey of 100 random U.S. adults about their support for background checks for purchasing guns. “Heads” means “supports” and “tails” means “opposes.” If the majority of Americans support background checks, then we will come across more people in our survey who tell us they support background checks. This shows up in our simulation as the appearance of more heads than tails.

Note that there is no guarantee that our sample will have exactly 85% heads. In fact, it doesn’t; it has 90% heads.

Again, keep in mind that we’re simulating the act of obtaining a random sample of 100 U.S. adults. If we get a different sample, we’ll get different results. (We set a different seed here. That ensures that this code chunk is randomly different from the one above.)

```
set.seed(123456)
rflip(100, prob = 0.85)
```

```
##
```

```
## Flipping 100 coins [ Prob(Heads) = 0.85 ] ...
##
## H H H H H H H H T H H H T T T T H H H H H H H H T T H H T H H H H
## T T H H H H T H H H H H H H H H T H T H H H H H H H H H H H H H H
## T H H H T H H H H H H T H H H H H H H H H H H T H H H
##
## Number of Heads: 81 [Proportion Heads: 0.81]
```

See, this time, only 81% came up heads, even though we expected 85%. That's how randomness works.

**Exercise 6(a)** Now imagine that 2000 people all go out and conduct surveys of 100 random U.S. adults, asking them about their support for background checks. Write some R code that simulates this. Plot a histogram of the results. (Hint: you'll need `do(2000) * in` there.) Use the proportion of supporters (`prop`), not the raw count of supporters (`heads`).

```
set.seed(1234)
# Add code here to simulate 2000 surveys of 100 U.S. adults.
```

```
# Plot the results in a histogram using proportions.
```

**Exercise 6(b)** Run another simulation, but this time, have each person survey 1000 adults and not just 100.

```
set.seed(1234)
# Add code here to simulate 2000 surveys of 1000 U.S. adults.
```

```
# Plot the results in a histogram using proportions.
```

**Exercise 6(c)** What changed when you surveyed 1000 people instead of 100? Please write up your answer here.

## 8.8 Sampling variability

We've seen that taking repeated samples (using the `do` command) leads to lots of different outcomes. That is randomness in action. We don't expect the results of each survey to be exactly the same every time the survey is administered.

But despite this randomness, there is an interesting pattern that we can observe. It has to do with the number of times we flip the coin. Since we're using coin

flips to simulate the act of conducting a survey, the number of coin flips is playing the role of the *sample size*. In other words, if we want to simulate a survey of U.S. adults with a sample size of 100, we simulate that by flipping 100 coins.

**Exercise 7** Go back and look at all the examples above. What do you notice about the range of values on the x-axis when the sample size is small versus large? (In other words, in what way are the histograms different when using `rflip(10, prob = ...)` or `rflip(100, prob = ...)` versus `rflip(1000, prob = ...)`? It's easier to compare histograms one to another when looking at the proportions instead of the raw head counts because proportions are always on the same scale from 0 to 1.)

Please write up your answer here.

## 8.9 Conclusion

Simulation is a tool for understanding what happens when a statistical process is repeated many times in a randomized way. The availability of fast computer processing makes simulation easy and accessible. Eventually, the goal will be to use simulation to answer important questions about data and the processes in the world that generate data. This is possible because, despite the ubiquitous presence of randomness, a certain order emerges when the number of samples is large enough. Even though there is sampling variability (different random outcomes each time we sample), there are patterns in that variability that can be exploited to make predictions.

## Chapter 9

# Introduction to randomization, Part 2

2.0

### Functions introduced in this chapter

`sample`, `specify`, `hypothesize`, `generate`, `calculate`, `visualize`,  
`shade_p_value`, `get_p_value`

## 9.1 Introduction

In this chapter, we'll learn more about randomization and simulation. Instead of flipping coins, though, we'll randomly shuffle data around in order to explore the effects of randomizing a predictor variable.

### 9.1.1 Install new packages

If you are using RStudio Workbench, you do not need to install any packages. (Any packages you need should already be installed by the server administrators.)

If you are using R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type the following commands at the Console:

```
install.packages("openintro")  
install.packages("infer")
```

### 9.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/09-intro\\_to\\_randomization\\_2.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/09-intro_to_randomization_2.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 9.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 9.2 Load packages

We'll load `tidyverse` as usual along with the `janitor` package to make tables (with `tabyl`). The `openintro` package has a data set called `sex_discrimination` that we will explore. Finally, the `infer` package will provide tools that we will use in nearly every chapter for the remainder of the book.

```
library(tidyverse)
library(janitor)
library(openintro)
```

```
## Loading required package: airports
```

```
## Loading required package: cherryblossom
```

```
## Loading required package: usdata
```

```
##
```

```
## Attaching package: 'openintro'
```

```
## The following object is masked from 'package:mosaic':
```

```
##
```

```
##      dotPlot
```

```
## The following objects are masked from 'package:lattice':
```

```
##
```

```
##      ethanol, lsegments
```

```
## The following object is masked from 'package:faraway':  
##  
##      orings  
  
library(infer)  
  
##  
## Attaching package: 'infer'  
  
## The following objects are masked from 'package:mosaic':  
##  
##      prop_test, t_test
```

### 9.3 Our research question

An interesting study was conducted in the 1970s that investigated gender discrimination in hiring.<sup>1</sup> The researchers brought in 48 male bank supervisors and asked them to evaluate personnel files. Based on their review, they were to determine if the person was qualified for promotion to branch manager. The trick is that all the files were identical, but half listed the candidate as male and half listed the candidate as female. The files were randomly assigned to the 48 supervisors.

The research question is whether the files supposedly belonging to males were recommended for promotion more than the files supposedly belonging to females.

**Exercise 1** Is the study described above an observational study or an experiment? How do you know?

Please write up your answer here.

**Exercise 2(a)** Identify the sample in the study. In other words, how many people were in the sample and what are the important characteristics common to those people.

Please write up your answer here.

**Exercise 2(b)** Identify the population of interest in the study. In other words, who is the sample supposed to represent? That is, what group of people that this study is trying to learn about?

Please write up your answer here.

---

<sup>1</sup>Rosen B and Jerdee T. 1974. Influence of sex role stereotypes on personnel decisions. *Journal of Applied Psychology* 59(1):9-14.

**Exercise 2(c)** In your opinion, does the sample from this study truly represent the population you identified above?

Please write up your answer here.

## 9.4 Exploratory data analysis

Here is the data:

```
sex_discrimination
```

```
## # A tibble: 48 x 2
##   sex    decision
##   <fct> <fct>
## 1 male  promoted
## 2 male  promoted
## 3 male  promoted
## 4 male  promoted
## 5 male  promoted
## 6 male  promoted
## 7 male  promoted
## 8 male  promoted
## 9 male  promoted
## 10 male promoted
## # ... with 38 more rows
```

```
glimpse(sex_discrimination)
```

```
## Rows: 48
## Columns: 2
## $ sex      <fct> male, male, male, male, male, male, male, male, male, m-
## $ decision <fct> promoted, promoted, promoted, promoted, promoted, promoted, p-
```

**Exercise 3** Which variable is the response variable and which variable is the predictor variable?

Please write up your answer here.

---

Here is a contingency table with `decision` as the row variable and `sex` as the column variable. (Recall that we always list the response variable first. That way, the column sums will show us how many are in each of the predictor groups.)



```

tabyl(sex_discrimination, decision, sex) %>%
  adorn_totals()

```

```

##      decision male female
##      promoted   21    14
## not promoted    3    10
##           Total   24    24

```

**Exercise 4** Create another contingency table of `decision` and `sex`, this time with percentages (*not* proportions) instead of counts. You'll probably have to go back to the "Categorical data" to review the syntax. (Hint: you should have three separate `adorn` functions on the lines following the `tabyl` command.)

```

# Add code here to create a contingency table of percentages

```

---

Although we can read off the percentages in the contingency table, we need to do computations using the proportions. (Remember that we use percentages to communicate with other human beings, but we do math with proportions.) Fortunately, the output of `tabyl` is a tibble! So we can manipulate and grab the elements we need.

Let's create and store the `tabyl` output with proportions. We don't need the marginal distribution, so we can dispense with `adorn_totals`.

```

decision_sex_tabyl <- tabyl(sex_discrimination, decision, sex) %>%
  adorn_percentages("col")
decision_sex_tabyl

```

```

##      decision male   female
##      promoted 0.875 0.5833333
## not promoted 0.125 0.4166667

```

**Exercise 5** Interpret these proportions in the context of the data. In other words, what do these proportions say about the male files that were recommended for promotion versus the female files recommended for promotion?

Please write up your answer here.

---

The real statistic of interest to us is the difference between these proportions. We can use the `mutate` command from `dplyr` variable compute the difference for us.

```
decision_sex_taby1 %>%
  mutate(diff = male - female)

##      decision male   female      diff
##      promoted 0.875 0.5833333 0.2916667
## not promoted 0.125 0.4166667 -0.2916667
```

As a matter of fact, once we know the difference in promotion rates, we don't really need the individual proportions anymore. The `transmute` verb is a version of `mutate` that gives us exactly what we want. It will create a new column just like `mutate`, but then it keeps only that new column. We'll call the resulting output `decision_sex_diff`.

```
decision_sex_diff <- decision_sex_taby1 %>%
  transmute(diff = male - female)
decision_sex_diff

##      diff
## 0.2916667
## -0.2916667
```

Notice the order of subtraction: we're doing the men's rates minus the women's rates.

This computes both the difference in promotion rates (in the first row) and the difference in not-promoted rates (in the second row). Let's just keep the first row, since we care more about promotion rates. (That's our success category.) We can use `slice` to grab the first row:

```
decision_sex_diff %>%
  slice(1)
```

```
##      diff
## 0.2916667
```

This means that there is a 29% difference between the male files that were promoted and the female files that were promoted. The difference was computed as males minus females, so the fact that the number is positive means that male files were *more* likely to recommended for promotion.

One way to see if there is evidence of an association between promotion decisions and sex is to assume, temporarily, that there is no association. If there were truly no association, then the difference between the promotion rates between the male files and female files should be 0%. Of course, the number of people promoted in the data was 35, an odd number, so the number of male files promoted and female files promoted cannot be the same. Therefore, the difference in proportions can't be exactly 0 in this data. Nevertheless, we would expect—under the assumption of no association—the number of male files promoted to be *close* to the number of female files promoted, giving a difference around 0%.

So how do we test the range of values that could arise from just chance alone?  
In other words, how do we explore sampling variability?

Let's see how permuting works in R. To begin with, look at the actual values of **sex** in our data:

```
sex_discrimination$sex
```

## [1]	male	male	male	male	male	male	male	male	male	male
## [11]	male	male	male	male	male	male	male	male	male	male
## [21]	male	male	male	male	female	female	female	female	female	female
## [31]	female	female	female	female	female	female	female	female	female	female
## [41]	female	female	female	female	female	female	female	female		
## Levels:	male	female								

All the males happen to be listed first, followed by all the females.

Now we permute all the values around (using the `sample` command). As explained in an earlier chapter, we will set the seed so that our results are reproducible.

```
set.seed(3141593)
sample(sex_discrimination$sex)
```

```
## [1] male   female male   male   female female female female female female
## [11] female female female female male   male   female male   female male
## [21] female female male   male   female female male   female male   male
## [31] male   male   male   female male   female male   male   male   male
## [41] female female female male   male   male   female male
## Levels: male female
```

Do it again without the seed, just to make sure it's truly random:

```
sample(sex_discrimination$sex)
```

```
## [1] male   male   male   female male   female male   female female female
## [11] female male   female male   female female female female male   male
## [21] female female female male   male   female male   male   male   female
## [31] male   male   male   male   male   female female female female male
## [41] female female male   male   male   female female male
## Levels: male female
```

## 9.6 Randomization

The idea here is to keep the promotion status the same for each file, but randomly permute the sex labels. There will still be the same number of male and female files, but now they will be randomly matched with promoted files and not promoted files. Since this new grouping into “males” and “females” is completely random and arbitrary, we expect the likelihood of promotion to be equal for both groups.

A more precise way of saying this is that the expected difference under the assumption of independent variables is 0%. If there were truly no association, then the percentage of people promoted would be independent of sex. However, sampling variability means that we are not likely to see an exact difference of 0%. (Also, as we mentioned earlier, the odd number of promotions means the difference will never be exactly 0% anyway in this data.) The real question, then, is how different could the difference be from 0% and still be reasonably possible due to random chance.

Let's perform a few random simulations. We'll walk through the steps one line at a time. The first thing we do is permute the `sex` column:

```
set.seed(3141593)
sex_discrimination %>%
  mutate(sex = sample(sex))
```

```
## # A tibble: 48 x 2
##   sex    decision
##   <fct> <fct>
## 1 male   promoted
## 2 female promoted
## 3 male   promoted
## 4 male   promoted
## 5 female promoted
## 6 female promoted
## 7 female promoted
## 8 female promoted
## 9 female promoted
## 10 female promoted
## # ... with 38 more rows
```

Then we follow the steps from earlier, generating a contingency table with proportions. This is accomplished by simply adding two lines of code to the previous code:

```
set.seed(3141593)
sex_discrimination %>%
  mutate(sex = sample(sex)) %>%
  tabyl(decision, sex) %>%
  adorn_percentages("col")
```

```
##      decision      male    female
##      promoted 0.6666667 0.7916667
##      not promoted 0.3333333 0.2083333
```

Note that the proportions in this table are different from the ones in the real data.

Then we calculate the difference between the male and female columns by adding a line with `transmute`:

```
set.seed(3141593)
sex_discrimination %>%
```

```
mutate(sex = sample(sex)) %>%
  tabyl(decision, sex) %>%
  adorn_percentages("col") %>%
  transmute(diff = male - female)
```

```
##      diff
## -0.125
##  0.125
```

In this case, the first row happens to be negative, but that's okay. This particular random shuffling had more females promoted than males. (Remember, though, that the permuted sex labels are now meaningless.)

Finally, we grab the entry in the first row with `slice`:

```
set.seed(3141593)
sex_discrimination %>%
  mutate(sex = sample(sex)) %>%
  tabyl(decision, sex) %>%
  adorn_percentages("col") %>%
  transmute(diff = male - female) %>%
  slice(1)
```

```
##      diff
## -0.125
```

We'll repeat this code a few more times, but without the seed, to get new random observations.

```
sex_discrimination %>%
  mutate(sex = sample(sex)) %>%
  tabyl(decision, sex) %>%
  adorn_percentages("col") %>%
  transmute(diff = male - female) %>%
  slice(1)
```

```
##      diff
## 0.04166667
```

```
sex_discrimination %>%
  mutate(sex = sample(sex)) %>%
  tabyl(decision, sex) %>%
  adorn_percentages("col") %>%
  transmute(diff = male - female) %>%
  slice(1)
```

```
## diff
## 0.125
```

```
sex_discrimination %>%
  mutate(sex = sample(sex)) %>%
  tabyl(decision, sex) %>%
  adorn_percentages("col") %>%
  transmute(diff = male - female) %>%
  slice(1)
```

```
## diff
## 0.125
```

```
sex_discrimination %>%
  mutate(sex = sample(sex)) %>%
  tabyl(decision, sex) %>%
  adorn_percentages("col") %>%
  transmute(diff = male - female) %>%
  slice(1)
```

```
## diff
## -0.2916667
```

Think carefully about what these random numbers mean. Each time we randomize, we get a simulated difference in the proportion of promotions between male files and female files. The `sample` part ensures that there is no actual relationship between promotion and sex among these randomized values. We expect each simulated difference to be close to zero, but we also expect deviations from zero due to randomness and chance.

## 9.7 The infer package

The above code examples show the nuts and bolts of permuting data around to break any association that might exist between two variables. However, to do a proper randomization, we need to repeat this process many, many times (just like how we flipped thousands of “coins” in the last chapter).

Here we introduce some code from the `infer` package that will help us automate this procedure. The added benefit of introducing `infer` now is that we will continue to use it in nearly every chapter of the book that follows.

Here is the code template, starting with setting the seed:

```

set.seed(3141593)
sims <- sex_discrimination %>%
  specify(decision ~ sex, success = "promoted") %>%
  hypothesize(null = "independence") %>%
  generate(reps = 1000, type = "permute") %>%
  calculate(stat = "diff in props", order = c("male", "female"))
sims

## Response: decision (factor)
## Explanatory: sex (factor)
## Null Hypothesis: independence
## # A tibble: 1,000 x 2
##   replicate    stat
##       <int>   <dbl>
## 1         1 -0.125
## 2         2 -0.125
## 3         3 -0.0417
## 4         4  0.0417
## 5         5  0.125
## 6         6 -0.0417
## 7         7 -0.0417
## 8         8  0.125
## 9         9  0.125
## 10        10  0.208
## # ... with 990 more rows

```

We will learn more about all these lines of code in future chapters. By the end of the course, running this type of analysis will be second nature. For now, you can copy and paste the code chunk above and make minor changes as you need. Here are the three things you will need to look out for for doing this with different data sets in the future:

1. The second line (after setting the seed) will be your new data set.
2. In the `specify` line, you will have a different response variable, predictor variable, and success condition that will depend on the context of your new data.
3. In the `calculate` line, you will have two different levels that you want to compare. Be careful to list them in the order in which you want to subtract them.

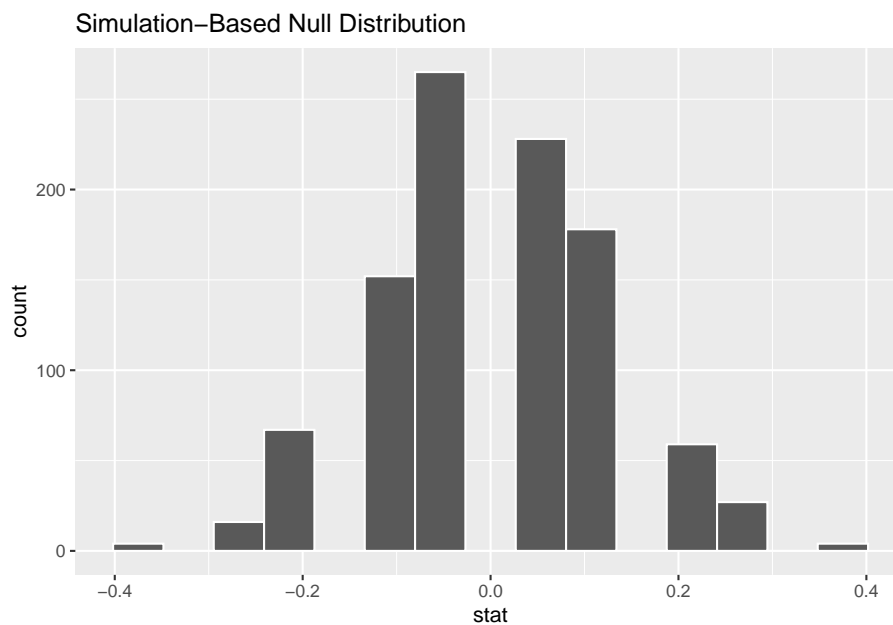
## 9.8 Plot results

A histogram will show us the range of possible values under the assumption of independence of the two variables. We can get one from our `infer` output using



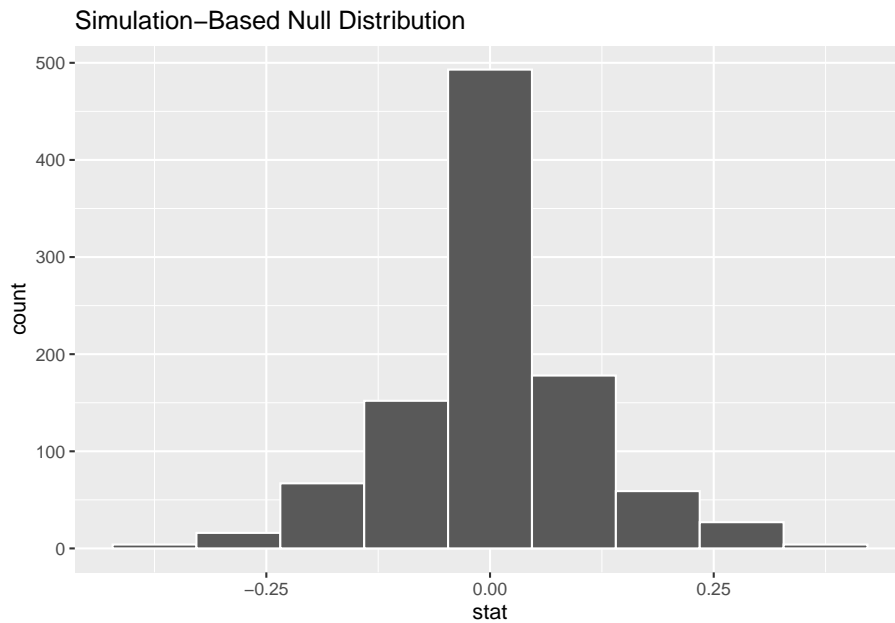
`visualize`. (This is a lot easier than building a histogram with `ggplot`!)

```
sims %>%  
  visualize()
```



The bins aren't great in the picture above. There is no way currently to set the binwidth or boundary as we've done before, but we can experiment with the total number of bins. 9 seems to be a good number.

```
sims %>%  
  visualize(bins = 9)
```



**Exercise 6** Why is the mode of the graph above at 0? This has been explained several different times in this chapter, but put it into your own words to make sure you understand the logic behind the randomization.

Please write up your answer here.

---

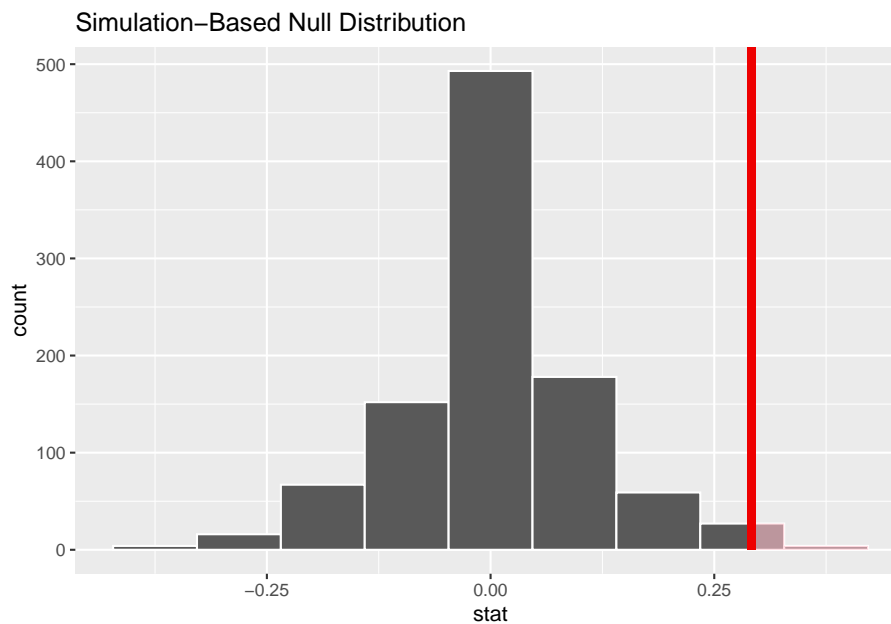
Let's compare these simulated values to the observed difference in the real data. We've computed the latter already, but let's use `infer` tools to find it. We'll give the answer a name, `obs_diff`.

```
obs_diff <- sex_discrimination %>%
  observe(decision ~ sex, success = "promoted",
           stat = "diff in props", order = c("male", "female"))
obs_diff
```

```
## Response: decision (factor)
## Explanatory: sex (factor)
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 0.292
```

Now we can graph the observed difference in the data alongside the simulated values under the assumption of independent variables. The name of the function `shade_p_value` is a little cryptic for now, but it will become clear within a few chapters.

```
sims %>%
  visualize(bins = 9) +
  shade_p_value(obs_stat = obs_diff, direction = "greater")
```



## 9.9 By chance?

How likely is it that the observed difference (or a difference even more extreme) could have resulted from chance alone? Because `sims` contains simulated results after permuting, the values in the `stat` column assume that promotion is independent of sex. In order to assess how plausible our observed difference is under that assumption, we want to find out how many of the simulated values are at least as big, if not bigger, than the observed difference, 0.292.

Look at the randomized differences sorted in decreasing order:

```
sims %>%
  arrange(desc(stat))
```

```
## Response: decision (factor)
## Explanatory: sex (factor)
## Null Hypothesis: independence
## # A tibble: 1,000 x 2
##   replicate  stat
##   <int> <dbl>
## 1      133 0.375
## 2      181 0.375
## 3      568 0.375
## 4      619 0.375
## 5       50 0.292
## 6       68 0.292
## 7       77 0.292
## 8       93 0.292
## 9      111 0.292
## 10     119 0.292
## # ... with 990 more rows
```

Of the 1000 simulations, the most extreme difference of 37.5% occurred four times, just by chance. That seems like a pretty extreme value when expecting a value of 0%, but the laws of probability tell us that extreme values will be observed from time to time, even if rarely. Also recall that the observed difference in the actual data was 29.2%. This specific value came up quite a bit in our simulated data. In fact, the 31st entry of the sorted data above is the last occurrence of the value 0.292. After that, the next higher larger value is 0.208.

So let's return to the original question. How many simulated values are as large—if not larger—than the observed difference? Apparently, 31 out of 1000, which is 0.031. In other words 3% of the simulated data is as extreme or more extreme than the actual difference in promotion rates between male files and female files in the real data. That's not very large. In other words, a difference like 29.2% could occur just by chance—like flipping 10 out of 10 heads or something like that. But it doesn't happen very often.

We can automate this calculation using the function `get_p_value` (similar to `shade_p_value` above) even though we don't yet know what “p value” means.

```
sims %>%
  get_p_value(obs_stat = obs_diff, direction = "greater")

## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1  0.031
```

**COPY/PASTE WARNING:** If the observed difference were negative, then extreme values of interest would be *less* than, say, -0.292, not greater than 0.292.

You must note if the observed difference is positive or negative and then use “greater” or “less” as appropriate!

Again, 0.031 is a small number. This shows us that if there were truly no association between promotion and sex, then our data is a rare event. (An observed difference this extreme or more extreme would only occur about 3% of the time by chance.)

Because the probability above is so small, it seems unlikely that our variables are independent. Therefore, it seems more likely that there is an association between promotion and sex. We have evidence of a statistically significant difference between the chance of getting recommended for promotion if the file indicates male versus female.

Because this is an experiment, it’s possible that a causal claim could be made. If everything in the application files was identical except the indication of gender, then it stands to reason that gender *explains* why more male files were promoted over female files. But all that depends on the experiment being a well-designed experiment.

**Exercise 7** Although we are not experts in experimental design, what concerns do you have about generalizing the results of this experiment to broad conclusions about sexism in the 1970s? (To be clear, I’m not saying that sexism wasn’t a broad problem in the 1970s. It surely was—and still is. I’m only asking you to opine as to why the results of this one study might not be conclusive in making an overly broad statement.)

Please write up your answer here.

## 9.10 Your turn

In this section, you’ll explore another famous data set related to the topic of gender discrimination. (Also from the 1970s!)

The following code will download admissions data from the six largest graduate departments at the University of California, Berkeley in 1973. We’ve seen the `read_csv` command before, but we’ve added some extra stuff in there to make sure all the columns get imported as factor variables (rather than having to convert them ourselves later).

```
ucb_admit <- read_csv("https://vectorposse.github.io/intro_stats/data/ucb_admit.csv",
                      col_types = list(
                        Admit = col_factor(),
                        Gender = col_factor(),
                        Dept = col_factor()))
```

```
ucb_admit
```

```
## # A tibble: 4,526 x 3
##   Admit    Gender Dept
##   <fct>   <fct> <fct>
## 1 Admitted Male    A
## 2 Admitted Male    A
## 3 Admitted Male    A
## 4 Admitted Male    A
## 5 Admitted Male    A
## 6 Admitted Male    A
## 7 Admitted Male    A
## 8 Admitted Male    A
## 9 Admitted Male    A
## 10 Admitted Male    A
## # ... with 4,516 more rows
```

```
glimpse(ucb_admit)
```

```
## Rows: 4,526
## Columns: 3
## $ Admit  <fct> Admitted, Admitted, Admitted, Admitted, Admitted, Admitted, Adm~
## $ Gender <fct> Male, Male, Male, Male, Male, Male, Male, Male, Male, Male, Mal~
## $ Dept   <fct> A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, A, ~
```

As you go through the exercises below, you should carefully copy and paste commands from earlier in the chapter, making the necessary changes.

**Remember that R is case sensitive!** In the `sex_discrimination` data, all the variables and levels started with lowercase letters. In the `ucb_admit` data, they all start with uppercase letters, so you'll need to be careful to change that after you copy and paste code examples from above.

**Exercise 8(a)** Is this data observational or experimental? How do you know?

Please write up your answer here.

**Exercise 8(b)** Exploratory data analysis: make two contingency tables with `Admit` as the response variable and `Gender` as the explanatory variable. One table should have counts and the other table should have percentages. (Both tables should include the marginal distribution at the bottom.)

```
# Add code here to make a contingency table with counts.
```

```
# Add code here to make a contingency table with percentages.
```

**Exercise 8(c)** Use `observe` from the `infer` package to calculate the observed difference in proportions between males who were admitted and females who were admitted. Do the subtraction in that order: males minus females. Store your output as `obs_diff2` so that it doesn't overwrite the variable `obs_diff` we created earlier.

```
# Add code here to calculate the observed difference.  
# Store this as obs_diff2.
```

**Exercise 8(d)** Simulate 1000 outcomes under the assumption that admission is independent of gender. Use the `specify`, `hypothesize`, `generate`, and `calculate` sequence from the `infer` package as above. Call the simulated data frame `sims2` so that it doesn't conflict with the earlier `sims`. Don't touch the `set.seed` command. That will ensure that all students get the same randomization.

```
set.seed(10101)  
# Add code here to simulate 1000 outcomes  
# under the independence assumption  
# and store the simulations in a data frame called sims2.
```

**Exercise 8(e)** Plot the simulated values in a histogram using the `visualize` verb from `infer`. When you first run the code, remove the `bins = 9` we had earlier and let `visualize` choose the number of bins. If you are satisfied with the graph, you don't need to specify a number of bins. If you are not satisfied, you can experiment with the number of bins until you find a number that seems reasonable.

Be sure to include a vertical line at the value of the observed difference using the `shade_p_value` command. Don't forget that the location of that line is `obs_diff2` now.

```
# Add code here to plot the results.
```

**Exercise 8(f)** Finally, comment on what you see. Based on the histogram above, is the observed difference in the data rare? In other words, under the assumption that admission and gender are independent, are we likely to see an observed difference as far away from zero as we actually see in the data? So

what is your conclusion then? Do you believe there was an association between admission and gender in the UC Berkeley admissions process in 1973?

Please write up your answer here.

## 9.11 Simpson’s paradox

The example above from UC Berkeley seems like an open and shut case. Male applicants were clearly admitted at a greater rate than female applicants. While we never expect the application rates to be *exactly* equal—even under the assumption that admission and gender are independent—the randomization exercise showed us that the observed data was *way* outside the range of possible differences that could have occurred just by chance.

But we also know this is observational data. Association is not causation.

**Exercise 9** Note that we didn’t say “correlation is not causation”. The latter is also true, but why does it not apply in this case? (Think about the conditions for correlation.)

Please write up your answer here.

---

Since we don’t have data from a carefully controlled experiment, we always have to be worried about lurking variables. Could there be a third variable apart from admission and gender that could be driving the association between them? In other words, the fact that males were admitted at a higher rate than females might be sexism, or it might be spurious.

Since we have access to a third variable, `Dept`, let’s analyze it as well. The `tabyl` command will happily take a third variable and create a *set* of contingency tables, one for each department.

Here are the tables with counts:

```
tabyl(ucb_admit, Admit, Gender, Dept) %>%
  adorn_totals()
```

```
## $A
##      Admit Male Female
## Admitted  512     89
## Rejected  313     19
##      Total  825    108
##
```



```
## $B
##      Admit Male Female
## Admitted 353     17
## Rejected 207     8
##      Total 560    25
##
## $C
##      Admit Male Female
## Admitted 120    202
## Rejected 205    391
##      Total 325    593
##
## $D
##      Admit Male Female
## Admitted 138    131
## Rejected 279    244
##      Total 417    375
##
## $E
##      Admit Male Female
## Admitted  53     94
## Rejected 138    299
##      Total 191    393
##
## $F
##      Admit Male Female
## Admitted  22     24
## Rejected 351    317
##      Total 373    341
```

And here are the tables with percentages:

```
tabyl(ucb_admit, Admit, Gender, Dept) %>%
  adorn_totals() %>%
  adorn_percentages("col") %>%
  adorn_pct_formatting()
```

```
## $A
##      Admit   Male Female
## Admitted 62.1% 82.4%
## Rejected 37.9% 17.6%
##      Total 100.0% 100.0%
##
## $B
##      Admit   Male Female
```

```

## Admitted 63.0% 68.0%
## Rejected 37.0% 32.0%
## Total 100.0% 100.0%
##
## $C
## Admit Male Female
## Admitted 36.9% 34.1%
## Rejected 63.1% 65.9%
## Total 100.0% 100.0%
##
## $D
## Admit Male Female
## Admitted 33.1% 34.9%
## Rejected 66.9% 65.1%
## Total 100.0% 100.0%
##
## $E
## Admit Male Female
## Admitted 27.7% 23.9%
## Rejected 72.3% 76.1%
## Total 100.0% 100.0%
##
## $F
## Admit Male Female
## Admitted 5.9% 7.0%
## Rejected 94.1% 93.0%
## Total 100.0% 100.0%

```

**Exercise 10** Look at the contingency tables with percentages. Examine each department individually. What do you notice about the admit rates (as percentages) between males and females for most of the departments listed? Identify the four departments where female admission rates were higher than male admission rates.

Please write up your answer here.

---

This is completely counterintuitive. How can males be admitted at a higher rate overall, and yet in most departments, females were admitted at a higher rate.

This phenomenon is often called *Simpson's Paradox*. Like almost everything in statistics, this is named after a person (Edward H. Simpson) who got the popular credit for writing about the phenomenon, but not being the person who actually discovered the phenomenon. (There does not appear to be a primeval reference

for the first person to have studied it. Similar observations had appeared in various sources more than 50 years before Simpson wrote his paper.)

**Exercise 11** Look at the contingency tables with counts. Focus on the four departments you identified above. What is true of the total number of male and female applicants for those four department (and not for the other two departments)?

Please write up your answer here.

**Exercise 12(a)** Now create a contingency table with percentages that uses `Admit` for the row variable and `Dept` as the column variable.

```
# Add code here to create a contingency table with percentages
# for Dept and Admit
```

**Exercise 12(b)** In the contingency table above, what's true about the admission rates for the four departments you identified above (and not true for the other two department)?

Please write up your answer here.

---

Your work in the previous exercises begins to paint a picture that explains what's going on with this "paradox". Males applied in greater numbers to departments with high acceptance rates. As a result, more male students overall got in to graduate school. Females applied in greater numbers to departments that were more selective. Overall, then, fewer females got in to graduate school. But on a department-by-department basis, female applicants were usually more likely to get accepted.

None of this suggests that sexism fails to exist. It doesn't even prove that sexism wasn't a factor in some departmental admission procedures. What it does suggest is that when we don't take into account possible lurking variables, we run the risk of oversimplifying issues that are potentially complex.

In our analysis of the UC Berkeley data, we've exhausted all the variables available to us in the data set. There remains the potential for *unmeasured confounders*, or variables that could still act as lurking variables, but we have no idea about them because they aren't in our data. This is an unavoidable peril of working with observational data. If we aren't careful to "control" for a reasonable set of possible lurking variables, we must be very careful when trying to make broad conclusions.

## 9.12 Conclusion

Here we used randomization to explore the idea of two variables being independent or associated. When we assume they are independent, we can explore the sampling variability of the differences that could occur by pure chance alone. We expect the difference to be zero, but we know that randomness will cause the simulated differences to have a range of values. Is the difference in the observed data far away from zero? In that case, we can say we have evidence that the variables are not independent; in other words, it is more likely that our variables are associated.

### 9.12.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1–2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.

## Chapter 10

# Hypothesis testing with randomization, Part 1

2.0

**Functions introduced in this chapter**

`drop_na`, `pull`

### 10.1 Introduction

Using a sample to deduce something about a population is called “statistical inference”. In this chapter, we’ll learn about one form of statistical inference called “hypothesis testing”. The focus will be on walking through the example from Part 2 of “Introduction to randomization” and recasting it here as a formal hypothesis test.

There are no new R commands here, but there are many new ideas that will require careful reading. You are not expected to be an expert on hypothesis testing after this one chapter. However, within the next few chapters, as we learn more about hypothesis testing and work through many more examples, the hope is that you will begin to assimilate and internalize the logic of inference and the steps of a hypothesis test.

#### 10.1.1 Install new packages

There are no new packages used in this chapter.

### 10.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/10-hypothesis\\_testing\\_with\\_randomization.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/10-hypothesis_testing_with_randomization.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 10.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 10.2 Load packages

We load `tidyverse` and `janitor`. We'll continue to explore the `infer` package for investigating statistical claims. We load the `openintro` package to access the `sex_discrimination` data (the one with the male bank managers promoting male files versus female files).

```
library(tidyverse)
library(janitor)
library(infer)
library(openintro)
```

## 10.3 Our research question

We return to the sex discrimination experiment from the last chapter. We are interested in finding out if there is an association between the recommendation to promote a candidate for branch manager and the gender listed on the file being evaluated by the male bank manager.

## 10.4 Hypothesis testing

The approach we used in Part 2 of “Introduction to randomization” was to assume that the two variables `decision` and `sex` were independent. From that assumption, we were able to compare the observed difference in promotion percentages between males and females from the actual data to the distribution of random values obtained by randomization. When the observed difference was

far enough away from zero, we concluded that the assumption of independence was probably false, giving us evidence that the two variables were associated after all.

This logic is formalized into a sequence of steps known as a *hypothesis test*. In this section, we will introduce a rubric for conducting a full and complete hypothesis test for the sex discrimination example. (This rubric also appears in the Appendix. If you need the rubric as a file, you can also download copies either as an `.Rmd` file [here](#) or as an `.nb.html` file [here](#).)

A hypothesis test can be organized into five parts:

1. Exploratory data analysis
2. Hypotheses
3. Model
4. Mechanics
5. Conclusion

Below, I'll address each of these steps.

### 10.4.1 Exploratory data analysis

Before we can answer questions using data, we need to understand our data.

Most data sets come with some information about the provenance and structure of the data. (Often this is called “metadata”.) Data provenance is the story of how the data was collected and for what purpose. Together with some information about the types of variables recorded, this is the who, what, when, where, why, and how. Without context, data is just a bunch of letters and numbers. You must understand the nature of the data in order to use the data. Information about the structure of the data is often recorded in a “code book”.

For data that you collect yourself, you'll already know all about it, although should probably write that stuff down in case other people want to use your data (or in case “future you” wants to use the data). For other data sets, you hope that other people have recorded information about how the data was collected and what is described in the data. When working with data sets in R as we do for these chapters, we've already seen that there are help files—sometimes more or less helpful. In some cases, you'll need to go beyond the brief explanations in the help file to investigate the data provenance. And for files we download from other places on the internet, we may have a lot of work to do.

**Exercise 1** What are some ethical issues you might want to consider when looking into the provenance of data? Have a discussion with a classmate and/or

do some internet sleuthing to see if you can identify one or two key issues that should be considered before you access or analyze data.

Please write up your answer here.

For exploring the raw data in front of us, we can use commands like `View` from the Console to see the data in spreadsheet form, although if we're using R Notebooks, we can just type the name of the data frame in a code chunk and run it to print the data in a form we can navigate and explore. There is also `glimpse` to explore the structure of the data (the variables and how they're coded), as well as other summary functions to get a quick sense of the variables.

Sometimes you have to prepare your data for analysis. A common example is converting categorical variables that should be coded as factor variables, but often are coded as character vectors, or are coded numerically (like "1" and "0" instead of "Yes" and "No"). Sometimes missing data is coded unusually (like "999") and that has to be fixed before trying to calculate statistics. "Cleaning" data is often a task that takes more time than analyzing it!

Finally, once the data is in a suitably tidy form, we can use visualizations like tables, graphs, and charts to understand the data better. Often, there are conditions about the shape of our data that have to be met before inference is appropriate, and this step can help diagnose problems that could arise in the inferential procedure. This is a good time to look for outliers, for example.

### 10.4.2 Hypotheses

We are trying to ask some question about a population of interest. However, all we have in our data is a sample of that population. The word inference comes from the verb "infer": we are trying to infer what might be true of a population just from examining a sample. It's also possible that our question involves comparing two or more populations to each other. In this case, we'll have multiple samples, one from each of our populations. For example, in our sex discrimination example, we are comparing two populations: male bank managers who consider male files for promotion, and male bank managers who consider female files for promotion. Our data gives us two samples who form only a part of the larger populations of interest.

To convince our audience that our analysis is correct, it makes sense to take a skeptical position. If we are trying to prove that there is an association between promotion and sex, we don't just declare it to be so. We start with a "null hypothesis", or an expression of the belief that there is no association. A null hypothesis always represents the "default" position that a skeptic might take. It codifies the idea that "there's nothing to see here."

Our job is to gather evidence to show that there is something interesting going on. The statement of interest to us is called the "alternative hypothesis". This is usually the thing we're trying to prove related to our research question.



We can perform *one-sided* tests or *two-sided* tests. A one-sided test is when we have a specific direction in mind for the effect. For example, if we are trying to prove that male files are *more* likely to be promoted than female files, then we would perform a one-sided test. On the other hand, if we only care about proving an association, then male files could be either more likely or less likely to be promoted than female files. (This is contrasted to the null that states that male files are *equally* likely to be promoted as female files.) If it seems weird to run a two-sided test, keep in mind that we want to give our statistical analysis a chance to prove an association regardless of the direction of the association. Wouldn't you be interested to know if it turned out that male files are, in fact, *less* likely to be promoted?

You can't cheat and look at the data first. In a normal research study out there in the real world, you develop hypotheses long before you collect data. So you have to decide to do a one-sided or two-sided test before you have the luxury of seeing your data pointing in one direction or the other.

Running a two-sided test is often a good default option. Again, this is because our analysis will allow us to show interesting effects in any direction.

We typically express hypotheses in two ways. First, we write down full sentences that express in the context of the problem what our null and alternative hypotheses are stating. Then, we express the same ideas as mathematical statements. This translation from words to math is important as it gives us the connection to the quantitative statistical analysis we need to perform. The null hypothesis will always be that some quantity is equal to ( $=$ ) the null value. The alternative hypothesis depends on whether we are conducting a one-sided test or a two-sided test. A one-sided test is mathematically saying that the quantity of interest is either greater than ( $>$ ) or less than ( $<$ ) the null value. A two-sided test always states that the quantity of interest is not equal to ( $\neq$ ) the null value. (Notice the math symbol enclosed in dollar signs in the previous sentence. In the HTML file, these symbols will appear correctly. In the R Notebook, you can hover the cursor anywhere between the dollar signs and the math symbol will show up. Alternatively, you can click somewhere between the dollar signs and hit Ctrl-Enter or Cmd-Enter, just like with inline R code.)

The most important thing to know is that the entire hypothesis test up until you reach the conclusion is conducted **under the assumption that the null hypothesis is true**. In other words, we pretend the whole time that our alternative hypothesis is false, and we carry out our analysis working under that assumption. This may seem odd, but it makes sense when you remember that the goal of inference is to try to convince a skeptic. Others will only believe your claim after you present evidence that suggests that the data is inconsistent with the claims made in the null.

### 10.4.3 Model

A model is an approximation—usually a simplification—of reality. In a hypothesis test, when we say “model” we are talking specifically about the “null model”. In other words, what is true about the population under the assumption of the null? If we sample from the population repeatedly, we find that there is some kind of distribution of values that can occur by pure chance alone. This is called the *sampling distribution model*. We have been learning about how to use randomization to understand the sampling distribution and how much sampling variability to expect, even when the null hypothesis is true.

Building a model is contingent upon certain assumptions being true. We cannot usually demonstrate directly that these assumptions are conclusively met; however, there are often conditions that can be checked with our data that can give us some confidence in saying that the assumptions are probably met. For example, there is no hope that we can infer anything from our sample unless that sample is close to a random sample of the population. There is rarely any direct evidence of having a properly random sample, and often, random samples are too much to ask for. There is almost never such a thing as a truly random sample of the population. Nevertheless, it is up to us to make the case that our sample is as representative of the population as possible. Additionally, we have to know that our sample comprises less than 10% of the size of the population. The reasons for this are somewhat technical and the 10% figure is just a rough guideline, but we should think carefully about this whenever we want our inference to be correct.

Those are just two examples. For the randomization tests we are running, those are the only two conditions we need to check. For other hypothesis tests in the future that use different types of models, we will need to check more conditions that correspond to the modeling assumptions we will need to make.

### 10.4.4 Mechanics

This is the nitty-gritty, nuts-and-bolts part of a hypothesis test. Once we have a model that tells us how data should behave under the assumption of the null hypothesis, we need to check how our data actually behaved. The measure of where our data is relative to the null model is called the *test statistic*. For example, if the null hypothesis states that there should be a difference of zero between promotion rates for males and females, then the test statistic would be the actual observed difference in our data between males and females.

Once we have a test statistic, we can plot it in the same graph as the null model. This gives us a visual sense of how rare or unusual our observed data is. The further our test statistic is from the center of the null model, the more evidence we have that our data would be very unusual if the null model were true. And that, in turn, gives us a reason not to believe the null model. When conducting

a two-sided test, we will actually graph locations on both side of the null value: the test statistic on one side of the null value and a point the same distance on the other side of the null value. This will acknowledge that we're interested in evidence of an effect in either direction.

Finally, we convert the visual evidence explained in the previous paragraph to a number called a *P-value*. This measures how likely it is to see our observed data—or data even more extreme—under the assumption of the null. A small P-value, then, means that if the null were really true, we wouldn't be very likely at all to see data like ours. That leaves us with little confidence that the null model is really true. (After all, we *did* see the data we gathered!) If the P-value is large—in other words, if the test statistic is closer to the middle of the null distribution—then our data is perfectly consistent with the null hypothesis. That doesn't mean the null is true, but it certainly does not give us evidence against the null.

A one-sided test will give us a P-value that only counts data more extreme than the observed data in the direction that we explicitly hypothesized. For example, if our alternative hypothesis was that male files are more likely to be promoted, then we would only look at the part of the model that showed differences with as many or more male promotions as our data showed. A two-sided P-value, by contrast, will count data that is extreme in either direction. This will include values on both sides of the distribution, which is why it's called a two-sided test. Computationally, it is usually easiest to calculate the one-sided P-value and just double it.<sup>1</sup>

Remember the statement made earlier that throughout the hypothesis testing process, **we work under the assumption that the null hypothesis is true**. The P-value is no exception. It tells us **under the assumption of the null** how likely we are to see data at least as extreme (if not even more extreme) as the data we actually saw.

### 10.4.5 Conclusion

The P-value we calculate in the Mechanics section allows us to determine what our decision will be relative to the null hypothesis. As explained above, when the P-value is small, that means we had data that would be very unlikely had the null been true. The sensible conclusion is then to “reject the null hypothesis.” On the other hand, if the data is consistent with the null hypothesis, then we “fail to reject the null hypothesis.”

How small does the P-value need to be before we are willing to reject the null hypothesis? That is a decision we have to make based on how much we are willing to risk an incorrect conclusion. A value that is widely used is 0.05; in other words, if  $P < 0.05$  we reject the null, and if  $P > 0.05$ , we fail to reject the

---

<sup>1</sup>This is not technically the most mathematically appropriate thing to do, but it's a reasonable approximation in many common situations.

null. However, for situations where we want to be conservative, we could choose this threshold to be much smaller. If we insist that the P-value be less than 0.01, for example, then we will only reject the null when we have a lot more evidence. The threshold we choose is called the “significance level”, denoted by the Greek letter alpha:  $\alpha$ . The value of  $\alpha$  must be chosen long before we compute our P-value so that we’re not tempted to cheat and change the value of  $\alpha$  to suit our P-value (and by doing so, quite literally, move the goalposts).

**Note that we never accept the null hypothesis.** The hypothesis testing procedure gives us no evidence in favor of the null. All we can say is that the evidence is either strong enough to warrant rejection of the null, or else it isn’t, in which case we can conclude nothing. If we can’t prove the null false, we are left not knowing much of anything at all.

The phrases “reject the null” or “fail to reject the null” are very statsy. Your audience may not be statistically trained. Besides, the *real* conclusion you care about concerns the research question of interest you posed at the beginning of this process, and that is built into the alternative hypothesis, not the null. Therefore, we need some statement that addresses the alternative hypothesis in words that a general audience will understand. I recommend the following templates:

- When you reject the null, you can safely say, “We have sufficient evidence that [restate the alternative hypothesis].”
- When you fail to reject the null, you can safely say, “We have insufficient evidence that [restate the alternative hypothesis].”

The last part of your conclusion should be an acknowledgement of the uncertainty in this process. Statistics tries to tame randomness, but in the end, randomness is always somewhat unpredictable. It is possible that we came to the wrong conclusion, not because we made mistakes in our computation, but because statistics just can’t be right 100% of the time when randomness is involved. Therefore, we need to explain to our audience that we may have made an error.

A *Type I* error is what happens when the null hypothesis is actually true, but our procedure rejects it anyway. This happens when we get an unrepresentative extreme sample for some reason. For example, perhaps there really is no association between promotion and sex. Even if that were true, we could accidentally survey a group of bank managers who—by pure chance alone—happen to recommend promotion more often for the male files. Our test statistic will be “accidentally” far from the null value, and we will mistakenly reject the null. Whenever we reject the null, we are at risk of making a Type I error. Given that we are conclusively stating a statistically significant finding, if that finding is wrong, this is a *false positive*, a term that is synonymous with a Type I error. The significance level  $\alpha$  discussed above is, in fact, the probability of making

a Type I error. (If the null is true, we will still reject the null if our P-value happens to be less than  $\alpha$ .)

On the other hand, the null may actually be false, and yet, we may not manage to gather enough evidence to disprove it. This can also happen due to an unusual sample—a sample that doesn’t conform to the “truth”. But there are other ways this can happen as well, most commonly when you have a small sample size (which doesn’t allow you to prove much of anything at all) or when the effect you’re trying to measure exists, but is so small that it is hard to distinguish from no effect at all (which is what the null postulates). In these cases, we are at risk of making a *Type II* error. Anytime we say that we fail to reject the null, we have to worry about the possibility of making a Type II error, also called a *false negative*.

## 10.5 Example

Below, we’ll model the process of walking through a complete hypothesis test, showing how we would address each step. Then, you’ll have a turn at doing the same thing for a different question. Unless otherwise stated, we will always assume a significance level of  $\alpha = 0.05$ . (In other words, we will reject the null if our computed P-value is less than 0.05, and we will fail to reject the null if our P-value is greater than or equal to 0.05.)

Note that there is some mathematical formatting. As mentioned before, this is done by enclosing such math in dollar signs. Don’t worry too much about the syntax; just mimic what you see in the example.

## 10.6 Exploratory data analysis

### 10.6.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

You can look at the help file by typing `?sex_discrimination` at the Console. (However, do not put that command here in a code chunk. The R Notebook has no way of displaying a help file when it’s processed.) You can also type that into the Help tab in the lower-right panel in RStudio.

The help file doesn’t say too much, but there is a “Source” at the bottom. We can do an internet search for “Rosen Jerdee Influence of sex role stereotypes on personnel decisions”. As many academics articles on the internet are, this one is pay-walled, so we can’t read it for free. If you go to school or work for an institution with a library, though, you may be able to access articles through your library services. Talk to a librarian if you’d like to access research articles.

As long as you have the citation details, librarians can often track down articles, and many are already accessible through library databases.

In this case, we can read the abstract for free. This tells us that the data we have is only one part of a larger set of experiments done.

This is also the place to comment on any ethical concerns you may have. For example, how was the data collected? Did the researchers follow ethical guidelines in the treatment of their subjects, like obtaining consent? Without accessing the full article, it's hard to know in this case. But do your best in each data analysis task you have to try to find out as much as possible about the data.

In this section, we'll also print the data set and use `glimpse` to summarize the variables.

```
sex_discrimination
```

```
## # A tibble: 48 x 2
##   sex    decision
##   <fct> <fct>
## 1 male  promoted
## 2 male  promoted
## 3 male  promoted
## 4 male  promoted
## 5 male  promoted
## 6 male  promoted
## 7 male  promoted
## 8 male  promoted
## 9 male  promoted
## 10 male promoted
## # ... with 38 more rows
```

```
glimpse(sex_discrimination)
```

```
## Rows: 48
## Columns: 2
## $ sex      <fct> male, male, male, male, male, male, male, male, male, male, m~
## $ decision <fct> promoted, promoted, promoted, promoted, promoted, promoted, p~
```

### 10.6.2 Prepare the data for analysis.

In this section, we do any tasks required to clean the data. This will often involve using `mutate`, either to convert other variable types to factors, or compute additional variables using existing columns. It may involve using `filter` to analyze only one part of the data we care about.

If there is missing data, this is the place to identify it and decide if you need to address it before starting your analysis. It's always important to check for missing data. It's not always necessary to address it now as many of the R functions we use will ignore rows with missing data.

The easiest way to detect missing data is to try deleting rows that are missing some data with `drop_na` and see if the number of rows changes:

```
sex_discrimination %>%
  drop_na()
```

```
## # A tibble: 48 x 2
##   sex    decision
##   <fct> <fct>
## 1 male   promoted
## 2 male   promoted
## 3 male   promoted
## 4 male   promoted
## 5 male   promoted
## 6 male   promoted
## 7 male   promoted
## 8 male   promoted
## 9 male   promoted
## 10 male  promoted
## # ... with 38 more rows
```

Since the result still has 48 rows, there are no missing values.

The `sex_discrimination` data is already squeaky clean, so we don't need to do anything here.

### 10.6.3 Make tables or plots to explore the data visually.

As we have two categorical variables, a contingency table is a good way of visualizing the distribution of both variables together. (Don't forget to include the marginal distribution and create two tables: one with counts and one with percentages!)

```
tabyl(sex_discrimination, decision, sex) %>%
  adorn_totals()
```

```
##      decision male female
##      promoted   21    14
## not promoted    3     10
##           Total   24    24
```

```

tabyl(sex_discrimination, decision, sex) %>%
  adorn_totals() %>%
  adorn_percentages("col") %>%
  adorn_pct_formatting()

```

```

##      decision   male female
##      promoted  87.5%  58.3%
## not promoted  12.5%  41.7%
##              Total 100.0% 100.0%

```

## 10.7 Hypotheses

### 10.7.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

There are technically two samples of interest here. All the data comes from a group of 48 bank managers recruited for the study, but one group of interest are bank managers who are evaluating male files, and the other group of interest are bank managers who are evaluating female files.

One of the contingency tables above shows the sample sizes for each group in the marginal distribution along the bottom of the table (i.e., the column sums). There are 24 managers with male files and 24 managers with female files.

The populations of interest are probably all bank managers evaluating male candidates and all bank managers evaluating female candidates, probably only in the U.S. (where the two researchers were based) and only during the 1970s.

### 10.7.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

(Note: The null hypothesis is indicated by the symbol  $H_0$ , often pronounced “H naught” or “H sub zero.” The alternative hypothesis is indicated by  $H_A$ , pronounced “H sub A.”)

$H_0$  : There is no association between decision and sex in hiring branch managers for banks in the 1970s.

$H_A$  : There is an association between decision and sex in hiring branch managers for banks in the 1970s.



### 10.7.3 Express the null and alternative hypotheses in symbols (when possible).

$$H_0 : p_{\text{promoted,male}} - p_{\text{promoted,female}} = 0$$

$$H_A : p_{\text{promoted,male}} - p_{\text{promoted,female}} \neq 0$$

Note: First, pay attention to the “success” condition (in this case, “promoted”). We could choose to measure either those promoted or those not promoted. The difference will be positive for one and negative for the other, so it really doesn’t matter which one we choose. Just make a choice and be consistent. Also pay close attention here to the order of the subtraction. Again, while it doesn’t matter conceptually, we need to make sure that the code we include later agrees with this order.

## 10.8 Model

### 10.8.1 Identify the sampling distribution model.

We will randomize to simulate the sampling distribution.

### 10.8.2 Check the relevant conditions to ensure that model assumptions are met.

- Random (for both groups)
  - We have no evidence that these are random samples of bank managers. We hope that they are representative. If the populations of interest are all bank managers in the U.S. evaluating either male candidates or female candidates, then we have some doubts as to how representative these samples are. It is likely that the bank managers were recruited from limited geographic areas based on the location of the researchers, and we know that geography could easily be a confounder for sex discrimination (because some areas of the country might be more prone to it than others). Despite our misgivings, we will proceed on with the analysis, but we will temper our expectations for grand, sweeping conclusions.
- 10% (for both groups)
  - Regardless of the intended populations, 24 bank managers evaluating male files and 24 bank managers evaluating female files are surely less than 10% of all bank managers under consideration.

## 10.9 Mechanics

### 10.9.1 Compute the test statistic.

We let `infer` do the work here:

```
obs_diff <- sex_discrimination %>%
  observe(decision ~ sex, success = "promoted",
          stat = "diff in props", order = c("male", "female"))
obs_diff

## Response: decision (factor)
## Explanatory: sex (factor)
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 0.292
```

Note: `obs_diff` is a tibble, albeit a small one, having only one column and one row. That tibble is what we need to feed into the visualization later. However, for reporting the value by itself, we have to pull it out of the tibble. We will do this below using the `pull` function. See the inline code in the next subsection.

### 10.9.2 Report the test statistic in context (when possible).

The observed difference in the proportion of promotion recommendations for male files versus female files is 0.2916667 (subtracting males minus females). Or, another way to say this: there is a 29.1666667% difference in the promotion rates between male files and female files.

### 10.9.3 Plot the null distribution.

Note: In this section, we will use the series of verbs from `infer` to generate all the information we need about the hypothesis test. We call that output `decision_sex_test` here, but you'll want to change it to another name for a different test. The recommended pattern is `response_predictor_test`.

Don't forget to set the seed. We are using randomization to permute the values of the predictor variable in order to break any association that might exist in the data. This will allow us to explore the sampling distribution created under the assumption of the null hypothesis.

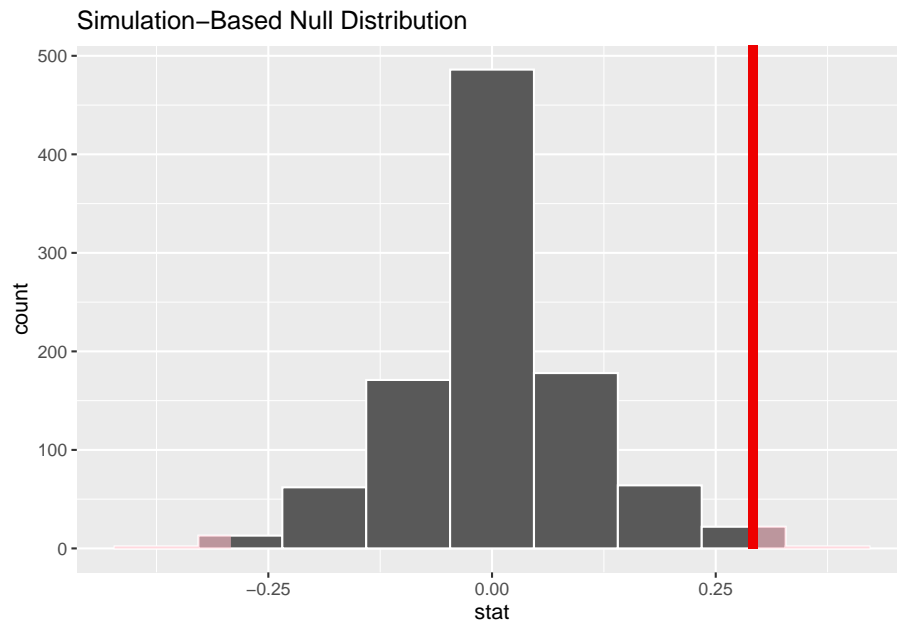
When you get to the `visualize` step, leave the number of bins out. (Just type `visualize()` with empty parentheses.) If you determine that the default

binning is not optimal, you can add back `bins` and experiment with the number. We know from the previous chapter that 9 bins is good here.

```
set.seed(9999)
decision_sex_test <- sex_discrimination %>%
  specify(decision ~ sex, success = "promoted") %>%
  hypothesize(null = "independence") %>%
  generate(reps = 1000, type = "permute") %>%
  calculate(stat = "diff in props", order = c("male", "female"))
decision_sex_test
```

```
## Response: decision (factor)
## Explanatory: sex (factor)
## Null Hypothesis: independence
## # A tibble: 1,000 x 2
##   replicate    stat
##       <int>   <dbl>
## 1         1 -0.0417
## 2         2  0.208
## 3         3  0.0417
## 4         4 -0.125
## 5         5 -0.0417
## 6         6 -0.208
## 7         7 -0.208
## 8         8  0.0417
## 9         9 -0.292
## 10        10  0.125
## # ... with 990 more rows
```

```
decision_sex_test %>%
  visualize(bins = 9) +
  shade_p_value(obs_stat = obs_diff, direction = "two-sided")
```



(You'll note that there is light gray shading in *both* tails above. This is because we are conducting a two-sided test, which means that we're interested in values that are more extreme than our observed difference in *both* directions.)

#### 10.9.4 Calculate the P-value.

```
P <- decision_sex_test %>%
  get_p_value(obs_stat = obs_diff, direction = "two-sided")
P
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1    0.048
```

Note: as with the test statistic above, the P-value appears above in a 1x1 tibble. That's fine for this step, but in the inline code below, we will need to use `pull` again to extract the value.

### 10.9.5 Interpret the P-value as a probability given the null.

The P-value is 0.048. If there were no association between decision and sex, there would be a 4.8% chance of seeing data at least as extreme as we saw.

Some important things here:

1. We include an interpretation for our P-value. Remember that the P-value is the probability—**under the assumption of the null hypothesis**—of seeing results as extreme or even more extreme than the data we saw.
2. The P-value is less than 0.05 (just barely). Remember that as we talk about the conclusion in the next section of the rubric.

## 10.10 Conclusion

### 10.10.1 State the statistical conclusion.

We reject the null hypothesis.

### 10.10.2 State (but do not overstate) a contextually meaningful conclusion.

There is sufficient evidence to suggest that there is an association between decision and sex in hiring branch managers for banks in the 1970s.

Note: the easiest thing to do here is just restate the alternative hypothesis. If we reject the null, then we have *sufficient* evidence for the alternative hypothesis. If we fail to reject the null, we have *insufficient* evidence for the alternative hypothesis. Either way, though, this contextually meaningful conclusion is all about the alternative hypothesis.

### 10.10.3 Express reservations or uncertainty about the generalizability of the conclusion.

We have some reservations about how generalizable this conclusion is due to the fact that we are lacking information about how representative our samples of bank managers were. We also point out that this experiment was conducted in the 1970s, so its conclusions are not valid for today.

Note: This would also be the place to point out any possible sources of bias or confounding that might be present, especially for observational studies.

#### 10.10.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.

As we rejected the null, we run the risk of committing a Type I error. It is possible that there is no association between decision and sex, but we've come across a sample in which male files were somehow more likely to be recommended for promotion.

---

After writing up your conclusions and acknowledging the possibility of a Type I or Type II error, the hypothesis test is complete. (At least for now. In the future, we will add one more step of computing a confidence interval.)

### 10.11 More on one-sided and two-sided tests

I want to emphasize again the difference between conducting a one-sided versus a two-sided test. You may recall that in “Introduction to simulation, Part 2”, we calculated this:

```
set.seed(9999)
sex_discrimination %>%
  specify(decision ~ sex, success = "promoted") %>%
  hypothesize(null = "independence") %>%
  generate(reps = 1000, type = "permute") %>%
  calculate(stat = "diff in props", order = c("male", "female")) %>%
  get_p_value(obs_stat = obs_diff, direction = "greater")

## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1    0.024
```

The justification was that, back then, we already suspected that male files were more likely to be promoted, and it appears that our evidence (the test statistic, or our observed difference) was pretty far in that direction. (Actually, we may get a slightly different number each time. Remember that we are randomizing. Therefore, we won't expect to get the exact same numbers each time.)

By way of contrast, in this chapter we computed the two-sided P-value:

```

set.seed(9999)
sex_discrimination %>%
  specify(decision ~ sex, success = "promoted") %>%
  hypothesize(null = "independence") %>%
  generate(reps = 1000, type = "permute") %>%
  calculate(stat = "diff in props", order = c("male", "female")) %>%
  get_p_value(obs_stat = obs_diff, direction = "two-sided")

## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1    0.048

```

The only change to the code is the word “two-sided” (versus “greater”) in the last line.

Our P-value in this chapter is twice as large as it could have been if we had run a one-sided test.

Doubling the P-value might mean that it no longer falls under the significance threshold  $\alpha = 0.05$  (although in this case, we still came in under 0.05). This raises an obvious question: why use two-sided tests at all? If the P-values are higher, that makes it less likely that we will reject the null, which means we won’t be able to prove our alternative hypothesis. Isn’t that a bad thing?

As a matter of fact, there are many researchers in the world who do think it’s a bad thing, and routinely do things like use one-sided tests to give them a better chance of getting small P-values. But this is not ethical. The point of research is to do good science, not prove your pet theories correct. There are many incentives in the world for a researcher to prove their theories correct (money, awards, career advancement, fame and recognition, legacy, etc.), but these should be secondary to the ultimate purpose of advancing knowledge. Sadly, many researchers out there have these priorities reversed. I do not claim that researchers set out to cheat; I suspect that the vast majority of researchers act in good faith. Nevertheless, the rewards associated with “successful” research cause cognitive biases that are hard to overcome. And “success” is often very narrowly defined as research that produces small P-values.

A better approach is to be conservative. For example, a two-sided test is not only more conservative because it produces higher P-values, but also because it answers a more general question. That is, it is scientifically interesting when an association goes in either direction (e.g. more male promotions, but also possibly more female promotions). This is why we recommended above using two-sided tests by default, and only using a one-sided test when there is a very strong research hypothesis that justifies it.

## 10.12 A reminder about failing to reject the null

It's also important to remember that when we fail to reject the null hypothesis, we are not saying that the null hypothesis is true. Neither are we saying it's false. Failure to reject the null is really a failure to conclude anything at all. But rather than looking at it as a failure, a more productive viewpoint is to see it as an opportunity for more research, possibly with larger sample sizes.

Even when we do reject the null, it is important not to see that as the end of the conversation. Too many times, a researcher publishes a “statistically significant” finding in a peer-reviewed journal, and then that result is taken as “Truth”. We should, instead, view statistical inference as incremental knowledge that works slowly to refine our state of scientific knowledge, as opposed to a collection of “facts” and “non-facts”.

## 10.13 Your turn

Now it's your turn to run a complete hypothesis test. Determine if males were admitted to the top six UC Berkeley grad programs at a higher rate than females. For purposes of this exercise, we will not take into account the `Dept` variable as we did in the last chapter when we discussed Simpson's Paradox. But as that is a potential source of confounding, be sure to mention it in the part of the rubric where you discuss reservations about your conclusion.

As always, use a significance level of  $\alpha = 0.05$ .

Here is the data import:

```
ucb_admit <- read_csv("https://vectorposse.github.io/intro_stats/data/ucb_admit.csv",
                      col_types = list(
                        Admit = col_factor(),
                        Gender = col_factor(),
                        Dept = col_factor()))
```

I have copied the template below. You need to fill in each step. Some of the steps will be the same or similar to steps in the example above. It is perfectly okay to copy and paste R code, making the necessary changes. It is **not** okay to copy and paste text. You need to put everything into your own words. Also, don't copy and paste the parts that are labeled as “Notes”. That is information to help you understand each step, but it's not part of the statistical analysis itself.

The template below is exactly the same as in the Appendix up to the part about confidence intervals which we haven't learned yet.



### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure. Please write up your answer here

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

Identify the sample (or samples) and a reasonable population (or populations) of interest. Please write up your answer here.

Express the null and alternative hypotheses as contextually meaningful full sentences.  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

Express the null and alternative hypotheses in symbols (when possible).  $H_0$  : *math*

$H_A$  : *math*

### Model

**Identify the sampling distribution model.** Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

**Mechanics**

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
set.seed(9999)
# Add code here to simulate the null distribution.
# Run 1000 reps like in the earlier example.
```

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

**Conclusion**

**State the statistical conclusion.** Please write up your answer here.

**State (but do not overstate) a contextually meaningful conclusion.**  
Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.**  
Please write up your answer here.

## 10.14 Conclusion

A hypothesis test is a formal set of steps—a procedure, if you will—for implementing the logic of inference. We take a skeptical position and assume a null hypothesis in contrast to the question of interest, the alternative hypothesis. We build a model under the assumption of the null hypothesis to see if our data is consistent with the null (in which case we fail to reject the null) or unusual/rare relative to the null (in which case we reject the null). We always work under the assumption of the null so that we can convince a skeptical audience using evidence. We also take care to acknowledge that statistical procedures can be wrong, and not to put too much credence in the results of any single set of data or single hypothesis test.

### 10.14.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.



## Chapter 11

# Hypothesis testing with randomization, Part 2

2.0

Functions introduced in this chapter

`factor`

### 11.1 Introduction

Now that we have learned about hypothesis testing, we'll explore a different example. Although the rubric for performing the hypothesis test will not change, the individual steps will be implemented in a different way due to the research question we're asking and the type of data used to answer it.

#### 11.1.1 Install new packages

If you are using RStudio Workbench, you do not need to install any packages. (Any packages you need should already be installed by the server administrators.)

If you are using R and RStudio on your own machine instead of accessing RStudio Workbench through a browser, you'll need to type the following command at the Console:

```
install.packages("MASS")
```

### 11.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/11-hypothesis\\_testing\\_with\\_randomization.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/11-hypothesis_testing_with_randomization.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 11.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 11.2 Load packages

In addition to `tidyverse` and `janitor`, we load the `MASS` package to access the `Melanoma` data on patients in Denmark with malignant melanoma, and the `infer` package for inference tools.

```
library(tidyverse)
library(janitor)
library(MASS)
```

```
##
## Attaching package: 'MASS'

## The following objects are masked from 'package:openintro':
##
##   housing, mammals

## The following object is masked from 'package:dplyr':
##
##   select
```

```
library(infer)
```

## 11.3 Our research question

We know that certain types of cancer are more common among females or males. Is there a sex bias among patients with malignant melanoma?

Let's jump into the “Exploratory data analysis” part of the rubric first.

## 11.4 Exploratory data analysis

### 11.4.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

You can look at the help file by typing `?Melanoma` at the Console. However, do not put that command here in a code chunk. The R Notebook has no way of displaying a help file when it's processed. Be careful: there's another data set called `melanoma` with a lower-case “m”. Make sure you are using an uppercase “M”.

There is a reference at the bottom of the help file.

**Exercise 1** Using the reference in the help file, do an internet search to find the source of this data. How can you tell that this reference is not, in fact, a reference to a study of cancer patients in Denmark?

Please write up your answer here.

---

From the exercise above, we can see that it will be very difficult, if not impossible, to discover anything useful about the true provenance of the data (unless you happen to have a copy of that textbook, which in theory provided another more primary source). We will not know, for example, how the data was collected and if the patients consented to having their data shared publicly. The data is suitably anonymized, though, so we don't have any serious concerns about the privacy of the data. Having said that, if a condition is rare enough, a dedicated research can often “de-anonymize” data by cross-referencing information in the data to other kinds of public records. But melanoma is not particularly rare. At any rate, all we can do is assume that the textbook authors obtained the data from a source that used proper procedures for collecting and handling the data.

We print the data frame:

Melanoma

```
##      time status sex age year thickness ulcer
## 1      10       3  1  76 1972       6.76     1
## 2      30       3  1  56 1968       0.65     0
## 3      35       2  1  41 1977       1.34     0
## 4      99       3  0  71 1968       2.90     0
## 5     185       1  1  52 1965      12.08     1
```

## 6	204	1	1	28	1971	4.84	1
## 7	210	1	1	77	1972	5.16	1
## 8	232	3	0	60	1974	3.22	1
## 9	232	1	1	49	1968	12.88	1
## 10	279	1	0	68	1971	7.41	1
## 11	295	1	0	53	1969	4.19	1
## 12	355	3	0	64	1972	0.16	1
## 13	386	1	0	68	1965	3.87	1
## 14	426	1	1	63	1970	4.84	1
## 15	469	1	0	14	1969	2.42	1
## 16	493	3	1	72	1971	12.56	1
## 17	529	1	1	46	1971	5.80	1
## 18	621	1	1	72	1972	7.06	1
## 19	629	1	1	95	1968	5.48	1
## 20	659	1	1	54	1972	7.73	1
## 21	667	1	0	89	1968	13.85	1
## 22	718	1	1	25	1967	2.34	1
## 23	752	1	1	37	1973	4.19	1
## 24	779	1	1	43	1967	4.04	1
## 25	793	1	1	68	1970	4.84	1
## 26	817	1	0	67	1966	0.32	0
## 27	826	3	0	86	1965	8.54	1
## 28	833	1	0	56	1971	2.58	1
## 29	858	1	0	16	1967	3.56	0
## 30	869	1	0	42	1965	3.54	0
## 31	872	1	0	65	1968	0.97	0
## 32	967	1	1	52	1970	4.83	1
## 33	977	1	1	58	1967	1.62	1
## 34	982	1	0	60	1970	6.44	1
## 35	1041	1	1	68	1967	14.66	0
## 36	1055	1	0	75	1967	2.58	1
## 37	1062	1	1	19	1966	3.87	1
## 38	1075	1	1	66	1971	3.54	1
## 39	1156	1	0	56	1970	1.34	1
## 40	1228	1	1	46	1973	2.24	1
## 41	1252	1	0	58	1971	3.87	1
## 42	1271	1	0	74	1971	3.54	1
## 43	1312	1	0	65	1970	17.42	1
## 44	1427	3	1	64	1972	1.29	0
## 45	1435	1	1	27	1969	3.22	0
## 46	1499	2	1	73	1973	1.29	0
## 47	1506	1	1	56	1970	4.51	1
## 48	1508	2	1	63	1973	8.38	1
## 49	1510	2	0	69	1973	1.94	0
## 50	1512	2	0	77	1973	0.16	0
## 51	1516	1	1	80	1968	2.58	1



## 52	1525	3	0	76	1970	1.29	1
## 53	1542	2	0	65	1973	0.16	0
## 54	1548	1	0	61	1972	1.62	0
## 55	1557	2	0	26	1973	1.29	0
## 56	1560	1	0	57	1973	2.10	0
## 57	1563	2	0	45	1973	0.32	0
## 58	1584	1	1	31	1970	0.81	0
## 59	1605	2	0	36	1973	1.13	0
## 60	1621	1	0	46	1972	5.16	1
## 61	1627	2	0	43	1973	1.62	0
## 62	1634	2	0	68	1973	1.37	0
## 63	1641	2	1	57	1973	0.24	0
## 64	1641	2	0	57	1973	0.81	0
## 65	1648	2	0	55	1973	1.29	0
## 66	1652	2	0	58	1973	1.29	0
## 67	1654	2	1	20	1973	0.97	0
## 68	1654	2	0	67	1973	1.13	0
## 69	1667	1	0	44	1971	5.80	1
## 70	1678	2	0	59	1973	1.29	0
## 71	1685	2	0	32	1973	0.48	0
## 72	1690	1	1	83	1971	1.62	0
## 73	1710	2	0	55	1973	2.26	0
## 74	1710	2	1	15	1973	0.58	0
## 75	1726	1	0	58	1970	0.97	1
## 76	1745	2	0	47	1973	2.58	1
## 77	1762	2	0	54	1973	0.81	0
## 78	1779	2	1	55	1973	3.54	1
## 79	1787	2	1	38	1973	0.97	0
## 80	1787	2	0	41	1973	1.78	1
## 81	1793	2	0	56	1973	1.94	0
## 82	1804	2	0	48	1973	1.29	0
## 83	1812	2	1	44	1973	3.22	1
## 84	1836	2	0	70	1972	1.53	0
## 85	1839	2	0	40	1972	1.29	0
## 86	1839	2	1	53	1972	1.62	1
## 87	1854	2	0	65	1972	1.62	1
## 88	1856	2	1	54	1972	0.32	0
## 89	1860	3	1	71	1969	4.84	1
## 90	1864	2	0	49	1972	1.29	0
## 91	1899	2	0	55	1972	0.97	0
## 92	1914	2	0	69	1972	3.06	0
## 93	1919	2	1	83	1972	3.54	0
## 94	1920	2	1	60	1972	1.62	1
## 95	1927	2	1	40	1972	2.58	1
## 96	1933	1	0	77	1972	1.94	0
## 97	1942	2	0	35	1972	0.81	0

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## 98	1955	2	0	46	1972	7.73	1
## 99	1956	2	0	34	1972	0.97	0
## 100	1958	2	0	69	1972	12.88	0
## 101	1963	2	0	60	1972	2.58	0
## 102	1970	2	1	84	1972	4.09	1
## 103	2005	2	0	66	1972	0.64	0
## 104	2007	2	1	56	1972	0.97	0
## 105	2011	2	0	75	1972	3.22	1
## 106	2024	2	0	36	1972	1.62	0
## 107	2028	2	1	52	1972	3.87	1
## 108	2038	2	0	58	1972	0.32	1
## 109	2056	2	0	39	1972	0.32	0
## 110	2059	2	1	68	1972	3.22	1
## 111	2061	1	1	71	1968	2.26	0
## 112	2062	1	0	52	1965	3.06	0
## 113	2075	2	1	55	1972	2.58	1
## 114	2085	3	0	66	1970	0.65	0
## 115	2102	2	1	35	1972	1.13	0
## 116	2103	1	1	44	1966	0.81	0
## 117	2104	2	0	72	1972	0.97	0
## 118	2108	1	0	58	1969	1.76	1
## 119	2112	2	0	54	1972	1.94	1
## 120	2150	2	0	33	1972	0.65	0
## 121	2156	2	0	45	1972	0.97	0
## 122	2165	2	1	62	1972	5.64	0
## 123	2209	2	0	72	1971	9.66	0
## 124	2227	2	0	51	1971	0.10	0
## 125	2227	2	1	77	1971	5.48	1
## 126	2256	1	0	43	1971	2.26	1
## 127	2264	2	0	65	1971	4.83	1
## 128	2339	2	0	63	1971	0.97	0
## 129	2361	2	1	60	1971	0.97	0
## 130	2387	2	0	50	1971	5.16	1
## 131	2388	1	1	40	1966	0.81	0
## 132	2403	2	0	67	1971	2.90	1
## 133	2426	2	0	69	1971	3.87	0
## 134	2426	2	0	74	1971	1.94	1
## 135	2431	2	0	49	1971	0.16	0
## 136	2460	2	0	47	1971	0.64	0
## 137	2467	1	0	42	1965	2.26	1
## 138	2492	2	0	54	1971	1.45	0
## 139	2493	2	1	72	1971	4.82	1
## 140	2521	2	0	45	1971	1.29	1
## 141	2542	2	1	67	1971	7.89	1
## 142	2559	2	0	48	1970	0.81	1
## 143	2565	1	1	34	1970	3.54	1

## 144	2570	2	0	44	1970	1.29	0
## 145	2660	2	0	31	1970	0.64	0
## 146	2666	2	0	42	1970	3.22	1
## 147	2676	2	0	24	1970	1.45	1
## 148	2738	2	0	58	1970	0.48	0
## 149	2782	1	1	78	1969	1.94	0
## 150	2787	2	1	62	1970	0.16	0
## 151	2984	2	1	70	1969	0.16	0
## 152	3032	2	0	35	1969	1.29	0
## 153	3040	2	0	61	1969	1.94	0
## 154	3042	1	0	54	1967	3.54	1
## 155	3067	2	0	29	1969	0.81	0
## 156	3079	2	1	64	1969	0.65	0
## 157	3101	2	1	47	1969	7.09	0
## 158	3144	2	1	62	1969	0.16	0
## 159	3152	2	0	32	1969	1.62	0
## 160	3154	3	1	49	1969	1.62	0
## 161	3180	2	0	25	1969	1.29	0
## 162	3182	3	1	49	1966	6.12	0
## 163	3185	2	0	64	1969	0.48	0
## 164	3199	2	0	36	1969	0.64	0
## 165	3228	2	0	58	1969	3.22	1
## 166	3229	2	0	37	1969	1.94	0
## 167	3278	2	1	54	1969	2.58	0
## 168	3297	2	0	61	1968	2.58	1
## 169	3328	2	1	31	1968	0.81	0
## 170	3330	2	1	61	1968	0.81	1
## 171	3338	1	0	60	1967	3.22	1
## 172	3383	2	0	43	1968	0.32	0
## 173	3384	2	0	68	1968	3.22	1
## 174	3385	2	0	4	1968	2.74	0
## 175	3388	2	1	60	1968	4.84	1
## 176	3402	2	1	50	1968	1.62	0
## 177	3441	2	0	20	1968	0.65	0
## 178	3458	3	0	54	1967	1.45	0
## 179	3459	2	0	29	1968	0.65	0
## 180	3459	2	1	56	1968	1.29	1
## 181	3476	2	0	60	1968	1.62	0
## 182	3523	2	0	46	1968	3.54	0
## 183	3667	2	0	42	1967	3.22	0
## 184	3695	2	0	34	1967	0.65	0
## 185	3695	2	0	56	1967	1.03	0
## 186	3776	2	1	12	1967	7.09	1
## 187	3776	2	0	21	1967	1.29	1
## 188	3830	2	1	46	1967	0.65	0
## 189	3856	2	0	49	1967	1.78	0

```
## 190 3872      2  0  35 1967      12.24      1
## 191 3909      2  1  42 1967       8.06      1
## 192 3968      2  0  47 1967       0.81      0
## 193 4001      2  0  69 1967       2.10      0
## 194 4103      2  0  52 1966       3.87      0
## 195 4119      2  1  52 1966       0.65      0
## 196 4124      2  0  30 1966       1.94      1
## 197 4207      2  1  22 1966       0.65      0
## 198 4310      2  1  55 1966       2.10      0
## 199 4390      2  0  26 1965       1.94      1
## 200 4479      2  0  19 1965       1.13      1
## 201 4492      2  1  29 1965       7.06      1
## 202 4668      2  0  40 1965       6.12      0
## 203 4688      2  0  42 1965       0.48      0
## 204 4926      2  0  50 1964       2.26      0
## 205 5565      2  0  41 1962       2.90      0
```

Use `glimpse` to examine the structure of the data:

```
glimpse(Melanoma)
```

```
## Rows: 205
## Columns: 7
## $ time      <int> 10, 30, 35, 99, 185, 204, 210, 232, 232, 279, 295, 355, 386, ~
## $ status    <int> 3, 3, 2, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 1, 1, ~
## $ sex       <int> 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, ~
## $ age       <int> 76, 56, 41, 71, 52, 28, 77, 60, 49, 68, 53, 64, 68, 63, 14, ~
## $ year      <int> 1972, 1968, 1977, 1968, 1965, 1971, 1972, 1974, 1968, 1971, ~
## $ thickness <dbl> 6.76, 0.65, 1.34, 2.90, 12.08, 4.84, 5.16, 3.22, 12.88, 7.41, ~
## $ ulcer     <int> 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ~
```

### 11.4.2 Prepare the data for analysis.

It appears that `sex` is coded as an integer. You will recall that we need to convert it to a factor variable since it is categorical, not numerical.

**Exercise 2** According to the help file, which number corresponds to which sex?

Please write up your answer here.

---

We can convert a numerical variable a couple of different ways. In Chapter 3, we used the `as_factor` command. That command works fine, but it doesn't give you a way to change the levels of the variable. In other words, if we used `as_factor` here, we would get a factor variable that still contained zeroes and ones.

Instead, we will use the `factor` command. It allows us to manually relabel the levels. The `levels` argument requires a vector (with `c`) of the current levels, and the `labels` argument requires a vector listing the new names you want to assign, as follows:

```
Melanoma <- Melanoma %>%
  mutate(sex_fct = factor(sex, levels = c(0, 1), labels = c("female", "male")))
glimpse(Melanoma)

## Rows: 205
## Columns: 8
## $ time      <int> 10, 30, 35, 99, 185, 204, 210, 232, 232, 279, 295, 355, 386, ~
## $ status    <int> 3, 3, 2, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 1, 1, ~
## $ sex       <int> 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, 1, ~
## $ age       <int> 76, 56, 41, 71, 52, 28, 77, 60, 49, 68, 53, 64, 68, 63, 14, ~
## $ year      <int> 1972, 1968, 1977, 1968, 1965, 1971, 1972, 1974, 1968, 1971, ~
## $ thickness <dbl> 6.76, 0.65, 1.34, 2.90, 12.08, 4.84, 5.16, 3.22, 12.88, 7.41, ~
## $ ulcer     <int> 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ~
## $ sex_fct   <fct> male, male, male, female, male, male, male, female, male, fe~
```

You should check to make sure the first few entries of `sex_fct` agree with the numbers in the `sex` variable according to the labels explained in the help file. (If not, it means that you put the `levels` in one order and the `labels` in a different order.)

### 11.4.3 Make tables or plots to explore the data visually.

We only have one categorical variable, so we only need a frequency table. Since we are concerned with proportions, we'll also look at a relative frequency table which the `tabyl` command provides for free.

```
tabyl(Melanoma, sex_fct) %>%
  adorn_totals()

## sex_fct  n  percent
## female 126 0.6146341
## male   79 0.3853659
## Total  205 1.0000000
```

## 11.5 The logic of inference and randomization

This is a good place to pause and remember why statistical inference is important. There are certainly more females than males in this data set. So why don't we just show the table above, declare females are more likely to have malignant melanoma, and then go home?

Think back to coin flips. Even though there was a 50% chance of seeing heads, did that mean that exactly half of our flips came up heads? No. We have to acknowledge *sampling variability*: even if the truth were 50%, when we sample, we could accidentally get more or less than 50%, just by pure chance alone. Perhaps these 205 patients just happen to have more females than average.

The key, then, is to figure out if 61.5% is *significantly* larger than 50%, or if a number like 61.5% (or one even more extreme) could easily come about from random chance.

As we know from the last chapter, we can run a formal hypothesis test to find out. As we do so, make note of the things that are the same and the things that have changed from the last hypothesis tests you ran. For example, we are not comparing two groups anymore. We have one group of patients, and all we're doing is measuring the percentage of this group that is female. It's tempting to think that we're comparing males and females, but that's not the case. We are not using sex to divide our data into two groups for the purpose of exploring whether some other variable differs between men and women. We just have one sample. "Female" and "Male" are simply categories in a single categorical variable. Also, because we are only asking about one variable (`sex_fct`), the mathematical form of the hypotheses will look a little different.

Because this is no longer a question about two variables being independent or associated, the "permuting" idea we've been using no longer makes sense. So what does make sense?

It helps to start by figuring out what our null hypothesis is. Remember, our question of interest is whether there is a sex bias in malignant melanoma. In other words, are there more or fewer females than males with malignant melanoma? As this is our research question, it will be the alternative hypothesis. So what is the null? What is the "default" situation in which nothing interesting is going on? Well, there would be no sex bias. In other words, there would be the same number of females and males with malignant melanoma. Or another way of saying that—with respect to the "success" condition of being female that we discussed earlier—is that females comprise 50% of all patients with malignant melanoma.

Okay, given our philosophy about the null hypothesis, let's take the skeptical position and assume that, indeed, 50% of all malignant melanoma patients in our population are female. Then let's take a sample of 205 patients. We can't get exactly 50% females from a sample of 205 (that would be 102.5 females!), so what numbers can we get?

Randomization will tell us. What kind of randomization? As we come across each patient in our sample, there is a 50% chance of them being female. So instead of sampling real patients, what if we just flipped a coin? A simulated coin flip will come up heads just as often as our patients will be female under the assumption of the null.

This brings us full circle, back to the first randomization idea we explored. We can simulate coin flips, graph our results, and calculate a P-value. More specifically, we'll flip a coin 205 times to represent sampling 205 patients. Then we'll repeat this procedure a bunch of times and establish a range of plausible percentages that can come about by chance from this procedure. Instead of doing coin flips with the `rflip` command as we did then, however, we'll use our new favorite friend, the `infer` package.

Let's dive back into the remaining steps of the formal hypothesis test.

## 11.6 Hypotheses

### 11.6.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

The sample consists of 205 patients from Denmark with malignant melanoma. Our population is presumably all patients with malignant melanoma, although in checking conditions below, we'll take care to discuss whether patients in Denmark are representative of patients elsewhere.

### 11.6.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

$H_0$  : Half of malignant melanoma patients are female.

$H_A$  : There is a sex bias among patients with malignant melanoma (meaning that females are either over-represented or under-represented).

### 11.6.3 Express the null and alternative hypotheses in symbols (when possible).

$H_0 : p_{female} = 0.5$

$H_A : p_{female} \neq 0.5$

## 11.7 Model

### 11.7.1 Identify the sampling distribution model.

We will randomize to simulate the sampling distribution.

### 11.7.2 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - As mentioned above, these 205 patients are not a random sample of all people with malignant melanoma. We don't even have any evidence that they are a random sample of melanoma patients in Denmark. Without such evidence, we have to hope that these 205 patients are representative of all patients who have malignant melanoma. Unless there's something special about Danes in terms of their genetics or diet or something like that, one could imagine that their physiology makes them just as susceptible to melanoma as anyone else. More specifically, though, our question is about females and males getting malignant melanoma. Perhaps there are more female sunbathers in Denmark than in other countries. That might make Danes unrepresentative in terms of the gender balance among melanoma patients. We should be cautious in interpreting any conclusion we might reach in light of these doubts.
- 10%
  - Whether in Denmark or not, given that melanoma is a fairly common form of cancer, I assume 205 is less than 10% of all patients with malignant melanoma.

## 11.8 Mechanics

### 11.8.1 Compute the test statistic.

```
female_prop <- Melanoma %>%
  observe(response = sex_fct, success = "female",
           stat = "prop")
female_prop
```

```
## Response: sex_fct (factor)
```



```
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 0.615
```

Note: Pay close attention to the difference in the `observe` command above. Unlike in the last chapter, we don't have any tildes. That's because there are not two variables involved. There is only one variable, which `observe` needs to see as the "response" variable. (Don't forget to use the factor version `sex_fct` and not `sex`!) We still have to specify a "success" condition. Since the hypotheses are about measuring females, we have to tell `observe` to calculate the proportion of females. Finally, the `stat` is no longer "diff in props" There are not two proportions with which to find a difference. There is just one proportion, hence, "prop".

### 11.8.2 Report the test statistic in context (when possible).

The observed percentage of females with melanoma in our sample is 61.4634146%.

Note: As explained in the last chapter, we have to use `pull` to pull out the number from the `female_prop` tibble.

### 11.8.3 Plot the null distribution.

Since this is the first step for which we need the simulated values, it will be convenient to run the simulation here. We'll need to set the seed as well.

```
set.seed(42)
melanoma_test <- Melanoma %>%
  specify(response = sex_fct, success = "female") %>%
  hypothesize(null = "point", p = 0.5) %>%
  generate(reps = 1000, type = "draw") %>%
  calculate(stat = "prop")
melanoma_test
```

```
## Response: sex_fct (factor)
## Null Hypothesis: point
## # A tibble: 1,000 x 2
##   replicate stat
##   <fct>      <dbl>
## 1 1          0.444
## 2 2          0.585
## 3 3          0.551
```

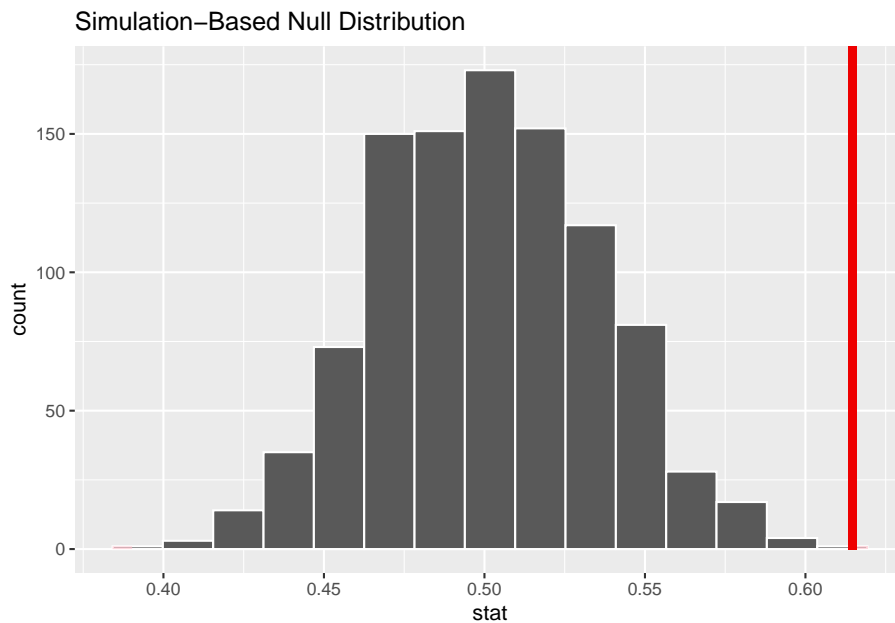
```
## 4 4      0.502
## 5 5      0.561
## 6 6      0.493
## 7 7      0.527
## 8 8      0.488
## 9 9      0.512
## 10 10     0.454
## # ... with 990 more rows
```

This list of proportions is the sampling distribution. It represents possible sample proportions of females with melanoma **under the assumption that the null is true**. In other words, even if the “true” proportion of female melanoma patients were 0.5, these are all values that can result from random samples.

In the `hypothesize` command, we use “point” to tell `infer` that we want the null to be centered at the point 0.5. In the `generate` command, we need to specify the `type` as “draw” instead of “permute”. We are not shuffling any values here; we are “drawing” values from a probability distribution like coin flips. Everything else in the command is pretty self-explanatory.

The value of our test statistic, `female_prop`, is 0.6146341. It appears in the right tail:

```
melanoma_test %>%
  visualize() +
  shade_p_value(obs_stat = female_prop, direction = "two-sided")
```



Although the line only appears on the right, keep in mind that we are conducting a two-sided test, so we are interested in values more extreme than the red line on the right, but also more extreme than a similarly placed line on the left.

**Exercise 3** The red line sits at about 0.615. If you were to draw a red line on the above histogram that represented a value equally distant from 0.5, but on the left instead of the right, where would that line be? Do a little arithmetic to figure it out and show your work.

Please write up your answer here.

#### 11.8.4 Calculate the P-value.

```
melanoma_test %>%
  get_p_value(obs_stat = female_prop, direction = "two-sided")

## Warning: Please be cautious in reporting a p-value of 0. This result is an
## approximation based on the number of `reps` chosen in the `generate()` step. See
## `?get_p_value()` for more information.

## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1      0
```

The P-value appears to be zero. Indeed, among the 1000 simulated values, we saw none that exceeded 0.615 and none that were less than 0.385. However, a true P-value can never be zero. If you did millions or billions of simulations (please don't try!), surely there would be one or two with even more extreme values. In cases when the P-value is really, really tiny, it is traditional to report  $P < 0.001$ . It is **incorrect** to say  $P = 0$ .

#### 11.8.5 Interpret the P-value as a probability given the null.

$P < 0.001$ . If there were no sex bias in malignant melanoma patients, there would be less than a 0.1% chance of seeing a percentage of females at least as extreme as the one we saw in our data.

Note: Don't forget to interpret the P-value in a contextually meaningful way. The P-value is the probability under the assumption of the null hypothesis of seeing data at least as extreme as the data we saw. In this context, that means that if we assume 50% of patients are female, it would be extremely rare to see more than 61.5% or less than 38.5% females in a sample of size 205.

## 11.9 Conclusion

### 11.9.1 State the statistical conclusion.

We reject the null hypothesis.

### 11.9.2 State (but do not overstate) a contextually meaningful conclusion.

There is sufficient evidence that there is a sex bias in patients who suffer from malignant melanoma.

### 11.9.3 Express reservations or uncertainty about the generalizability of the conclusion.

We have no idea how these patients were sampled. Are these all the patients in Denmark with malignant melanoma over a certain period of time? Were they part of a convenience sample? As a result of our uncertainty about the sampling process, we can't be sure if the results generalize to a larger population, either in Denmark or especially outside of Denmark.

**Exercise 4** Can you find on the internet any evidence that females do indeed suffer from malignant melanoma more often than males (not just in Denmark, but anywhere)?

Please write up your answer here.

### 11.9.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.

As we rejected the null, we run the risk of making a Type I error. If we have made such an error, that would mean that patients with malignant melanoma are equally likely to be male or female, but that we got a sample with an unusual number of female patients.

## 11.10 Your turn

Determine if the percentage of patients in Denmark with malignant melanoma who also have an ulcerated tumor (measured with the `ulcer` variable) is significantly different from 50%.

As before, you have the outline of the rubric for inference below. Some of the steps will be the same or similar to steps in the example above. It is perfectly okay to copy and paste R code, making the necessary changes. It is **not** okay to copy and paste text. You need to put everything into your own words.

The template below is exactly the same as in the appendix (Rubric for inference) up to the part about confidence intervals which we haven't learned yet.

### Exploratory data analysis

```
# Add code here to understand the data.
```

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

Identify the sample (or samples) and a reasonable population (or populations) of interest. Please write up your answer here.

Express the null and alternative hypotheses as contextually meaningful full sentences.  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

**Express the null and alternative hypotheses in symbols (when possible).**  $H_0 :$  *math*

$H_A :$  *math*

**Model**

**Identify the sampling distribution model.** Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

**Mechanics**

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
set.seed(42)
# Add code here to simulate the null distribution.
# Run 1000 simulations like in the earlier example.
```

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

### Conclusion

**State the statistical conclusion.** Please write up your answer here.

**State (but do not overstate) a contextually meaningful conclusion.** Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.** Please write up your answer here.

## 11.11 Conclusion

Now you have seen two fully-worked examples of hypothesis tests using randomization, and you have created two more examples on your own. Hopefully, the logic of inference and the process of running a formal hypothesis test are starting to make sense.

Keep in mind that the outline of steps will not change. However, the way each step is carried out will vary from problem to problem. Not only does the context change (one example involved sex discrimination, the other melanoma patients), but the statistics you compute also change (one example compared proportions from two samples and the other only had one proportion from a single sample). Pay close attention to the research question and the data that will be used to answer that question. That will be the only information you have to help you know which hypothesis test applies.

### 11.11.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.

4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.



## Chapter 12

# Confidence intervals

2.0

### Functions introduced in this chapter

`get_confidence_interval`, `shade_confidence_interval`, `fct_collapse`

## 12.1 Introduction

Sampling variability means that we can never trust a single sample to identify a population parameter exactly. Instead of simply trusting a point estimate, we can look at the entire sampling distribution to create an interval of plausible values called a confidence interval. By making our intervals wide enough, we hope to have some chance of capturing the true population value. Like hypothesis tests, confidence intervals are a form of inference because they use a sample to deduce something about the population. Along the way, we will also learn about a new form of randomization called *bootstrapping*.

### 12.1.1 Install new packages

There are no new packages used in this chapter.

### 12.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/12-confidence\\_intervals.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/12-confidence_intervals.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 12.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 12.2 Load packages

We load the standard `tidyverse`, `janitor`, and `infer` packages. We’ll also need the `openintro` package later in the chapter for the `hsb2` and the `smoking` data set.

```
library(tidyverse)
library(janitor)
library(infer)
library(openintro)
```

## 12.3 Bootstrapping

Imagine you obtain a random sample of 200 high school seniors from across the U.S. Suppose 32 of them attend private school. As a sample statistic, we have

$$\hat{p} = 32/200 = 0.16$$

In other words, 16% of the students in the sample attended private school.

If our sample is representative, we might guess that the true population parameter  $p$  is also close to 0.16, but we’re not really sure:

$$p \approx 0.16?$$

And what about the sampling variability? A few chapters ago, we flipped coins. A “weighted” coin flipped 200 times can give us a “new” (fake) sample, and doing that a thousand times (or even more) can give us a lot of new samples to see what range of values is possible. But what would we use as the probability of heads for the weighted coin? It would be a bad idea to use 0.16 because that would assume that the population proportion agreed exactly with the one sample we happen to have. It worked in a hypothesis test because we had a value of  $p$  we assumed was true in the guise of a null hypothesis. But in general,

if I simply want to estimate a population parameter with a sample statistic, I have no such information to use. So coin flipping is out.

An alternative that is available to us is a procedure called *bootstrapping*. The idea sounds weird, but it's pretty simple: instead of building fake samples, what if we tried to build a fake population? And then, what if we took repeated samples from it?

How would we build a fake population? Imagine making many, many copies of our sample until we had thousands or even millions of students. In fact, we can think of an infinite number of copies of our sample if we want. Sure, this fake population isn't exactly like the real population of all high school seniors. But if our sample is representative, we might hope that lots of copies of our sample would approximate the population we care about.

Computationally, it's a lot of work to copy our sample thousands or millions of times. And we certainly can't work with an infinite number of copies. Fortunately, we can use a shortcut. It's called *sampling with replacement*.

Normal sampling is usually *without replacement*, meaning that once we have sampled an individual, they are not eligible to be sampled again. We don't want to survey Billy and then later in our study, survey Billy again.

In sampling *with replacement*, we put Billy back in the pool and make him eligible to be sampled again. This is the same thing as having access to an infinite population. Remember that our fake population is just many, many copies of our sample. So in that fake population, there are many, many Billy clones that could end up in our sample. So rather than cloning Billy many, many times, let's just put Billy back in the group any time he's sampled.

We need to see this in action. We have a random sample of 200 students obtained by the National Center of Education Statistics in their "High School and Beyond" survey. This is stored in the `hsb2` data set from the `openintro` package. Here are the school types for these students, stored in the variable `schtyp`:

```
hsb2$schtyp
```

```
## [1] public public public public public public public public public public
## [10] public public public public public public public public public private
## [19] public public public public public public public public public public
## [28] private private public public public private public private public
## [37] private public public public private private public public public
## [46] public public public private public public public public private
## [55] public public public public private public private public public
## [64] public private public public public public public public public
## [73] public public public public public public public public public
## [82] public private public public public public public public public
```

```
## [91] public public public public public public public public public public
## [100] private public public public public public public public public public
## [109] private private public public public public public private public public
## [118] public public public private public public public public public public
## [127] public public public public public public public public public public
## [136] public private public public private public public public public public
## [145] public private public private public public public public public public
## [154] public public public public public public public public public public
## [163] private public public public public public public private public public
## [172] public public public public public public public public public public
## [181] public private public public public public public public public private
## [190] public public private private public private private public private
## [199] public public
## Levels: public private
```

Let's sample an individual from our sample:

```
set.seed(6)
sample(hsb2$schtyp, size = 1)
```

```
## [1] public
## Levels: public private
```

That was one of the public school students from among the 200 students in our sample. Here's another one:

```
set.seed(7)
sample(hsb2$schtyp, size = 1)
```

```
## [1] private
## Levels: public private
```

That was one of the private school students.

We can do this 200 times. Now, if we sample *without* replacement, all we get back are the original students, just listed in a different order. Think about why: we're just picking one student at a time. But since they don't get replaced, eventually, every student will get chosen. We're choosing 200 students, but there are only 200 students from which to choose.

```
set.seed(8)
sample_without_replacement1 <- sample(hsb2$schtyp, size = 200)
sample_without_replacement1
```

```
## [1] public public public public public public public public public public
## [10] public public public public public public public private public
## [19] public public public public public public public public public public
## [28] public public private public public public public public public public
## [37] public public public public public public public private public
## [46] public public public public private private private public public
## [55] private private public public public public public public public private
## [64] public public private public public public public public public public
## [73] public public public public public public public public public public
## [82] public public public public public public public public public public
## [91] public public public public public public public public public public
## [100] public public private public public public public public public public
## [109] public public public public public public public public public public
## [118] public public public private public public private public public
## [127] private private public public public public private public private
## [136] private public public public public public public public public public
## [145] public public public public public public public public public private
## [154] public public public public public private public private private
## [163] public public public public private public public private public
## [172] private private public public public public public public public public
## [181] public private public private public public public private private
## [190] public public public public public public public public public private
## [199] public private
## Levels: public private
```

```
tabyl(sample_without_replacement1)
```

```
## sample_without_replacement1  n percent
##                public 168    0.84
##                private  32    0.16
```

```
set.seed(9)
sample_without_replacement2 <- sample(hsb2$schtyp, size = 200)
sample_without_replacement2
```

```
## [1] public public public private public private public private public
## [10] public public private private private private private public public
## [19] private public public public public public public public public
## [28] public public public public public public public public public
## [37] public public public public public public public private public
## [46] public public public public private public private public public
## [55] public private public public public public public public public
## [64] private public public public public public public public public
## [73] public public public public private public public public public
```

```
## [82] private private public public public public public public private
## [91] public private public public public private public public public
## [100] public public private private public public public public public
## [109] public private public public private public private public public
## [118] public public public public public private public public public
## [127] public public public public public public public public public
## [136] public public public public public public private public public
## [145] public private public public public public public public public
## [154] public public public public public public public public private
## [163] private public public public public public public public public
## [172] public public public public public public public public public
## [181] private public public public public public private public public
## [190] public public public public public public public public public
## [199] public public
## Levels: public private
```

```
tabyl(sample_without_replacement2)
```

```
## sample_without_replacement2  n percent
##                public 168    0.84
##                private  32    0.16
```

The two lists above consist of the same 200 students, just drawn in a different order.

On the other hand, if we sample *with* replacement, then students can get chosen more than once. (Remember, we’re equating “getting chosen more than once” with “sampling from an infinite population and choosing a clone”.) Now, the number of private school students we see might not be 32.

Each of the following samples is called a *bootstrap sample*. Notice that we’ve added the argument `replace = TRUE` to the `sample` function:

```
set.seed(10)
sample_with_replacement1 <- sample(hsb2$schtyp, size = 200, replace = TRUE)
sample_with_replacement1
```

```
## [1] private public public public public private public public public
## [10] public public public public public public public public public
## [19] private public public public public private private private public
## [28] public private public public public private public public public
## [37] public public public public public public private public public
## [46] public public public public public public private public public
## [55] public public public public public public public public public
## [64] public public private public private public public public private
```

```
## [73] public public public public public public public public public
## [82] public public public public private public public public public
## [91] public private public private public private public public public
## [100] public public public private private public public public public
## [109] public public public public public private private public public
## [118] private public public private public public private public public
## [127] public public public private private private public public private
## [136] public public public public public public public public public
## [145] public public public public public public public public public
## [154] public public public public public public public private public
## [163] public public public private private public private private private
## [172] public public public public public public private public public
## [181] public public public public public public private public public
## [190] public public public public public public public public public
## [199] public public
## Levels: public private
```

```
tabyl(sample_with_replacement1)
```

```
## sample_with_replacement1  n percent
##                public 164    0.82
##                private  36    0.18
```

That bootstrap sample proportion is 0.18, not 0.16.

```
set.seed(11)
sample_with_replacement2 <- sample(hsb2$schtyp, size = 200, replace = TRUE)
sample_with_replacement2
```

```
## [1] public public public public public private public public private
## [10] public public private public public public public public public
## [19] public public public public public public public public public
## [28] public public public public public public public public public
## [37] public public public public public public public public public
## [46] public public public public public public public public private
## [55] public public public public public public public public public
## [64] public public public public public public public public public
## [73] public public public public private public public public public
## [82] public private private public public private public public public
## [91] public private public public public public private public private
## [100] public public private public public public public public public
## [109] public public private public public private public public public
## [118] private private public public public public private public private
## [127] public private public private public public public private public
```

```
## [136] private public public public private private private public private
## [145] public public private public public private public public public
## [154] private private public public public public public public private
## [163] private public public public public public private public private
## [172] public public public private private public private public public
## [181] public public public public public public public public public
## [190] public private public private public public public private public
## [199] public public
## Levels: private public
```

```
tabyl(sample_with_replacement2)
```

```
## sample_with_replacement2  n percent
##                public 160      0.8
##                private  40      0.2
```

That bootstrap sample proportion is 0.2.

Now we're getting some sampling variability!

If we do this many, many times, we get a whole collection of sample proportions. The distribution of all those sample proportions, obtained with bootstrap samples (samples drawn with replacement), is called the *bootstrap sampling distribution*.

## 12.4 Computing a bootstrap sampling distribution

The `infer` package can compute bootstrap samples and, hence, produce a bootstrap sampling distribution. The code looks a whole like the code you already know for hypothesis testing:

```
private_boot <- hsb2 %>%
  specify(response = schtyp, success = "private") %>%
  generate(reps = 1000, type = "bootstrap") %>%
  calculate(stat = "prop")
private_boot
```

```
## Response: schtyp (factor)
## # A tibble: 1,000 x 2
##   replicate  stat
##       <int> <dbl>
## 1         1 0.185
```

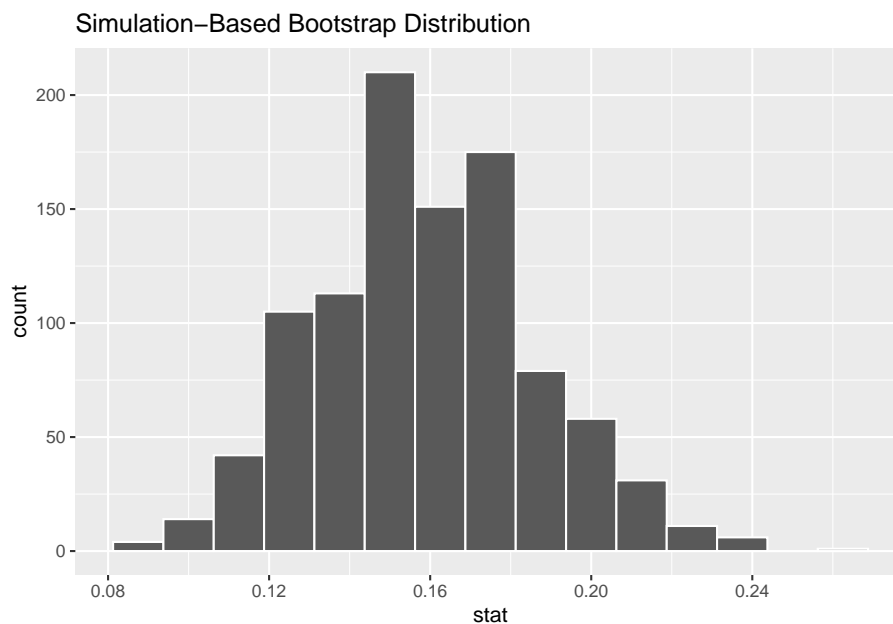


```
## 2      2 0.185
## 3      3 0.15
## 4      4 0.135
## 5      5 0.145
## 6      6 0.175
## 7      7 0.15
## 8      8 0.195
## 9      9 0.18
## 10     10 0.185
## # ... with 990 more rows
```

We simply changed the `type` to “bootstrap”.

Now we visualize like normal:

```
private_boot %>%
  visualize()
```



(We can change the number of bins if we want, but this number looks pretty good.)

## 12.5 Confidence intervals

The histogram above simulates what might happen if we took many samples from our infinite “fake” population consisting of many copies of our original,

actual sample data. On the lower end, we might see something like 8% private school students. On the upper end, we could see 25% or more private school students.

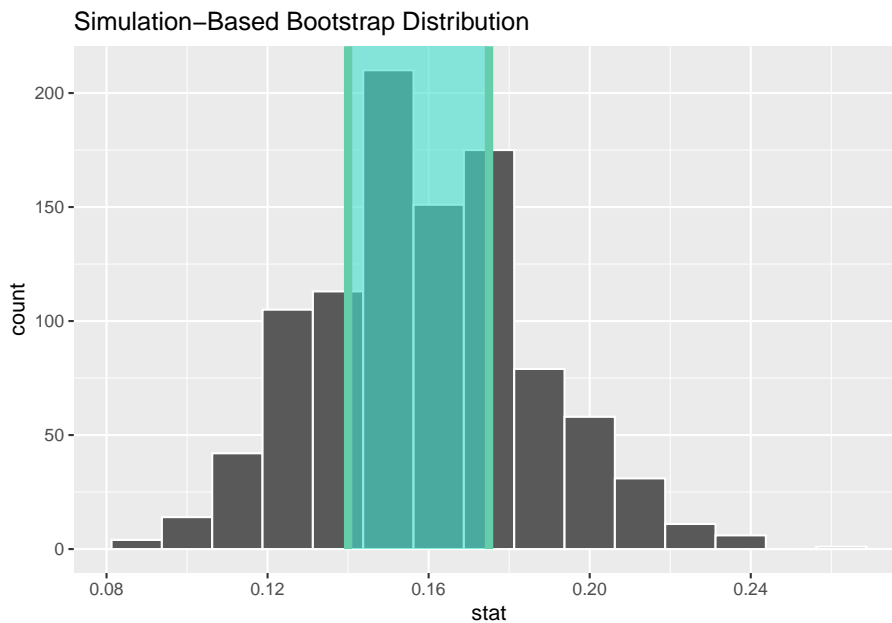
In the chapter about numerical data, we computed the IQR (interquartile range), which was the difference between the 25th percentile and the 75th percentile. The IQR was then the range of the middle 50% of the data. Let's use `infer` tools to calculate the middle 50% of the above distribution:

```
private_50 <- private_boot %>%
  get_confidence_interval(level = 0.5)
private_50
```

```
## # A tibble: 1 x 2
##   lower_ci upper_ci
##   <dbl>    <dbl>
## 1     0.14     0.175
```

The middle 50% ranges from 14% up to 17.5%. We can also visualize this:

```
private_boot %>%
  visualise() +
  shade_confidence_interval(endpoints = private_50)
```



In other words, when we go out to gather a sample from our (fake infinite) population of high school seniors, about half of the time, we expect the percentage of private students to be somewhere between 14% and 17.5%. The other half of the time, we will sample a value outside that range.

This is a confidence interval. More specifically, this is a 50% confidence interval. This is the range of values we expect sample proportions to be in approximately half of the samples we might gather from our (fake infinite) population.

Now don't forget the goal. What we are really trying to find is the value  $p$ , the true population parameter. We want to know what proportion of high school seniors attend private school in the whole population of all high school seniors in the U.S.

For mathematical reasons that are outside the scope of this course, it turns out that the sampling variability in the bootstrap distribution around  $\hat{p}$  is very similar to the sampling variability of the sample proportion  $\hat{p}$  around the true value  $p$ . We bootstrapped our way to the picture above using one actual sample with about 16% private school students. A different sample of high school seniors would give us different bootstrap samples, producing a slightly different bootstrap distribution from the one above. But it, too, will have a shaded region like the histogram above. Every actual sample we might obtain in the real world would give us a bootstrap distribution with a different shaded region. But the amazing fact is this: about half of those shaded regions will actually contain the true population parameter  $p$ .

Think about the value  $p$  like a fish hidden in a murky lake. The sample proportion  $\hat{p}$  is our attempt at fishing. We drop a hook down at the value  $\hat{p}$  and pull it right back up. It's not very likely that we caught the fish, although we hope that we were close. Alas, the sample proportion is almost never exactly equal to the true proportion  $p$ . But what if we cast a net instead? That net is the shaded range of values in our confidence interval. That range of values might catch the fish.

The difference between statistics and fishing is that, in the latter, when we pull up the net, we can see if we successfully caught the fish. In the former, all we can say is that there is some probability that the net caught the fish, but you're not able to look inside the net to know for sure.

So the confidence interval we created above might have caught the true value  $p$ . But then again, it might not have. There's only a 50% chance we captured the true value in the range 14% to 17.5% that we computed from our specific sample with its accompanying bootstrap samples. Most researchers would be displeased with only a 50% success rate. So can we do better?

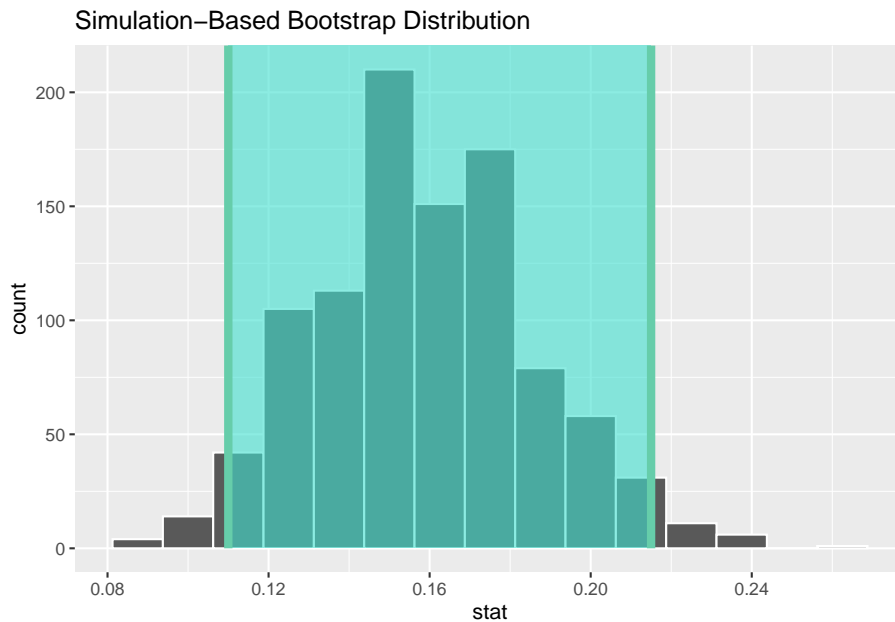
How much better do we want to do? This is a subjective question with no definitive answer. Many people say they want to be 95% confident that the confidence interval they build will capture the true population parameter. Let's modify our code to do that:

```
private_95 <- private_boot %>%
  get_confidence_interval(level = 0.95)
private_95
```

```
## # A tibble: 1 x 2
##   lower_ci upper_ci
##   <dbl>    <dbl>
## 1     0.11     0.215
```

The middle 95% ranges from 11% up to 21.5%. We can also visualize this:

```
private_boot %>%
  visualise() +
  shade_confidence_interval(endpoints = private_95)
```



The interpretation is that when you go collect many samples, the confidence intervals you produce using the bootstrap procedure described above will capture the true population proportion 95% of the time.

**Exercise 1** Why is a 95% confidence interval wider than a 50% confidence interval? In other words, why should our desire to be 95% confident in capturing the true value of  $p$  result in an interval that is wider than if we only wanted to be 50% confident?

Please write up your answer here.

**Exercise 2** Being more confident seems like a good thing. In fact, we might want a 99% confidence interval. Compute and visualize a 99% confidence interval for proportion of private school students.

```
# Add code here to compute a 99% confidence interval
```

```
# Add code here to visualize a 99% confidence interval
```

**Exercise 3** Can you think of any downside to using higher and higher confidence levels? As a hint, think about the following completely true sentence: “I am 100% confident that the true proportion of high school seniors attending private school is somewhere between 0% and 100%.”

Please write up your answer here.

---

While 50% is clearly too low for a confidence level, as seen above, there is no particular reason that we need to compute a 95% confidence interval either. There is some consensus in the scientific community here: 95% has evolved to become a generally agreed-upon standard. But we could compute a 90% confidence interval or a 99% confidence interval (as you did above), or any other type of interval. Having said that, if you choose other intervals besides these three, people might wonder if you’re up to something.<sup>1</sup>

## 12.6 Conditions

Don’t forget that there are always assumptions we make when relying on any kind of statistical inference. Before computing a confidence interval for a proportion, we must verify that certain conditions are satisfied. But these conditions are not new. We already know from hypothesis testing what is required for good inference from a sample. These are the “Random” and the “10%” conditions.

- Random
  - The sample must be random (or hopefully representative).
- 10%
  - The sample size must be less than 10% of the size of the population.

Both conditions are met for the data in the High School and Beyond survey.

---

<sup>1</sup>A contrary position is proffered by Richard McElreath, an evolutionary ecologist and author of the amazing book *Statistical Rethinking*. He uses 89% and 97% intervals to highlight the absurdity of regarding 95% as a magic number that has some kind of deep, special meaning.

## 12.7 Rubric for confidence intervals

Typically, you will be asked to report a confidence interval after performing a hypothesis test. Whereas a hypothesis test gives you a “decision criterion” (using data to make a decision to reject the null or fail to reject the null), a confidence interval gives you an estimate of the “effect size” (a range of plausible values for the population parameter).

As such, there is a section in the Rubric for inference that shows the steps of calculating and reporting a confidence interval. They are as follows:

1. Check the relevant conditions to ensure that model assumptions are met.
2. Calculate and graph the confidence interval.
3. State (but do not overstate) a contextually meaningful interpretation.
4. If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test.
5. When comparing two groups, comment on the effect size and the practical significance of the result.

## 12.8 Example

Here is a worked example. (Unless otherwise stated, we always use a 95% confidence level.)

Some of the students in the “High School and Beyond” survey attended vocational programs. This data is stored in the `prog` variable. Using a confidence interval, estimate what percentage of all high school seniors attend vocational programs.

We will need to do a little data cleaning before we can address this question. There are actually three types of programs: “general”, “academic”, and “vocational”. The `infer` commands will only work when a categorical variable has two levels. We are thinking of “general” and “academic” together as more like a combined “other” category. We can fix this by creating a new factor variable with `mutate`. Inside that `mutate`, we will use the `fct_collapse` function to collapse two of the levels into one as follows:

```
hsb2 <- hsb2 %>%  
  mutate(prog2 = fct_collapse(prog,  
                                vocational = "vocational",  
                                other = c("general", "academic")))  
glimpse(hsb2)
```

```
## Rows: 200
```

```
## Columns: 12
## $ id      <int> 70, 121, 86, 141, 172, 113, 50, 11, 84, 48, 75, 60, 95, 104, 3~
## $ gender  <chr> "male", "female", "male", "male", "male", "male", "male", "mal~
## $ race    <chr> "white", "white", "white", "white", "white", "white", "african~
## $ ses     <fct> low, middle, high, high, middle, middle, middle, middle, middl~
## $ schtyp  <fct> public, public, public, public, public, public, public, public~
## $ prog    <fct> general, vocational, general, vocational, academic, academic, ~
## $ read    <int> 57, 68, 44, 63, 47, 44, 50, 34, 63, 57, 60, 57, 73, 54, 45, 42~
## $ write   <int> 52, 59, 33, 44, 52, 52, 59, 46, 57, 55, 46, 65, 60, 63, 57, 49~
## $ math    <int> 41, 53, 54, 47, 57, 51, 42, 45, 54, 52, 51, 51, 71, 57, 50, 43~
## $ science <int> 47, 63, 58, 53, 53, 63, 53, 39, 58, 50, 53, 63, 61, 55, 31, 50~
## $ socst   <int> 57, 61, 31, 56, 61, 61, 61, 36, 51, 51, 61, 61, 71, 46, 56, 56~
## $ prog2   <fct> other, vocational, other, vocational, other, other, other, oth~
```

Inspect the variables `prog` and `prog2` above to make sure that the recoding was successful. Then be sure to use `prog2` and not `prog` everywhere.

### 12.8.1 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - The sample is a random sample of high school seniors from the U.S. as the survey was conducted by the National Center of Education Statistics, a reputable government organization.
- 10%
  - The sample size is 200, which is much less than 10% of the population of all U.S. high school seniors.

### 12.8.2 Calculate and graph the confidence interval.

```
vocational_boot <- hsb2 %>%
  specify(response = prog2, success = "vocational") %>%
  generate(reps = 1000, type = "bootstrap") %>%
  calculate(stat = "prop")
vocational_boot
```

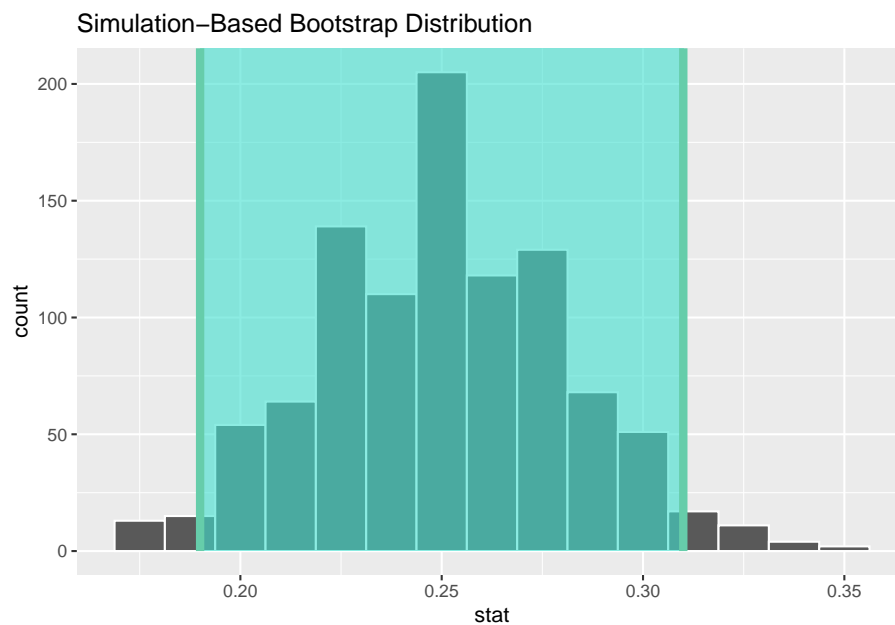
```
## Response: prog2 (factor)
## # A tibble: 1,000 x 2
##   replicate stat
##       <int> <dbl>
```

```
## 1      1 0.335
## 2      2 0.25
## 3      3 0.17
## 4      4 0.24
## 5      5 0.245
## 6      6 0.245
## 7      7 0.2
## 8      8 0.24
## 9      9 0.265
## 10     10 0.25
## # ... with 990 more rows
```

```
vocational_ci <- vocational_boot %>%
  get_confidence_interval(level = 0.95)
vocational_ci
```

```
## # A tibble: 1 x 2
##   lower_ci upper_ci
##   <dbl>    <dbl>
## 1     0.19     0.31
```

```
vocational_boot %>%
  visualize() +
  shade_confidence_interval(endpoints = vocational_ci)
```





### 12.8.3 State (but do not overstate) a contextually meaningful interpretation.

We are 95% confident that the true percentage of U.S. high school seniors who attend a vocational program is captured in the interval (19%, 31%).

Note: we use inline code to grab the values of the endpoints of the confidence interval. We also multiply by 100 to report percentages instead of proportions.

### 12.8.4 If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test.

In this chapter, we haven't run a hypothesis test, so this step is irrelevant for us here. However, in future chapters, we will incorporate this step into the rubric and see how the confidence interval relates to the conclusion of a hypothesis test.

### 12.8.5 When comparing two groups, comment on the effect size and the practical significance of the result.

This step will also become more clear in future chapters. It only applies to situations where you are attempting to find a difference between two groups. In this example, we're simply using a sample statistic to estimate a single population parameter.

## 12.9 Your turn

Use the `smoking` data set from the `openintro` package. What percentage of the population of the U.K. smokes tobacco? (The information you need is in the `smoke` variable.) Use a 95% confidence interval.

**Check the relevant conditions to ensure that model assumptions are met.**

- Random
  - [Check condition here.]
- 10%
  - [Check condition here.]

```
# Add code here to create the bootstrap sampling distribution.
```

```
# Add code here to calculate the confidence interval.
```

```
# Add code here to graph the confidence interval.
```

**Calculate and graph the confidence interval.**

**State (but do not overstate) a contextually meaningful interpretation.**  
Please write up your answer here.

---

(We will ignore the last two last steps in the rubric. We haven't run a hypothesis test and we're not comparing smoking between two groups.)

## 12.10 Interpreting confidence intervals

Confidence intervals are notoriously difficult to interpret.<sup>2</sup>

Here are several *wrong* interpretations of a 95% confidence interval:

- 95% of the data lies in the interval.
- There is a 95% chance that the sample proportion lies in the interval.
- There is a 95% chance that the population parameter lies in the interval.

We'll take a closer look at these incorrect claims in a moment. First, let's see how confidence intervals work using simulation.

In order to simulate, we'll have to pretend temporarily that we know a true population parameter. Let's use the example of a candidate who has the support of 64% of voters. In other words,  $p = 0.64$ . We go out and get a sample of voters, let's say 50. From that sample we construct a 95% confidence interval

---

<sup>2</sup>Several studies have given surveys to statistics students, teachers, and researchers, and find that even these people often misinterpret confidence intervals. See, for example, this paper: <http://www.ejwagenmakers.com/inpress/HoekstraEtAlPBR.pdf>

by bootstrapping. Most of the time, 64% (the true value!) should be in our interval. But sometimes it won't be. We can get an unusual sample that is far away from 64%, just by pure chance alone. (Perhaps we accidentally run into a bunch of people who oppose our candidate.)

Okay, let's do it again. Get a new sample and calculate a new confidence interval. This sample will likely result in a different sample proportion than the first sample. Therefore, the confidence interval will be located in a different place. Does it contain 64%? Most of the time, we expect it to. Occasionally, it will not.

We can do this over and over again through the magic of simulation! Here's what this simulation looks like in R. The following code is quite technical, although you will recognize bits and pieces of it. Don't worry about it. You won't need to generate code like this on your own. Just look at the pretty picture in the output below the code.

```
set.seed(11111)

# The true population proportion is 0.64
true_val <- 0.64
# The sample size is 50
sample_size <- 50
# Set confidence level
our_level <- 0.95
# Set number of intervals to simulate
sim_num <- 100

# Get a random sample of size n.
# Compute the test statistic and the bootstrap confidence interval.
# Put both into a single tibble.
simulate_ci <- function(n, level = 0.95) {
  sample_data <-
    factor(rbinom(n, size = 1, prob = true_val)) %>%
    tibble(data = .)
  stat <- sample_data %>%
    observe(response = data, success = "1", stat = "prop")
  ci <- sample_data %>%
    specify(response = data, success = "1") %>%
    generate(reps = 1000, type = "bootstrap") %>%
    calculate(stat = "prop") %>%
    get_confidence_interval(level = our_level)
  bind_cols(stat, ci) %>%
    return()
}

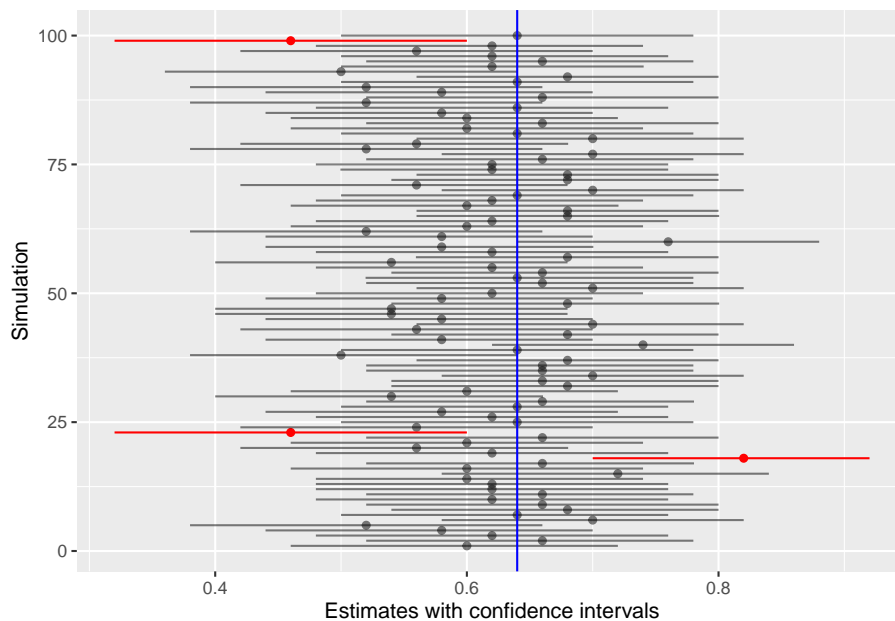
# Simulate 100 random samples (each of size 50)
```

```

# Assign a color based on whether the intervals contain the true proportion
ci <- map_dfr(rep(sample_size, times = sim_num), simulate_ci, level = our_level) %>%
  mutate(row_num = row_number()) %>%
  mutate(color = ifelse(lower_ci <= true_val & true_val <= upper_ci,
                        "black", "red"),
         alpha = ifelse(color == "black", 0.5, 1))

# Plot all the simulated intervals
ggplot(ci, aes(x = stat, y = row_num,
              color = color, alpha = alpha)) +
  geom_point() +
  scale_color_manual(values = c("black", "red"), guide = "none") +
  geom_segment(aes(x = lower_ci, xend = upper_ci, yend = row_num)) +
  geom_vline(xintercept = true_val, color = "blue") +
  scale_alpha_identity() +
  labs(y = "Simulation", x = "Estimates with confidence intervals")

```



Each sample gives us a slightly different estimate, and therefore, a different confidence interval as well.

For each of the 100 simulated intervals, most of them (the black ones) do capture the true value of 0.64 (the blue vertical line). Occasionally they don't (the red ones). We expect 5 red intervals, but since randomness is involved, it won't necessarily be exactly 5. (Here there were only 3 bad intervals.)

This is the key to interpreting confidence intervals. The “95%” in a 95% con-

fidence interval means that if we were to collect many random samples, about 95% of them would contain the true population parameter and about 5% would not.

So let's revisit the erroneous statements from the beginning of this section and correct the misconceptions.

- ~~95% of the data lies in the interval.~~
  - This doesn't even make sense. Our data is categorical. The confidence interval is a range of plausible values for the proportion of successes in the sample.
- ~~There is a 95% chance that the sample proportion lies in the interval.~~
  - No. There is essentially a 100% chance that the sample proportion lies in the interval. Most of the time, the sample proportion is very close to the center of the interval. When we bootstrap, the "infinite population" we are simulating has the same population proportion as the sample we started with. (After all, the infinite population is just many copies of the sample we started with.) Therefore, samples from that infinite population should be more or less centered around the sample proportion.
- ~~There is a 95% chance that the population parameter lies in the interval.~~
  - This is wrong in a more subtle way. The problem here is that it takes our interval as being fixed and special, and then tries to declare that of all possible population parameters, we have a 95% chance of the true one landing in our interval. The logic is backwards. The population parameter is the fixed truth. It doesn't wander around and land in our interval sometimes and not at other times. It is our confidence interval that wanders; it is just one of many intervals we could have obtained from random sampling. When we say, "We are 95% confident that...", we are just using a convenient shorthand for, "If we were to repeat the process of sampling and creating confidence intervals many times, about 95% of those times would produce an interval that happens to capture the actual population proportion." But we're lazy and we don't want to say that every time.

## 12.11 Conclusion

A confidence interval is a form of statistical inference that gives us a range of numbers in which we hope to capture the true population parameter. Of course, we can't be certain of that. If we repeatedly collect samples, the expectation is that 95% of those samples will produce confidence intervals that capture the true population parameter, but that also means that 5% will not. We'll never

know if our sample was one of the 95% that worked, or one of the 5% that did not. And even if we get one of the intervals that worked, all we have is a range of values and it's impossible to determine which of those values is the true population parameter. Because it's statistics, we just have to live with that uncertainty.

### 12.11.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1–2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.

# Chapter 13

## Normal models

2.0

### Functions introduced in this chapter

`pdist`, `diff`, `qdist`, `scale`, `geom_qq`

### 13.1 Introduction

In this chapter we will learn how to work with normal models. In addition to learning about theoretical normal distributions, we will also develop QQ plots to assess the normality of data.

#### 13.1.1 Install new packages

There are no new packages used in this chapter.

#### 13.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/13-normal\\_models.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/13-normal_models.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 13.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 13.2 Load packages

In addition to `tidyverse`, we return to the `mosaic` package to produce some nice visualizations of normal models.

```
library(tidyverse)
library(mosaic)
```

## 13.3 The Central Limit Theorem

An important aspect of all the simulations that we’ve done so far—assuming that we’ve run a large enough number of them—is that their histograms all look like bell curves. This fact is known as the “Central Limit Theorem”. Under some basic assumptions that we’ll discuss in a later chapter, this will be typical of many of our simulated null distributions.

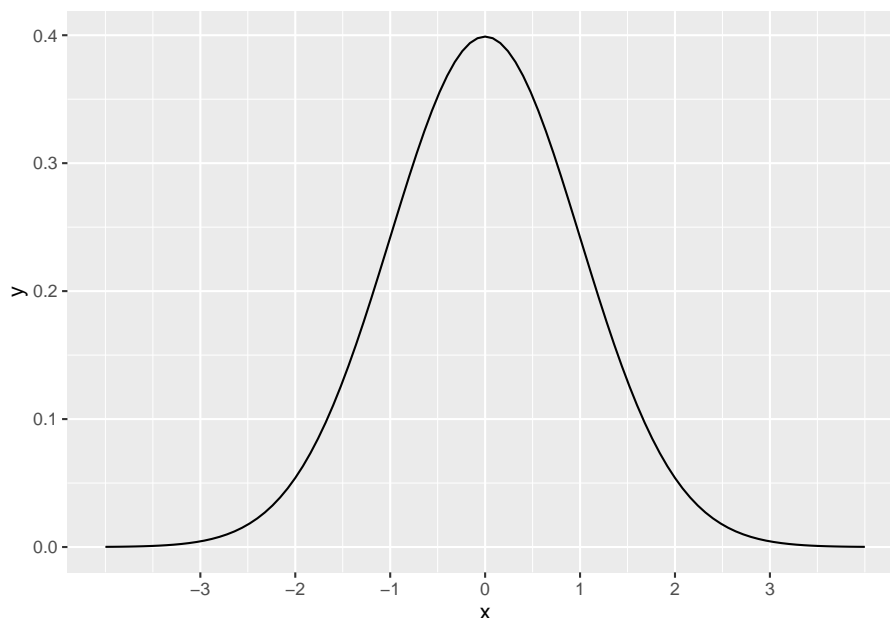
So rather than running a simulation each time we want to conduct a hypothesis test, we could also assume that the null distribution *is* a bell curve. The rest of this chapter will teach you how to work with the “normal distribution,” which is just the mathematically correct term for a bell curve.

## 13.4 Normal models

The normal distribution looks like this:

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
ggplot(data.frame(x = c(-4, 4)), aes(x)) +
  stat_function(fun = dnorm) +
  scale_x_continuous(breaks = -3:3)
```





The curve pictured above is called the *standard normal distribution*. It has a mean of 0 and a standard deviation of 1. Mathematically, this is written as

$$N(\mu = 0, \sigma = 1),$$

or usually just

$$N(0, 1).$$

We use this bell curve shape to model data that is unimodal, symmetric, and without outliers. A statistical “model” is a simplification or an idealization. Reality is, of course, never perfectly bell-shaped. Real data is not exactly symmetric with one clear peak in the middle. Nevertheless, an abstract model can give us good answers if used properly.

As an example of this, systolic blood pressure (SBP, measured in millimeters of mercury, or mmHg) is more-or-less normally distributed in women ages 30–44 in the U.S. and Canada, with a mean of 114 and a standard deviation of 14.<sup>1</sup>

If we were to plot a histogram with the SBP of every woman between the ages of 30 and 44 in the U.S. and Canada, it would have the shape of a normal distribution, but instead of being centered at 0 like the graph above, this one would be centered at 114. Mathematically, we write

---

<sup>1</sup>Statistics from the World Health Organization: <http://www.who.int/publications/cra/chapters/volume1/0281-0390.pdf>

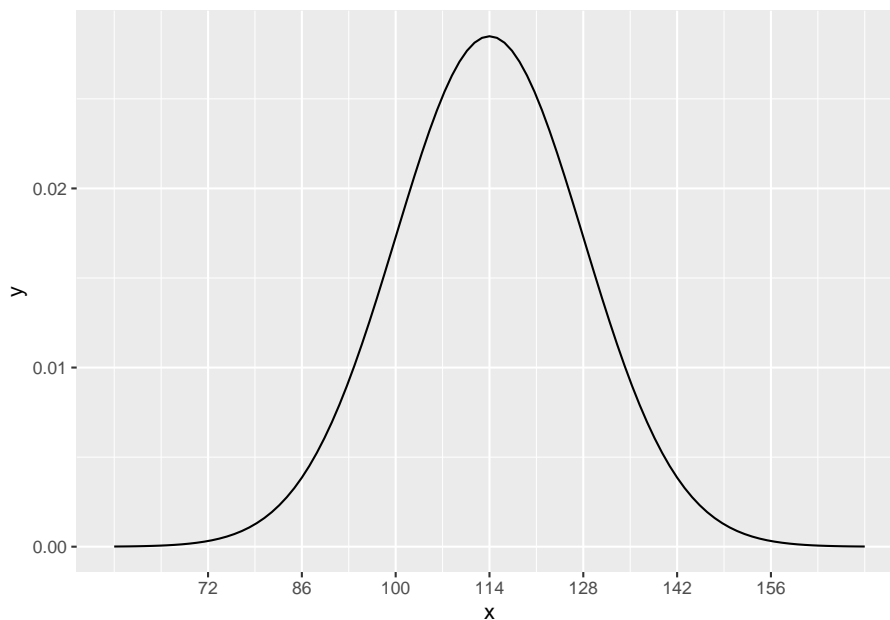
$$N(\mu = 114, \sigma = 14),$$

or

$$N(114, 14).$$

The graph now looks like this:

```
# Again, don't worry about the syntax here.
ggplot(data.frame(x = c(58, 170)), aes(x)) +
  stat_function(fun = dnorm, args = list(mean = 114, sd = 14)) +
  scale_x_continuous(breaks = c(72, 86, 100, 114, 128, 142, 156))
```

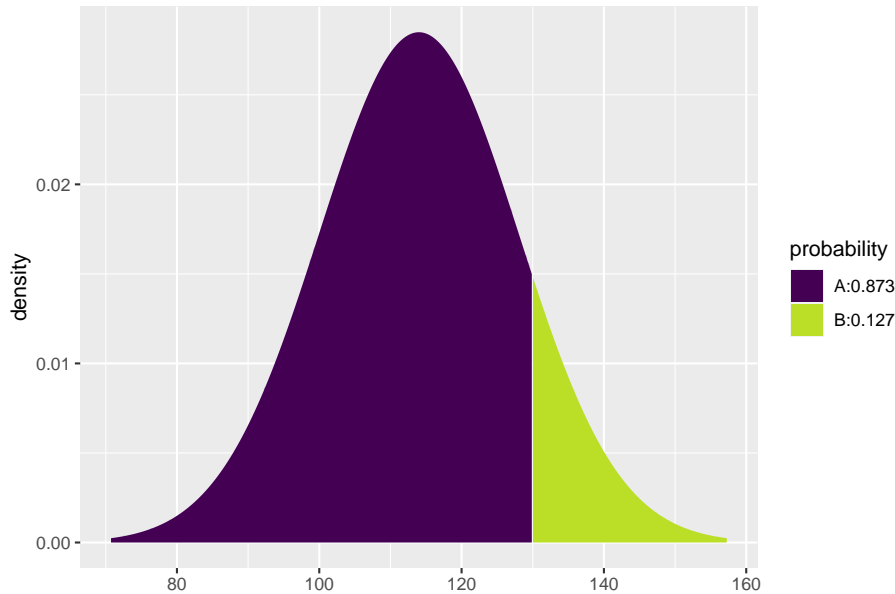


## 13.5 Predictions using normal models

Using this information, we can estimate the percentage of such women who are expected to have any range of SBP without having access to all such data.

For example, what percentage of women ages 30–44 in the U.S. and Canada are expected to have SBP under 130 mmHg? The `pdist` command from the `mosaic` package will not only help us with this calculation, but it also offers a nice visual representation depending on the arguments we supply to the function:

```
pdist("norm", q = 130, mean = 114, sd = 14)
```



```
## [1] 0.873451
```

In the notebook view, you have to switch back and forth between the two boxes below the code chunk (above the graph) to see the number versus the graph. In the HTML output, however, both the number and the plot are visible.

For situations where we really just want to see the number, we can always add `plot = FALSE` to the function:

```
pdist("norm", q = 130, mean = 114, sd = 14, plot = FALSE)
```

```
## [1] 0.873451
```

The other pieces of the `pdist` function are pretty intuitive: `"norm"` (and it has to be in quotes) indicates that we want a normal model, `q` is the value of interest to us, and `mean` and `sd` are self-evident. The numerical output gives the area under the curve to the left of our value of interest. This area is 0.873451; in other words, about 87.3% of women are expected to have SBP less than 130.

If you use this command inline, the pretty picture is not generated, just the value. For example, look at the following sentence (remembering that you can click anywhere inside the inline R code and hit Ctrl-Enter or Cmd-Enter):

The model predicts that 87.3451046% of women ages 30–44 in the U.S. and Canada will have systolic blood pressure under 130 mmHg.

Note that the above code multiplied the result of the `pdist` command by 100. This is important because the full sentence interpretation is meant to be read by human beings, and human beings tend to report these kinds of numbers as percentages and not decimals.<sup>2</sup>

It's also important that you include the phrase, "The model predicts..." or something like that. Without that part, the claim is likely false. It would be too definitive. Remember that a model is just an approximation or simplification of reality. We're not claiming we've found the "True" number. All we know is that if the model is roughly correct, we can predict the true value.

Here's another question: how many women are predicted to have SBP *greater* than 130? If 87.3% of women have SBP under 130, then 12.7% must have SBP over 130. Why? Because all women have to add up to 100%!

Therefore, all we have to do to solve this problem is subtract the number we obtained in the previous question from 1. (Remember that 1 = 100%.)

The model predicts that 12.6548954% of women ages 30–44 in the U.S. and Canada will have systolic blood pressure over 130 mmHg.

Don't forget to include parentheses. We need to multiply the whole expression by 100.

Now, here's a more complicated question: what percentage of women are predicted to have SBP between 110 mmHg and 130 mmHg?

Recall that the proportion of women predicted to have SBP less than 130 mmHg was 0.873. But this is also counting women with SBP under 110 mmHg, whom we now want to exclude. The proportion of women with SBP under 110 is found with the following code:

```
pdist("norm", q = 110, mean = 114, sd = 14, plot = FALSE)
```

```
## [1] 0.3875485
```

Therefore, all we have to do is calculate 0.873 minus 0.388:

The model predicts that 48.5902564% of women ages 30–44 in the U.S. and Canada will have systolic blood pressure between 110 mmHg and 130 mmHg.

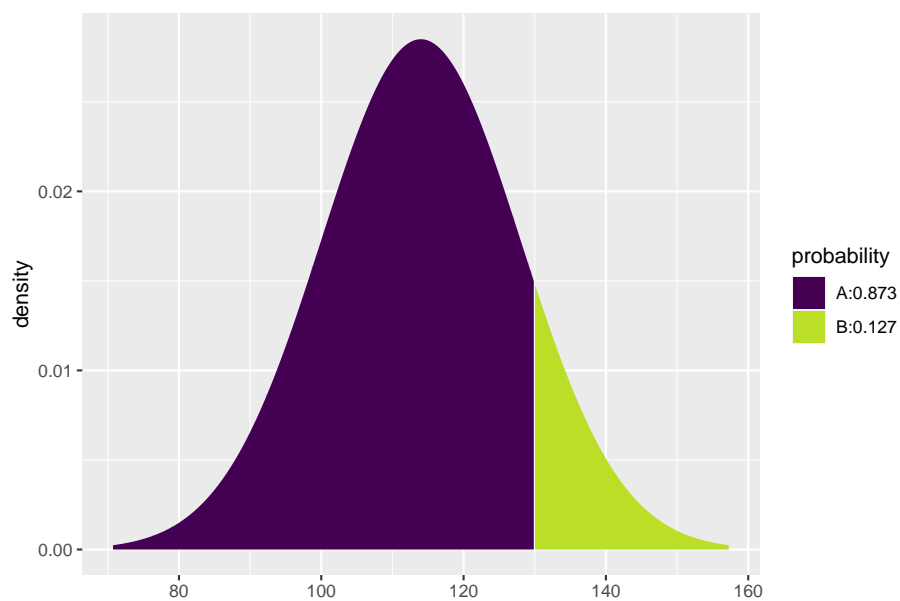
---

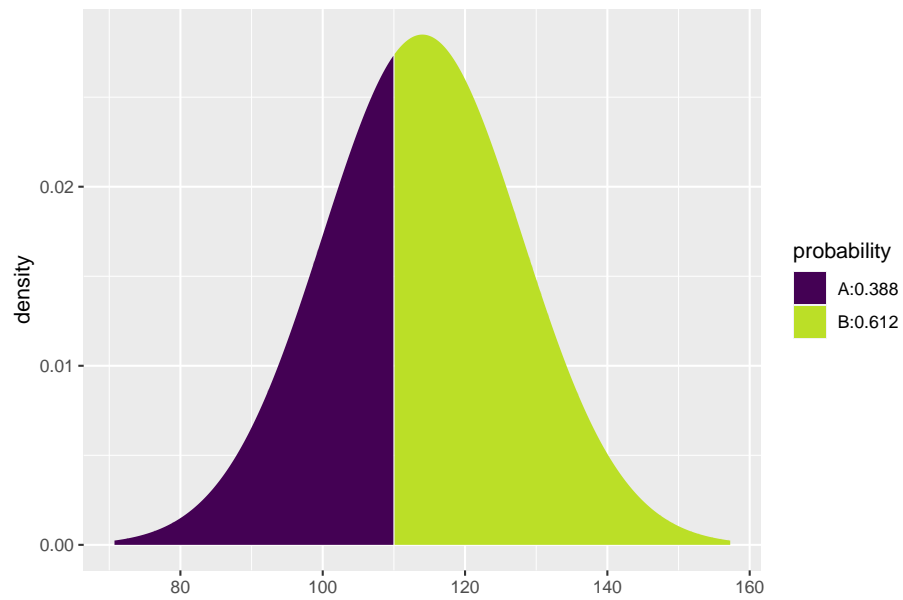
<sup>2</sup>When you preview this in HTML, you'll see a ridiculous number of decimal places that R reports. It's a bit of a hassle to try to change it, so we'll just ignore the issue.

(Again, don't forget the parentheses.)

What about the pretty picture? Unfortunately, this doesn't work so well:

```
pdist("norm", q = 130, mean = 114, sd = 14) -  
  pdist("norm", q = 110, mean = 114, sd = 14)
```



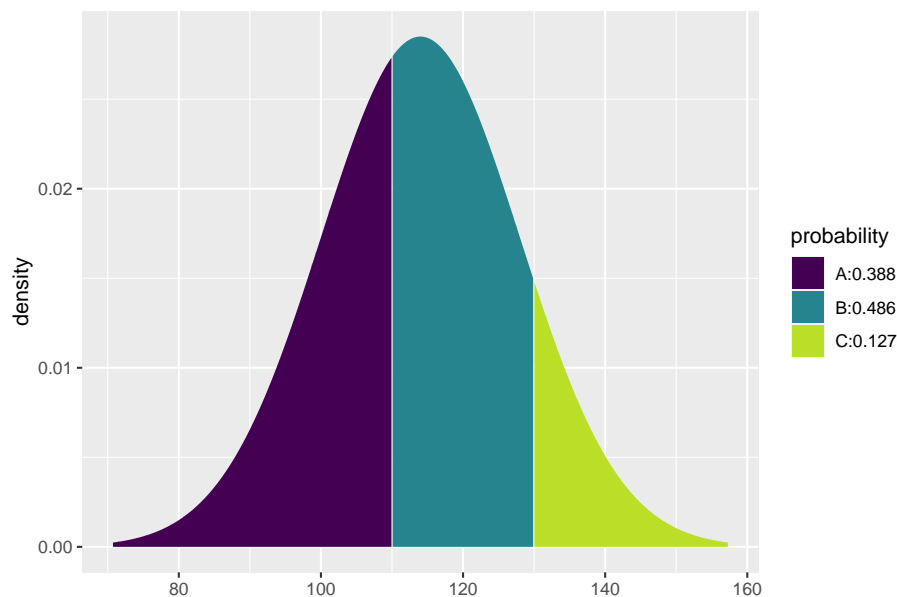


```
## [1] 0.4859026
```

The code is bulky and it prints two pictures, neither of which are quite right for our question.

Instead, let's observe that the `pdist` command can include both values (110 and 130) using the vector notation `c`:

```
pdist("norm", q = c(110, 130), mean = 114, sd = 14)
```



```
## [1] 0.3875485 0.8734510
```

Now the picture looks great and you can see the proportion you desire in the area between the two lines at 110 and 130.

This doesn't work so well for the numerical output though. Observe:

```
pdist("norm", q = c(110, 130), mean = 114, sd = 14, plot = FALSE)
```

```
## [1] 0.3875485 0.8734510
```

There are two numbers shown, but neither is the correct answer. This command shows the percentages below 110 and below 130, respectively, but not the area in between 110 and 130. We still have to subtract. However, R can do this for us easily with the `diff` command:

```
pdist("norm", q = c(110, 130), mean = 114, sd = 14, plot = FALSE) %>%
  diff()
```

```
## [1] 0.4859026
```

Again, for inline R code, you don't need to specify `plot = FALSE`:

The model predicts that 48.5902564% of women ages 30–44 in the U.S. and Canada will have systolic blood pressure between 110 mmHg and 130 mmHg.

For the following exercises, we'll use a running example of IQ scores. Keep in mind that, at best, IQ scores fail to measure anything like “intelligence” (<https://www.sciencedaily.com/releases/2012/12/121219133334.htm>). At worse, IQ tests (and other forms of standardized testing) have been used to perpetuate systemic racism and inequality (<https://www.nea.org/advocating-for-change/new-from-nea/racist-beginnings-standardized-testing>).

IQ scores—whatever they actually measure—are standardized so that they have a mean of 100 and a standard deviation of 16. For each exercise, use the `pdist` to draw the right picture and then state your answer in a contextually meaningful full sentence using inline R code. Don't forget to use the phrase “The model predicts...” and report numbers as percentages, not decimals.

**Exercise 1(a)** What percentage of people would you expect to have IQ scores over 80?

```
# Add code here to draw the model.
```

Please write up your answer here.

**Exercise 1(b)** What percentage of people would you expect to have IQ scores under 90?

```
# Add code here to draw the model.
```

Please write up your answer here.

**Exercise 1(c)** What percentage of people would you expect to have IQ scores between 112 and 132?

```
# Add code here to draw the model.
```

Please write up your answer here.

## 13.6 Percentiles

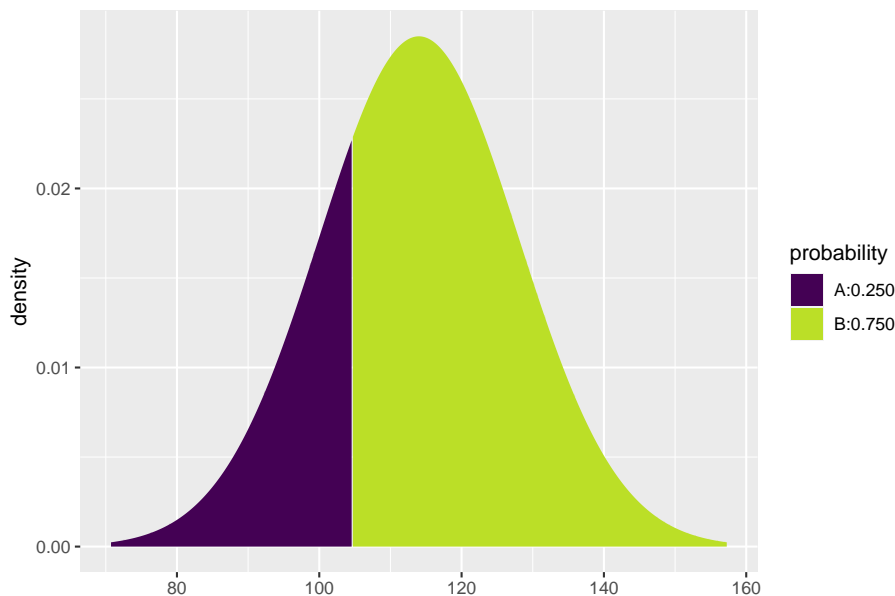
Often, the question is reversed: instead of getting a value and being asked what percentage of the population falls above or below it, we are given a percentile and asked about the value to which it corresponds.



Here is an example using systolic blood pressure: what is the cutoff value of SBP for the lowest 25% of women ages 30–44 in the U.S. and Canada? In other words, what is the 25th percentile of SBP for this group of women?

The command we need is `qdist`. It looks a lot like `pdist`. Observe:

```
qdist("norm", p = 0.25, mean = 114, sd = 14)
```



```
## [1] 104.5571
```

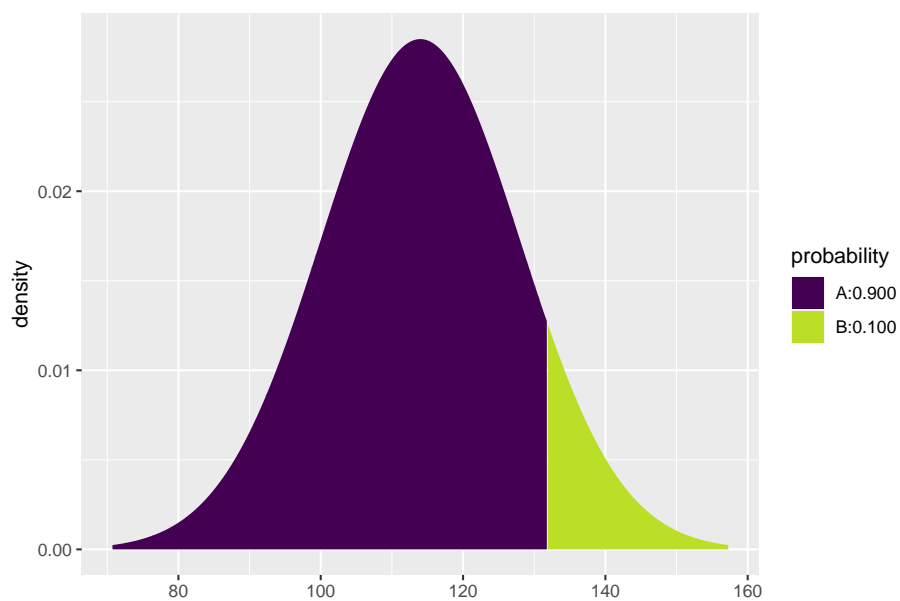
The only change here is that one of the arguments is `p` instead of `q`, and the value of `p` is a proportion (between 0 and 1) instead of a value of SBP. The *output* is now an SBP value.

Here it is inline:

The model predicts that the 25th percentile for SBP in women ages 30–44 in the U.S. and Canada is 104.5571435 mmHg.

What if we asked about the highest 10% of women? All you have to do is remember that the top 10% is actually the 90th percentile.

```
qdist("norm", p = 0.9, mean = 114, sd = 14)
```

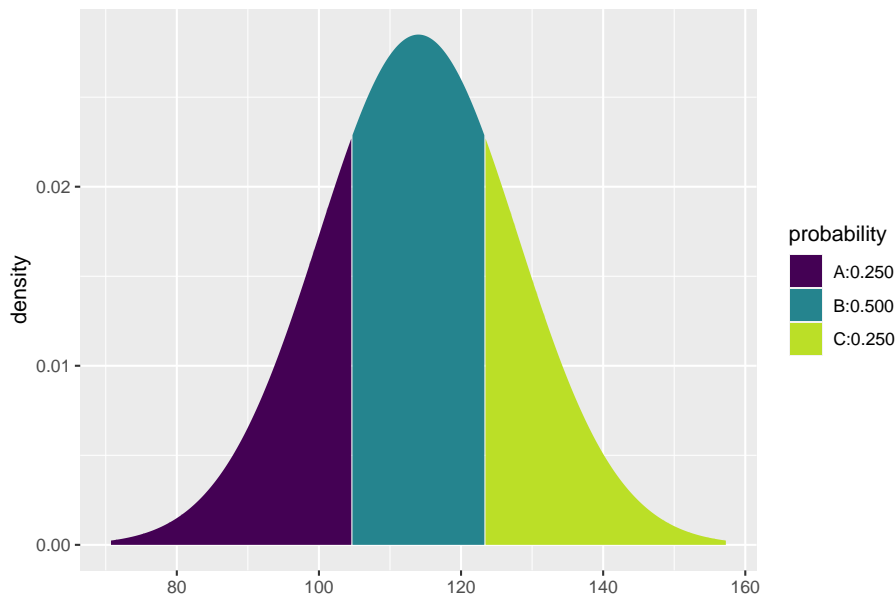


```
## [1] 131.9417
```

The model predicts that the top 10% of SBP in women ages 30–44 in the U.S. and Canada have SBP higher than 131.9417219 mmHg.

Finally, what if we want the middle 50%? This is trickier. The middle 50% lies between the 25th percentile and the 75th percentile. Observe the syntax below:

```
qdist("norm", p = c(0.25, 0.75), mean = 114, sd = 14)
```



```
## [1] 104.5571 123.4429
```

Therefore, the model predicts that the middle 50% of SBP for women ages 30–44 in the U.S. and Canada lies between 104.5571435 mmHg and 123.4428565 mmHg.

We did something tricky in the inline code above. Because the `qdist` command produces two values (one at the 25th percentile and one at the 75th percentile), we can grab each value separately by appending `[1]` or `[2]` to the end of the command.

For the exercises below, we'll continue to use IQ scores (mean of 100 and standard deviation of 16). Use the `qdist` command to draw the right picture and then state your answer in a contextually meaningful full sentence. Don't forget to use the phrase "The model predicts..."

**Exercise 2(a)** What cutoff value bounds the highest 5% of IQ scores?

```
# Add code here to draw the model.
```

Please write up your answer here.

**Exercise 2(b)** What cutoff value bounds the lowest 30% of IQ scores?

```
# Add code here to draw the model.
```

Please write up your answer here.

**Exercise 2(c)** What cutoff values bound the middle 80% of IQ scores?

```
# Add code here to draw the model.
```

Please write up your answer here.

## 13.7 Z scores

Sometimes it is easier to refer to a value in terms of how many standard deviations it lies from the mean. For example, a systolic blood pressure of 100 is 14 mmHg below the mean, but since the standard deviation is 14 mmHg, this means that 100 is one standard deviation below the mean. This distance from the mean in terms of standard deviations is called a *z score*.

We calculate z scores using the following formula:

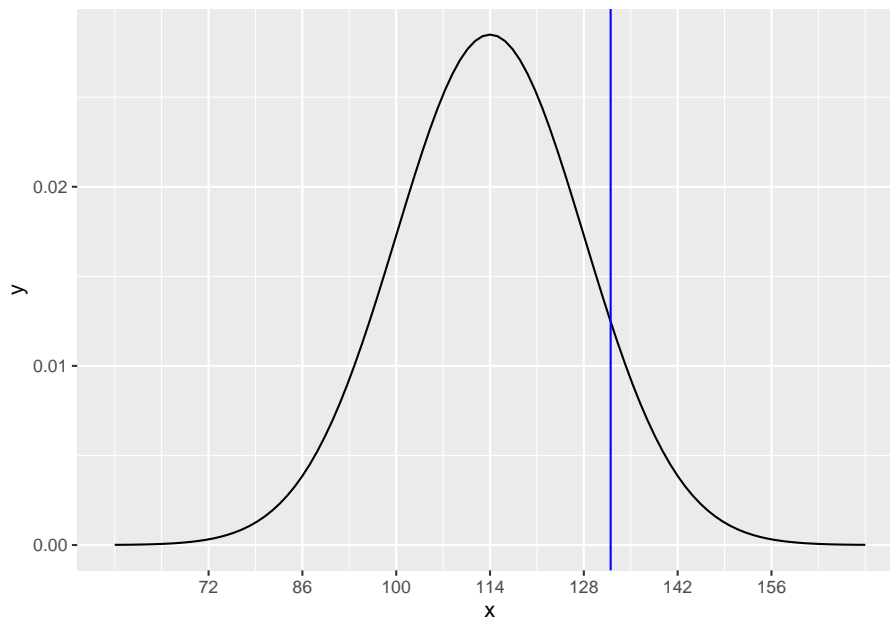
$$z = \frac{x - \mu}{\sigma}.$$

In our example, if we wanted to know the z score for an SBP of 100, we just plug all the numbers into the formula above:

$$z = \frac{100 - 114}{14} = -1.$$

What is the z score for an SBP of 132? Look at the graph of the normal model  $N(114, 14)$ :

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
ggplot(data.frame(x = c(58, 170)), aes(x)) +
  stat_function(fun = dnorm, args = list(mean = 114, sd = 14)) +
  scale_x_continuous(breaks = c(72, 86, 100, 114, 128, 142, 156)) +
  geom_vline(xintercept = 132, color = "blue")
```



We can see that 132 lies between 128 and 142, which are 1 and 2 standard deviations above the mean, respectively. The exact z score is

$$z = \frac{132 - 114}{14} = 1.285714.$$

The `scale` function from R also computes z scores. Just note that the function takes arguments `center` and `scale`, not `mean` and `sd`.

```
scale(x = 100, center = 114, scale = 14)
```

```
##      [,1]
## [1,]   -1
## attr(,"scaled:center")
## [1] 114
## attr(,"scaled:scale")
## [1] 14
```

```
scale(x = 132, center = 114, scale = 14)
```

```
##      [,1]
## [1,] 1.285714
## attr(,"scaled:center")
## [1] 114
```

```
## attr("scaled:scale")
## [1] 14
```

Also note that the function spits about a bunch of extra crap we don't care about. This goes away for inline code. Go ahead and preview the HTML file now so you can see the effect in the following sentence:

The z score for 100 is -1 and the z score for 132 is 1.2857143.

**Exercise 3** If IQ scores have a mean of 100 and a standard deviation of 16, what are the z scores for the following IQ scores? Write up your answers as full sentences using inline R code.

- 80

Please write up your answer here.

- 102

Please write up your answer here.

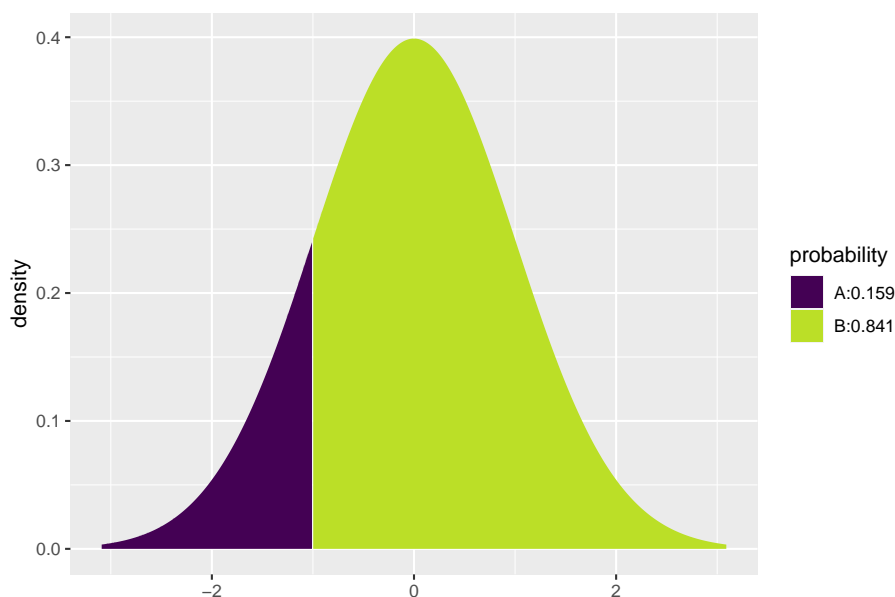
- 130

Please write up your answer here.

---

Working with z scores also makes it easier to work with normal models. The default settings for `pdist` and `qdist` are `mean = 0` and `sd = 1`. That saves you some typing. So, for example, we calculated above that an SBP of 100 has a z score of -1. What percentage of women are expected to have SBP lower than 100?

```
pdist("norm", q = -1)
```



```
## [1] 0.1586553
```

The model predicts that 15.8655254% of women ages 30–44 in the U.S. and Canada will have SBP less than 100.

**Exercise 4** Albert Einstein supposedly had an IQ of 160. Calculate the z score for his IQ and then use that z score to figure out what percentage of the population is predicted to have higher IQ than Einstein. Use full sentences and inline R code to express your answer.

Please write up your answer here.

## 13.8 QQ plots

All of the work we do with normal models assumes that a normal model is appropriate. When we want to summarize data using a normal model, this means that the data distribution should be reasonably unimodal, symmetric, and with no serious outliers.

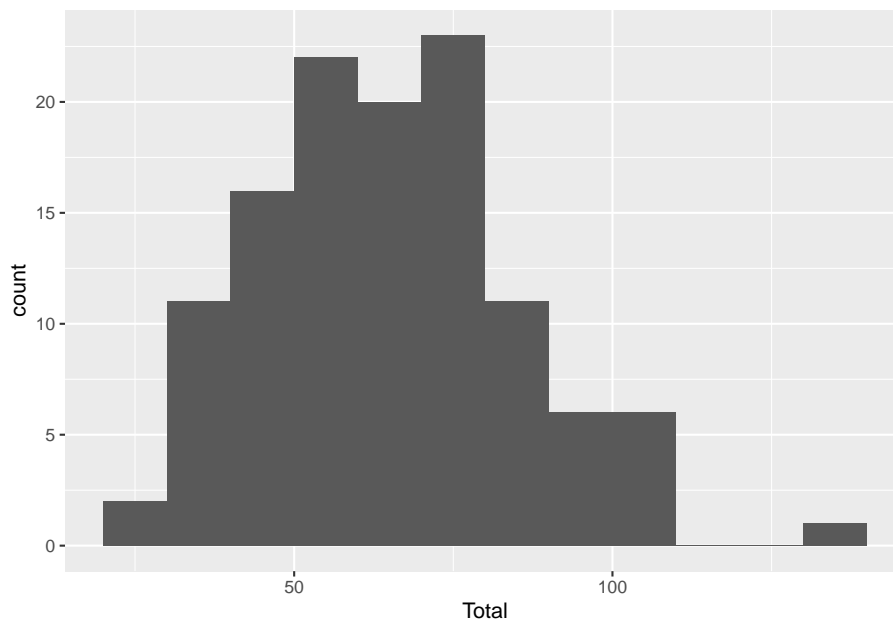
We can, of course, use a histogram to check this. But a histogram can be highly sensitive to the choice of bins. Furthermore, for small sample sizes, histograms look “chunky”, making it hard to test this assumption.

An easier way to check normality is to use a *quantile-quantile plot*, typically called a *QQ plot* or sometimes a *normal probability plot*. We won't get into the technicalities of how this plot works. Suffice it to say that if data is normally distributed, the points of a QQ plot should lie along a diagonal line.

Here is an example. The total snowfall in Grand Rapids, Michigan has been recorded every year since 1893. This data is included with the *mosaic* package in the data frame `SnowGR`. A histogram (with reasonable binning) shows that the data is nearly normal.

```
ggplot(SnowGR, aes(x = Total)) +  
  geom_histogram(binwidth = 10, boundary = 50)
```

```
## Warning: Removed 1 rows containing non-finite values (stat_bin).
```



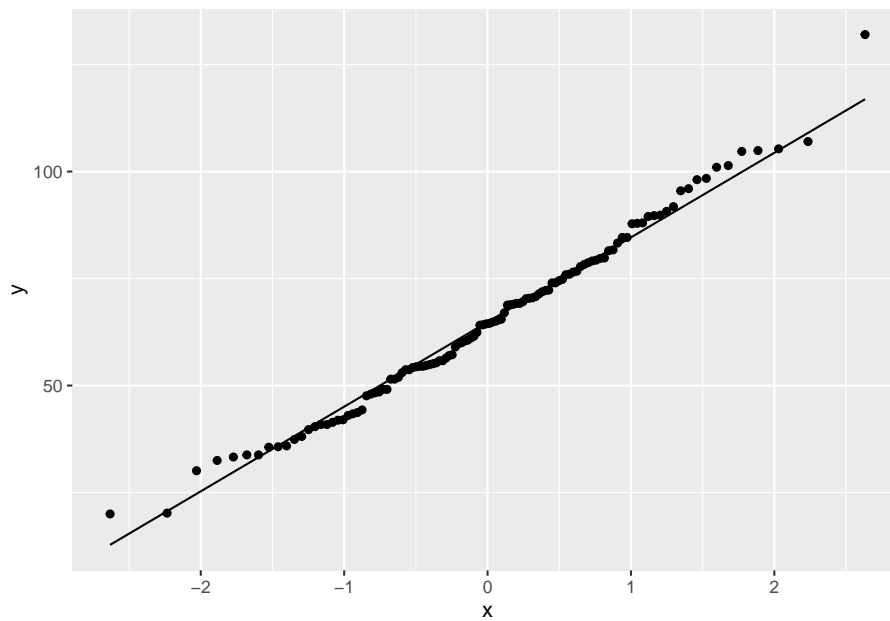
Here is the QQ plot for the same data. Notice that the aesthetics are a little different; instead of `x`, we have to use `sample`.

```
ggplot(SnowGR, aes(sample = Total)) +  
  geom_qq() +  
  geom_qq_line()
```

```
## Warning: Removed 1 rows containing non-finite values (stat_qq).
```

```
## Warning: Removed 1 rows containing non-finite values (stat_qq_line).
```





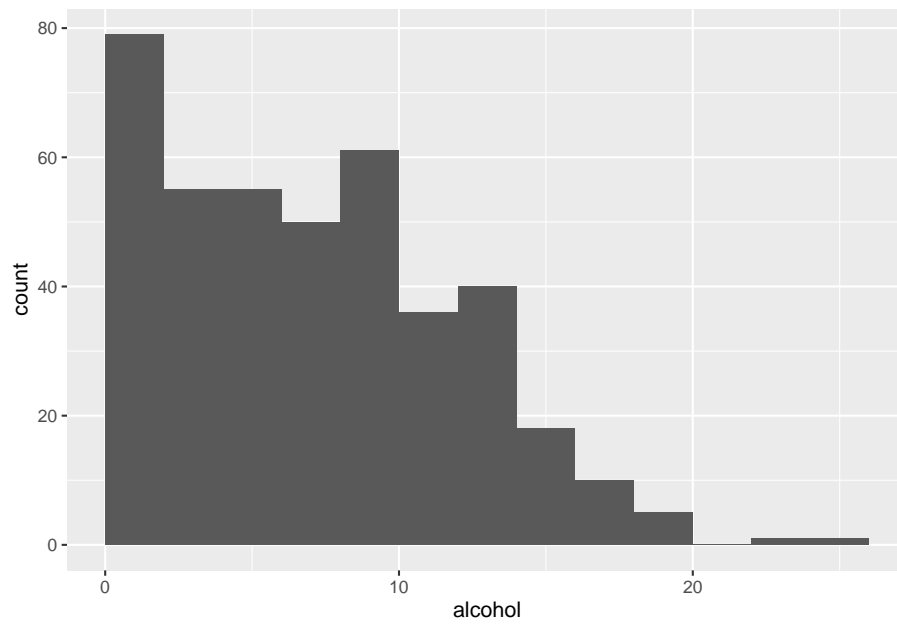
(The warning is because there is one missing value in the data.)

The `geom_qq()` layer plots the dots and the `geom_qq_line()` layer plots a diagonal line that the dots should more or less follow.

Other than a few points here and there, the bulk of the data is lined up nicely. There's a minor outlier, and that can be seen in both the histogram and the QQ plot.

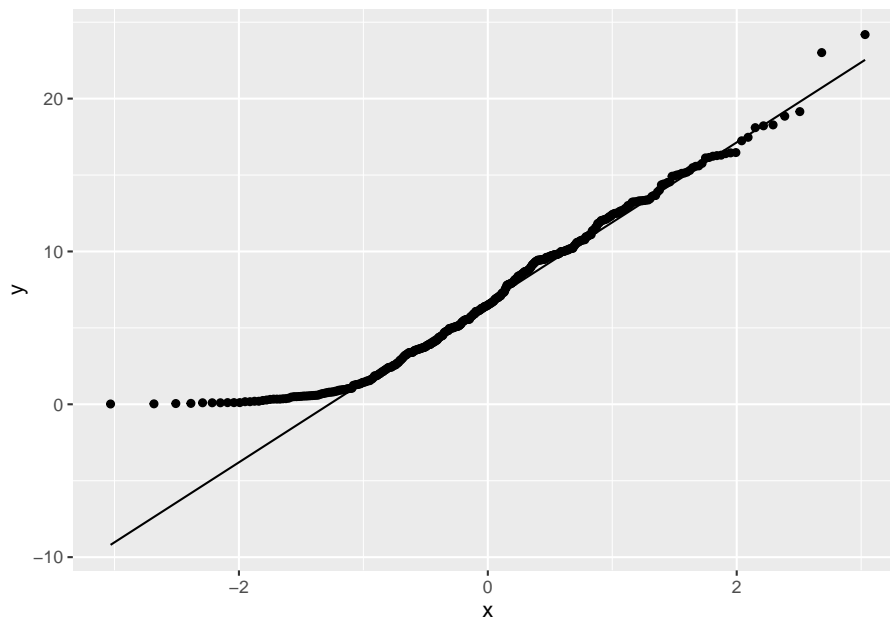
Contrast that with skewed data. For example, the `Alcohol` data set contains per capita consumption (in liters) of alcohol for various countries over several years. The alcohol consumption variable is highly skewed, as one can see in the histogram.

```
ggplot(Alcohol, aes(x = alcohol)) +  
  geom_histogram(binwidth = 2, boundary = 0)
```



It is also apparent in the QQ plot that the data is not normally distributed.

```
ggplot(Alcohol, aes(sample = alcohol)) +  
  geom_qq() +  
  geom_qq_line()
```



The path of dots is sharply curved, indicating a lack of normality.

**Exercise 5(a)** Find a data set with a numerical variable that is nearly normal in its distribution. (It can be something we’ve already seen in a past chapter, or if you’re really ambitious, you’re welcome to find a new data set.) Plot both a histogram and a QQ plot to demonstrate that the data is nearly normal. No need for a written response. Just plot the graphs.

Be aware that if you use a data set from a package, you may have to add `library(PACKAGE)` to your code. (You replace the word `PACKAGE` with whatever package you need.)

```
# Add code here to plot a histogram.
```

```
# Add code here to plot a QQ plot.
```

**Exercise 5(b)** Now find a data set with a numerical variable that is skewed in its distribution. Plot both a histogram and a QQ plot to demonstrate that the data is not normal. Again, no need for a written response. Just plot the graphs.

```
# Add code here to plot a histogram.
```

```
# Add code here to plot a QQ plot.
```

## 13.9 Conclusion

The normal model is ubiquitous in statistics, so understanding how to use it to make predictions is critical. When certain assumptions are met (that will be discussed in a future chapter), we can use the normal model to make predictions. The use of  $z$  scores allows us to measure distances from the mean in terms of standard deviations, giving us a scale in which data from different contexts are comparable as long as such measurements are normally distributed. A QQ plot helps us check that assumption.

### 13.9.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.

## Chapter 14

# Sampling distribution models

2.0

### Functions introduced in this chapter

No new R functions are introduced here.

## 14.1 Introduction

In this chapter, we'll revisit the idea of a sampling distribution model. We've already seen how useful it can be to simulate the process of simulating samples from a population and looking at the distribution of values that can occur by chance (i.e., sampling variability). We've also had some experience working with normal models. Under certain assumptions, we can use normal models to approximate our simulated sampling distributions.

### 14.1.1 Install new packages

There are no new packages used in this chapter.

### 14.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/14-sampling\\_distribution\\_models.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/14-sampling_distribution_models.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 14.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 14.2 Load packages

We load the standard `tidyverse` package. The `mosaic` package will provide coin flips.

```
library(tidyverse)
library(mosaic)
```

## 14.3 Sampling variability and sample size

We know that when we sample from a population, our sample is “wrong”: even when the sample is representative of the population, we don’t actually expect our sample statistic to agree exactly with the population parameter of interest. Our prior simulations have demonstrated this. They are centered on the “true” value (for example, in a hypothesis test, the “true” value is the assumed null value), but there is some spread due to sampling variability.

Let’s explore this idea a little further, this time considering how sample size plays a role in sampling variability.

Suppose that a certain candidate in an election actually has 64% of the support of registered voters. We conduct a poll of 10 random people, gathering a representative (though not very large) sample of voters.

We can simulate this task in R by using the `rflip` command from the `mosaic` package. Remember that the default for a coin flip is a 50% probability of heads, so we have to change that if we want to model a candidate with 64% support.

```
set.seed(13579)
rflip(10, prob = 0.64)
```

```
##
## Flipping 10 coins [ Prob(Heads) = 0.64 ] ...
##
```

```
## H T H T H H H T T H
##
## Number of Heads: 6 [Proportion Heads: 0.6]
```

You can think of the above command as taking one random sample of size 10 and getting a certain number of “successes”, where a “success” is a person who votes for our candidate—here encoded as “heads”. In other words, of the 10 people in this particular sample, we surveyed 6 people who said they were voting for our candidate and 4 people who were not.

Using the `do` command, we can simulate many samples, all of size 10. Let’s take 1000 samples and store them in a variable called `sims_1000_10`.

```
set.seed(13579)
sims_1000_10 <- do(1000) * rflip(10, prob = 0.64)
sims_1000_10
```

```
##      n heads tails prop
## 1     10      5     5 0.5
## 2     10      6     4 0.6
## 3     10      8     2 0.8
## 4     10      7     3 0.7
## 5     10      8     2 0.8
## 6     10      7     3 0.7
## 7     10      5     5 0.5
## 8     10      7     3 0.7
## 9     10      6     4 0.6
## 10    10      6     4 0.6
## 11    10      6     4 0.6
## 12    10      7     3 0.7
## 13    10      6     4 0.6
## 14    10      8     2 0.8
## 15    10      6     4 0.6
## 16    10      9     1 0.9
## 17    10      5     5 0.5
## 18    10      6     4 0.6
## 19    10      9     1 0.9
## 20    10      5     5 0.5
## 21    10      5     5 0.5
## 22    10      5     5 0.5
## 23    10      4     6 0.4
## 24    10      6     4 0.6
## 25    10      9     1 0.9
## 26    10      4     6 0.4
## 27    10      8     2 0.8
## 28    10      8     2 0.8
```

##	29	10	8	2	0.8
##	30	10	3	7	0.3
##	31	10	8	2	0.8
##	32	10	8	2	0.8
##	33	10	5	5	0.5
##	34	10	4	6	0.4
##	35	10	7	3	0.7
##	36	10	6	4	0.6
##	37	10	5	5	0.5
##	38	10	5	5	0.5
##	39	10	6	4	0.6
##	40	10	8	2	0.8
##	41	10	7	3	0.7
##	42	10	6	4	0.6
##	43	10	8	2	0.8
##	44	10	7	3	0.7
##	45	10	5	5	0.5
##	46	10	9	1	0.9
##	47	10	8	2	0.8
##	48	10	9	1	0.9
##	49	10	8	2	0.8
##	50	10	6	4	0.6
##	51	10	5	5	0.5
##	52	10	7	3	0.7
##	53	10	9	1	0.9
##	54	10	7	3	0.7
##	55	10	7	3	0.7
##	56	10	7	3	0.7
##	57	10	5	5	0.5
##	58	10	8	2	0.8
##	59	10	4	6	0.4
##	60	10	7	3	0.7
##	61	10	5	5	0.5
##	62	10	6	4	0.6
##	63	10	5	5	0.5
##	64	10	8	2	0.8
##	65	10	6	4	0.6
##	66	10	7	3	0.7
##	67	10	7	3	0.7
##	68	10	4	6	0.4
##	69	10	7	3	0.7
##	70	10	7	3	0.7
##	71	10	7	3	0.7
##	72	10	3	7	0.3
##	73	10	6	4	0.6
##	74	10	6	4	0.6



## 75	10	5	5	0.5
## 76	10	7	3	0.7
## 77	10	6	4	0.6
## 78	10	5	5	0.5
## 79	10	4	6	0.4
## 80	10	9	1	0.9
## 81	10	5	5	0.5
## 82	10	8	2	0.8
## 83	10	5	5	0.5
## 84	10	7	3	0.7
## 85	10	8	2	0.8
## 86	10	4	6	0.4
## 87	10	6	4	0.6
## 88	10	6	4	0.6
## 89	10	8	2	0.8
## 90	10	8	2	0.8
## 91	10	6	4	0.6
## 92	10	8	2	0.8
## 93	10	8	2	0.8
## 94	10	5	5	0.5
## 95	10	7	3	0.7
## 96	10	9	1	0.9
## 97	10	8	2	0.8
## 98	10	5	5	0.5
## 99	10	8	2	0.8
## 100	10	8	2	0.8
## 101	10	6	4	0.6
## 102	10	6	4	0.6
## 103	10	5	5	0.5
## 104	10	5	5	0.5
## 105	10	8	2	0.8
## 106	10	5	5	0.5
## 107	10	6	4	0.6
## 108	10	8	2	0.8
## 109	10	5	5	0.5
## 110	10	6	4	0.6
## 111	10	7	3	0.7
## 112	10	9	1	0.9
## 113	10	8	2	0.8
## 114	10	6	4	0.6
## 115	10	9	1	0.9
## 116	10	7	3	0.7
## 117	10	8	2	0.8
## 118	10	4	6	0.4
## 119	10	9	1	0.9
## 120	10	6	4	0.6

##	121	10	6	4	0.6
##	122	10	8	2	0.8
##	123	10	5	5	0.5
##	124	10	6	4	0.6
##	125	10	7	3	0.7
##	126	10	7	3	0.7
##	127	10	5	5	0.5
##	128	10	4	6	0.4
##	129	10	4	6	0.4
##	130	10	4	6	0.4
##	131	10	5	5	0.5
##	132	10	5	5	0.5
##	133	10	7	3	0.7
##	134	10	5	5	0.5
##	135	10	8	2	0.8
##	136	10	7	3	0.7
##	137	10	6	4	0.6
##	138	10	5	5	0.5
##	139	10	8	2	0.8
##	140	10	5	5	0.5
##	141	10	8	2	0.8
##	142	10	6	4	0.6
##	143	10	3	7	0.3
##	144	10	5	5	0.5
##	145	10	5	5	0.5
##	146	10	7	3	0.7
##	147	10	7	3	0.7
##	148	10	8	2	0.8
##	149	10	7	3	0.7
##	150	10	6	4	0.6
##	151	10	10	0	1.0
##	152	10	8	2	0.8
##	153	10	7	3	0.7
##	154	10	4	6	0.4
##	155	10	5	5	0.5
##	156	10	9	1	0.9
##	157	10	6	4	0.6
##	158	10	10	0	1.0
##	159	10	6	4	0.6
##	160	10	7	3	0.7
##	161	10	8	2	0.8
##	162	10	7	3	0.7
##	163	10	6	4	0.6
##	164	10	7	3	0.7
##	165	10	6	4	0.6
##	166	10	8	2	0.8

##	167	10	4	6	0.4
##	168	10	7	3	0.7
##	169	10	6	4	0.6
##	170	10	8	2	0.8
##	171	10	6	4	0.6
##	172	10	7	3	0.7
##	173	10	4	6	0.4
##	174	10	5	5	0.5
##	175	10	6	4	0.6
##	176	10	7	3	0.7
##	177	10	4	6	0.4
##	178	10	4	6	0.4
##	179	10	7	3	0.7
##	180	10	8	2	0.8
##	181	10	7	3	0.7
##	182	10	4	6	0.4
##	183	10	7	3	0.7
##	184	10	5	5	0.5
##	185	10	4	6	0.4
##	186	10	3	7	0.3
##	187	10	5	5	0.5
##	188	10	6	4	0.6
##	189	10	6	4	0.6
##	190	10	7	3	0.7
##	191	10	7	3	0.7
##	192	10	6	4	0.6
##	193	10	6	4	0.6
##	194	10	6	4	0.6
##	195	10	8	2	0.8
##	196	10	9	1	0.9
##	197	10	7	3	0.7
##	198	10	4	6	0.4
##	199	10	6	4	0.6
##	200	10	8	2	0.8
##	201	10	5	5	0.5
##	202	10	8	2	0.8
##	203	10	5	5	0.5
##	204	10	6	4	0.6
##	205	10	9	1	0.9
##	206	10	6	4	0.6
##	207	10	6	4	0.6
##	208	10	3	7	0.3
##	209	10	4	6	0.4
##	210	10	5	5	0.5
##	211	10	6	4	0.6
##	212	10	8	2	0.8

##	213	10	7	3	0.7
##	214	10	6	4	0.6
##	215	10	7	3	0.7
##	216	10	6	4	0.6
##	217	10	6	4	0.6
##	218	10	7	3	0.7
##	219	10	5	5	0.5
##	220	10	6	4	0.6
##	221	10	7	3	0.7
##	222	10	9	1	0.9
##	223	10	6	4	0.6
##	224	10	9	1	0.9
##	225	10	4	6	0.4
##	226	10	7	3	0.7
##	227	10	5	5	0.5
##	228	10	6	4	0.6
##	229	10	6	4	0.6
##	230	10	7	3	0.7
##	231	10	6	4	0.6
##	232	10	6	4	0.6
##	233	10	8	2	0.8
##	234	10	6	4	0.6
##	235	10	7	3	0.7
##	236	10	6	4	0.6
##	237	10	8	2	0.8
##	238	10	5	5	0.5
##	239	10	7	3	0.7
##	240	10	6	4	0.6
##	241	10	4	6	0.4
##	242	10	4	6	0.4
##	243	10	7	3	0.7
##	244	10	7	3	0.7
##	245	10	6	4	0.6
##	246	10	2	8	0.2
##	247	10	7	3	0.7
##	248	10	7	3	0.7
##	249	10	6	4	0.6
##	250	10	7	3	0.7
##	251	10	8	2	0.8
##	252	10	7	3	0.7
##	253	10	7	3	0.7
##	254	10	8	2	0.8
##	255	10	7	3	0.7
##	256	10	6	4	0.6
##	257	10	8	2	0.8
##	258	10	7	3	0.7

## 259	10	7	3	0.7
## 260	10	5	5	0.5
## 261	10	7	3	0.7
## 262	10	5	5	0.5
## 263	10	5	5	0.5
## 264	10	7	3	0.7
## 265	10	5	5	0.5
## 266	10	4	6	0.4
## 267	10	7	3	0.7
## 268	10	8	2	0.8
## 269	10	8	2	0.8
## 270	10	4	6	0.4
## 271	10	8	2	0.8
## 272	10	6	4	0.6
## 273	10	7	3	0.7
## 274	10	9	1	0.9
## 275	10	8	2	0.8
## 276	10	4	6	0.4
## 277	10	8	2	0.8
## 278	10	6	4	0.6
## 279	10	6	4	0.6
## 280	10	7	3	0.7
## 281	10	9	1	0.9
## 282	10	10	0	1.0
## 283	10	8	2	0.8
## 284	10	9	1	0.9
## 285	10	9	1	0.9
## 286	10	7	3	0.7
## 287	10	6	4	0.6
## 288	10	8	2	0.8
## 289	10	6	4	0.6
## 290	10	5	5	0.5
## 291	10	7	3	0.7
## 292	10	7	3	0.7
## 293	10	5	5	0.5
## 294	10	6	4	0.6
## 295	10	5	5	0.5
## 296	10	5	5	0.5
## 297	10	4	6	0.4
## 298	10	8	2	0.8
## 299	10	9	1	0.9
## 300	10	6	4	0.6
## 301	10	5	5	0.5
## 302	10	5	5	0.5
## 303	10	9	1	0.9
## 304	10	5	5	0.5

##	305	10	5	5	0.5
##	306	10	6	4	0.6
##	307	10	6	4	0.6
##	308	10	9	1	0.9
##	309	10	9	1	0.9
##	310	10	6	4	0.6
##	311	10	7	3	0.7
##	312	10	8	2	0.8
##	313	10	7	3	0.7
##	314	10	8	2	0.8
##	315	10	3	7	0.3
##	316	10	7	3	0.7
##	317	10	6	4	0.6
##	318	10	7	3	0.7
##	319	10	7	3	0.7
##	320	10	8	2	0.8
##	321	10	8	2	0.8
##	322	10	9	1	0.9
##	323	10	8	2	0.8
##	324	10	7	3	0.7
##	325	10	7	3	0.7
##	326	10	8	2	0.8
##	327	10	7	3	0.7
##	328	10	7	3	0.7
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##	332	10	7	3	0.7
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##	334	10	6	4	0.6
##	335	10	8	2	0.8
##	336	10	5	5	0.5
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##	340	10	7	3	0.7
##	341	10	6	4	0.6
##	342	10	4	6	0.4
##	343	10	5	5	0.5
##	344	10	7	3	0.7
##	345	10	7	3	0.7
##	346	10	7	3	0.7
##	347	10	6	4	0.6
##	348	10	7	3	0.7
##	349	10	6	4	0.6
##	350	10	8	2	0.8

##	351	10	5	5	0.5
##	352	10	10	0	1.0
##	353	10	5	5	0.5
##	354	10	7	3	0.7
##	355	10	7	3	0.7
##	356	10	5	5	0.5
##	357	10	7	3	0.7
##	358	10	7	3	0.7
##	359	10	5	5	0.5
##	360	10	8	2	0.8
##	361	10	8	2	0.8
##	362	10	6	4	0.6
##	363	10	6	4	0.6
##	364	10	6	4	0.6
##	365	10	5	5	0.5
##	366	10	6	4	0.6
##	367	10	5	5	0.5
##	368	10	7	3	0.7
##	369	10	8	2	0.8
##	370	10	4	6	0.4
##	371	10	4	6	0.4
##	372	10	6	4	0.6
##	373	10	7	3	0.7
##	374	10	6	4	0.6
##	375	10	6	4	0.6
##	376	10	8	2	0.8
##	377	10	5	5	0.5
##	378	10	7	3	0.7
##	379	10	6	4	0.6
##	380	10	6	4	0.6
##	381	10	4	6	0.4
##	382	10	4	6	0.4
##	383	10	6	4	0.6
##	384	10	8	2	0.8
##	385	10	5	5	0.5
##	386	10	6	4	0.6
##	387	10	7	3	0.7
##	388	10	6	4	0.6
##	389	10	8	2	0.8
##	390	10	8	2	0.8
##	391	10	6	4	0.6
##	392	10	5	5	0.5
##	393	10	8	2	0.8
##	394	10	5	5	0.5
##	395	10	6	4	0.6
##	396	10	6	4	0.6

##	397	10	5	5	0.5
##	398	10	4	6	0.4
##	399	10	7	3	0.7
##	400	10	7	3	0.7
##	401	10	9	1	0.9
##	402	10	6	4	0.6
##	403	10	6	4	0.6
##	404	10	5	5	0.5
##	405	10	8	2	0.8
##	406	10	5	5	0.5
##	407	10	9	1	0.9
##	408	10	7	3	0.7
##	409	10	6	4	0.6
##	410	10	6	4	0.6
##	411	10	9	1	0.9
##	412	10	4	6	0.4
##	413	10	4	6	0.4
##	414	10	7	3	0.7
##	415	10	7	3	0.7
##	416	10	6	4	0.6
##	417	10	5	5	0.5
##	418	10	6	4	0.6
##	419	10	6	4	0.6
##	420	10	6	4	0.6
##	421	10	7	3	0.7
##	422	10	8	2	0.8
##	423	10	6	4	0.6
##	424	10	7	3	0.7
##	425	10	8	2	0.8
##	426	10	5	5	0.5
##	427	10	8	2	0.8
##	428	10	8	2	0.8
##	429	10	6	4	0.6
##	430	10	5	5	0.5
##	431	10	4	6	0.4
##	432	10	7	3	0.7
##	433	10	6	4	0.6
##	434	10	6	4	0.6
##	435	10	9	1	0.9
##	436	10	5	5	0.5
##	437	10	5	5	0.5
##	438	10	6	4	0.6
##	439	10	6	4	0.6
##	440	10	7	3	0.7
##	441	10	6	4	0.6
##	442	10	8	2	0.8



## 443	10	6	4	0.6
## 444	10	5	5	0.5
## 445	10	7	3	0.7
## 446	10	6	4	0.6
## 447	10	5	5	0.5
## 448	10	7	3	0.7
## 449	10	6	4	0.6
## 450	10	5	5	0.5
## 451	10	9	1	0.9
## 452	10	8	2	0.8
## 453	10	8	2	0.8
## 454	10	5	5	0.5
## 455	10	6	4	0.6
## 456	10	5	5	0.5
## 457	10	8	2	0.8
## 458	10	8	2	0.8
## 459	10	8	2	0.8
## 460	10	5	5	0.5
## 461	10	7	3	0.7
## 462	10	5	5	0.5
## 463	10	5	5	0.5
## 464	10	8	2	0.8
## 465	10	4	6	0.4
## 466	10	6	4	0.6
## 467	10	6	4	0.6
## 468	10	8	2	0.8
## 469	10	8	2	0.8
## 470	10	6	4	0.6
## 471	10	6	4	0.6
## 472	10	10	0	1.0
## 473	10	4	6	0.4
## 474	10	8	2	0.8
## 475	10	6	4	0.6
## 476	10	6	4	0.6
## 477	10	9	1	0.9
## 478	10	7	3	0.7
## 479	10	7	3	0.7
## 480	10	5	5	0.5
## 481	10	7	3	0.7
## 482	10	5	5	0.5
## 483	10	5	5	0.5
## 484	10	8	2	0.8
## 485	10	7	3	0.7
## 486	10	7	3	0.7
## 487	10	6	4	0.6
## 488	10	6	4	0.6

##	489	10	6	4	0.6
##	490	10	8	2	0.8
##	491	10	8	2	0.8
##	492	10	2	8	0.2
##	493	10	5	5	0.5
##	494	10	8	2	0.8
##	495	10	7	3	0.7
##	496	10	8	2	0.8
##	497	10	5	5	0.5
##	498	10	7	3	0.7
##	499	10	7	3	0.7
##	500	10	9	1	0.9
##	501	10	6	4	0.6
##	502	10	4	6	0.4
##	503	10	6	4	0.6
##	504	10	5	5	0.5
##	505	10	4	6	0.4
##	506	10	7	3	0.7
##	507	10	7	3	0.7
##	508	10	5	5	0.5
##	509	10	6	4	0.6
##	510	10	6	4	0.6
##	511	10	7	3	0.7
##	512	10	6	4	0.6
##	513	10	3	7	0.3
##	514	10	7	3	0.7
##	515	10	7	3	0.7
##	516	10	6	4	0.6
##	517	10	6	4	0.6
##	518	10	6	4	0.6
##	519	10	6	4	0.6
##	520	10	8	2	0.8
##	521	10	6	4	0.6
##	522	10	8	2	0.8
##	523	10	8	2	0.8
##	524	10	7	3	0.7
##	525	10	8	2	0.8
##	526	10	7	3	0.7
##	527	10	7	3	0.7
##	528	10	5	5	0.5
##	529	10	6	4	0.6
##	530	10	8	2	0.8
##	531	10	6	4	0.6
##	532	10	4	6	0.4
##	533	10	5	5	0.5
##	534	10	5	5	0.5

## 535	10	4	6	0.4
## 536	10	7	3	0.7
## 537	10	6	4	0.6
## 538	10	9	1	0.9
## 539	10	7	3	0.7
## 540	10	4	6	0.4
## 541	10	7	3	0.7
## 542	10	3	7	0.3
## 543	10	10	0	1.0
## 544	10	5	5	0.5
## 545	10	7	3	0.7
## 546	10	8	2	0.8
## 547	10	5	5	0.5
## 548	10	6	4	0.6
## 549	10	7	3	0.7
## 550	10	7	3	0.7
## 551	10	5	5	0.5
## 552	10	7	3	0.7
## 553	10	5	5	0.5
## 554	10	7	3	0.7
## 555	10	6	4	0.6
## 556	10	7	3	0.7
## 557	10	6	4	0.6
## 558	10	5	5	0.5
## 559	10	6	4	0.6
## 560	10	7	3	0.7
## 561	10	5	5	0.5
## 562	10	6	4	0.6
## 563	10	5	5	0.5
## 564	10	7	3	0.7
## 565	10	7	3	0.7
## 566	10	6	4	0.6
## 567	10	4	6	0.4
## 568	10	5	5	0.5
## 569	10	6	4	0.6
## 570	10	4	6	0.4
## 571	10	8	2	0.8
## 572	10	7	3	0.7
## 573	10	7	3	0.7
## 574	10	7	3	0.7
## 575	10	8	2	0.8
## 576	10	6	4	0.6
## 577	10	5	5	0.5
## 578	10	8	2	0.8
## 579	10	5	5	0.5
## 580	10	6	4	0.6

##	581	10	6	4	0.6
##	582	10	7	3	0.7
##	583	10	7	3	0.7
##	584	10	8	2	0.8
##	585	10	7	3	0.7
##	586	10	7	3	0.7
##	587	10	6	4	0.6
##	588	10	5	5	0.5
##	589	10	8	2	0.8
##	590	10	8	2	0.8
##	591	10	8	2	0.8
##	592	10	6	4	0.6
##	593	10	7	3	0.7
##	594	10	6	4	0.6
##	595	10	7	3	0.7
##	596	10	5	5	0.5
##	597	10	6	4	0.6
##	598	10	6	4	0.6
##	599	10	8	2	0.8
##	600	10	10	0	1.0
##	601	10	5	5	0.5
##	602	10	4	6	0.4
##	603	10	9	1	0.9
##	604	10	7	3	0.7
##	605	10	8	2	0.8
##	606	10	7	3	0.7
##	607	10	5	5	0.5
##	608	10	4	6	0.4
##	609	10	7	3	0.7
##	610	10	7	3	0.7
##	611	10	7	3	0.7
##	612	10	8	2	0.8
##	613	10	6	4	0.6
##	614	10	7	3	0.7
##	615	10	7	3	0.7
##	616	10	7	3	0.7
##	617	10	7	3	0.7
##	618	10	5	5	0.5
##	619	10	6	4	0.6
##	620	10	7	3	0.7
##	621	10	6	4	0.6
##	622	10	6	4	0.6
##	623	10	6	4	0.6
##	624	10	6	4	0.6
##	625	10	8	2	0.8
##	626	10	7	3	0.7

## 627	10	4	6	0.4
## 628	10	6	4	0.6
## 629	10	5	5	0.5
## 630	10	4	6	0.4
## 631	10	8	2	0.8
## 632	10	5	5	0.5
## 633	10	7	3	0.7
## 634	10	6	4	0.6
## 635	10	5	5	0.5
## 636	10	6	4	0.6
## 637	10	7	3	0.7
## 638	10	8	2	0.8
## 639	10	6	4	0.6
## 640	10	5	5	0.5
## 641	10	6	4	0.6
## 642	10	9	1	0.9
## 643	10	9	1	0.9
## 644	10	4	6	0.4
## 645	10	8	2	0.8
## 646	10	8	2	0.8
## 647	10	7	3	0.7
## 648	10	8	2	0.8
## 649	10	9	1	0.9
## 650	10	7	3	0.7
## 651	10	5	5	0.5
## 652	10	5	5	0.5
## 653	10	6	4	0.6
## 654	10	8	2	0.8
## 655	10	5	5	0.5
## 656	10	8	2	0.8
## 657	10	9	1	0.9
## 658	10	8	2	0.8
## 659	10	9	1	0.9
## 660	10	7	3	0.7
## 661	10	6	4	0.6
## 662	10	8	2	0.8
## 663	10	6	4	0.6
## 664	10	7	3	0.7
## 665	10	7	3	0.7
## 666	10	8	2	0.8
## 667	10	6	4	0.6
## 668	10	7	3	0.7
## 669	10	6	4	0.6
## 670	10	10	0	1.0
## 671	10	5	5	0.5
## 672	10	7	3	0.7

##	673	10	7	3	0.7
##	674	10	8	2	0.8
##	675	10	7	3	0.7
##	676	10	4	6	0.4
##	677	10	5	5	0.5
##	678	10	7	3	0.7
##	679	10	3	7	0.3
##	680	10	6	4	0.6
##	681	10	6	4	0.6
##	682	10	6	4	0.6
##	683	10	6	4	0.6
##	684	10	7	3	0.7
##	685	10	7	3	0.7
##	686	10	4	6	0.4
##	687	10	6	4	0.6
##	688	10	6	4	0.6
##	689	10	6	4	0.6
##	690	10	6	4	0.6
##	691	10	8	2	0.8
##	692	10	8	2	0.8
##	693	10	7	3	0.7
##	694	10	6	4	0.6
##	695	10	8	2	0.8
##	696	10	7	3	0.7
##	697	10	8	2	0.8
##	698	10	8	2	0.8
##	699	10	5	5	0.5
##	700	10	9	1	0.9
##	701	10	6	4	0.6
##	702	10	7	3	0.7
##	703	10	7	3	0.7
##	704	10	6	4	0.6
##	705	10	7	3	0.7
##	706	10	8	2	0.8
##	707	10	5	5	0.5
##	708	10	7	3	0.7
##	709	10	6	4	0.6
##	710	10	6	4	0.6
##	711	10	7	3	0.7
##	712	10	7	3	0.7
##	713	10	8	2	0.8
##	714	10	4	6	0.4
##	715	10	6	4	0.6
##	716	10	5	5	0.5
##	717	10	8	2	0.8
##	718	10	6	4	0.6

## 719	10	6	4	0.6
## 720	10	4	6	0.4
## 721	10	7	3	0.7
## 722	10	6	4	0.6
## 723	10	9	1	0.9
## 724	10	7	3	0.7
## 725	10	5	5	0.5
## 726	10	7	3	0.7
## 727	10	6	4	0.6
## 728	10	6	4	0.6
## 729	10	5	5	0.5
## 730	10	8	2	0.8
## 731	10	7	3	0.7
## 732	10	6	4	0.6
## 733	10	5	5	0.5
## 734	10	6	4	0.6
## 735	10	5	5	0.5
## 736	10	4	6	0.4
## 737	10	7	3	0.7
## 738	10	7	3	0.7
## 739	10	4	6	0.4
## 740	10	7	3	0.7
## 741	10	8	2	0.8
## 742	10	6	4	0.6
## 743	10	6	4	0.6
## 744	10	7	3	0.7
## 745	10	10	0	1.0
## 746	10	4	6	0.4
## 747	10	8	2	0.8
## 748	10	7	3	0.7
## 749	10	7	3	0.7
## 750	10	4	6	0.4
## 751	10	9	1	0.9
## 752	10	7	3	0.7
## 753	10	7	3	0.7
## 754	10	9	1	0.9
## 755	10	5	5	0.5
## 756	10	8	2	0.8
## 757	10	5	5	0.5
## 758	10	8	2	0.8
## 759	10	4	6	0.4
## 760	10	8	2	0.8
## 761	10	7	3	0.7
## 762	10	8	2	0.8
## 763	10	6	4	0.6
## 764	10	8	2	0.8

##	765	10	3	7	0.3
##	766	10	9	1	0.9
##	767	10	7	3	0.7
##	768	10	6	4	0.6
##	769	10	3	7	0.3
##	770	10	4	6	0.4
##	771	10	6	4	0.6
##	772	10	6	4	0.6
##	773	10	5	5	0.5
##	774	10	4	6	0.4
##	775	10	5	5	0.5
##	776	10	7	3	0.7
##	777	10	5	5	0.5
##	778	10	8	2	0.8
##	779	10	8	2	0.8
##	780	10	6	4	0.6
##	781	10	7	3	0.7
##	782	10	6	4	0.6
##	783	10	6	4	0.6
##	784	10	6	4	0.6
##	785	10	7	3	0.7
##	786	10	7	3	0.7
##	787	10	6	4	0.6
##	788	10	6	4	0.6
##	789	10	8	2	0.8
##	790	10	6	4	0.6
##	791	10	9	1	0.9
##	792	10	5	5	0.5
##	793	10	8	2	0.8
##	794	10	4	6	0.4
##	795	10	6	4	0.6
##	796	10	5	5	0.5
##	797	10	6	4	0.6
##	798	10	6	4	0.6
##	799	10	7	3	0.7
##	800	10	3	7	0.3
##	801	10	4	6	0.4
##	802	10	6	4	0.6
##	803	10	5	5	0.5
##	804	10	7	3	0.7
##	805	10	8	2	0.8
##	806	10	7	3	0.7
##	807	10	7	3	0.7
##	808	10	4	6	0.4
##	809	10	6	4	0.6
##	810	10	8	2	0.8



## 811	10	4	6	0.4
## 812	10	7	3	0.7
## 813	10	9	1	0.9
## 814	10	7	3	0.7
## 815	10	7	3	0.7
## 816	10	6	4	0.6
## 817	10	5	5	0.5
## 818	10	8	2	0.8
## 819	10	6	4	0.6
## 820	10	6	4	0.6
## 821	10	5	5	0.5
## 822	10	8	2	0.8
## 823	10	6	4	0.6
## 824	10	4	6	0.4
## 825	10	5	5	0.5
## 826	10	3	7	0.3
## 827	10	7	3	0.7
## 828	10	9	1	0.9
## 829	10	8	2	0.8
## 830	10	7	3	0.7
## 831	10	6	4	0.6
## 832	10	5	5	0.5
## 833	10	8	2	0.8
## 834	10	6	4	0.6
## 835	10	8	2	0.8
## 836	10	5	5	0.5
## 837	10	10	0	1.0
## 838	10	5	5	0.5
## 839	10	4	6	0.4
## 840	10	7	3	0.7
## 841	10	7	3	0.7
## 842	10	7	3	0.7
## 843	10	4	6	0.4
## 844	10	7	3	0.7
## 845	10	7	3	0.7
## 846	10	7	3	0.7
## 847	10	6	4	0.6
## 848	10	8	2	0.8
## 849	10	6	4	0.6
## 850	10	5	5	0.5
## 851	10	7	3	0.7
## 852	10	7	3	0.7
## 853	10	4	6	0.4
## 854	10	7	3	0.7
## 855	10	8	2	0.8
## 856	10	2	8	0.2

##	857	10	9	1	0.9
##	858	10	6	4	0.6
##	859	10	7	3	0.7
##	860	10	5	5	0.5
##	861	10	7	3	0.7
##	862	10	6	4	0.6
##	863	10	5	5	0.5
##	864	10	7	3	0.7
##	865	10	8	2	0.8
##	866	10	4	6	0.4
##	867	10	4	6	0.4
##	868	10	5	5	0.5
##	869	10	4	6	0.4
##	870	10	4	6	0.4
##	871	10	5	5	0.5
##	872	10	6	4	0.6
##	873	10	4	6	0.4
##	874	10	5	5	0.5
##	875	10	7	3	0.7
##	876	10	10	0	1.0
##	877	10	6	4	0.6
##	878	10	7	3	0.7
##	879	10	5	5	0.5
##	880	10	9	1	0.9
##	881	10	7	3	0.7
##	882	10	5	5	0.5
##	883	10	5	5	0.5
##	884	10	8	2	0.8
##	885	10	6	4	0.6
##	886	10	5	5	0.5
##	887	10	7	3	0.7
##	888	10	7	3	0.7
##	889	10	6	4	0.6
##	890	10	7	3	0.7
##	891	10	9	1	0.9
##	892	10	7	3	0.7
##	893	10	5	5	0.5
##	894	10	8	2	0.8
##	895	10	6	4	0.6
##	896	10	5	5	0.5
##	897	10	6	4	0.6
##	898	10	6	4	0.6
##	899	10	6	4	0.6
##	900	10	8	2	0.8
##	901	10	8	2	0.8
##	902	10	7	3	0.7

##	903	10	7	3	0.7
##	904	10	3	7	0.3
##	905	10	9	1	0.9
##	906	10	4	6	0.4
##	907	10	6	4	0.6
##	908	10	9	1	0.9
##	909	10	7	3	0.7
##	910	10	7	3	0.7
##	911	10	8	2	0.8
##	912	10	4	6	0.4
##	913	10	6	4	0.6
##	914	10	7	3	0.7
##	915	10	8	2	0.8
##	916	10	5	5	0.5
##	917	10	7	3	0.7
##	918	10	5	5	0.5
##	919	10	9	1	0.9
##	920	10	7	3	0.7
##	921	10	6	4	0.6
##	922	10	6	4	0.6
##	923	10	8	2	0.8
##	924	10	6	4	0.6
##	925	10	7	3	0.7
##	926	10	5	5	0.5
##	927	10	5	5	0.5
##	928	10	5	5	0.5
##	929	10	4	6	0.4
##	930	10	6	4	0.6
##	931	10	3	7	0.3
##	932	10	5	5	0.5
##	933	10	7	3	0.7
##	934	10	7	3	0.7
##	935	10	9	1	0.9
##	936	10	7	3	0.7
##	937	10	6	4	0.6
##	938	10	6	4	0.6
##	939	10	7	3	0.7
##	940	10	7	3	0.7
##	941	10	7	3	0.7
##	942	10	7	3	0.7
##	943	10	9	1	0.9
##	944	10	8	2	0.8
##	945	10	7	3	0.7
##	946	10	7	3	0.7
##	947	10	5	5	0.5
##	948	10	5	5	0.5

##	949	10	7	3	0.7
##	950	10	7	3	0.7
##	951	10	6	4	0.6
##	952	10	4	6	0.4
##	953	10	7	3	0.7
##	954	10	5	5	0.5
##	955	10	8	2	0.8
##	956	10	6	4	0.6
##	957	10	8	2	0.8
##	958	10	6	4	0.6
##	959	10	7	3	0.7
##	960	10	6	4	0.6
##	961	10	9	1	0.9
##	962	10	6	4	0.6
##	963	10	5	5	0.5
##	964	10	5	5	0.5
##	965	10	6	4	0.6
##	966	10	7	3	0.7
##	967	10	7	3	0.7
##	968	10	8	2	0.8
##	969	10	7	3	0.7
##	970	10	7	3	0.7
##	971	10	7	3	0.7
##	972	10	4	6	0.4
##	973	10	9	1	0.9
##	974	10	6	4	0.6
##	975	10	6	4	0.6
##	976	10	8	2	0.8
##	977	10	7	3	0.7
##	978	10	7	3	0.7
##	979	10	8	2	0.8
##	980	10	3	7	0.3
##	981	10	9	1	0.9
##	982	10	4	6	0.4
##	983	10	5	5	0.5
##	984	10	6	4	0.6
##	985	10	9	1	0.9
##	986	10	5	5	0.5
##	987	10	4	6	0.4
##	988	10	8	2	0.8
##	989	10	6	4	0.6
##	990	10	5	5	0.5
##	991	10	9	1	0.9
##	992	10	7	3	0.7
##	993	10	6	4	0.6
##	994	10	5	5	0.5

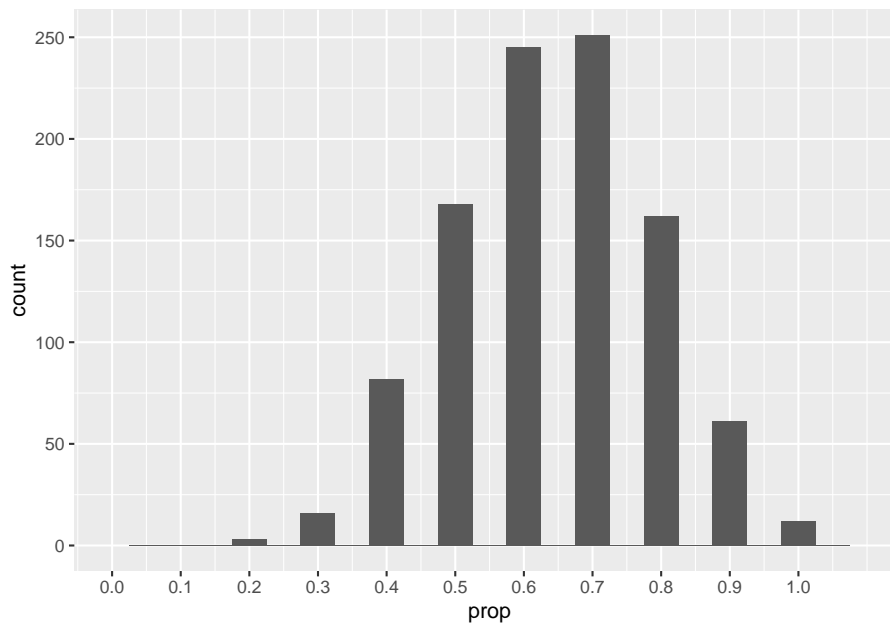
```
## 995 10      6      4 0.6
## 996 10      6      4 0.6
## 997 10      5      5 0.5
## 998 10     10      0 1.0
## 999 10      6      4 0.6
## 1000 10     7      3 0.7
```

Note that with 10 people, it is impossible to get a 64% success rate in any given sample. (That would be 6.4 people!) Nevertheless, we can see that many of the samples gave us around 5–8 successes, as we’d expect when the true population rate is 64%. Also, the mean number of successes across all simulations is 6.414, which is very close to 6.4.

Instead of focusing on the total number of successes, let’s use the proportion of successes in each sample. We can graph our simulated proportions, just as we’ve done in previous chapters. (The fancy stuff in `scale_x_continuous` is just making sure that the x-axis goes from 0 to 1 and that the tick marks appear as multiples of 0.1.)

```
ggplot(sims_1000_10, aes(x = prop)) +
  geom_histogram(binwidth = 0.05) +
  scale_x_continuous(limits = c(0, 1.1),
                    breaks = seq(0, 1, 0.1))
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



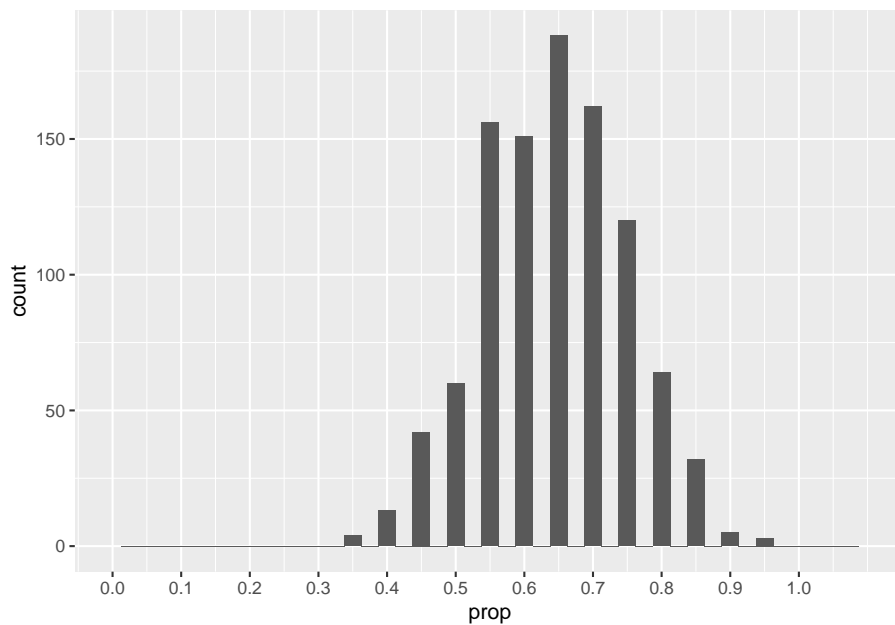
Because each sample has size 10, the proportion of successes can only be multiples of 0.1. Although the distribution is somewhat normally shaped, it is discrete (no values in between the bars) and there is an appreciable left skew.

What happens if we increase the sample size to 20? (The binwidth has to change to see the discrete bars.)

```
set.seed(13579)
sims_1000_20 <- do(1000) * rflip(20, prob = 0.64)
```

```
ggplot(sims_1000_20, aes(x = prop)) +
  geom_histogram(binwidth = 0.025) +
  scale_x_continuous(limits = c(0, 1.1),
                     breaks = seq(0, 1, 0.1))
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



**Exercise 1** Explain how the distribution of simulations has changed going from a sample size of 10 to a sample size of 20.

Please write up your answer here.

**Exercise 2(a)** Run a set of simulations yourself, this time with samples of size 50. Use the same number of simulations (1000) and the same `ggplot` code from above (especially the `scale_x_continuous` option) so that the x-axis is scaled identically to the previous cases, but change the binwidth to 0.01.

```
set.seed(13579)
# Add code here to simulate 1000 random samples of size 50 and plot them.
```

**Exercise 2(b)** Explain how the distribution of simulations has changed going from a sample size of 10 to 20 to 50.

Please write up your answer here.

## 14.4 The sampling distribution model and the standard error

In the last chapter on normal models, we mentioned briefly the Central Limit Theorem and the fact that under certain assumptions, our simulations would look normally distributed. More concretely, the Central Limit Theorem tells us that as our sample size increases, the distribution of sample proportions looks more and more like a normal model. This model is called the *sampling distribution model* because it describes how many different samples from a population should be distributed.

Which normal model do we use? In other words, what is the mean and standard deviation of a normal model that describes a simulation of repeated samples?

The simulations above are all centered at the same place, 0.64. This is no surprise. If the true population proportion is 0.64, then we expect most of our samples to be around 64% (even if, as above, it is actually impossible to get exactly 64% in any given sample).

But what about the standard deviation? It seems to be changing with each sample size.

**Exercise 3** Looking at your simulations above, how does the standard deviation appear to change as the sample size increases? Intuitively, why do you think this happens? (Hint: think about the relationship between larger sample sizes and accuracy.)

Please write up your answer here.

---

The standard deviation of a sampling distribution is usually called the *standard error*. (The use of the word “error” in statistics does not mean that anyone made a mistake. A better word for error would be “uncertainty” or even just “variability”.)

There is some complicated mathematics involved in figuring out the standard error, so I’ll just tell you what it is. If  $p$  is the true population proportion, then the standard error is

$$\sqrt{\frac{p(1-p)}{n}}.$$

Therefore, if the sample size is large enough, the sampling distribution model is nearly normal, and the correct normal model is

$$N\left(p, \sqrt{\frac{p(1-p)}{n}}\right).$$

In our election example, we can calculate the standard error for a sample of size 10:

$$\sqrt{\frac{p(1-p)}{n}} = \sqrt{\frac{0.64(1-0.64)}{10}} = 0.152.$$

We can do this easily using inline R code. (Remember that R is nothing more than a glorified calculator.) If a candidate has 64% of the vote and we take a sample of size 10, the standard error is 0.1517893. In other words, the sampling distribution model is

$$N(0.64, 0.152).$$

For a sample of size 20, the standard error is 0.1073313 and the sampling distribution model is

$$N(0.64, 0.107).$$

**Exercise 4** Calculate the standard error for the example above, but this time using a sample size of 50. Give your answer as a contextually meaningful full sentence using inline R code.

Please write up your answer here.



## 14.5 Conditions

Like anything in statistics, there are assumptions that have to be met before applying any technique. We must check that certain conditions are true before we can reasonably make the necessary assumptions required by our model.

When we want to use a normal model, we have to make sure the sampling distribution model is truly normal (or nearly normal).

First, we need our samples to be random. Clearly, when samples are not random, there is a danger of bias, and then all bets are off. Of course, in real life hardly any sample will be truly random, so being representative is the most we can usually hope for.

Second, our sample size must be less than 10% of the population size. The reasons for this are somewhat technical, and 10% is a rough guideline. The idea is that if we are sampling, we need our sample not to be a significant chunk of the population.

These two conditions are always important when sampling. Together, they help ensure that the mathematical assumption of independence is met. In other words, when these two conditions are met, there is a better chance that the data from one member of our sample will not influence nor be influenced by the data from another member.

For applying normal models, there is one more condition. It is called the “success/failure” condition. We need for the total number of successes to be at least 10 and, similarly, for the total number of failures to be at least 10.

Go back and consider our first simulated sample. The true rate of success in the population was presumed to be 64%. Given that we were sampling only 10 individuals, this implies that, on average, we would expect 6.4 people out of 10 to vote for the candidate. And likewise, that means that we would expect 3.6 people to vote against the candidate. (Clearly, it is impossible in any given sample to get 6.4 votes for, or 3.6 votes against. But *on average*, this is what we expect.) In fact, since the sample size was 10, there was no way that we could meet the success/failure condition. When we plotted the histogram of simulated proportions, we saw the problem: with such small numbers, the histogram was skewed, and not normal.

We check the success/failure condition by calculating  $np$  and  $n(1 - p)$ :  $n$  is the sample size and  $p$  is the proportion of successes. Therefore,  $np$  is the total number of successes. Since  $1 - p$  is the proportion of failures,  $n(1 - p)$  is the total number of failures. Each of the numbers  $np$  and  $n(1 - p)$  needs to be bigger than 10.

In our example,  $n = 10$  (the sample size), and  $p = 0.64$  (the probability of success). So

$$np = 10(0.64) = 6.4$$

and

$$n(1 - p) = 10(1 - 0.64) = 10(0.36) = 3.6.$$

Neither of these numbers is bigger than 10.

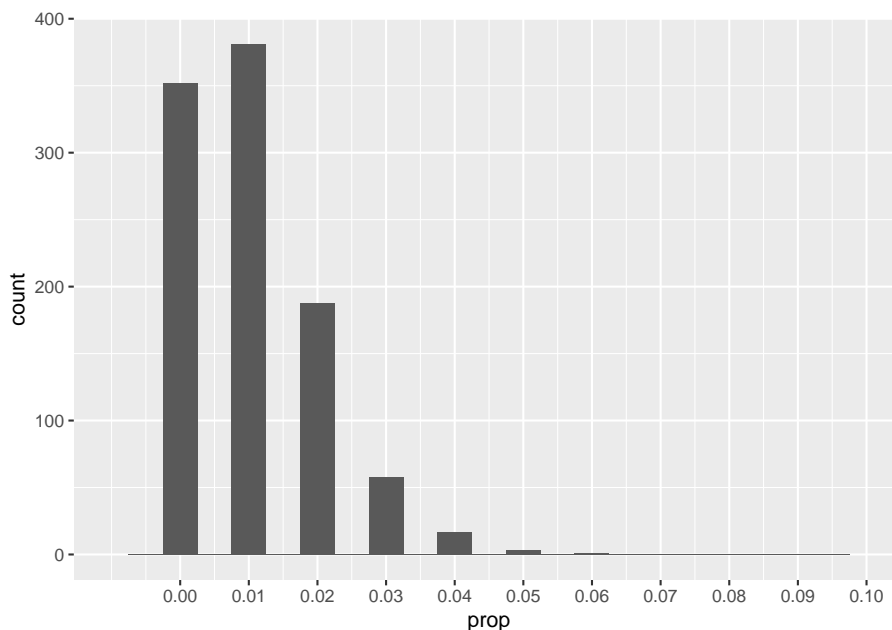
Notice that when  $n$  is large, the quantities  $np$  and  $n(1 - p)$  will also tend to be large. This is the content of the Central Limit Theorem: when sample sizes grow, the sampling distribution model becomes more and more normal.

There is something else going on too. Suppose that  $n = 100$  but  $p = 0.01$ . The sample seems quite large, but let's look at the sampling distribution through a simulation.

```
set.seed(13579)
sims_1000_100 <- do(1000) * rflip(100, prob = 0.01)
```

```
ggplot(sims_1000_100, aes(x = prop)) +
  geom_histogram(binwidth = 0.005) +
  scale_x_continuous(limits = c(-0.01, 0.1),
                    breaks = seq(0, 0.1, 0.01))
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



(Note that the x-axis scale is much smaller than it was before.)

**Exercise 5** What’s the problem here? Despite having a fairly large sample size, why is this distribution so skewed?

Please write up your answer here.

---

In this scenario, the success/failure condition fails because

$$np = (100)(0.01) = 1 \not\geq 10.$$

In other words, in a typical sample, we expect 1 success and 99 failures.

**Exercise 6** Going back to the election example (in which the candidate has 64% of the vote), check that a sample size of 50 does satisfy the success/failure condition.

Please write up your answer here.

## 14.6 Using the model to make predictions

Once we know that a normal model is appropriate, we can employ all the tools we’ve previously developed to work with normal models, notably `pdist` and `qdist`.

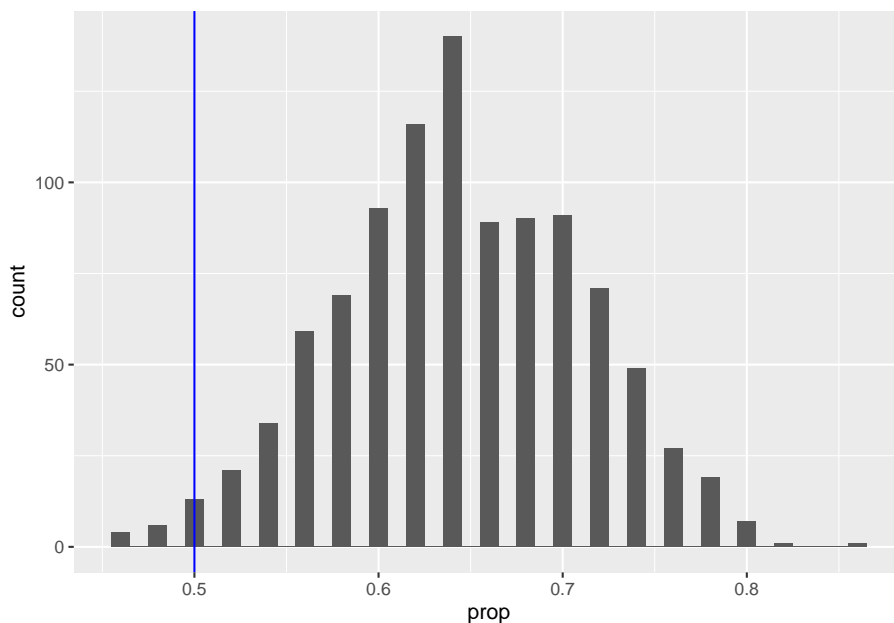
For example, we know that samples can be “wrong” due to sampling variability. Even though we know the candidate has 64% support, most surveys are not going to give us back that exact number.

Could a survey of 50 random voters accidentally predict defeat for the candidate even though the candidate will actually win with 64% support?

Let’s simulate:

```
set.seed(13579)
survey_sim <- do(1000) * rflip(50, prob = 0.64)
```

```
ggplot(survey_sim, aes(x = prop)) +
  geom_histogram(binwidth = 0.01) +
  geom_vline(xintercept = 0.5, color = "blue")
```



It looks like there are at least a few simulated samples that could come in less than 50% by chance.

Let's check the conditions to see if we can use a normal model:

- Random
  - We are told that our 50 voters are a random sample.
- 10%
  - It is safe to assume there are more than 500 voters for this election.
- Success/failure
  - The number of expected successes is 32 and the expected number of failures is 18. These are both greater than 10.

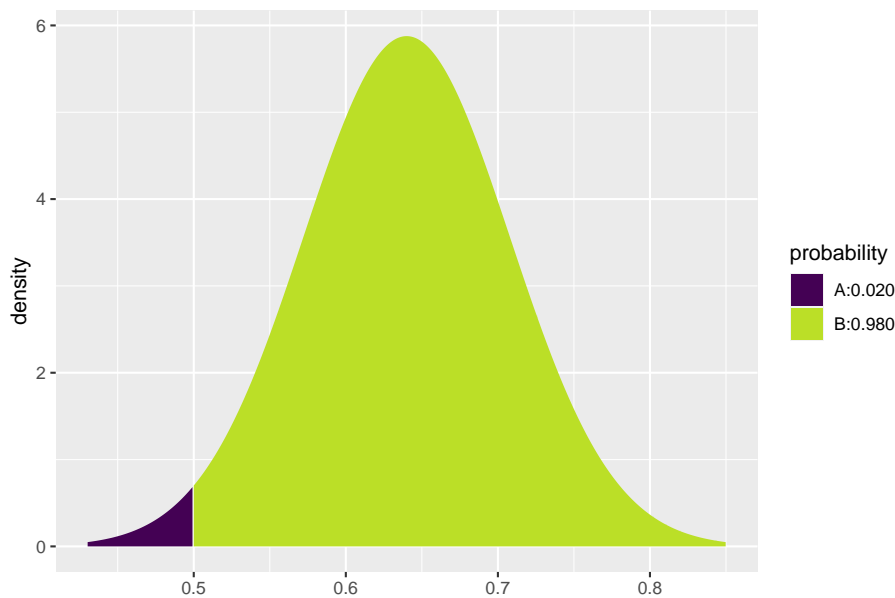
Since the conditions are satisfied, our sampling distribution model can be approximated with a normal model. The standard error is 0.0678823. Therefore, our normal model is

$$N(0.64, 0.068).$$

Back to our original question. How likely is it that a random survey of 50 voters predicts defeat for the candidate? Well, any survey that comes in less than 50% will make it look like the candidate is going to lose. So we simply need to figure

out how much of the sampling distribution lies below 50%. This is made simple with the `pdist` command. Note that we'll get a more accurate answer if we include the formula for the standard error, rather than rounding it off as 0.068.

```
pdist("norm", q = 0.5,
      mean = 0.64, sd = sqrt(0.64 * (1 - 0.64) / 50))
```



```
## [1] 0.01958508
```

From the picture, we can see that there is only about a 2% chance that one of our surveys of 50 voters could predict defeat. Using inline code, we calculate it as 1.9585083%. The vast majority of the time, then, when we go out and take such a survey, the results will show the candidate in the lead. It will likely not say exactly 64%; there is still a relatively wide range of values that seem to be possible outcomes of such surveys. Nevertheless, this range of values is mostly above 50%. Nevertheless, there is a small chance that the survey will give us the “wrong” answer and predict defeat for the candidate.<sup>1</sup>

**Exercise 7(a)** Suppose we are testing a new drug that is intended to reduce cholesterol levels in patients with high cholesterol. Also suppose that the drug

<sup>1</sup>Most polls in the 2016 presidential election predicted a win for Hillary Clinton, so they also gave the wrong answer. It’s possible that some of them were accidentally wrong due to sampling variability, but a much more likely explanation for their overall failure was bias.

works for 83% of such patients. When testing our drug, we use a suitably random sample of 143 individuals with high cholesterol.

First, simulate the sampling distribution using 1000 samples, each of size 143. Plot the resulting sampling distribution.

```
set.seed(13579)
# Add code here to simulate 1000 samples of size 143
# and plot the resulting distribution.
```

**Exercise 7(b)** Next, check the conditions that would allow you to use a normal model as a sampling distribution model. I've given you an outline below:

- Random
  - [Check condition here.]
- 10%
  - [Check condition here.]
- Success/failure
  - [Check condition here.]

**Exercise 7(c)** If the conditions are met, we can use a normal model as the sampling distribution model. What are the mean and standard error of this model? (You should use inline R code to calculate and report the standard error.)

Please write up your answer here.

**Exercise 7(d)** Market analysis shows that unless the drug is effective in more than 85% of patients, doctors won't prescribe it. Secretly, we know that the true rate of effectiveness is 83%, but the manufacturer doesn't know that yet. They only have access to their drug trial data in which they had 143 patients with high cholesterol.

Using the normal model you just developed, determine how likely the drug trial data will be to show the drug as "effective" according to the 85% standard. In other words, how often will our sample give us a result that is 85% or higher (even though secretly we know the true effectiveness is only 83%)? Report your answer in a contextually-meaningful full sentence using inline R code. (Hint: you'll need to use the `pdist` command.)

Please write up your answer here.

## 14.7 Conclusion

It is very easy to work with normal models. Therefore, when we want to study sampling variability, it is useful to have a normal model as a sampling distribution model. The standard error is a measure of how variable random samples can be. Such variability naturally decreases as our sample size grows. (This makes sense: larger samples give us more precise estimates of the true population, so they should be “closer” to the true population value.) Once conditions are checked, we can use normal models to make predictions about what we are likely to see when we sample from the population.

### 14.7.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.





# Chapter 15

## Inference for one proportion

2.0

### Functions introduced in this chapter

No new R functions are introduced here.

### 15.1 Introduction

Our earlier work with simulations showed us that when the number of successes and failures is large enough, we can use a normal model as our sampling distribution model.

We revisit hypothesis tests for a single proportion, but now, instead of running a simulation to compute the P-value, we take the shortcut of computing the P-value directly from a normal model.

There are no new concepts here. All we are doing is revisiting the rubric for inference and making the necessary changes.

#### 15.1.1 Install new packages

There are no new packages used in this chapter.

#### 15.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter

as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/15-inference\\_for\\_one\\_proportion.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/15-inference_for_one_proportion.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 15.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 15.2 Load packages

We load the standard `tidyverse`, `janitor` and `infer` packages as well as the `openintro` package to access data on heart transplant candidates. We’ll include `mosaic` for one spot below when we compare the results of `infer` to the results of graphing a normal distribution using `qdist`.

```
library(tidyverse)
library(janitor)
library(infer)
library(openintro)
library(mosaic)
```

## 15.3 Revisiting the rubric for inference

Instead of running a simulation, we are going to assume that the sampling distribution can be modeled with a normal model as long as the conditions for using a normal model are met.

Although the rubric has not changed, the use of a normal model changes quite a bit about the way we go through the other steps. For example, we won’t have simulated values to give us a histogram of the null model. Instead, we’ll go straight to graphing a normal model. We won’t compute the percent of our simulated samples that are at least as extreme as our test statistic to get the P-value. The P-value from a normal model is found directly from shading the model.

What follows is a fully-worked example of inference for one proportion. After the hypothesis test (sometimes called a one-proportion z-test for reasons that will become clear), we also follow up by computing a confidence interval. **From now on, we will consider inference to consist of a hypothesis test and a confidence interval.** Whenever you’re asked a question that requires

statistical inference, you should follow both the rubric steps for a hypothesis test and for a confidence interval.

The example below will pause frequently for commentary on the steps, especially where their execution will be different from what you’ve seen before when you used simulation. When it’s your turn to work through another example on your own, you should follow the outline of the rubric, but you should **not** copy and paste the commentary that accompanies it.

## 15.4 Research question

Data from the Stanford University Heart Transplant Study is located in the `openintro` package in a data frame called `heart_transplant`. From the help file we learn, “Each patient entering the program was designated officially a heart transplant candidate, meaning that he was gravely ill and would most likely benefit from a new heart.” Survival rates are not good for this population, although they are better for those who receive a heart transplant. Do heart transplant recipients still have less than a 50% chance of survival?

## 15.5 Exploratory data analysis

### 15.5.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

Start by typing `?heart_transplant` at the Console or searching for `heart_transplant` in the Help tab to read the help file.

**Exercise 1** Click on the link under “Source” in the help file. Why is this not helpful for determining the provenance of the data?

Now try to do an internet search to find the original research article from 1974. Why is this search process also not likely to help you determine the provenance of the data?

Please write up your answer here.

---

Now that we have learned everything we can reasonably learn about the data, we print it out and look at the variables.

```
heart_transplant
```

```
## # A tibble: 103 x 8
##       id acceptyear   age survived survtime prior transplant wait
##   <int>    <int> <int> <fct>      <int> <fct> <fct>      <int>
## 1    15      68    53 dead         1 no    control      NA
## 2    43      70    43 dead         2 no    control      NA
## 3    61      71    52 dead         2 no    control      NA
## 4    75      72    52 dead         2 no    control      NA
## 5     6      68    54 dead         3 no    control      NA
## 6    42      70    36 dead         3 no    control      NA
## 7    54      71    47 dead         3 no    control      NA
## 8    38      70    41 dead         5 no    treatment     5
## 9    85      73    47 dead         5 no    control      NA
## 10     2      68    51 dead         6 no    control      NA
## # ... with 93 more rows
```

```
glimpse(heart_transplant)
```

```
## Rows: 103
## Columns: 8
## $ id      <int> 15, 43, 61, 75, 6, 42, 54, 38, 85, 2, 103, 12, 48, 102, 35, ~
## $ acceptyear <int> 68, 70, 71, 72, 68, 70, 71, 70, 73, 68, 67, 68, 71, 74, 70, ~
## $ age      <int> 53, 43, 52, 52, 54, 36, 47, 41, 47, 51, 39, 53, 56, 40, 43, ~
## $ survived <fct> dead, dead, dead, dead, dead, dead, dead, dead, dead, dead, dead, ~
## $ survtime <int> 1, 2, 2, 2, 3, 3, 3, 5, 5, 6, 6, 8, 9, 11, 12, 16, 16, 16, ~
## $ prior    <fct> no, no, no, no, no, no, no, no, no, no, no, no, no, no, no, ~
## $ transplant <fct> control, control, control, control, control, control, control, contr~
## $ wait     <int> NA, NA, NA, NA, NA, NA, NA, NA, 5, NA, NA, NA, NA, NA, NA, ~
```

Commentary: The variable of interest is `survived`, which is coded as a factor variable with two categories, “alive” and “dead”. Keep in mind that because we are interested in survival rates, the “alive” condition will be considered the “success” condition.

There are 103 patients, but we are not considering all these patients. Our sample should consist of only those patients who actually received the transplant. The following table shows that only 69 patients were in the “treatment” group (meaning that they received a heart transplant).

```
tabyl(heart_transplant, transplant) %>%
  adorn_totals()
```

```
## transplant    n    percent
```

```
##      control   34 0.3300971
##      treatment  69 0.6699029
##           Total 103 1.0000000
```

### 15.5.2 Prepare the data for analysis.

**CAUTION:** If you are copying and pasting from this example to use for another research question, the following code chunk is specific to this research question and not applicable in other contexts.

We need to use `filter` so we get only the patients who actually received the heart transplant.

```
# Do not copy and paste this code for future work
heart_transplant2 <- heart_transplant %>%
  filter(transplant == "treatment")
heart_transplant2
```

```
## # A tibble: 69 x 8
##       id acceptyear  age survived survtime prior transplant wait
##   <int>    <int> <int> <fct>      <int> <fct> <fct>      <int>
## 1    38      70   41 dead         5 no    treatment     5
## 2    95      73   40 dead        16 no    treatment     2
## 3     3      68   54 dead        16 no    treatment     1
## 4    74      72   29 dead        17 no    treatment     5
## 5    20      69   55 dead        28 no    treatment     1
## 6    70      72   52 dead        30 no    treatment     5
## 7     4      68   40 dead        39 no    treatment    36
## 8   100      74   35 alive        39 yes   treatment    38
## 9    16      68   56 dead        43 no    treatment    20
## 10   45      71   36 dead        45 no    treatment     1
## # ... with 59 more rows
```

Commentary: don't forget the double equal sign (`==`) that checks whether the `treatment` variable is equal to the value "treatment". (See the Chapter 5 if you've forgotten how to use `filter`.)

Again, this step isn't something you need to do for other research questions. This question is peculiar because it asks only about patients who received a heart transplant, and that only involves a subset of the data we have in the `heart_transplant` data frame.

### 15.5.3 Make tables or plots to explore the data visually.

Making sure that we refer from now on to the `heart_transplant2` data frame and not the original `heart_transplant` data frame:

```
tabyl(heart_transplant2, survived) %>%
  adorn_totals()
```

```
## survived n percent
## alive 24 0.3478261
## dead 45 0.6521739
## Total 69 1.0000000
```

## 15.6 Hypotheses

### 15.6.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

The sample consists of 69 heart transplant recipients in a study at Stanford University. The population of interest is presumably all heart transplants recipients.

### 15.6.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

$H_0$  : Heart transplant recipients have a 50% chance of survival.

$H_A$  : Heart transplant recipients have less than a 50% chance of survival.

Commentary: It is slightly unusual that we are conducting a one-sided test. The standard default is typically a two-sided test. However, it is not for us to choose: the proposed research question is unequivocal in hypothesizing “less than 50%” survival.

### 15.6.3 Express the null and alternative hypotheses in symbols (when possible).

$H_0 : p_{\text{alive}} = 0.5$

$H_A : p_{\text{alive}} < 0.5$

## 15.7 Model

### 15.7.1 Identify the sampling distribution model.

We will use a normal model.

Commentary: In past chapters, we have simulated the sampling distribution or applied some kind of randomization to simulate the effect of the null hypothesis. The point of this chapter is that we can—when the conditions are met—substitute a normal model to replace the unimodal and symmetric histogram that resulted from randomization and simulation.

### 15.7.2 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - Since the 69 patients are from a study at Stanford, we do not have a random sample of all heart transplant recipients. We hope that the patients recruited to this study were physiologically similar to other heart patients so that they are a representative sample. Without more information, we have no real way of knowing.
- 10%
  - 69 patients are definitely less than 10% of all heart transplant recipients.
- Success/failure

$$np_{\text{alive}} = 69(0.5) = 34.5 \geq 10$$

$$n(1 - p_{\text{alive}}) = 69(0.5) = 34.5 \geq 10$$

Commentary: Notice something interesting here. Why did we not use the 24 patients who survived and the 45 who died as the successes and failures? In other words, why did we use  $np_{\text{alive}}$  and  $n(1 - p_{\text{alive}})$  instead of  $n\hat{p}_{\text{alive}}$  and  $n(1 - \hat{p}_{\text{alive}})$ ?

Remember the logic of inference and the philosophy of the null hypothesis. To convince the skeptics, we must assume the null hypothesis throughout the process. It's only after we present sufficient evidence that can we reject the null and fall back on the alternative hypothesis that encapsulates our research question.

Therefore, under the assumption of the null, the sampling distribution is the *null distribution*, meaning that it's centered at 0.5. All work we do with the normal model, including checking conditions, must use the null model with  $p_{\text{alive}} = 0.5$ .

That's also why the numbers don't have to be whole numbers. If the null states that of the 69 patients, 50% are expected to survive, then we expect 50% of 69, or 34.5, to survive. Of course, you can't have half of a survivor. But these are not *actual* survivors. Rather, they are the expected number of survivors in a group of 69 patients *on average* under the assumption of the null.

## 15.8 Mechanics

### 15.8.1 Compute the test statistic.

```

alive_prop <- heart_transplant2 %>%
  specify(response = survived, succes = "alive") %>%
  calculate(stat = "prop")
alive_prop

## Response: survived (factor)
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 0.348

```

We'll also compute the corresponding z score.

```

alive_z <- heart_transplant2 %>%
  specify(response = survived, succes = "alive") %>%
  hypothesize(null = "point", p = 0.5) %>%
  calculate(stat = "z")
alive_z

## Response: survived (factor)
## Null Hypothesis: point
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 -2.53

```

Commentary: The sample proportion code is straightforward and we've seen it before. To get the z score, we also have to tell **infer** what the null hypothesis is so that it knows where the center of our normal distribution will be. In the **hypothesize** function, we tell **infer** to use a “point” null hypothesis with **p** = 0.5. All this means is that the null is a specific point: 0.5. (Contrast this to hypothesis tests with two variables when we had **null** = “independence”.)

We can confirm the calculation of the z score manually. It's easiest to compute the standard error first. Recall that the standard error is

$$SE = \sqrt{\frac{p_{alive}(1 - p_{alive})}{n}} = \sqrt{\frac{0.5(1 - 0.5)}{69}}$$



**Remember that are working under the assumption of the null hypothesis.** This means that we use  $p_{alive} = 0.5$  everywhere in the formula for the standard error.

We can do the math in R and store our result as `SE`.

```
SE <- sqrt(0.5*(1 - 0.5)/69)
SE
```

```
## [1] 0.06019293
```

Then our z score is

$$z = \frac{(\hat{p}_{alive} - p_{alive})}{SE} = \frac{(\hat{p}_{alive} - p_{alive})}{\sqrt{\frac{p_{alive}(1-p_{alive})}{n}}} = \frac{(0.348 - 0.5)}{\sqrt{\frac{0.5(1-0.5)}{69}}} = -2.53.$$

Using the values of `alive_prop` and `SE`:

```
z <- (alive_prop - 0.5)/SE
z
```

```
##          stat
## 1 -2.528103
```

Both the sample proportion  $\hat{p}_{alive}$  (stored above as `alive_prop`) and the corresponding z-score can be considered the “test statistic”. If we use  $\hat{p}_{alive}$  as the test statistic, then we’re considering the null model to be

$$N\left(0.5, \sqrt{\frac{0.5(1-0.5)}{69}}\right).$$

If we use `z` as the test statistic, then we’re considering the null model to be the *standard normal model*:

$$N(0, 1).$$

The standard normal model is more intuitive and easier to work with, both conceptually and in R. Generally, then, we will consider `z` as the test statistic so that we can consider our null model to be the standard normal model. For example, knowing that our test statistic is two and a half standard deviations to the left of the null value already tells us a lot. We can anticipate a small P-value leading to rejection of the null. Nevertheless, for this type of hypothesis test, we’ll compute both in this section of the rubric.

### 15.8.2 Report the test statistic in context (when possible).

The test statistic is 0.3478261. In other words, 34.7826087% of heart transplant recipients were alive at the end of the study.

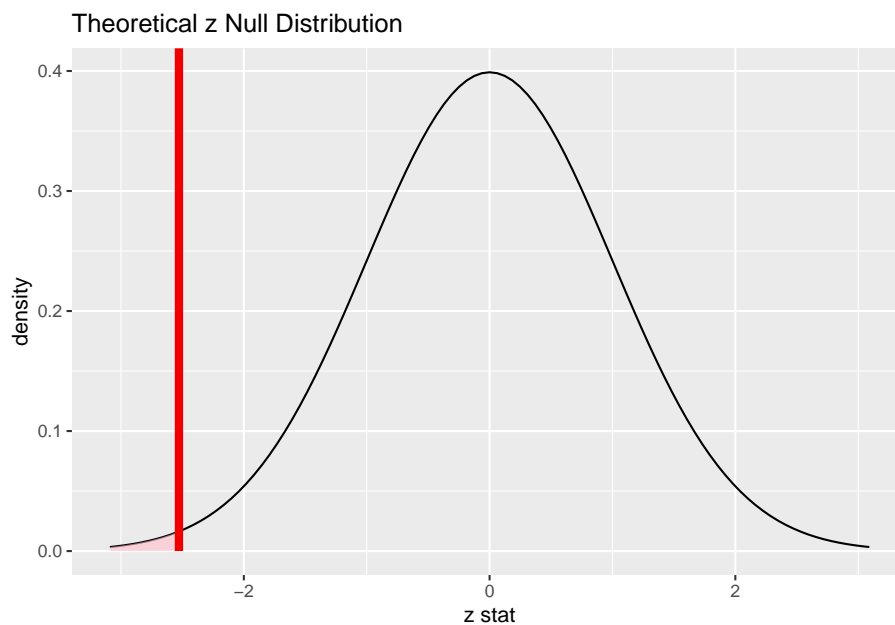
The z score is -2.5281029. The proportion of survivors is about 2.5 standard errors below the null value.

### 15.8.3 Plot the null distribution.

```
alive_test <- heart_transplant2 %>%  
  specify(response = survived, success = "alive") %>%  
  hypothesize(null = "point", p = 0.5) %>%  
  assume(distribution = "z")  
alive_test
```

```
## A Z distribution.
```

```
alive_test %>%  
  visualize() +  
  shade_p_value(obs_stat = alive_z, direction = "less")
```



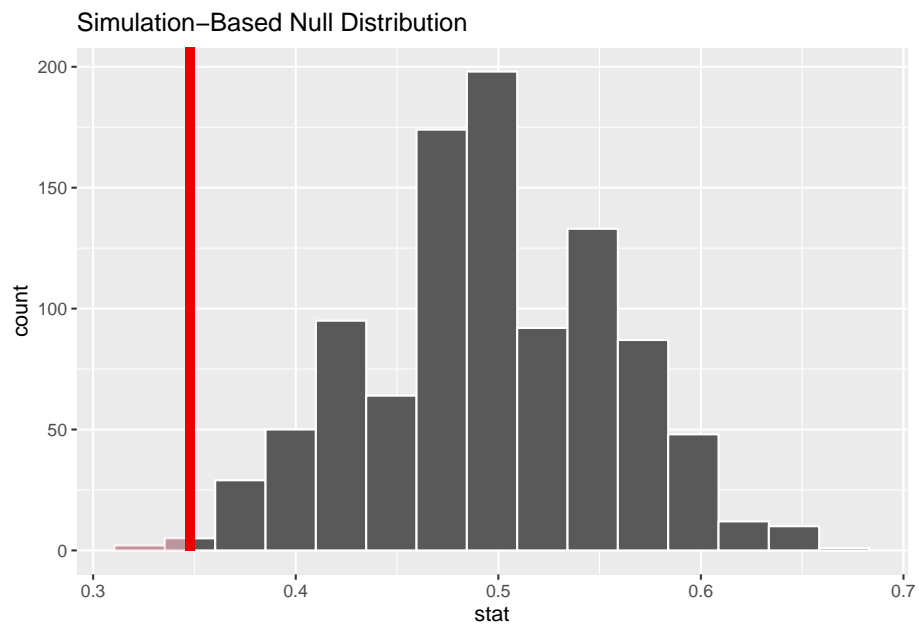
Commentary: In past chapters, we have used the **generate** verb to get many repetitions (usually 1000) of some kind of random process to simulate the sampling distribution model. In this chapter, we have used the verb **assume** instead to assume that the sampling distribution is a normal model. As long as the conditions hold, this is a reasonable assumption. This also means that we don't have to use **set.seed** as there is no random process to reproduce.

Compare the graph above to what we would see if we simulated the sampling distribution. (Now we do need **set.seed**!)

```
set.seed(6789)
alive_test_draw <- heart_transplant2 %>%
  specify(response = survived, success = "alive") %>%
  hypothesize(null = "point", p = 0.5) %>%
  generate(reps = 1000, type = "draw") %>%
  calculate(stat = "prop")
alive_test_draw
```

```
## Response: survived (factor)
## Null Hypothesis: point
## # A tibble: 1,000 x 2
##   replicate stat
##   <fct>      <dbl>
## 1 1         0.493
## 2 2         0.406
## 3 3         0.435
## 4 4         0.580
## 5 5         0.522
## 6 6         0.507
## 7 7         0.580
## 8 8         0.435
## 9 9         0.551
## 10 10        0.435
## # ... with 990 more rows
```

```
alive_test_draw %>%
  visualize() +
  shade_p_value(obs_stat = alive_prop, direction = "less")
```



This is essentially the same picture, although the model above is centered on the null value 0.5 instead of the z score of 0. This also means that the `obs_stat` had to be the sample proportion `alive_prop` and not the z score `alive_z`.

#### 15.8.4 Calculate the P-value.

```
alive_test_p <- alive_test %>%
  get_p_value(obs_stat = alive_z, direction = "less")
alive_test_p
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1 0.00573
```

Commentary: compare this to the P-value we get from simulating random draws:

```
alive_test_draw %>%
  get_p_value(obs_stat = alive_prop, direction = "less")
```

```
## # A tibble: 1 x 1
```

```
##    p_value
##    <dbl>
## 1    0.007
```

The values are not exactly the same. And a new simulation with a different seed would likely give another slightly different P-value. The takeaway here is that the P-value itself has some uncertainty, so you should never take the value too seriously.

### 15.8.5 Interpret the P-value as a probability given the null.

The P-value is 0.005734. If there were truly a 50% chance of survival among heart transplant patients, there would only be a 0.5734037% chance of seeing data at least as extreme as we saw.

## 15.9 Conclusion

### 15.9.1 State the statistical conclusion.

We reject the null hypothesis.

### 15.9.2 State (but do not overstate) a contextually meaningful conclusion.

We have sufficient evidence that heart transplant recipients have less than a 50% chance of survival.

### 15.9.3 Express reservations or uncertainty about the generalizability of the conclusion.

Because we know nearly nothing about the provenance of the data, it's hard to generalize the conclusion. We know the data is from 1974, so it's also very likely that survival rates for heart transplant patients then are not the same as they are today. The most we could hope for is that the Stanford data was representative for heart transplant patients in 1974. Our sample size (69) is also quite small.

### 15.9.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.

As we rejected the null, we run the risk of making a Type I error. It is possible that the null is true and that there is a 50% chance of survival for these patients, but we got an unusual sample that appears to have a much smaller chance of survival.

## 15.10 Confidence interval

### 15.10.1 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - Same as above.
- 10%
  - Same as above.
- Success/failure
  - There were 24 patients who survived and 45 who died in our sample. Both are larger than 10.

Commentary: In the “Confidence interval” section of the rubric, there is no need to recheck conditions that have already been checked. The sample has not changed; if it met the “Random” and “10%” conditions before, it will meet them now.

So why recheck the success/failure condition?

Keep in mind that in a hypothesis test, we temporarily assume the null is true. The null states that  $p = 0.5$  and the resulting null distribution is, therefore, centered at  $p = 0.5$ . The success/failure condition is a condition that applies to the normal model we’re using, and for a hypothesis test, that’s the null model.

By contrast, a confidence interval is making no assumption about the “true” value of  $p$ . The inferential goal of a confidence interval is to try to capture the true value of  $p$ , so we certainly cannot make any assumptions about it. Therefore, we go back to the original way we learned about the success/failure condition. That is, we check the actual number of successes and failures.

## 15.10.2 Calculate and graph the confidence interval.

```

alive_ci <- alive_test %>%
  get_confidence_interval(point_estimate = alive_prop, level = 0.95)
alive_ci

```

```

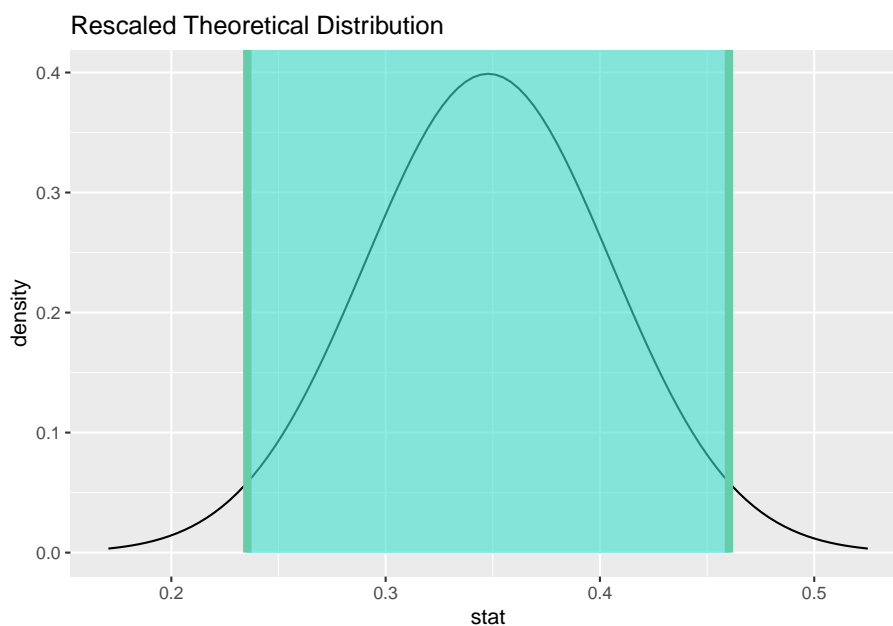
## # A tibble: 1 x 2
##   lower_ci upper_ci
##   <dbl>    <dbl>
## 1    0.235    0.460

```

```

alive_test %>%
  visualize() +
  shade_confidence_interval(endpoints = alive_ci)

```



Commentary: when we use a theoretical normal distribution, we have to compute the confidence interval a different way.

When we bootstrapped, we had many repetitions of a process that resulted in a sampling distribution. From all those, we could find the 2.5th percentile and the 97.5th percentile. Although we let the computer do it for us, the process is straightforward enough that we could do it by hand if we needed to. Just put all 1000 bootstrapped values in order, then go to the 25th and 975th position in the list.

We don't have a list of 1000 values when we use an abstract curve to represent our sampling distribution. Nevertheless, we can find the 2.5th percentile and the 97.5th percentile using the area under the normal curve as we saw in the last two chapters. We can do this “manually” with the `qdist` command, but we need the standard error first.

Didn't we calculate this earlier?

$$SE = \sqrt{\frac{p_{alive}(1 - p_{alive})}{n}} = \sqrt{\frac{0.5(1 - 0.5)}{69}}$$

Well...sort of. The value of  $p_{alive}$  here is the value of the null hypothesis from the hypothesis test above. *However*, the hypothesis test is done. For a confidence interval, we have no information about any “null” value. There is no null anymore. It's irrelevant.

So what is the standard error for a confidence interval? Since we don't have  $p_{alive}$ , the best we can do is replace it with  $\hat{p}_{alive}$ :

$$SE = \sqrt{\frac{\hat{p}_{alive}(1 - \hat{p}_{alive})}{n}} = \sqrt{\frac{0.3478261(1 - 0.3478261)}{69}}.$$

We can let R do the heavy lifting here:

```
SE2 <- sqrt(alive_prop * (1 - alive_prop) / 69)
SE2
```

```
##          stat
## 1 0.05733743
```

And now this number can go into `qdist` as our standard deviation:

```
qdist("norm", p = c(0.025, 0.975), mean = 0.3478261, sd = 0.05733743, plot = FALSE)

## [1] 0.2354468 0.4602054
```

The numbers above are identical to the ones computed by the `infer` commands.

### 15.10.3 State (but do not overstate) a contextually meaningful interpretation.

We are 95% confident that the true percentage of heart transplant recipients who survive is captured in the interval (23.5446784%, 46.020539%).

Commentary: Note that when we state our contextually meaningful conclusion, we also convert the decimal proportions to percentages. Humans like percentages a lot better.



**15.10.4 If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test.**

We are not running a two-sided test, so this step is not applicable.

**15.10.5 When comparing two groups, comment on the effect size and the practical significance of the result.**

This is not applicable here because we are not comparing two groups. We are looking at the survival percentage in only one group of patients, those who had a heart transplant.

## 15.11 Your turn

Follow the rubric to answer the following research question:

Some heart transplant candidates have already had a prior surgery. Use the variable `prior` in the `heart_transplant` data set to determine if fewer than 50% of patients have had a prior surgery. (To be clear, you are being asked to perform a one-sided test again.) **Be sure to use the full `heart_transplant` data, not the modified `heart_transplant2` from the previous example.**

The rubric outline is reproduced below. You may refer to the worked example above and modify it accordingly. Remember to strip out all the commentary. That is just exposition for your benefit in understanding the steps, but is not meant to form part of the formal inference process.

Another word of warning: the copy/paste process is not a substitute for your brain. You will often need to modify more than just the names of the tibbles and variables to adapt the worked examples to your own work. For example, if you run a two-sided test instead of a one-sided test, there are a few places that have to be adjusted accordingly. Understanding the sampling distribution model and the computation of the P-value goes a long way toward understanding the changes that must be made. Do not blindly copy and paste code without understanding what it does. And you should **never** copy and paste text. All the sentences and paragraphs you write are expressions of your own analysis. They must reflect your own understanding of the inferential process.

**Also, so that your answers here don't mess up the code chunks above, use new variable names everywhere. In particular, you should use `prior_test` (instead of `alive_test`) to store the results of your hypothesis test. Make other corresponding changes as necessary, like `prior_test_p` instead of `alive_test_p`, for example.**

### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure. Please write up your answer here.

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

Identify the sample (or samples) and a reasonable population (or populations) of interest. [Remember that you are using the full heart\_transplant data, so your sample size should be larger here than in the example above.]

Please write up your answer here.

Express the null and alternative hypotheses as contextually meaningful full sentences.  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

Express the null and alternative hypotheses in symbols (when possible).  $H_0$  : *math*

$H_A$  : *math*

**Model**

**Identify the sampling distribution model.** Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.** [Remember that you are using the full `heart_transplant` data, so the number of successes and failures will be different here than in the example above.]

Please write up your answer here. (Some conditions may require R code as well.)

**Mechanics** [Be sure to use `heart_transplant` everywhere and not `heart_transplant2`!]

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

**Conclusion**

**State the statistical conclusion.** Please write up your answer here.

**State (but do not overstate) a contextually meaningful conclusion.**  
Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.**  
Please write up your answer here.

**Confidence interval**

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

```
# Add code here to calculate the confidence interval.
```

**Calculate the confidence interval.**

**State (but do not overstate) a contextually meaningful interpretation.**  
Please write up your answer here.

**If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test.** [Not always applicable.]  
Please write up your answer here.

**When comparing two groups, comment on the effect size and the practical significance of the result.** [Not always applicable.] Please write up your answer here.

## 15.12 Conclusion

When certain conditions are met, we can use a theoretical normal model—a perfectly symmetric bell curve—as a sampling distribution model in hypothesis testing. Because this does not require drawing many samples, it is faster and cleaner than simulation. Of course, on modern computing devices, drawing even thousands of simulated samples is not very time consuming, and the code we write doesn’t really change much. Given the additional success/failure condition that has to met, it’s worth considering the pros and cons of using a normal model instead of simulating the sampling distribution. Similarly, confidence intervals can be obtained directly from the percentiles of the normal model without the need to obtain bootstrapped samples.

### 15.12.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.



## Chapter 16

# Inference for two proportions

2.0

### Functions introduced in this chapter

No new R functions are introduced here.

### 16.1 Introduction

In this chapter, we revisit the idea of inference for two proportions, but this time using a normal model as the sampling distribution model.

#### 16.1.1 Install new packages

There are no new packages used in this chapter.

#### 16.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/16-inference\\_for\\_two\\_proportions.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/16-inference_for_two_proportions.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 16.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 16.2 Load packages

We load the standard `tidyverse`, `janitor` and `infer` packages as well as the `MASS` package for the `Melanoma` data.

```
library(tidyverse)
library(janitor)
library(infer)
library(MASS)
```

## 16.3 Research question

In an earlier chapter, we used the data set `Melanoma` from the `MASS` package to explore the possibility of a sex bias among patients with melanoma. A related question is whether male or females are more likely to die from melanoma. In this case, we are thinking of `status` as the response variable and `sex` as the predictor variable.

## 16.4 The sampling distribution model for two proportions

When we simulated a sampling distribution using randomization (shuffling the values of the predictor variable), it looked like the simulated sampling distribution was roughly normal. Therefore, we should be able to use a normal model in place of randomization when we want to perform statistical inference.

The question is, “Which normal model?” In other words, what is the mean and standard deviation we should use?

Since we have two groups, let’s call the true proportion of success  $p_1$  for group 1 and  $p_2$  for group 2. Therefore, the true difference between groups 1 and 2 in the population is  $p_1 - p_2$ . If we sample repeatedly from groups 1 and 2 and form many sample differences  $\hat{p}_1 - \hat{p}_2$ , we should expect most of the values  $\hat{p}_1 - \hat{p}_2$  to be close to the true difference  $p_1 - p_2$ . In other words, the sampling distribution is centered at a mean of  $p_1 - p_2$ .

What about the standard error? This is much more technical and complicated. Here is the formula, whose derivation is outside the scope of the course:



$$\sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}.$$

So the somewhat complicated normal model is

$$N\left(p_1 - p_2, \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}\right).$$

When we ran hypothesis tests for one proportion, the true proportion  $p$  was assumed to be known, set equal to some null value. Therefore, we could calculate the standard error  $\sqrt{\frac{p(1-p)}{n}}$  under the assumption of the null.

We also have a null hypothesis for two proportions. When comparing two groups, the default assumption is that the two groups are the same. This translates into the mathematical statement  $p_1 - p_2 = 0$  (i.e., there is no difference between  $p_1$  and  $p_2$ ).

But there is a problem here. Although we are assuming something about the difference  $p_1 - p_2$ , we are not assuming anything about the actual values of  $p_1$  and  $p_2$ . For example, both groups could be 0.3, or 0.6, or 0.92, or whatever, and the difference between the groups would still be zero.

Without values of  $p_1$  and  $p_2$ , we cannot plug anything into the standard error formula above. One easy “cheat” is to just use the sample values  $\hat{p}_1$  and  $\hat{p}_2$ :

$$SE = \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}.$$

There is a more sophisticated way to address this called “pooling”. This more advanced concept is covered in an optional appendix to this chapter.

## 16.5 Inference for two proportions

Below is a fully-worked example of inference (hypothesis test and confidence interval) for two proportions. When you work your own example, you can thoughtfully copy and paste the R code, making changes as necessary.

The example below will pause frequently for commentary on the steps, especially where their execution will be different from what you’ve seen before when you used randomization. When it’s your turn to work through another example on your own, you should follow the outline of the rubric, but you should **not** copy and paste the commentary that accompanies it.

## 16.6 Exploratory data analysis

### 16.6.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

Type `?Melanoma` at the Console to read the help file. We discussed this data back in Chapter 11 and determined that it was difficult, if not impossible, to discover anything useful about the true provenance of the data. We can, at least, print the data out and examine the variables

Melanoma

```
##      time status sex age year thickness ulcer sex_fct
## 1      10      3  1  76 1972      6.76      1    male
## 2      30      3  1  56 1968      0.65      0    male
## 3      35      2  1  41 1977      1.34      0    male
## 4      99      3  0  71 1968      2.90      0  female
## 5     185      1  1  52 1965     12.08      1    male
## 6     204      1  1  28 1971      4.84      1    male
## 7     210      1  1  77 1972      5.16      1    male
## 8     232      3  0  60 1974      3.22      1  female
## 9     232      1  1  49 1968     12.88      1    male
## 10    279      1  0  68 1971      7.41      1  female
## 11    295      1  0  53 1969      4.19      1  female
## 12    355      3  0  64 1972      0.16      1  female
## 13    386      1  0  68 1965      3.87      1  female
## 14    426      1  1  63 1970      4.84      1    male
## 15    469      1  0  14 1969      2.42      1  female
## 16    493      3  1  72 1971     12.56      1    male
## 17    529      1  1  46 1971      5.80      1    male
## 18    621      1  1  72 1972      7.06      1    male
## 19    629      1  1  95 1968      5.48      1    male
## 20    659      1  1  54 1972      7.73      1    male
## 21    667      1  0  89 1968     13.85      1  female
## 22    718      1  1  25 1967      2.34      1    male
## 23    752      1  1  37 1973      4.19      1    male
## 24    779      1  1  43 1967      4.04      1    male
## 25    793      1  1  68 1970      4.84      1    male
## 26    817      1  0  67 1966      0.32      0  female
## 27    826      3  0  86 1965      8.54      1  female
## 28    833      1  0  56 1971      2.58      1  female
## 29    858      1  0  16 1967      3.56      0  female
## 30    869      1  0  42 1965      3.54      0  female
## 31    872      1  0  65 1968      0.97      0  female
```

## 32	967	1	1	52	1970	4.83	1	male
## 33	977	1	1	58	1967	1.62	1	male
## 34	982	1	0	60	1970	6.44	1	female
## 35	1041	1	1	68	1967	14.66	0	male
## 36	1055	1	0	75	1967	2.58	1	female
## 37	1062	1	1	19	1966	3.87	1	male
## 38	1075	1	1	66	1971	3.54	1	male
## 39	1156	1	0	56	1970	1.34	1	female
## 40	1228	1	1	46	1973	2.24	1	male
## 41	1252	1	0	58	1971	3.87	1	female
## 42	1271	1	0	74	1971	3.54	1	female
## 43	1312	1	0	65	1970	17.42	1	female
## 44	1427	3	1	64	1972	1.29	0	male
## 45	1435	1	1	27	1969	3.22	0	male
## 46	1499	2	1	73	1973	1.29	0	male
## 47	1506	1	1	56	1970	4.51	1	male
## 48	1508	2	1	63	1973	8.38	1	male
## 49	1510	2	0	69	1973	1.94	0	female
## 50	1512	2	0	77	1973	0.16	0	female
## 51	1516	1	1	80	1968	2.58	1	male
## 52	1525	3	0	76	1970	1.29	1	female
## 53	1542	2	0	65	1973	0.16	0	female
## 54	1548	1	0	61	1972	1.62	0	female
## 55	1557	2	0	26	1973	1.29	0	female
## 56	1560	1	0	57	1973	2.10	0	female
## 57	1563	2	0	45	1973	0.32	0	female
## 58	1584	1	1	31	1970	0.81	0	male
## 59	1605	2	0	36	1973	1.13	0	female
## 60	1621	1	0	46	1972	5.16	1	female
## 61	1627	2	0	43	1973	1.62	0	female
## 62	1634	2	0	68	1973	1.37	0	female
## 63	1641	2	1	57	1973	0.24	0	male
## 64	1641	2	0	57	1973	0.81	0	female
## 65	1648	2	0	55	1973	1.29	0	female
## 66	1652	2	0	58	1973	1.29	0	female
## 67	1654	2	1	20	1973	0.97	0	male
## 68	1654	2	0	67	1973	1.13	0	female
## 69	1667	1	0	44	1971	5.80	1	female
## 70	1678	2	0	59	1973	1.29	0	female
## 71	1685	2	0	32	1973	0.48	0	female
## 72	1690	1	1	83	1971	1.62	0	male
## 73	1710	2	0	55	1973	2.26	0	female
## 74	1710	2	1	15	1973	0.58	0	male
## 75	1726	1	0	58	1970	0.97	1	female
## 76	1745	2	0	47	1973	2.58	1	female
## 77	1762	2	0	54	1973	0.81	0	female

## 78	1779	2	1	55	1973	3.54	1	male
## 79	1787	2	1	38	1973	0.97	0	male
## 80	1787	2	0	41	1973	1.78	1	female
## 81	1793	2	0	56	1973	1.94	0	female
## 82	1804	2	0	48	1973	1.29	0	female
## 83	1812	2	1	44	1973	3.22	1	male
## 84	1836	2	0	70	1972	1.53	0	female
## 85	1839	2	0	40	1972	1.29	0	female
## 86	1839	2	1	53	1972	1.62	1	male
## 87	1854	2	0	65	1972	1.62	1	female
## 88	1856	2	1	54	1972	0.32	0	male
## 89	1860	3	1	71	1969	4.84	1	male
## 90	1864	2	0	49	1972	1.29	0	female
## 91	1899	2	0	55	1972	0.97	0	female
## 92	1914	2	0	69	1972	3.06	0	female
## 93	1919	2	1	83	1972	3.54	0	male
## 94	1920	2	1	60	1972	1.62	1	male
## 95	1927	2	1	40	1972	2.58	1	male
## 96	1933	1	0	77	1972	1.94	0	female
## 97	1942	2	0	35	1972	0.81	0	female
## 98	1955	2	0	46	1972	7.73	1	female
## 99	1956	2	0	34	1972	0.97	0	female
## 100	1958	2	0	69	1972	12.88	0	female
## 101	1963	2	0	60	1972	2.58	0	female
## 102	1970	2	1	84	1972	4.09	1	male
## 103	2005	2	0	66	1972	0.64	0	female
## 104	2007	2	1	56	1972	0.97	0	male
## 105	2011	2	0	75	1972	3.22	1	female
## 106	2024	2	0	36	1972	1.62	0	female
## 107	2028	2	1	52	1972	3.87	1	male
## 108	2038	2	0	58	1972	0.32	1	female
## 109	2056	2	0	39	1972	0.32	0	female
## 110	2059	2	1	68	1972	3.22	1	male
## 111	2061	1	1	71	1968	2.26	0	male
## 112	2062	1	0	52	1965	3.06	0	female
## 113	2075	2	1	55	1972	2.58	1	male
## 114	2085	3	0	66	1970	0.65	0	female
## 115	2102	2	1	35	1972	1.13	0	male
## 116	2103	1	1	44	1966	0.81	0	male
## 117	2104	2	0	72	1972	0.97	0	female
## 118	2108	1	0	58	1969	1.76	1	female
## 119	2112	2	0	54	1972	1.94	1	female
## 120	2150	2	0	33	1972	0.65	0	female
## 121	2156	2	0	45	1972	0.97	0	female
## 122	2165	2	1	62	1972	5.64	0	male
## 123	2209	2	0	72	1971	9.66	0	female

## 124 2227	2	0	51	1971	0.10	0	female
## 125 2227	2	1	77	1971	5.48	1	male
## 126 2256	1	0	43	1971	2.26	1	female
## 127 2264	2	0	65	1971	4.83	1	female
## 128 2339	2	0	63	1971	0.97	0	female
## 129 2361	2	1	60	1971	0.97	0	male
## 130 2387	2	0	50	1971	5.16	1	female
## 131 2388	1	1	40	1966	0.81	0	male
## 132 2403	2	0	67	1971	2.90	1	female
## 133 2426	2	0	69	1971	3.87	0	female
## 134 2426	2	0	74	1971	1.94	1	female
## 135 2431	2	0	49	1971	0.16	0	female
## 136 2460	2	0	47	1971	0.64	0	female
## 137 2467	1	0	42	1965	2.26	1	female
## 138 2492	2	0	54	1971	1.45	0	female
## 139 2493	2	1	72	1971	4.82	1	male
## 140 2521	2	0	45	1971	1.29	1	female
## 141 2542	2	1	67	1971	7.89	1	male
## 142 2559	2	0	48	1970	0.81	1	female
## 143 2565	1	1	34	1970	3.54	1	male
## 144 2570	2	0	44	1970	1.29	0	female
## 145 2660	2	0	31	1970	0.64	0	female
## 146 2666	2	0	42	1970	3.22	1	female
## 147 2676	2	0	24	1970	1.45	1	female
## 148 2738	2	0	58	1970	0.48	0	female
## 149 2782	1	1	78	1969	1.94	0	male
## 150 2787	2	1	62	1970	0.16	0	male
## 151 2984	2	1	70	1969	0.16	0	male
## 152 3032	2	0	35	1969	1.29	0	female
## 153 3040	2	0	61	1969	1.94	0	female
## 154 3042	1	0	54	1967	3.54	1	female
## 155 3067	2	0	29	1969	0.81	0	female
## 156 3079	2	1	64	1969	0.65	0	male
## 157 3101	2	1	47	1969	7.09	0	male
## 158 3144	2	1	62	1969	0.16	0	male
## 159 3152	2	0	32	1969	1.62	0	female
## 160 3154	3	1	49	1969	1.62	0	male
## 161 3180	2	0	25	1969	1.29	0	female
## 162 3182	3	1	49	1966	6.12	0	male
## 163 3185	2	0	64	1969	0.48	0	female
## 164 3199	2	0	36	1969	0.64	0	female
## 165 3228	2	0	58	1969	3.22	1	female
## 166 3229	2	0	37	1969	1.94	0	female
## 167 3278	2	1	54	1969	2.58	0	male
## 168 3297	2	0	61	1968	2.58	1	female
## 169 3328	2	1	31	1968	0.81	0	male

```
## 170 3330      2   1  61 1968      0.81      1   male
## 171 3338      1   0  60 1967      3.22      1  female
## 172 3383      2   0  43 1968      0.32      0  female
## 173 3384      2   0  68 1968      3.22      1  female
## 174 3385      2   0   4 1968      2.74      0  female
## 175 3388      2   1  60 1968      4.84      1   male
## 176 3402      2   1  50 1968      1.62      0   male
## 177 3441      2   0  20 1968      0.65      0  female
## 178 3458      3   0  54 1967      1.45      0  female
## 179 3459      2   0  29 1968      0.65      0  female
## 180 3459      2   1  56 1968      1.29      1   male
## 181 3476      2   0  60 1968      1.62      0  female
## 182 3523      2   0  46 1968      3.54      0  female
## 183 3667      2   0  42 1967      3.22      0  female
## 184 3695      2   0  34 1967      0.65      0  female
## 185 3695      2   0  56 1967      1.03      0  female
## 186 3776      2   1  12 1967      7.09      1   male
## 187 3776      2   0  21 1967      1.29      1  female
## 188 3830      2   1  46 1967      0.65      0   male
## 189 3856      2   0  49 1967      1.78      0  female
## 190 3872      2   0  35 1967     12.24      1  female
## 191 3909      2   1  42 1967      8.06      1   male
## 192 3968      2   0  47 1967      0.81      0  female
## 193 4001      2   0  69 1967      2.10      0  female
## 194 4103      2   0  52 1966      3.87      0  female
## 195 4119      2   1  52 1966      0.65      0   male
## 196 4124      2   0  30 1966      1.94      1  female
## 197 4207      2   1  22 1966      0.65      0   male
## 198 4310      2   1  55 1966      2.10      0   male
## 199 4390      2   0  26 1965      1.94      1  female
## 200 4479      2   0  19 1965      1.13      1  female
## 201 4492      2   1  29 1965      7.06      1   male
## 202 4668      2   0  40 1965      6.12      0  female
## 203 4688      2   0  42 1965      0.48      0  female
## 204 4926      2   0  50 1964      2.26      0  female
## 205 5565      2   0  41 1962      2.90      0  female
```

```
glimpse(Melanoma)
```

```
## Rows: 205
## Columns: 8
## $ time      <int> 10, 30, 35, 99, 185, 204, 210, 232, 232, 279, 295, 355, 386, ~
## $ status    <int> 3, 3, 2, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 1, 1, ~
## $ sex       <int> 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, ~
## $ age       <int> 76, 56, 41, 71, 52, 28, 77, 60, 49, 68, 53, 64, 68, 63, 14, ~
```

```
## $ year      <int> 1972, 1968, 1977, 1968, 1965, 1971, 1972, 1974, 1968, 1971, ~
## $ thickness <dbl> 6.76, 0.65, 1.34, 2.90, 12.08, 4.84, 5.16, 3.22, 12.88, 7.41~
## $ ulcer     <int> 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ~
## $ sex_fct   <fct> male, male, male, female, male, male, male, male, female, male, fe~
```

### 16.6.2 Prepare the data for analysis.

The two variables of interest are **status** and **sex**. We are considering them as categorical variables, but they are recorded numerically in the data frame. We convert them to proper factor variables and put them in their own data frame using the help file to identify the levels and labels we need.

There is a minor hitch with **status**. The help file shows three categories: 1. died from melanoma, 2. alive, 3. dead from other causes. For two-proportion inference, it would be better to have two categories only, a success category and a failure category. Since our research question asks about deaths due to melanoma, the “success” condition is the one numbered 1 in the help file, “died from melanoma”. That means we need to combine the other two categories into a single failure category. Perhaps we should call it “other”. You can accomplish this by simply repeating the “other” label more than once in the **factor** command:

```
Melanoma <- Melanoma %>%
  mutate(sex_fct = factor(sex,
                          levels = c(0, 1),
                          labels = c("female", "male")),
         status_fct = factor(status,
                              levels = c(1, 2, 3),
                              labels = c("died from melanoma", "other", "other")))
glimpse(Melanoma)
```

```
## Rows: 205
## Columns: 9
## $ time      <int> 10, 30, 35, 99, 185, 204, 210, 232, 232, 279, 295, 355, 386~
## $ status    <int> 3, 3, 2, 3, 1, 1, 1, 3, 1, 1, 1, 3, 1, 1, 1, 1, 1, 1, ~
## $ sex       <int> 1, 1, 1, 0, 1, 1, 1, 0, 1, 0, 0, 0, 0, 1, 0, 1, 1, 1, 1, ~
## $ age       <int> 76, 56, 41, 71, 52, 28, 77, 60, 49, 68, 53, 64, 68, 63, 14, ~
## $ year      <int> 1972, 1968, 1977, 1968, 1965, 1971, 1972, 1974, 1968, 1971, ~
## $ thickness <dbl> 6.76, 0.65, 1.34, 2.90, 12.08, 4.84, 5.16, 3.22, 12.88, 7.4~
## $ ulcer     <int> 1, 0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ~
## $ sex_fct   <fct> male, male, male, female, male, male, male, male, female, male, f~
## $ status_fct <fct> other, other, other, other, died from melanoma, died from m~
```

**Exercise 1** Observe the new variables `sex_fct` and `status_fct` in the `glimpse` output above. How can we check that the categories got assigned correctly and match the original `sex` and `status` variables?

Please write up your answer here.

### 16.6.3 Make tables or plots to explore the data visually.

As these are two categorical variables, we should look at contingency tables (both counts and percentages). The variable `status` is the response and `sex` is the predictor.

```
tabyl(Melanoma, status_fct, sex_fct) %>%
  adorn_totals()
```

```
##           status_fct female male
## died from melanoma      28    29
##           other        98    50
##           Total       126    79
```

```
tabyl(Melanoma, status_fct, sex_fct) %>%
  adorn_totals() %>%
  adorn_percentages("col") %>%
  adorn_pct_formatting()
```

```
##           status_fct female   male
## died from melanoma  22.2%  36.7%
##           other    77.8%  63.3%
##           Total  100.0% 100.0%
```

Commentary: You can see why column percentages are necessary in a contingency table. There are 28 females and 29 males who died from melanoma, almost a tie. However, there are more females (126) than there are males (79) who have melanoma in this data set. So the *proportion* of males who died from melanoma is quite a bit larger.

## 16.7 Hypotheses

### 16.7.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

There are two samples: 126 female patients and 79 male patients in Denmark with malignant melanoma. In order for these samples to be representative of



their respective populations, we should probably restrict our conclusions to the population of all females and males in Denmark with malignant melanoma, although we might be able to make the case that these females and males could be representative of people in other countries who have malignant melanoma.

### 16.7.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

$H_0$  : There is no difference between the rate at which women and men in Denmark die from malignant melanoma.

$H_A$  : There is a difference between the rate at which women and men in Denmark die from malignant melanoma.

OR

$H_0$  : In Denmark, death from malignant melanoma is independent of sex.

$H_A$  : In Denmark, death from malignant melanoma is associated with sex.

Commentary: Either of these forms is correct. The former makes it a little easier to figure out how to express the hypotheses mathematically in the next step. The latter reminds us that the `hypothesize` step of the `infer` pipeline will require a null of `independence`.

### 16.7.3 Express the null and alternative hypotheses in symbols (when possible).

$$H_0 : p_{died,F} - p_{died,M} = 0$$

$$H_A : p_{died,F} - p_{died,M} \neq 0$$

Commentary: The order in which you subtract is irrelevant to the inferential process. However, you should be sure that any future steps respect the order you choose here. To be on the safe side, it's always best to subtract in the order in which the factor was created. So in the contingency tables above, females are listed first, and that's because "female" was the first label we used when we created the `sex_fct` variable. So we'll subtract females minus males throughout the remaining steps.

## 16.8 Model

### 16.8.1 Identify the sampling distribution model.

We will use a normal model.

### 16.8.2 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - As observed in a previous chapter when we used this data set before, We have no information about how these samples were obtained. We hope the 126 female patients and 79 male patients are representative of other Danish patients with malignant melanoma.
- 10%
  - We don't know exactly how many people in Denmark suffer from malignant melanoma, but we could imagine over time it's more than 1260 females and 790 males.
- Success/Failure
  - Checking the contingency table above (the one with counts), we see the numbers 28 and 98 (the successes and failures among females), and 29 and 50 (the successes and failures among males). These are all larger than 10.

Commentary: Ideally, for the success/failure condition we would like to check  $n_1 p_1$ ,  $n_1(1 - p_1)$ ,  $n_2 p_2$ , and  $n_2(1 - p_2)$ ; however, the null makes no claim about the values of  $p_1$  and  $p_2$ . We do the next best thing and estimate these by substituting the sample proportions  $\hat{p}_1$  and  $\hat{p}_2$ . But  $n_1 \hat{p}_1$  and  $n_2 \hat{p}_2$  are just the raw counts of successes in each group. Likewise,  $n_1(1 - \hat{p}_1)$  and  $n_2(1 - \hat{p}_2)$  are just the raw counts of failures in each group. That's why we can just read them off the contingency table.

For a more sophisticated approach, one could also use “pooled proportions”. See the optional appendix to this chapter for more information.

## 16.9 Mechanics

### 16.9.1 Compute the test statistic.

```
obs_diff <- Melanoma %>%
  observe(status_fct ~ sex_fct, success = "died from melanoma",
          stat = "diff in props", order = c("female", "male"))
obs_diff

## Response: status_fct (factor)
## Explanatory: sex_fct (factor)
```

```
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 -0.145
```

The test statistic is the difference of proportions in the sample,  $\hat{p}_{died,F} - \hat{p}_{died,M}$ :

$$\hat{p}_{died,F} - \hat{p}_{died,M} = 0.222 - 0.367 = -0.145$$

As a z-score:

```
obs_diff_z <- Melanoma %>%
  observe(status_fct ~ sex_fct, success = "died from melanoma",
          stat = "z", order = c("female", "male"))
obs_diff_z
```

```
## Response: status_fct (factor)
## Explanatory: sex_fct (factor)
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 -2.25
```

Commentary: We can confirm the value of the z-score manually just to make sure we understand where it comes from.

The standard error looks like the following:

$$SE = \sqrt{\frac{\hat{p}_{died,F}(1 - \hat{p}_{died,F})}{n_F} + \frac{\hat{p}_{died,M}(1 - \hat{p}_{died,M})}{n_M}}$$

Plugging in the numbers from the exploratory data analysis output:

$$SE = \sqrt{\frac{0.222(1 - 0.222)}{126} + \frac{0.367(1 - 0.367)}{79}}$$

In R,

```
sqrt(0.222 * (1 - 0.222) / 126 + 0.367 * (1 - 0.367) / 79)
```

```
## [1] 0.06566131
```

Now our z-score formula is

$$z = \frac{(\hat{p}_{died,F} - \hat{p}_{died,M}) - (p_{died,F} - p_{died,M})}{SE}$$

The first term in the numerator  $(\hat{p}_{died,F} - \hat{p}_{died,M})$  is our test statistic, -0.145. The second term in the numerator  $(p_{died,F} - p_{died,M})$  is zero according to the null hypothesis. Plugging all that in, along with the value of SE, gives

$$z = \frac{-0.145 - 0}{0.066} \approx -2.2$$

Other than a little rounding error (since we rounded everything in sight to three decimal places instead of keeping more precision), this is what the `infer` output also reported.

### 16.9.2 Report the test statistic in context (when possible).

In our sample, there is a -14.4866385% difference between the rate at which women and men in Denmark die from malignant melanoma (meaning that males died at a higher rate).

The test statistic has a z score of -2.2530721. The difference in proportions between the rate at which women and men in Denmark die from malignant melanoma lies a bit more than 2 standard errors to the left of the null value.

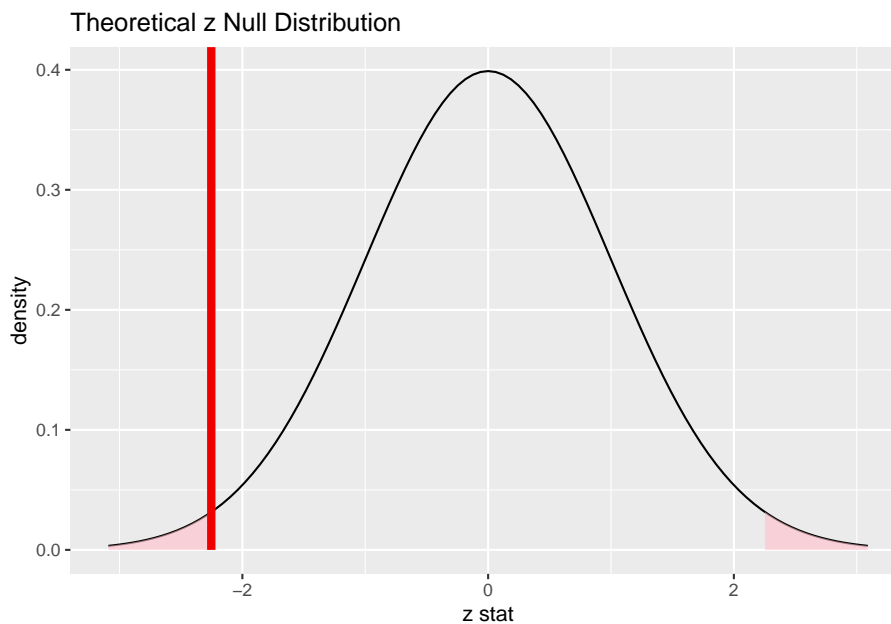
Commentary: Note the phrase “meaning that males died at a higher rate”. If you are looking at a difference, you must indicate the direction of the difference. Without that, we would know that there was a difference, but we would have no idea whether women or men die more from malignant melanoma. Once we know that we are subtracting female minus male, then given the values are negative, we can infer that males die from malignant melanoma more often than females in these samples.

### 16.9.3 Plot the null distribution.

```
status_sex_test <- Melanoma %>%
  specify(status_fct ~ sex_fct, success = "died from melanoma") %>%
  hypothesize(null = "independence") %>%
  assume(distribution = "z")
status_sex_test
```

```
## A Z distribution.
```

```
status_sex_test %>%
  visualize() +
  shade_p_value(obs_stat = obs_diff_z, direction = "two-sided")
```



Commentary: Remember that this is a two-sided test. The red line above is the location of the test statistic, but both tails are shaded and count toward the P-value.

#### 16.9.4 Calculate the P-value.

```
status_sex_test_p <- status_sex_test %>%
  get_p_value(obs_stat = obs_diff_z, direction = "two-sided")
status_sex_test_p
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1 0.0243
```

### **16.9.5 Interpret the P-value as a probability given the null.**

The P-value is 0.0242546. If there were truly no difference between the rate at which women and men in Denmark die from malignant melanoma, there is only a 2.4254604% chance of seeing a difference in our data at least as extreme as what we saw.

## **16.10 Conclusion**

### **16.10.1 State the statistical conclusion.**

We reject the null hypothesis.

### **16.10.2 State (but do not overstate) a contextually meaningful conclusion.**

We have sufficient evidence to suggest that there is a difference between the rate at which women and men in Denmark die from malignant melanoma.

### **16.10.3 Express reservations or uncertainty about the generalizability of the conclusion.**

We echo the same concerns we had back in Chapter 11 when we first saw this data. We have no idea how these patients were sampled. Are these all the patients in Denmark with malignant melanoma over a certain period of time? Were they part of a convenience sample? As a result of our uncertainty about the sampling process, we can't be sure if the results generalize to a larger population, either in Denmark or especially outside of Denmark.

### **16.10.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.**

If we have made a Type I error, then there would actually be no difference between the rate at which women and men in Denmark die from malignant melanoma, but our samples showed a significant difference.

## 16.11 Confidence interval

### 16.11.1 Check the relevant conditions to ensure that model assumptions are met.

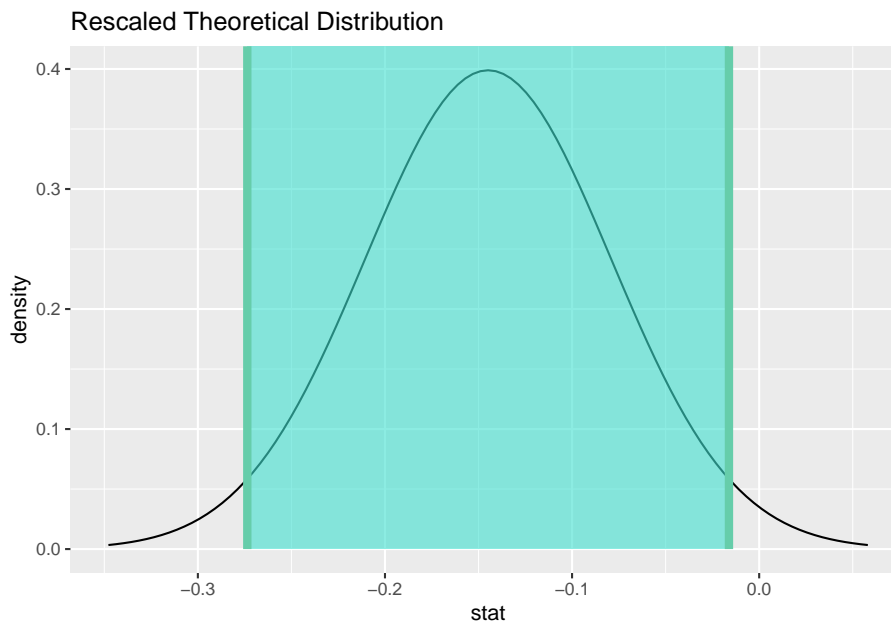
None of the conditions have changed, so they don't need to be rechecked.

### 16.11.2 Calculate and graph the confidence interval.

```
status_sex_ci <- status_sex_test %>%  
  get_confidence_interval(point_estimate = obs_diff, level = 0.95)  
status_sex_ci
```

```
## # A tibble: 1 x 2  
##   lower_ci upper_ci  
##   <dbl>    <dbl>  
## 1   -0.274  -0.0162
```

```
status_sex_test %>%  
  visualize() +  
  shade_confidence_interval(endpoints = status_sex_ci)
```



### 16.11.3 State (but do not overstate) a contextually meaningful interpretation.

We are 95% confident that the true difference between the rate at which women and men die from malignant melanoma is captured in the interval (-27.3579265%, -1.6153506%). (This difference is measured by calculating female minus male.)

Commentary: Note the addition of that last sentence. As we mentioned before, if you are looking at a difference, you must indicate the direction of the difference. We know that we are subtracting female minus male, so given that the values are negative, we can infer that males die from malignant melanoma more often than females—at least according to this confidence interval.

### 16.11.4 If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test.

The confidence interval does not contain the null value of zero. Since zero is not a plausible value for the true difference between the rate at which women and men die from malignant melanoma, it makes sense that we rejected the null hypothesis.

### 16.11.5 When comparing two groups, comment on the effect size and the practical significance of the result.

At the most extreme end of the confidence interval, -27.3579265% is a very large difference between females and males. If this outer value is close to the truth, males are at much more risk of melanoma than females (at least in Denmark at the time of the study). The other end of the confidence interval, -1.6153506%, is a negligible difference. If that number were close to the truth, it's not clear that the true difference would have practical significance in the real world.

Commentary: The P-value for the hypothesis test indicated that the results are *statistically significant*. But what does that really mean? It means that if the null were true, the probability of getting samples of females and males whose melanoma rates differed by -14.4866385%—or something more extreme in either direction—would be quite small. Our conclusion to reject the null follows as a logical consequence.

So we can be somewhat confident that there is a difference between females and males. But how much of a difference? A small difference can be statistically significant, and yet be completely irrelevant in the real world. A 1% difference in melanoma rates might not be enough to enact extra preventative measures



for men, for example. On the other hand, a 27% difference is huge, and might result in a campaign targeted at men specifically due to the extra risk.

In other words, we cannot just rest on a conclusion of statistical significance. A difference might exist, but so what? We also need to know if that difference is *practically significant*? Are there any practical, real-world consequences due to the magnitude of the difference? There is no cutoff for practical significance. This is determined in the context of the problem, preferably using expert guidance. There are policy considerations, cost-benefit analyses, risk assessments, and a host of other considerations that are made when determining if a result is practically significant.

A big part of this process that is often neglected is the role of uncertainty. Our point estimate was -14.4866385%. But that number, by itself, is not that meaningful. That is but one estimate coming from one set of samples. The range of plausible values, according to the confidence interval, is -27.3579265% to -1.6153506%. This is a huge range, and there are very different consequences to society if the difference is -27.3579265% versus -1.6153506%.

## 16.12 Your turn

Go through the rubric to determine if females and males in Denmark who are diagnosed with malignant melanoma suffer from ulcerated tumors at different rates.

The rubric outline is reproduced below. You may refer to the worked example above and modify it accordingly. Remember to strip out all the commentary. That is just exposition for your benefit in understanding the steps, but is not meant to form part of the formal inference process.

Another word of warning: the copy/paste process is not a substitute for your brain. You will often need to modify more than just the names of the data frames and variables to adapt the worked examples to your own work. Do not blindly copy and paste code without understanding what it does. And you should **never** copy and paste text. All the sentences and paragraphs you write are expressions of your own analysis. They must reflect your own understanding of the inferential process.

**Also, so that your answers here don't mess up the code chunks above, use new variable names everywhere. In particular, you should use `ulcer_sex` everywhere instead of `status_sex`**

### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure. Please write up your answer here

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

Identify the sample (or samples) and a reasonable population (or populations) of interest. Please write up your answer here.

Express the null and alternative hypotheses as contextually meaningful full sentences.  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

Express the null and alternative hypotheses in symbols (when possible).  $H_0$  : *math*

$H_A$  : *math*

### Model

Identify the sampling distribution model. Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

### Mechanics

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

### Conclusion

**State the statistical conclusion.** Please write up your answer here.

**State (but do not overstate) a contextually meaningful conclusion.** Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.** Please write up your answer here.

### Confidence interval

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

```
# Add code here to calculate the confidence interval.
```

```
# Add code here to graph the confidence interval.
```

**Calculate and graph the confidence interval.**

**State (but do not overstate) a contextually meaningful interpretation.** Please write up your answer here.

**If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test. [Not always applicable.]** Please write up your answer here.

**When comparing two groups, comment on the effect size and the practical significance of the result. [Not always applicable.]** Please write up your answer here.

## 16.13 Conclusion

Just like with one proportion, when certain conditions are met, the difference between two proportions follow a normal model. Rather than simulating a bunch of different sample differences under the assumption of independent variables,

we can just replace all that with a relatively simple normal model with mean zero and a standard error based on the sample proportions of successes and failures in the two samples. From that normal model, we obtain P-values and confidence intervals as before.

### 16.13.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.

## 16.14 Optional appendix: Pooling

Earlier, we mentioned that that we cannot calculate the “true” standard error directly because the null hypothesis does not give us  $p_1$  and  $p_2$ . (The null only addresses the value of the difference  $p_1 - p_2$ .) We dealt with this by simply substituting  $\hat{p}_1$  for  $p_1$  and  $\hat{p}_2$  for  $p_2$ .

There is, however, one assumption from the null we can still salvage that will improve our test. Since the null hypothesis assumes that the two groups are the same, let’s compute a single overall success rate for both samples together. In other words, if the two groups aren’t different, let’s just pool them into one single group and calculate the successes for the whole group.

This is called a *pooled proportion*. It’s straightforward to compute: just take the total number of successes in both groups and divide by the total size of both groups. Here is the formula:

$$\hat{p}_{pooled} = \frac{successes_1 + successes_2}{n_1 + n_2}.$$

Occasionally, we are not given the raw number of successes in each group, but rather, the proportion of successes in each group,  $\hat{p}_1$  and  $\hat{p}_2$ . The simple fix is to recompute the raw count of successes as  $n_1\hat{p}_1$  and  $n_2\hat{p}_2$ . Here is what it looks like in the formula:

$$\hat{p}_{pooled} = \frac{n_1 \hat{p}_1 + n_2 \hat{p}_2}{n_1 + n_2}.$$

The normal model can still have a mean of  $p_1 - p_2$ . (We usually assume this is 0 in the null hypothesis.) But its standard error will use the pooled proportion:

$$N \left( p_1 - p_2, \sqrt{\frac{\hat{p}_{pooled}(1 - \hat{p}_{pooled})}{n_1} + \frac{\hat{p}_{pooled}(1 - \hat{p}_{pooled})}{n_2}} \right).$$

Not only can we use the pooled proportion in the standard error, but in fact we can use it anywhere we assume the null. For example, the success/failure condition is also subject to the assumption of the null, so we could use the pooled proportion there too.

For a confidence interval, things are different. There is no null hypothesis in effect while computing a confidence interval, so there is no assumption that would justify pooling.

The standard error in the one-proportion interval is

$$\sqrt{\frac{\hat{p}(1 - \hat{p})}{n}}$$

which just substitutes  $\hat{p}$  for  $p$ . We do the same for the standard error in the two-proportion case:

$$SE = \sqrt{\frac{\hat{p}_1(1 - \hat{p}_1)}{n_1} + \frac{\hat{p}_2(1 - \hat{p}_2)}{n_2}}.$$

## Chapter 17

# Chi-square goodness-of-fit test

2.0

**Functions introduced in this chapter:**

`chisq.test`

### 17.1 Introduction

In this assignment we will learn how to run the chi-square goodness-of-fit test. A chi-square goodness-of-fit test is similar to a test for a single proportion except, instead of two categories (success/failure), we now try to understand the distribution among three or more categories.

#### 17.1.1 Install new packages

There are no new packages used in this chapter.

#### 17.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/17-chi\\_square\\_goodness\\_of\\_fit.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/17-chi_square_goodness_of_fit.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 17.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 17.2 Load packages

We load the standard `tidyverse`, `janitor`, and `infer` packages and the `openintro` package for the `hsb2` data.

```
library(tidyverse)
library(janitor)
library(infer)
```

## 17.3 Research question

We use a classic data set `mtcars` from a 1974 Motor Trend magazine to examine the distribution of the number of engine cylinders (with values 4, 6, or 8). We’ll assume that this data set is representative of all cars from 1974.

In recent years, 4-cylinder vehicles and 6-cylinder vehicles have comprised about 38% of the market each, with nearly all the rest (24%) being 8-cylinder cars. (This ignores a very small number of cars manufactured with 3- or 5-cylinder engines.) Were car engines in 1974 manufactured according to the same distribution?

Here is the structure of the data:

```
glimpse(mtcars)
```

```
## Rows: 32
## Columns: 11
## $ mpg   <dbl> 21.0, 21.0, 22.8, 21.4, 18.7, 18.1, 14.3, 24.4, 22.8, 19.2, 17.8, ~
## $ cyl   <dbl> 6, 6, 4, 6, 8, 6, 8, 4, 4, 6, 6, 8, 8, 8, 8, 8, 4, 4, 4, 4, 8, ~
## $ disp  <dbl> 160.0, 160.0, 108.0, 258.0, 360.0, 225.0, 360.0, 146.7, 140.8, 16~
## $ hp    <dbl> 110, 110, 93, 110, 175, 105, 245, 62, 95, 123, 123, 180, 180, 180~
## $ drat  <dbl> 3.90, 3.90, 3.85, 3.08, 3.15, 2.76, 3.21, 3.69, 3.92, 3.92, 3.92, ~
## $ wt    <dbl> 2.620, 2.875, 2.320, 3.215, 3.440, 3.460, 3.570, 3.190, 3.150, 3.~
```



```
## $ qsec <dbl> 16.46, 17.02, 18.61, 19.44, 17.02, 20.22, 15.84, 20.00, 22.90, 18~
## $ vs <dbl> 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0,~
## $ am <dbl> 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, 0,~
## $ gear <dbl> 4, 4, 4, 3, 3, 3, 3, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 4, 4, 4, 3, 3,~
## $ carb <dbl> 4, 4, 1, 1, 2, 1, 4, 2, 2, 4, 4, 3, 3, 3, 4, 4, 4, 1, 2, 1, 1, 2,~
```

Note that the variable of interest `cyl` is not coded as a factor variable. Let's convert `cyl` to a factor variable first and add it to a new data frame called `mtcars2`. (Since the levels are already called 4, 6, and 8, we do not need to specify levels or labels.) Be sure to remember to use `mtcars2` from here on out, and not the original `mtcars`.

```
mtcars2 <- mtcars %>%
  mutate(cyl_fct = factor(cyl))
mtcars2
```

##	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb	cyl_fct
## Mazda RX4	21.0	6	160.0	110	3.90	2.620	16.46	0	1	4	4	6
## Mazda RX4 Wag	21.0	6	160.0	110	3.90	2.875	17.02	0	1	4	4	6
## Datsun 710	22.8	4	108.0	93	3.85	2.320	18.61	1	1	4	1	4
## Hornet 4 Drive	21.4	6	258.0	110	3.08	3.215	19.44	1	0	3	1	6
## Hornet Sportabout	18.7	8	360.0	175	3.15	3.440	17.02	0	0	3	2	8
## Valiant	18.1	6	225.0	105	2.76	3.460	20.22	1	0	3	1	6
## Duster 360	14.3	8	360.0	245	3.21	3.570	15.84	0	0	3	4	8
## Merc 240D	24.4	4	146.7	62	3.69	3.190	20.00	1	0	4	2	4
## Merc 230	22.8	4	140.8	95	3.92	3.150	22.90	1	0	4	2	4
## Merc 280	19.2	6	167.6	123	3.92	3.440	18.30	1	0	4	4	6
## Merc 280C	17.8	6	167.6	123	3.92	3.440	18.90	1	0	4	4	6
## Merc 450SE	16.4	8	275.8	180	3.07	4.070	17.40	0	0	3	3	8
## Merc 450SL	17.3	8	275.8	180	3.07	3.730	17.60	0	0	3	3	8
## Merc 450SLC	15.2	8	275.8	180	3.07	3.780	18.00	0	0	3	3	8
## Cadillac Fleetwood	10.4	8	472.0	205	2.93	5.250	17.98	0	0	3	4	8
## Lincoln Continental	10.4	8	460.0	215	3.00	5.424	17.82	0	0	3	4	8
## Chrysler Imperial	14.7	8	440.0	230	3.23	5.345	17.42	0	0	3	4	8
## Fiat 128	32.4	4	78.7	66	4.08	2.200	19.47	1	1	4	1	4
## Honda Civic	30.4	4	75.7	52	4.93	1.615	18.52	1	1	4	2	4
## Toyota Corolla	33.9	4	71.1	65	4.22	1.835	19.90	1	1	4	1	4
## Toyota Corona	21.5	4	120.1	97	3.70	2.465	20.01	1	0	3	1	4
## Dodge Challenger	15.5	8	318.0	150	2.76	3.520	16.87	0	0	3	2	8
## AMC Javelin	15.2	8	304.0	150	3.15	3.435	17.30	0	0	3	2	8
## Camaro Z28	13.3	8	350.0	245	3.73	3.840	15.41	0	0	3	4	8
## Pontiac Firebird	19.2	8	400.0	175	3.08	3.845	17.05	0	0	3	2	8
## Fiat X1-9	27.3	4	79.0	66	4.08	1.935	18.90	1	1	4	1	4
## Porsche 914-2	26.0	4	120.3	91	4.43	2.140	16.70	0	1	5	2	4
## Lotus Europa	30.4	4	95.1	113	3.77	1.513	16.90	1	1	5	2	4

```
## Ford Pantera L      15.8   8 351.0 264 4.22 3.170 14.50 0 1   5   4   8
## Ferrari Dino       19.7   6 145.0 175 3.62 2.770 15.50 0 1   5   6   6
## Maserati Bora      15.0   8 301.0 335 3.54 3.570 14.60 0 1   5   8   8
## Volvo 142E        21.4   4 121.0 109 4.11 2.780 18.60 1 1   4   2   4
```

```
glimpse(mtcars2)
```

```
## Rows: 32
## Columns: 12
## $ mpg      <dbl> 21.0, 21.0, 22.8, 21.4, 18.7, 18.1, 14.3, 24.4, 22.8, 19.2, 17.~
## $ cyl      <dbl> 6, 6, 4, 6, 8, 6, 8, 4, 4, 6, 6, 8, 8, 8, 8, 8, 4, 4, 4, 4, ~
## $ disp     <dbl> 160.0, 160.0, 108.0, 258.0, 360.0, 225.0, 360.0, 146.7, 140.8, ~
## $ hp       <dbl> 110, 110, 93, 110, 175, 105, 245, 62, 95, 123, 123, 180, 180, ~
## $ drat     <dbl> 3.90, 3.90, 3.85, 3.08, 3.15, 3.15, 2.76, 3.21, 3.69, 3.92, 3.92, 3.~
## $ wt       <dbl> 2.620, 2.875, 2.320, 3.215, 3.440, 3.460, 3.570, 3.190, 3.150, ~
## $ qsec     <dbl> 16.46, 17.02, 18.61, 19.44, 17.02, 20.22, 15.84, 20.00, 22.90, ~
## $ vs       <dbl> 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, ~
## $ am       <dbl> 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, ~
## $ gear     <dbl> 4, 4, 4, 3, 3, 3, 3, 4, 4, 4, 4, 3, 3, 3, 3, 3, 4, 4, 4, 4, 3, ~
## $ carb     <dbl> 4, 4, 1, 1, 2, 1, 4, 2, 2, 4, 4, 3, 3, 3, 4, 4, 4, 1, 2, 1, 1, ~
## $ cyl_fct  <fct> 6, 6, 4, 6, 8, 6, 8, 4, 4, 6, 6, 8, 8, 8, 8, 8, 8, 4, 4, 4, 4, ~
```

## 17.4 Chi-squared

When we have three or more categories in a categorical variable, it is natural to ask how the observed counts in each category compare to the counts that we expect to see under the assumption of some null hypothesis. In other words, we’re assuming that there is some “true” distribution to which we are going to compare our data. Sometimes, this null comes from substantive expert knowledge. (For example, we will be comparing the 1974 distribution to a known distribution from recent years.) Sometimes we’re interested to see if our data deviates from a null distribution that predicts an equal number of observations in each category.

First of all, what is the actual distribution of cylinders in our data? Here’s a frequency table.

```
tabyl(mtcars2, cyl_fct) %>%
  adorn_totals() %>%
  adorn_pct_formatting()
```

```
## cyl_fct  n percent
##      4 11   34.4%
```

##	6	7	21.9%
##	8	14	43.8%
##	Total	32	100.0%

The counts of our frequency table are the “observed” values, usually denoted by the letter  $O$  (uppercase “O”, which is a little unfortunate, because it also looks like a zero).

What are the expected counts? Well, since there are 32 cars, we need to multiply 32 by the percentages listed in the research question. For 4-cylinder and 6-cylinder cars, if the distribution of engines in 1974 were the same as today, there would be  $32 \times 0.38$  or about 12.2 cars we would expect to see in our sample that have 4-cylinder engines, and the same for 6-cylinder cars. For 8-cylinder cars, we expect  $32 \times 0.24$  or about 7.7 cars in our sample to have 8-cylinder engines. These “expected” counts are usually denoted by the letter  $E$ .

Why aren’t the expected counts whole numbers? In any given data set, of course, we will see a whole number of cars with 4, 6, or 8 cylinders. However, since we’re looking only at expected counts, they are the average over lots of possible sets of 32 cars under the assumption of the null. We don’t need for these averages to be whole numbers.

How should the deviation between the data and the null distribution be measured? We could simply look at the difference between the observed counts and the expected counts  $O - E$ . However, there will be some positive values (cells where we have more than the expected number of cars) and some negative values (cells where we have fewer than the expected number of cars). These will all cancel out.

If this sounds vaguely familiar, it is because we encountered the same problem with the formula for the standard deviation. The differences  $y - \bar{y}$  had the same issue. Do you recall the solution in that case? It was to square these values, making them all positive.

So instead of  $O - E$ , we will consider  $(O - E)^2$ . Finally, to make sure that cells with large expected values don’t dominate, we divide by  $E$ :

$$\frac{(O - E)^2}{E}.$$

This puts each cell on equal footing. Now that we have a reasonable measure of the deviation between observed and expected counts for each cell, we define  $\chi^2$  (“chi-squared”, pronounced “kye-squared”—rhymes with “die-scared”, or if that’s too dark, how about “pie-shared”<sup>1</sup>) as the sum of all these fractions, one for each cell:

---

<sup>1</sup>Rhyming is fun!

$$\chi^2 = \sum \frac{(O - E)^2}{E}.$$

A  $\chi^2$  value of zero would indicate perfect agreement between observed and expected values. As the  $\chi^2$  value gets larger and larger, this indicates more and more deviation between observed and expected values.

As an example, for our data, we calculate chi-squared as follows:

$$\chi^2 = \frac{(11 - 12.2)^2}{12.2} + \frac{(7 - 12.2)^2}{12.2} + \frac{(14 - 7.7)^2}{7.7} \approx 7.5.$$

Or we could just do it in R with the `infer` package. To do so, we have to state explicitly the proportions that correspond to the null hypothesis. In this case, since the order of entries in the frequency table is 4-cylinder, 6-cylinder, then 8-cylinder, we need to give `infer` a vector of entries `c("4" = 0.38, "6" = 0.38, "8" = 0.24)` that represents the 38%, 38%, and 24% expected for 4, 6, and 8 cylinders respectively.

```
obs_chisq <- mtcars2 %>%
  specify(response = cyl_fct) %>%
  hypothesize(null = "point",
              p = c("4" = 0.38,
                    "6" = 0.38,
                    "8" = 0.24)) %>%
  calculate(stat = "chisq")
obs_chisq
```

```
## Response: cyl_fct (factor)
## Null Hypothesis: point
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1  7.50
```

## 17.5 The chi-square distribution

We know that even if the true distribution were 38%, 38%, 24%, we would not see exactly 12.2, 12.2, 7.7 in a sample of 32 cars. (In fact, the “true” distribution is physically impossible because these are not whole numbers!) So what kinds of numbers could we get?

Let’s do a quick simulation to find out.

Under the assumption of the null, there should be a 38%, 38%, and 24% chance of seeing 4, 6, or 8 cylinders, respectively. To get a sense of the extent of sampling variability, we could use the `sample` command to see what happens in a sample of size 32 taken from a population where the true percentages are 38%, 38%, and 24%.

```
set.seed(99999)
sample1 <- sample(c(4, 6, 8), size = 32, replace = TRUE,
  prob = c(0.38, 0.38, 0.24))
sample1
```

```
## [1] 6 8 4 8 6 6 8 4 8 6 8 6 6 4 6 8 4 6 6 8 8 6 6 8 4 8 6 4 4 4 6 4
```

```
sample1 %>%
  table()
```

```
## .
## 4 6 8
## 9 13 10
```

```
sample2 <- sample(c(4, 6, 8), size = 32, replace = TRUE,
  prob = c(0.38, 0.38, 0.24))
sample2
```

```
## [1] 6 8 8 8 4 4 8 4 8 6 8 4 4 6 6 6 6 4 4 4 6 4 4 4 4 8 4 4 8 4 4 4 8
```

```
sample2 %>%
  table()
```

```
## .
## 4 6 8
## 16 7 9
```

```
sample3 <- sample(c(4, 6, 8), size = 32, replace = TRUE,
  prob = c(0.38, 0.38, 0.24))
sample3
```

```
## [1] 8 6 4 6 6 6 6 4 6 6 6 4 6 4 8 8 6 8 8 8 4 6 8 4 8 6 6 6 6 8 4 6
```

```
sample3 %>%
  table()
```

```
## .
## 4 6 8
## 7 16 9
```

We can calculate the chi-squared value for each of these samples to get a sense of the possibilities. The `chisq.test` command from base R is a little unusual because it requires a frequency table (generated from the `table` command) as input. We will never use the `chisq.test` command directly because we will always use `infer` to do this work. But just to see some examples:

```
sample1 %>%
  table() %>%
  chisq.test()
```

```
##
## Chi-squared test for given probabilities
##
## data: .
## X-squared = 0.8125, df = 2, p-value = 0.6661
```

```
sample2 %>%
  table() %>%
  chisq.test()
```

```
##
## Chi-squared test for given probabilities
##
## data: .
## X-squared = 4.1875, df = 2, p-value = 0.1232
```

```
sample3 %>%
  table() %>%
  chisq.test()
```

```
##
## Chi-squared test for given probabilities
##
## data: .
## X-squared = 4.1875, df = 2, p-value = 0.1232
```

**Exercise 1** Look more carefully at the three random samples above. Why does sample 1 have a chi-squared closer to 0 while samples 2 and 3 have a chi-squared values that are a little larger? (Hint: look at the counts of 4s, 6s, and

8s in those samples. How do those counts compare to the expected number of 4s, 6s, and 8s?)

Please write up your answer here.

---

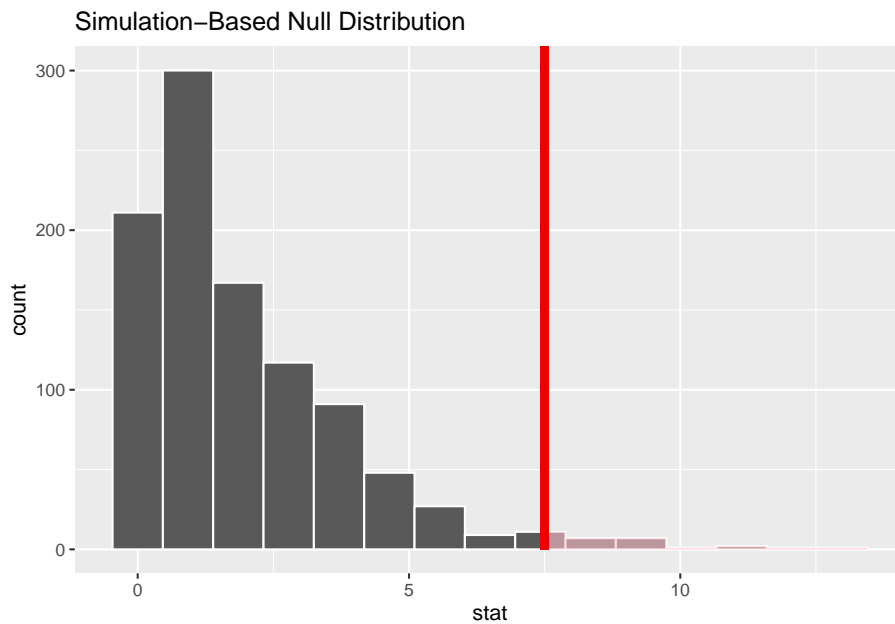
The `infer` pipeline below (the `generate` command specifically) takes the values “4”, “6”, or “8” and grabs them at random according to the probabilities specified until it has 32 values. In other words, it will randomly select “4” about 38% of the time, “6” about 38% of the time, and “8” about 24% of the time, until it gets a list of 32 total cars. Then it will calculate the chi-squared value for that simulated set of 32 cars. But because randomness is involved, the simulated samples are subject to sampling variability and the chi-square values obtained will differ from each other. This is exactly what we did above with the `sample` command and the `chisq` command, but the benefit now is that we get 1000 random samples very quickly.

```
set.seed(99999)
cyl_test_sim <- mtcars2 %>%
  specify(response = cyl_fct) %>%
  hypothesize(null = "point",
              p = c("4" = 0.38,
                    "6" = 0.38,
                    "8" = 0.24)) %>%
  generate(reps = 1000, type = "draw") %>%
  calculate(stat = "chisq")
cyl_test_sim
```

```
## Response: cyl_fct (factor)
## Null Hypothesis: point
## # A tibble: 1,000 x 2
##   replicate stat
##   <fct>      <dbl>
## 1 1          1.58
## 2 2          3.63
## 3 3          3.63
## 4 4          0.669
## 5 5          2.31
## 6 6          0.648
## 7 7          4.13
## 8 8          7.08
## 9 9          0.648
## 10 10         0.669
## # ... with 990 more rows
```

The “stat” column above contains 1000 random values of  $\chi^2$ . Let’s graph these values and include the chi-squared value for our actual data in the same graph.

```
cyl_test_sim %>%
  visualize() +
  shade_p_value(obs_chisq, direction = "greater")
```



A few things are apparent:

1. The values are all positive. (The leftmost bar is sitting at 0, but it represents values greater than zero.) This makes sense when you remember that each piece of the  $\chi^2$  calculation was positive. This is different from our earlier simulations that looked like normal models. (Z scores can be positive or negative, but not  $\chi^2$ .)
2. This is a severely right-skewed graph. Although most values are near zero, the occasional unusual sample can have a large value of  $\chi^2$ .
3. You can see that our sample (the red line) is pretty far to the right. It is an unusual value given the assumption of the null hypothesis. In fact, we can count the proportion of sampled values that are to the right of the red line:



```
cyl_test_sim %>%
  get_p_value(obs_chisq, direction = "greater")
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1    0.021
```

This is the simulated P-value. Keep this number in mind when we calculate the P-value using a sampling distribution model below.

## 17.6 Chi-square as a sampling distribution model

Just like there was a mathematical model for our simulated data before (the normal model back then), there is also a mathematical model for this type of simulated data. It's called (not surprisingly) the *chi-square distribution*.

There is one new idea, though. Although all normal models have the same bell shape, there are many different chi-square models. This is because the number of cells can change the sampling distribution. Our engine cylinder example has three cells (corresponding to the categories “4”, “6”, and “8”). But what if there were 10 categories? The shape of the chi-square model would be different.

The terminology used by statisticians to distinguish these models is *degrees of freedom*, abbreviated *df*. The reason for this name and the mathematics behind it are somewhat technical. Suffice it to say for now that if there are  $c$  cells, you use  $c-1$  degrees of freedom. For our car example, there are 3 cylinder categories, so  $df = 2$ .

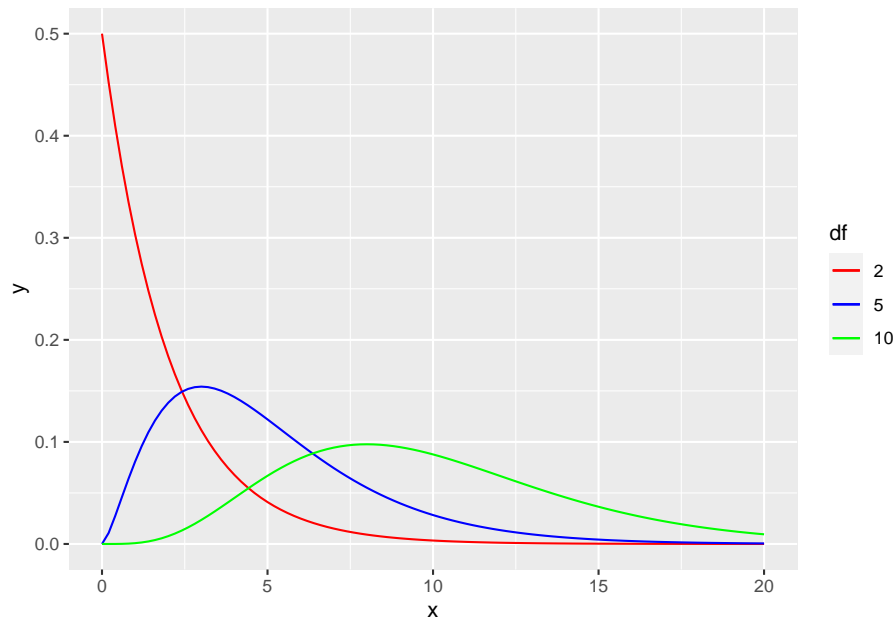
Look at the graph below that shows the theoretical chi-square models for varying degrees of freedom.

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
ggplot(data.frame(x = c(0, 20)), aes(x)) +
  stat_function(fun = dchisq, args = list(df = 2),
    aes(color = "2")) +
  stat_function(fun = dchisq, args = list(df = 5),
    aes(color = "5" )) +
  stat_function(fun = dchisq, args = list(df = 10),
    aes(color = "10")) +
  scale_color_manual(name = "df",
    values = c("2" = "red",
```

```

      "5" = "blue",
      "10" = "green"),
  breaks = c("2", "5", "10"))

```



The red curve (corresponding to  $df = 2$ ) looks a lot like our simulation above. But as the degrees of freedom increase, the mode shifts further to the right.

## 17.7 Chi-square goodness-of-fit test

The formal inferential procedure for examining whether data from a categorical variable fits a proposed distribution in the population is called a *chi-square goodness-of-fit test*.

We can use the chi-square model as the sampling distribution as long as the sample size is large enough. This is checked by calculating that the expected cell counts (not the observed cell counts!) are at least 5 in each cell.

The following `infer` pipeline will run a hypothesis test using the theoretical chi-squared distribution with 2 degrees of freedom.

```

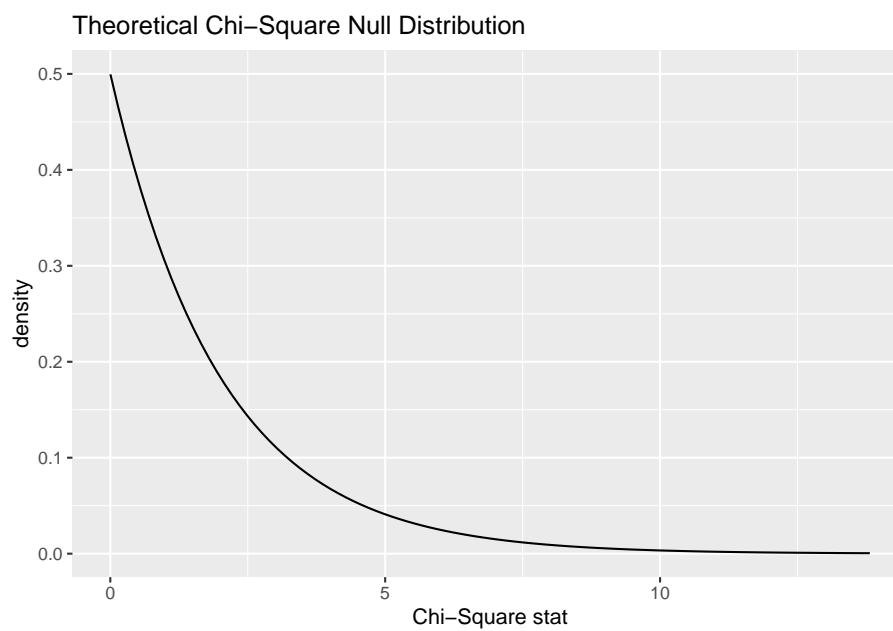
cyl_test <- mtcars2 %>%
  specify(response = cyl_fct) %>%
  assume(distribution = "chisq")
cyl_test

```

```
## A Chi-squared distribution with 2 degrees of freedom.
```

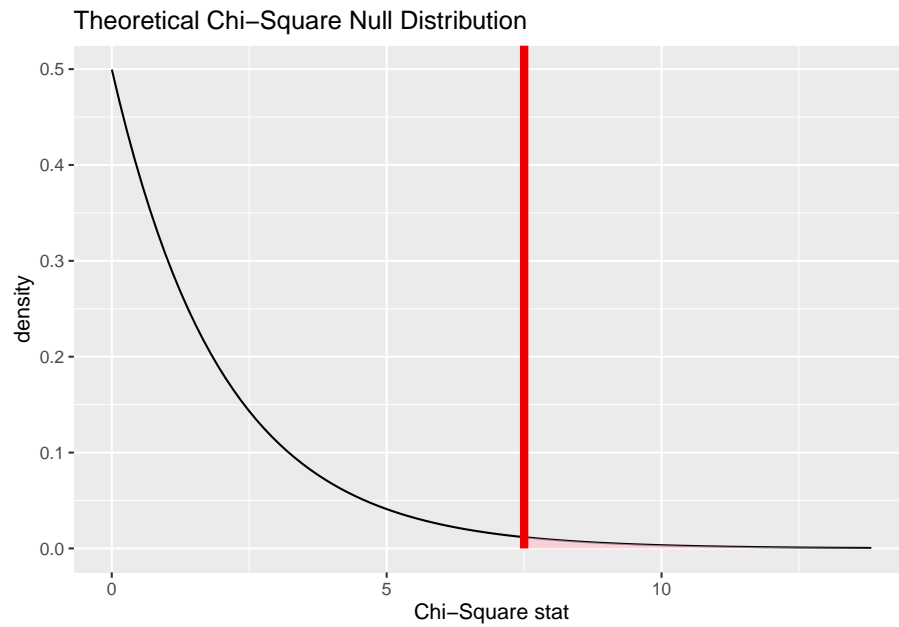
Here is the theoretical distribution:

```
cyl_test %>%  
  visualize()
```



And here it is will our test statistic (the chi-squared value for our observed data) marked:

```
cyl_test %>%  
  visualize() +  
  shade_p_value(obs_chisq, direction = "greater")
```



Finally, here is the P-value associated with the shaded area to the right of the test statistic:

```
cyl_test %>%
  get_p_value(obs_chisq, direction = "greater")
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1  0.0235
```

Note that this P-value is quite similar to the P-value derived from the simulation earlier.

We'll walk through the engine cylinder example from top to bottom using the rubric. Most of this is just repeating work we've already done, but showing this work in the context of the rubric will help you as you take over in the "Your Turn" section later.

## 17.8 Exploratory data analysis

### 17.8.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

Type `?mtcars` at the Console to read the help file. *Motor Trend* is a reputable publication and, therefore, we do not doubt the accuracy of the data. It's not clear, however, why these specific 32 cars were chosen and if they reflect a representative sample of cars on the road in 1974.

`mtcars`

##	mpg	cyl	disp	hp	drat	wt	qsec	vs	am	gear	carb
## Mazda RX4	21.0	6	160.0	110	3.90	2.620	16.46	0	1	4	4
## Mazda RX4 Wag	21.0	6	160.0	110	3.90	2.875	17.02	0	1	4	4
## Datsun 710	22.8	4	108.0	93	3.85	2.320	18.61	1	1	4	1
## Hornet 4 Drive	21.4	6	258.0	110	3.08	3.215	19.44	1	0	3	1
## Hornet Sportabout	18.7	8	360.0	175	3.15	3.440	17.02	0	0	3	2
## Valiant	18.1	6	225.0	105	2.76	3.460	20.22	1	0	3	1
## Duster 360	14.3	8	360.0	245	3.21	3.570	15.84	0	0	3	4
## Merc 240D	24.4	4	146.7	62	3.69	3.190	20.00	1	0	4	2
## Merc 230	22.8	4	140.8	95	3.92	3.150	22.90	1	0	4	2
## Merc 280	19.2	6	167.6	123	3.92	3.440	18.30	1	0	4	4
## Merc 280C	17.8	6	167.6	123	3.92	3.440	18.90	1	0	4	4
## Merc 450SE	16.4	8	275.8	180	3.07	4.070	17.40	0	0	3	3
## Merc 450SL	17.3	8	275.8	180	3.07	3.730	17.60	0	0	3	3
## Merc 450SLC	15.2	8	275.8	180	3.07	3.780	18.00	0	0	3	3
## Cadillac Fleetwood	10.4	8	472.0	205	2.93	5.250	17.98	0	0	3	4
## Lincoln Continental	10.4	8	460.0	215	3.00	5.424	17.82	0	0	3	4
## Chrysler Imperial	14.7	8	440.0	230	3.23	5.345	17.42	0	0	3	4
## Fiat 128	32.4	4	78.7	66	4.08	2.200	19.47	1	1	4	1
## Honda Civic	30.4	4	75.7	52	4.93	1.615	18.52	1	1	4	2
## Toyota Corolla	33.9	4	71.1	65	4.22	1.835	19.90	1	1	4	1
## Toyota Corona	21.5	4	120.1	97	3.70	2.465	20.01	1	0	3	1
## Dodge Challenger	15.5	8	318.0	150	2.76	3.520	16.87	0	0	3	2
## AMC Javelin	15.2	8	304.0	150	3.15	3.435	17.30	0	0	3	2
## Camaro Z28	13.3	8	350.0	245	3.73	3.840	15.41	0	0	3	4
## Pontiac Firebird	19.2	8	400.0	175	3.08	3.845	17.05	0	0	3	2
## Fiat X1-9	27.3	4	79.0	66	4.08	1.935	18.90	1	1	4	1
## Porsche 914-2	26.0	4	120.3	91	4.43	2.140	16.70	0	1	5	2
## Lotus Europa	30.4	4	95.1	113	3.77	1.513	16.90	1	1	5	2
## Ford Pantera L	15.8	8	351.0	264	4.22	3.170	14.50	0	1	5	4
## Ferrari Dino	19.7	6	145.0	175	3.62	2.770	15.50	0	1	5	6
## Maserati Bora	15.0	8	301.0	335	3.54	3.570	14.60	0	1	5	8

```
## Volvo 142E          21.4    4 121.0 109 4.11 2.780 18.60  1  1    4    2
```

```
glimpse(mtcars)
```

```
## Rows: 32
## Columns: 11
## $ mpg <dbl> 21.0, 21.0, 22.8, 21.4, 18.7, 18.1, 14.3, 24.4, 22.8, 19.2, 17.8, ~
## $ cyl <dbl> 6, 6, 4, 6, 8, 6, 8, 4, 4, 6, 6, 8, 8, 8, 8, 8, 4, 4, 4, 4, 8, ~
## $ disp <dbl> 160.0, 160.0, 108.0, 258.0, 360.0, 225.0, 360.0, 146.7, 140.8, 16~
## $ hp <dbl> 110, 110, 93, 110, 175, 105, 245, 62, 95, 123, 123, 180, 180, 180~
## $ drat <dbl> 3.90, 3.90, 3.85, 3.08, 3.15, 2.76, 3.21, 3.69, 3.92, 3.92, 3.92, ~
## $ wt <dbl> 2.620, 2.875, 2.320, 3.215, 3.440, 3.460, 3.570, 3.190, 3.150, 3.~
## $ qsec <dbl> 16.46, 17.02, 18.61, 19.44, 17.02, 20.22, 15.84, 20.00, 22.90, 18~
## $ vs <dbl> 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, 0, ~
## $ am <dbl> 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 0, ~
## $ gear <dbl> 4, 4, 4, 3, 3, 3, 3, 4, 4, 4, 4, 3, 3, 3, 3, 3, 4, 4, 4, 3, 3, ~
## $ carb <dbl> 4, 4, 1, 1, 2, 1, 4, 2, 2, 4, 4, 3, 3, 3, 4, 4, 4, 1, 2, 1, 1, 2, ~
```

### 17.8.2 Prepare the data for analysis.

```
# Although we've already done this above,
# we include it here again for completeness.
mtcars2 <- mtcars %>%
  mutate(cyl_fct = factor(cyl))
mtcars2
```

```
##          mpg  cyl  disp  hp drat   wt  qsec vs am gear carb  cyl_fct
## Mazda RX4      21.0    6 160.0 110 3.90 2.620 16.46 0  1    4    4        6
## Mazda RX4 Wag  21.0    6 160.0 110 3.90 2.875 17.02 0  1    4    4        6
## Datsun 710     22.8    4 108.0  93 3.85 2.320 18.61 1  1    4    1        4
## Hornet 4 Drive  21.4    6 258.0 110 3.08 3.215 19.44 1  0    3    1        6
## Hornet Sportabout 18.7    8 360.0 175 3.15 3.440 17.02 0  0    3    2        8
## Valiant        18.1    6 225.0 105 2.76 3.460 20.22 1  0    3    1        6
## Duster 360     14.3    8 360.0 245 3.21 3.570 15.84 0  0    3    4        8
## Merc 240D      24.4    4 146.7  62 3.69 3.190 20.00 1  0    4    2        4
## Merc 230       22.8    4 140.8  95 3.92 3.150 22.90 1  0    4    2        4
## Merc 280       19.2    6 167.6 123 3.92 3.440 18.30 1  0    4    4        6
## Merc 280C      17.8    6 167.6 123 3.92 3.440 18.90 1  0    4    4        6
## Merc 450SE     16.4    8 275.8 180 3.07 4.070 17.40 0  0    3    3        8
## Merc 450SL     17.3    8 275.8 180 3.07 3.730 17.60 0  0    3    3        8
## Merc 450SLC    15.2    8 275.8 180 3.07 3.780 18.00 0  0    3    3        8
## Cadillac Fleetwood 10.4    8 472.0 205 2.93 5.250 17.98 0  0    3    4        8
## Lincoln Continental 10.4    8 460.0 215 3.00 5.424 17.82 0  0    3    4        8
```

```
## Chrysler Imperial    14.7    8 440.0 230 3.23 5.345 17.42 0 0    3    4    8
## Fiat 128              32.4    4  78.7  66 4.08 2.200 19.47 1 1    4    1    4
## Honda Civic           30.4    4  75.7  52 4.93 1.615 18.52 1 1    4    2    4
## Toyota Corolla        33.9    4  71.1  65 4.22 1.835 19.90 1 1    4    1    4
## Toyota Corona         21.5    4 120.1  97 3.70 2.465 20.01 1 0    3    1    4
## Dodge Challenger      15.5    8 318.0 150 2.76 3.520 16.87 0 0    3    2    8
## AMC Javelin           15.2    8 304.0 150 3.15 3.435 17.30 0 0    3    2    8
## Camaro Z28            13.3    8 350.0 245 3.73 3.840 15.41 0 0    3    4    8
## Pontiac Firebird      19.2    8 400.0 175 3.08 3.845 17.05 0 0    3    2    8
## Fiat X1-9             27.3    4  79.0  66 4.08 1.935 18.90 1 1    4    1    4
## Porsche 914-2         26.0    4 120.3  91 4.43 2.140 16.70 0 1    5    2    4
## Lotus Europa          30.4    4  95.1 113 3.77 1.513 16.90 1 1    5    2    4
## Ford Pantera L        15.8    8 351.0 264 4.22 3.170 14.50 0 1    5    4    8
## Ferrari Dino          19.7    6 145.0 175 3.62 2.770 15.50 0 1    5    6    6
## Maserati Bora          15.0    8 301.0 335 3.54 3.570 14.60 0 1    5    8    8
## Volvo 142E            21.4    4 121.0 109 4.11 2.780 18.60 1 1    4    2    4
```

```
glimpse(mtcars2)
```

```
## Rows: 32
## Columns: 12
## $ mpg      <dbl> 21.0, 21.0, 22.8, 21.4, 18.7, 18.1, 14.3, 24.4, 22.8, 19.2, 17~
## $ cyl      <dbl> 6, 6, 4, 6, 8, 6, 8, 4, 4, 6, 6, 8, 8, 8, 8, 8, 4, 4, 4, 4, ~
## $ disp     <dbl> 160.0, 160.0, 108.0, 258.0, 360.0, 225.0, 360.0, 146.7, 140.8, ~
## $ hp       <dbl> 110, 110, 93, 110, 175, 105, 245, 62, 95, 123, 123, 180, 180, ~
## $ drat     <dbl> 3.90, 3.90, 3.85, 3.08, 3.15, 2.76, 3.21, 3.69, 3.92, 3.92, 3.~
## $ wt       <dbl> 2.620, 2.875, 2.320, 3.215, 3.440, 3.460, 3.570, 3.190, 3.150, ~
## $ qsec     <dbl> 16.46, 17.02, 18.61, 19.44, 17.02, 20.22, 15.84, 20.00, 22.90, ~
## $ vs       <dbl> 0, 0, 1, 1, 0, 1, 0, 1, 1, 1, 1, 0, 0, 0, 0, 0, 1, 1, 1, 1, ~
## $ am       <dbl> 1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, ~
## $ gear     <dbl> 4, 4, 4, 3, 3, 3, 3, 4, 4, 4, 4, 3, 3, 3, 3, 3, 4, 4, 4, 4, 3, ~
## $ carb     <dbl> 4, 4, 1, 1, 2, 1, 4, 2, 2, 4, 4, 3, 3, 3, 4, 4, 4, 1, 2, 1, 1, ~
## $ cyl_fct  <fct> 6, 6, 4, 6, 8, 6, 8, 4, 4, 6, 6, 8, 8, 8, 8, 8, 8, 4, 4, 4, 4, ~
```

### 17.8.3 Make tables or plots to explore the data visually.

```
tabyl(mtcars2, cyl_fct) %>%
  adorn_totals() %>%
  adorn_pct_formatting()
```

```
## cyl_fct  n percent
##      4 11   34.4%
```

##	6	7	21.9%
##	8	14	43.8%
##	Total	32	100.0%

## 17.9 Hypotheses

### 17.9.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

The sample is a set of 32 cars from a 1974 Motor Trends magazine. The population is all cars from 1974.

### 17.9.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

$H_0$  : In 1974, the proportion of cars with 4, 6, and 8 cylinders was 38%, 38%, and 24%, respectively.

$H_A$  : In 1974, the proportion of cars with 4, 6, and 8 cylinders was not 38%, 38%, and 24%.

### 17.9.3 Express the null and alternative hypotheses in symbols (when possible).

$$H_0 : p_4 = 0.38, p_6 = 0.38, p_8 = 0.24$$

There is no easy way to express the alternate hypothesis in symbols because any deviation in any of the categories can lead to rejection of the null. You can't just say  $p_4 \neq 0.38, p_6 \neq 0.38, p_8 \neq 0.24$  because one of these categories might have the correct proportion with the other two different and that would still be consistent with the alternative hypothesis.

So the only requirement here is to express the null in symbols.

## 17.10 Model

### 17.10.1 Identify the sampling distribution model.

We use a  $\chi^2$  model with 2 degrees of freedom.

Commentary: Unlike the normal model, there are infinitely many different  $\chi^2$  models, so you have to specify the degrees of freedom when you identify it as the sampling distribution model.



### 17.10.2 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - We do not know how Motor Trends magazine sampled these 32 cars, so we're not sure if this list is random or representative of all cars from 1974. We should be cautious in our conclusions.
- 10%
  - As long as there are at least 320 different car models, we are okay. This sounds like a lot, so this condition might not quite be met. Again, we need to be careful. (Also note that the population is not all automobiles manufactured in 1974. It is all *types* of automobile manufactured in 1974. There's a big difference.)
- Expected cell counts
  - This condition says that under the null, we should see at least 5 cars in each category. The expected counts are  $32(0.38) = 12.2$ ,  $32(0.38) = 12.2$ , and  $32(0.24) = 7.7$ . So this condition is met.

Commentary: The expected counts condition is necessary for using the theoretical chi-squared distribution. If we were using simulation instead, we would not need this condition.

## 17.11 Mechanics

### 17.11.1 Compute the test statistic.

```
obs_chisq <- mtcars2 %>%
  specify(response = cyl_fct) %>%
  hypothesize(null = "point",
              p = c("4" = 0.38,
                    "6" = 0.38,
                    "8" = 0.24)) %>%
  calculate(stat = "chisq")
obs_chisq
```

```
## Response: cyl_fct (factor)
## Null Hypothesis: point
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1  7.50
```

### 17.11.2 Report the test statistic in context (when possible).

The value of  $\chi^2$  is 7.5010965.

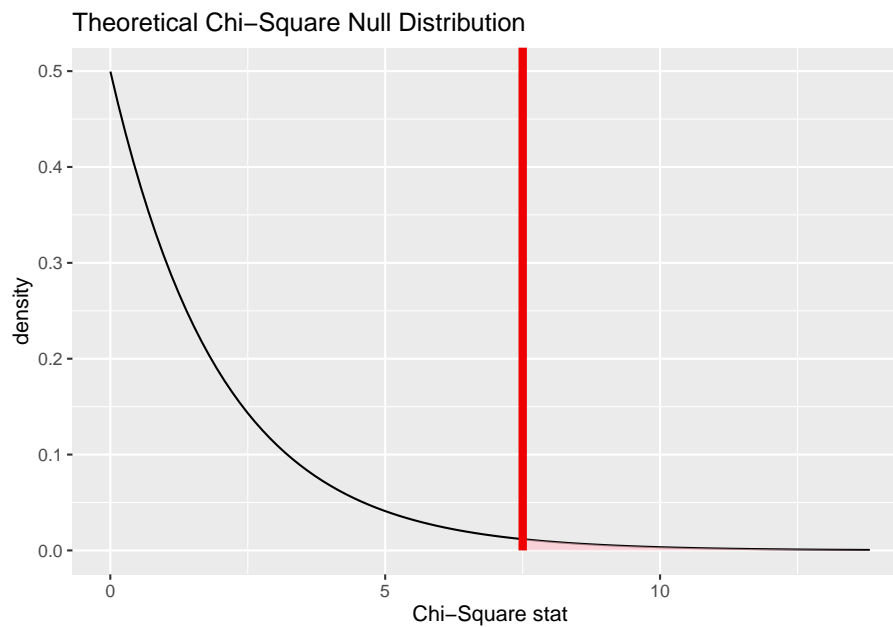
Commentary: The  $\chi^2$  test statistic is, of course, the same value we computed manually by hand earlier. Also, the formula for  $\chi^2$  is a complicated function of observed and expected values, making it difficult to say anything about this number in the context of cars and engine cylinders. So even though the requirement is to “report the test statistic in context,” there’s not much one can say here other than just to report the test statistic.

### 17.11.3 Plot the null distribution.

```
cyl_test <- mtcars2 %>%  
  specify(response = cyl_fct) %>%  
  assume(distribution = "chisq")  
cyl_test
```

```
## A Chi-squared distribution with 2 degrees of freedom.
```

```
cyl_test %>%  
  visualize() +  
  shade_p_value(obs_chisq, direction = "greater")
```



Commentary: We will use the theoretical distribution

#### 17.11.4 Calculate the P-value.

```
cyl_test_p <- cyl_test %>%
  get_p_value(obs_chisq, direction = "greater")
cyl_test_p
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1  0.0235
```

#### 17.11.5 Interpret the P-value as a probability given the null.

The P-value is 0.0235048558887484. If the true distribution of cars in 1974 were 38% 4-cylinder, 38% 6-cylinder, and 24% 8-cylinder, there would be a 2.35048558887484% chance of seeing data at least as extreme as what we saw.

## 17.12 Conclusion

### 17.12.1 State the statistical conclusion.

We reject the null.

### 17.12.2 State (but do not overstate) a contextually meaningful conclusion.

There is sufficient evidence that in 1974, the distribution of cars was not 38% 4-cylinder, 38% 6-cylinder, and 24% 8-cylinder.

### 17.12.3 Express reservations or uncertainty about the generalizability of the conclusion.

As long as we restrict our attention to cars in 1974, we are pretty safe, although we are still uncertain if the sample we had was representative of all cars in 1974.

### 17.12.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.

If we made a Type I error, that would mean the true distribution of cars in 1974 was 38% 4-cylinder, 38% 6-cylinder, and 24% 8-cylinder, but our sample showed otherwise.

## 17.13 Confidence interval

There is no confidence interval for a chi-square test. Since our test is not about measuring some parameter of interest (like  $p$  or  $p_1 - p_2$ ), there is no interval to produce.

## 17.14 Your turn

Use the `hsb2` data and determine if the proportion of high school students who attend general programs, academic programs, and vocational programs is 15%, 60%, and 25% respectively.

The rubric outline is reproduced below. You may refer to the worked example above and modify it accordingly. Remember to strip out all the commentary.

That is just exposition for your benefit in understanding the steps, but is not meant to form part of the formal inference process.

Another word of warning: the copy/paste process is not a substitute for your brain. You will often need to modify more than just the names of the data frames and variables to adapt the worked examples to your own work. Do not blindly copy and paste code without understanding what it does. And you should **never** copy and paste text. All the sentences and paragraphs you write are expressions of your own analysis. They must reflect your own understanding of the inferential process.

**Also, so that your answers here don't mess up the code chunks above, use new variable names everywhere.**

### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure. Please write up your answer here

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

Identify the sample (or samples) and a reasonable population (or populations) of interest. Please write up your answer here.

**Express the null and alternative hypotheses as contextually meaningful full sentences.**  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

**Express the null and alternative hypotheses in symbols (when possible).**  $H_0$  : *math*

$H_A$  : *math*

**Model**

**Identify the sampling distribution model.** Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

**Mechanics**

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

### Conclusion

**State the statistical conclusion.** Please write up your answer here.

**State (but do not overstate) a contextually meaningful conclusion.** Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.** Please write up your answer here.

## 17.15 Conclusion

When a categorical variable has three or more categories, we can run a chi-square goodness-of-fit test to determine if the distribution of counts across those categories matches some pre-specified null hypothesis. The key new mathematical tool we need is the chi-square distribution, a way of measuring the deviation between observed counts and expected counts according to the null.

### 17.15.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.





## Chapter 18

# Chi-square test for independence

2.0

### Functions introduced in this chapter:

No new R functions are introduced here.

## 18.1 Introduction

In this chapter we will learn how to run the chi-square test for independence.

A chi-square test for independence tests the relationship between two categorical variables. This is an extension of the test for two proportions, except now applied in situations where either the predictor or response variables (or both) have three or more categories.

### 18.1.1 Install new packages

There are no new packages used in this chapter.

### 18.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/18-chi\\_square\\_test\\_for\\_independence.F](https://vectorposse.github.io/intro_stats/chapter_downloads/18-chi_square_test_for_independence.F)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 18.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 18.2 Load packages

We load the standard `tidyverse`, `janitor`, and `infer` packages. We also use the `MASS` package for the `birthwt` data, and the `openintro` package for the smoking data.

```
library(tidyverse)
library(janitor)
library(infer)
library(MASS)
library(openintro)
```

## 18.3 Research question

Are mothers from certain racial groups more or less likely to have low birth weight babies? In other words, are low birth weight and race associated?

Let’s look at the data. The `birthwt` data was collected at Baystate Medical Center, Springfield, Mass during 1986. In terms of addressing the research question, we are, of course, limited to conclusions about women in that area of the country in the mid-1980s.

```
birthwt
```

##	low	age	lwt	race	smoke	ptl	ht	ui	ftv	bwt
## 85	0	19	182	2	0	0	0	1	0	2523
## 86	0	33	155	3	0	0	0	0	3	2551
## 87	0	20	105	1	1	0	0	0	1	2557
## 88	0	21	108	1	1	0	0	1	2	2594
## 89	0	18	107	1	1	0	0	1	0	2600
## 91	0	21	124	3	0	0	0	0	0	2622
## 92	0	22	118	1	0	0	0	0	1	2637
## 93	0	17	103	3	0	0	0	0	1	2637

## 94	0	29	123	1	1	0	0	0	1	2663
## 95	0	26	113	1	1	0	0	0	0	2665
## 96	0	19	95	3	0	0	0	0	0	2722
## 97	0	19	150	3	0	0	0	0	1	2733
## 98	0	22	95	3	0	0	1	0	0	2751
## 99	0	30	107	3	0	1	0	1	2	2750
## 100	0	18	100	1	1	0	0	0	0	2769
## 101	0	18	100	1	1	0	0	0	0	2769
## 102	0	15	98	2	0	0	0	0	0	2778
## 103	0	25	118	1	1	0	0	0	3	2782
## 104	0	20	120	3	0	0	0	1	0	2807
## 105	0	28	120	1	1	0	0	0	1	2821
## 106	0	32	121	3	0	0	0	0	2	2835
## 107	0	31	100	1	0	0	0	1	3	2835
## 108	0	36	202	1	0	0	0	0	1	2836
## 109	0	28	120	3	0	0	0	0	0	2863
## 111	0	25	120	3	0	0	0	1	2	2877
## 112	0	28	167	1	0	0	0	0	0	2877
## 113	0	17	122	1	1	0	0	0	0	2906
## 114	0	29	150	1	0	0	0	0	2	2920
## 115	0	26	168	2	1	0	0	0	0	2920
## 116	0	17	113	2	0	0	0	0	1	2920
## 117	0	17	113	2	0	0	0	0	1	2920
## 118	0	24	90	1	1	1	0	0	1	2948
## 119	0	35	121	2	1	1	0	0	1	2948
## 120	0	25	155	1	0	0	0	0	1	2977
## 121	0	25	125	2	0	0	0	0	0	2977
## 123	0	29	140	1	1	0	0	0	2	2977
## 124	0	19	138	1	1	0	0	0	2	2977
## 125	0	27	124	1	1	0	0	0	0	2922
## 126	0	31	215	1	1	0	0	0	2	3005
## 127	0	33	109	1	1	0	0	0	1	3033
## 128	0	21	185	2	1	0	0	0	2	3042
## 129	0	19	189	1	0	0	0	0	2	3062
## 130	0	23	130	2	0	0	0	0	1	3062
## 131	0	21	160	1	0	0	0	0	0	3062
## 132	0	18	90	1	1	0	0	1	0	3062
## 133	0	18	90	1	1	0	0	1	0	3062
## 134	0	32	132	1	0	0	0	0	4	3080
## 135	0	19	132	3	0	0	0	0	0	3090
## 136	0	24	115	1	0	0	0	0	2	3090
## 137	0	22	85	3	1	0	0	0	0	3090
## 138	0	22	120	1	0	0	1	0	1	3100
## 139	0	23	128	3	0	0	0	0	0	3104
## 140	0	22	130	1	1	0	0	0	0	3132
## 141	0	30	95	1	1	0	0	0	2	3147

## 142	0	19	115	3	0	0	0	0	0	3175
## 143	0	16	110	3	0	0	0	0	0	3175
## 144	0	21	110	3	1	0	0	1	0	3203
## 145	0	30	153	3	0	0	0	0	0	3203
## 146	0	20	103	3	0	0	0	0	0	3203
## 147	0	17	119	3	0	0	0	0	0	3225
## 148	0	17	119	3	0	0	0	0	0	3225
## 149	0	23	119	3	0	0	0	0	2	3232
## 150	0	24	110	3	0	0	0	0	0	3232
## 151	0	28	140	1	0	0	0	0	0	3234
## 154	0	26	133	3	1	2	0	0	0	3260
## 155	0	20	169	3	0	1	0	1	1	3274
## 156	0	24	115	3	0	0	0	0	2	3274
## 159	0	28	250	3	1	0	0	0	6	3303
## 160	0	20	141	1	0	2	0	1	1	3317
## 161	0	22	158	2	0	1	0	0	2	3317
## 162	0	22	112	1	1	2	0	0	0	3317
## 163	0	31	150	3	1	0	0	0	2	3321
## 164	0	23	115	3	1	0	0	0	1	3331
## 166	0	16	112	2	0	0	0	0	0	3374
## 167	0	16	135	1	1	0	0	0	0	3374
## 168	0	18	229	2	0	0	0	0	0	3402
## 169	0	25	140	1	0	0	0	0	1	3416
## 170	0	32	134	1	1	1	0	0	4	3430
## 172	0	20	121	2	1	0	0	0	0	3444
## 173	0	23	190	1	0	0	0	0	0	3459
## 174	0	22	131	1	0	0	0	0	1	3460
## 175	0	32	170	1	0	0	0	0	0	3473
## 176	0	30	110	3	0	0	0	0	0	3544
## 177	0	20	127	3	0	0	0	0	0	3487
## 179	0	23	123	3	0	0	0	0	0	3544
## 180	0	17	120	3	1	0	0	0	0	3572
## 181	0	19	105	3	0	0	0	0	0	3572
## 182	0	23	130	1	0	0	0	0	0	3586
## 183	0	36	175	1	0	0	0	0	0	3600
## 184	0	22	125	1	0	0	0	0	1	3614
## 185	0	24	133	1	0	0	0	0	0	3614
## 186	0	21	134	3	0	0	0	0	2	3629
## 187	0	19	235	1	1	0	1	0	0	3629
## 188	0	25	95	1	1	3	0	1	0	3637
## 189	0	16	135	1	1	0	0	0	0	3643
## 190	0	29	135	1	0	0	0	0	1	3651
## 191	0	29	154	1	0	0	0	0	1	3651
## 192	0	19	147	1	1	0	0	0	0	3651
## 193	0	19	147	1	1	0	0	0	0	3651
## 195	0	30	137	1	0	0	0	0	1	3699

## 196	0	24	110	1	0	0	0	0	1	3728
## 197	0	19	184	1	1	0	1	0	0	3756
## 199	0	24	110	3	0	1	0	0	0	3770
## 200	0	23	110	1	0	0	0	0	1	3770
## 201	0	20	120	3	0	0	0	0	0	3770
## 202	0	25	241	2	0	0	1	0	0	3790
## 203	0	30	112	1	0	0	0	0	1	3799
## 204	0	22	169	1	0	0	0	0	0	3827
## 205	0	18	120	1	1	0	0	0	2	3856
## 206	0	16	170	2	0	0	0	0	4	3860
## 207	0	32	186	1	0	0	0	0	2	3860
## 208	0	18	120	3	0	0	0	0	1	3884
## 209	0	29	130	1	1	0	0	0	2	3884
## 210	0	33	117	1	0	0	0	1	1	3912
## 211	0	20	170	1	1	0	0	0	0	3940
## 212	0	28	134	3	0	0	0	0	1	3941
## 213	0	14	135	1	0	0	0	0	0	3941
## 214	0	28	130	3	0	0	0	0	0	3969
## 215	0	25	120	1	0	0	0	0	2	3983
## 216	0	16	95	3	0	0	0	0	1	3997
## 217	0	20	158	1	0	0	0	0	1	3997
## 218	0	26	160	3	0	0	0	0	0	4054
## 219	0	21	115	1	0	0	0	0	1	4054
## 220	0	22	129	1	0	0	0	0	0	4111
## 221	0	25	130	1	0	0	0	0	2	4153
## 222	0	31	120	1	0	0	0	0	2	4167
## 223	0	35	170	1	0	1	0	0	1	4174
## 224	0	19	120	1	1	0	0	0	0	4238
## 225	0	24	116	1	0	0	0	0	1	4593
## 226	0	45	123	1	0	0	0	0	1	4990
## 4	1	28	120	3	1	1	0	1	0	709
## 10	1	29	130	1	0	0	0	1	2	1021
## 11	1	34	187	2	1	0	1	0	0	1135
## 13	1	25	105	3	0	1	1	0	0	1330
## 15	1	25	85	3	0	0	0	1	0	1474
## 16	1	27	150	3	0	0	0	0	0	1588
## 17	1	23	97	3	0	0	0	1	1	1588
## 18	1	24	128	2	0	1	0	0	1	1701
## 19	1	24	132	3	0	0	1	0	0	1729
## 20	1	21	165	1	1	0	1	0	1	1790
## 22	1	32	105	1	1	0	0	0	0	1818
## 23	1	19	91	1	1	2	0	1	0	1885
## 24	1	25	115	3	0	0	0	0	0	1893
## 25	1	16	130	3	0	0	0	0	1	1899
## 26	1	25	92	1	1	0	0	0	0	1928
## 27	1	20	150	1	1	0	0	0	2	1928

## 28	1	21	200	2	0	0	0	1	2	1928
## 29	1	24	155	1	1	1	0	0	0	1936
## 30	1	21	103	3	0	0	0	0	0	1970
## 31	1	20	125	3	0	0	0	1	0	2055
## 32	1	25	89	3	0	2	0	0	1	2055
## 33	1	19	102	1	0	0	0	0	2	2082
## 34	1	19	112	1	1	0	0	1	0	2084
## 35	1	26	117	1	1	1	0	0	0	2084
## 36	1	24	138	1	0	0	0	0	0	2100
## 37	1	17	130	3	1	1	0	1	0	2125
## 40	1	20	120	2	1	0	0	0	3	2126
## 42	1	22	130	1	1	1	0	1	1	2187
## 43	1	27	130	2	0	0	0	1	0	2187
## 44	1	20	80	3	1	0	0	1	0	2211
## 45	1	17	110	1	1	0	0	0	0	2225
## 46	1	25	105	3	0	1	0	0	1	2240
## 47	1	20	109	3	0	0	0	0	0	2240
## 49	1	18	148	3	0	0	0	0	0	2282
## 50	1	18	110	2	1	1	0	0	0	2296
## 51	1	20	121	1	1	1	0	1	0	2296
## 52	1	21	100	3	0	1	0	0	4	2301
## 54	1	26	96	3	0	0	0	0	0	2325
## 56	1	31	102	1	1	1	0	0	1	2353
## 57	1	15	110	1	0	0	0	0	0	2353
## 59	1	23	187	2	1	0	0	0	1	2367
## 60	1	20	122	2	1	0	0	0	0	2381
## 61	1	24	105	2	1	0	0	0	0	2381
## 62	1	15	115	3	0	0	0	1	0	2381
## 63	1	23	120	3	0	0	0	0	0	2410
## 65	1	30	142	1	1	1	0	0	0	2410
## 67	1	22	130	1	1	0	0	0	1	2410
## 68	1	17	120	1	1	0	0	0	3	2414
## 69	1	23	110	1	1	1	0	0	0	2424
## 71	1	17	120	2	0	0	0	0	2	2438
## 75	1	26	154	3	0	1	1	0	1	2442
## 76	1	20	105	3	0	0	0	0	3	2450
## 77	1	26	190	1	1	0	0	0	0	2466
## 78	1	14	101	3	1	1	0	0	0	2466
## 79	1	28	95	1	1	0	0	0	2	2466
## 81	1	14	100	3	0	0	0	0	2	2495
## 82	1	23	94	3	1	0	0	0	0	2495
## 83	1	17	142	2	0	0	1	0	0	2495
## 84	1	21	130	1	1	0	1	0	3	2495

```
glimpse(birthwt)
```

```
## Rows: 189
## Columns: 10
## $ low   <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0~
## $ age   <int> 19, 33, 20, 21, 18, 21, 22, 17, 29, 26, 19, 19, 22, 30, 18, 18, ~
## $ lwt   <int> 182, 155, 105, 108, 107, 124, 118, 103, 123, 113, 95, 150, 95, 1~
## $ race  <int> 2, 3, 1, 1, 1, 3, 1, 3, 1, 1, 3, 3, 3, 1, 1, 2, 1, 3, 1, 3, 1~
## $ smoke <int> 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0~
## $ ptl   <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0~
## $ ht    <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0~
## $ ui    <int> 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0~
## $ ftv   <int> 0, 3, 1, 2, 0, 0, 1, 1, 1, 0, 0, 1, 0, 2, 0, 0, 0, 3, 0, 1, 2~
## $ bwt   <int> 2523, 2551, 2557, 2594, 2600, 2622, 2637, 2637, 2663, 2665, 2722~
```

The `low` variable is an indicator of birth weight less than 2.5 kg. So even though birth weight is numerical, we have a convenient categorical variable that serves as a marker of low birth weight, gathering all low birth weight babies into a single group. The `race` variable is categorical, coded as 1 = white, 2 = black, 3 = other.

Neither variable appears in the data frame as a factor variable, so we will need to change that. The new tibble will be called `birthwt2`.

```
birthwt2 <- birthwt %>%
  mutate(low_fct = factor(low, levels = c(0, 1),
                           labels = c("no", "yes")),
         race_fct = factor(race, levels = c(1, 2, 3),
                           labels = c("white", "black", "other")))
birthwt2
```

```
##      low age lwt race smoke ptl ht ui ftv  bwt low_fct race_fct
## 85    0  19 182    2     0   0  0  1   0 2523      no    black
## 86    0  33 155    3     0   0  0  0   3 2551      no    other
## 87    0  20 105    1     1   0  0  0   1 2557      no    white
## 88    0  21 108    1     1   0  0  1   2 2594      no    white
## 89    0  18 107    1     1   0  0  1   0 2600      no    white
## 91    0  21 124    3     0   0  0  0   0 2622      no    other
## 92    0  22 118    1     0   0  0  0   1 2637      no    white
## 93    0  17 103    3     0   0  0  0   1 2637      no    other
## 94    0  29 123    1     1   0  0  0   1 2663      no    white
## 95    0  26 113    1     1   0  0  0   0 2665      no    white
## 96    0  19  95    3     0   0  0  0   0 2722      no    other
## 97    0  19 150    3     0   0  0  0   1 2733      no    other
## 98    0  22  95    3     0   0  1  0   0 2751      no    other
```

## 99	0	30	107	3	0	1	0	1	2	2750	no	other
## 100	0	18	100	1	1	0	0	0	0	2769	no	white
## 101	0	18	100	1	1	0	0	0	0	2769	no	white
## 102	0	15	98	2	0	0	0	0	0	2778	no	black
## 103	0	25	118	1	1	0	0	0	3	2782	no	white
## 104	0	20	120	3	0	0	0	1	0	2807	no	other
## 105	0	28	120	1	1	0	0	0	1	2821	no	white
## 106	0	32	121	3	0	0	0	0	2	2835	no	other
## 107	0	31	100	1	0	0	0	1	3	2835	no	white
## 108	0	36	202	1	0	0	0	0	1	2836	no	white
## 109	0	28	120	3	0	0	0	0	0	2863	no	other
## 111	0	25	120	3	0	0	0	1	2	2877	no	other
## 112	0	28	167	1	0	0	0	0	0	2877	no	white
## 113	0	17	122	1	1	0	0	0	0	2906	no	white
## 114	0	29	150	1	0	0	0	0	2	2920	no	white
## 115	0	26	168	2	1	0	0	0	0	2920	no	black
## 116	0	17	113	2	0	0	0	0	1	2920	no	black
## 117	0	17	113	2	0	0	0	0	1	2920	no	black
## 118	0	24	90	1	1	1	0	0	1	2948	no	white
## 119	0	35	121	2	1	1	0	0	1	2948	no	black
## 120	0	25	155	1	0	0	0	0	1	2977	no	white
## 121	0	25	125	2	0	0	0	0	0	2977	no	black
## 123	0	29	140	1	1	0	0	0	2	2977	no	white
## 124	0	19	138	1	1	0	0	0	2	2977	no	white
## 125	0	27	124	1	1	0	0	0	0	2922	no	white
## 126	0	31	215	1	1	0	0	0	2	3005	no	white
## 127	0	33	109	1	1	0	0	0	1	3033	no	white
## 128	0	21	185	2	1	0	0	0	2	3042	no	black
## 129	0	19	189	1	0	0	0	0	2	3062	no	white
## 130	0	23	130	2	0	0	0	0	1	3062	no	black
## 131	0	21	160	1	0	0	0	0	0	3062	no	white
## 132	0	18	90	1	1	0	0	1	0	3062	no	white
## 133	0	18	90	1	1	0	0	1	0	3062	no	white
## 134	0	32	132	1	0	0	0	0	4	3080	no	white
## 135	0	19	132	3	0	0	0	0	0	3090	no	other
## 136	0	24	115	1	0	0	0	0	2	3090	no	white
## 137	0	22	85	3	1	0	0	0	0	3090	no	other
## 138	0	22	120	1	0	0	1	0	1	3100	no	white
## 139	0	23	128	3	0	0	0	0	0	3104	no	other
## 140	0	22	130	1	1	0	0	0	0	3132	no	white
## 141	0	30	95	1	1	0	0	0	2	3147	no	white
## 142	0	19	115	3	0	0	0	0	0	3175	no	other
## 143	0	16	110	3	0	0	0	0	0	3175	no	other
## 144	0	21	110	3	1	0	0	1	0	3203	no	other
## 145	0	30	153	3	0	0	0	0	0	3203	no	other
## 146	0	20	103	3	0	0	0	0	0	3203	no	other



## 147	0	17	119	3	0	0	0	0	0	3225	no	other
## 148	0	17	119	3	0	0	0	0	0	3225	no	other
## 149	0	23	119	3	0	0	0	0	2	3232	no	other
## 150	0	24	110	3	0	0	0	0	0	3232	no	other
## 151	0	28	140	1	0	0	0	0	0	3234	no	white
## 154	0	26	133	3	1	2	0	0	0	3260	no	other
## 155	0	20	169	3	0	1	0	1	1	3274	no	other
## 156	0	24	115	3	0	0	0	0	2	3274	no	other
## 159	0	28	250	3	1	0	0	0	6	3303	no	other
## 160	0	20	141	1	0	2	0	1	1	3317	no	white
## 161	0	22	158	2	0	1	0	0	2	3317	no	black
## 162	0	22	112	1	1	2	0	0	0	3317	no	white
## 163	0	31	150	3	1	0	0	0	2	3321	no	other
## 164	0	23	115	3	1	0	0	0	1	3331	no	other
## 166	0	16	112	2	0	0	0	0	0	3374	no	black
## 167	0	16	135	1	1	0	0	0	0	3374	no	white
## 168	0	18	229	2	0	0	0	0	0	3402	no	black
## 169	0	25	140	1	0	0	0	0	1	3416	no	white
## 170	0	32	134	1	1	1	0	0	4	3430	no	white
## 172	0	20	121	2	1	0	0	0	0	3444	no	black
## 173	0	23	190	1	0	0	0	0	0	3459	no	white
## 174	0	22	131	1	0	0	0	0	1	3460	no	white
## 175	0	32	170	1	0	0	0	0	0	3473	no	white
## 176	0	30	110	3	0	0	0	0	0	3544	no	other
## 177	0	20	127	3	0	0	0	0	0	3487	no	other
## 179	0	23	123	3	0	0	0	0	0	3544	no	other
## 180	0	17	120	3	1	0	0	0	0	3572	no	other
## 181	0	19	105	3	0	0	0	0	0	3572	no	other
## 182	0	23	130	1	0	0	0	0	0	3586	no	white
## 183	0	36	175	1	0	0	0	0	0	3600	no	white
## 184	0	22	125	1	0	0	0	0	1	3614	no	white
## 185	0	24	133	1	0	0	0	0	0	3614	no	white
## 186	0	21	134	3	0	0	0	0	2	3629	no	other
## 187	0	19	235	1	1	0	1	0	0	3629	no	white
## 188	0	25	95	1	1	3	0	1	0	3637	no	white
## 189	0	16	135	1	1	0	0	0	0	3643	no	white
## 190	0	29	135	1	0	0	0	0	1	3651	no	white
## 191	0	29	154	1	0	0	0	0	1	3651	no	white
## 192	0	19	147	1	1	0	0	0	0	3651	no	white
## 193	0	19	147	1	1	0	0	0	0	3651	no	white
## 195	0	30	137	1	0	0	0	0	1	3699	no	white
## 196	0	24	110	1	0	0	0	0	1	3728	no	white
## 197	0	19	184	1	1	0	1	0	0	3756	no	white
## 199	0	24	110	3	0	1	0	0	0	3770	no	other
## 200	0	23	110	1	0	0	0	0	1	3770	no	white
## 201	0	20	120	3	0	0	0	0	0	3770	no	other

## 202	0	25	241	2	0	0	1	0	0	3790	no	black
## 203	0	30	112	1	0	0	0	0	1	3799	no	white
## 204	0	22	169	1	0	0	0	0	0	3827	no	white
## 205	0	18	120	1	1	0	0	0	2	3856	no	white
## 206	0	16	170	2	0	0	0	0	4	3860	no	black
## 207	0	32	186	1	0	0	0	0	2	3860	no	white
## 208	0	18	120	3	0	0	0	0	1	3884	no	other
## 209	0	29	130	1	1	0	0	0	2	3884	no	white
## 210	0	33	117	1	0	0	0	1	1	3912	no	white
## 211	0	20	170	1	1	0	0	0	0	3940	no	white
## 212	0	28	134	3	0	0	0	0	1	3941	no	other
## 213	0	14	135	1	0	0	0	0	0	3941	no	white
## 214	0	28	130	3	0	0	0	0	0	3969	no	other
## 215	0	25	120	1	0	0	0	0	2	3983	no	white
## 216	0	16	95	3	0	0	0	0	1	3997	no	other
## 217	0	20	158	1	0	0	0	0	1	3997	no	white
## 218	0	26	160	3	0	0	0	0	0	4054	no	other
## 219	0	21	115	1	0	0	0	0	1	4054	no	white
## 220	0	22	129	1	0	0	0	0	0	4111	no	white
## 221	0	25	130	1	0	0	0	0	2	4153	no	white
## 222	0	31	120	1	0	0	0	0	2	4167	no	white
## 223	0	35	170	1	0	1	0	0	1	4174	no	white
## 224	0	19	120	1	1	0	0	0	0	4238	no	white
## 225	0	24	116	1	0	0	0	0	1	4593	no	white
## 226	0	45	123	1	0	0	0	0	1	4990	no	white
## 4	1	28	120	3	1	1	0	1	0	709	yes	other
## 10	1	29	130	1	0	0	0	1	2	1021	yes	white
## 11	1	34	187	2	1	0	1	0	0	1135	yes	black
## 13	1	25	105	3	0	1	1	0	0	1330	yes	other
## 15	1	25	85	3	0	0	0	1	0	1474	yes	other
## 16	1	27	150	3	0	0	0	0	0	1588	yes	other
## 17	1	23	97	3	0	0	0	1	1	1588	yes	other
## 18	1	24	128	2	0	1	0	0	1	1701	yes	black
## 19	1	24	132	3	0	0	1	0	0	1729	yes	other
## 20	1	21	165	1	1	0	1	0	1	1790	yes	white
## 22	1	32	105	1	1	0	0	0	0	1818	yes	white
## 23	1	19	91	1	1	2	0	1	0	1885	yes	white
## 24	1	25	115	3	0	0	0	0	0	1893	yes	other
## 25	1	16	130	3	0	0	0	0	1	1899	yes	other
## 26	1	25	92	1	1	0	0	0	0	1928	yes	white
## 27	1	20	150	1	1	0	0	0	2	1928	yes	white
## 28	1	21	200	2	0	0	0	1	2	1928	yes	black
## 29	1	24	155	1	1	1	0	0	0	1936	yes	white
## 30	1	21	103	3	0	0	0	0	0	1970	yes	other
## 31	1	20	125	3	0	0	0	1	0	2055	yes	other
## 32	1	25	89	3	0	2	0	0	1	2055	yes	other

## 33	1	19	102	1	0	0	0	0	2	2082	yes	white
## 34	1	19	112	1	1	0	0	1	0	2084	yes	white
## 35	1	26	117	1	1	1	0	0	0	2084	yes	white
## 36	1	24	138	1	0	0	0	0	0	2100	yes	white
## 37	1	17	130	3	1	1	0	1	0	2125	yes	other
## 40	1	20	120	2	1	0	0	0	3	2126	yes	black
## 42	1	22	130	1	1	1	0	1	1	2187	yes	white
## 43	1	27	130	2	0	0	0	1	0	2187	yes	black
## 44	1	20	80	3	1	0	0	1	0	2211	yes	other
## 45	1	17	110	1	1	0	0	0	0	2225	yes	white
## 46	1	25	105	3	0	1	0	0	1	2240	yes	other
## 47	1	20	109	3	0	0	0	0	0	2240	yes	other
## 49	1	18	148	3	0	0	0	0	0	2282	yes	other
## 50	1	18	110	2	1	1	0	0	0	2296	yes	black
## 51	1	20	121	1	1	1	0	1	0	2296	yes	white
## 52	1	21	100	3	0	1	0	0	4	2301	yes	other
## 54	1	26	96	3	0	0	0	0	0	2325	yes	other
## 56	1	31	102	1	1	1	0	0	1	2353	yes	white
## 57	1	15	110	1	0	0	0	0	0	2353	yes	white
## 59	1	23	187	2	1	0	0	0	1	2367	yes	black
## 60	1	20	122	2	1	0	0	0	0	2381	yes	black
## 61	1	24	105	2	1	0	0	0	0	2381	yes	black
## 62	1	15	115	3	0	0	0	1	0	2381	yes	other
## 63	1	23	120	3	0	0	0	0	0	2410	yes	other
## 65	1	30	142	1	1	1	0	0	0	2410	yes	white
## 67	1	22	130	1	1	0	0	0	1	2410	yes	white
## 68	1	17	120	1	1	0	0	0	3	2414	yes	white
## 69	1	23	110	1	1	1	0	0	0	2424	yes	white
## 71	1	17	120	2	0	0	0	0	2	2438	yes	black
## 75	1	26	154	3	0	1	1	0	1	2442	yes	other
## 76	1	20	105	3	0	0	0	0	3	2450	yes	other
## 77	1	26	190	1	1	0	0	0	0	2466	yes	white
## 78	1	14	101	3	1	1	0	0	0	2466	yes	other
## 79	1	28	95	1	1	0	0	0	2	2466	yes	white
## 81	1	14	100	3	0	0	0	0	2	2495	yes	other
## 82	1	23	94	3	1	0	0	0	0	2495	yes	other
## 83	1	17	142	2	0	0	1	0	0	2495	yes	black
## 84	1	21	130	1	1	0	1	0	3	2495	yes	white

```
glimpse(birthwt2)
```

```
## Rows: 189
```

```
## Columns: 12
```

```
## $ low      <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0~
```

```
## $ age      <int> 19, 33, 20, 21, 18, 21, 22, 17, 29, 26, 19, 19, 22, 30, 18, 1~
```

```
## $ lwt      <int> 182, 155, 105, 108, 107, 124, 118, 103, 123, 113, 95, 150, 95~
## $ race     <int> 2, 3, 1, 1, 1, 3, 1, 3, 1, 1, 3, 3, 3, 3, 1, 1, 2, 1, 3, 1, 3~
## $ smoke    <int> 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0~
## $ ptl      <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0~
## $ ht       <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0~
## $ ui       <int> 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0~
## $ ftv      <int> 0, 3, 1, 2, 0, 0, 1, 1, 1, 0, 0, 1, 0, 2, 0, 0, 0, 3, 0, 1, 2~
## $ bwt      <int> 2523, 2551, 2557, 2594, 2600, 2622, 2637, 2637, 2663, 2665, 2~
## $ low_fct  <fct> no, no, no, no, no, no, no, no, no, no, no, no, no, no, no, n~
## $ race_fct <fct> black, other, white, white, white, other, white, other, white~
```

## 18.4 Chi-square test for independence

In a previous chapter, we learned about the chi-square goodness-of-fit test. With a single categorical variable, we summarized data in a frequency table. Each cell of the table had an observed count from the data that we compared to an expected count from the assumption of a null hypothesis. The chi-square statistic measured the discrepancy between observed and expected.

With two categorical variables, we use a contingency table instead of a frequency table. But the principle of the chi-square statistic is the same: each cell in the contingency table has an observed count and an expected count. This forms the basis of a chi-square test for independence.

Below is the contingency table for these two variables. Normally, we only care about column totals because we care how the response variable (here, `low_fct`) is distributed in each group of the predictor variable (i.e., each racial group). But for the calculation of chi-squared, we will need both row and column totals.

```
tabyl(birthwt2, low_fct, race_fct) %>%
  adorn_totals(where = c("row", "col"))
```

```
##   low_fct white black other Total
##      no    73    15    42   130
##     yes    23    11    25    59
##   Total    96    26    67   189
```

A test for independence has a simple null hypothesis: the two variables are independent. This gives us a way to compute expected counts. To see how, look at the sum of all the normal weight babies ( $73 + 15 + 42 = 130$ ) and all the low birth weight babies ( $23 + 11 + 25 = 59$ ). In other words, if race is ignored, there were 130 normal weight babies and 59 low birth weight babies out of 189 total babies. 59 of 189 is 0.31217 or 31.217%, and 130 of 189 is 0.68783 or 68.783%.

Now, if low birth weight and race are truly independent, it shouldn't matter if the mothers were white, black, or some other race. In other words, of 96 white mothers, we should still expect 68.783% of them to have normal weight babies and 31.217% of them to have low birth weight babies. 68.783% of 96 is 66.032. **This is the expected cell count for normal birth weight babies of white women.** 31.217% of 96 is 29.968. **This is the expected cell count for low birth weight babies of white women.** The same analysis can be done for the next two columns as well.

**Exercise 1** Complete the list of expected cell counts in the table above. In other words, apply the percentages 68.783% and 31.217% to the totals of the “black” and “other” columns. Put them in the table below:

---

	white	black	other
no	66.032	?	?
yes	29.968	?	?

---

Unlike the goodness-of-fit test that requires one to specify expected counts for each cell, the test for independence uses only the data to determine the expected counts. For any given cell, if  $R$  is the row total,  $C$  is the column total, and  $n$  is the grand total (the sample size), the expected count in any cell is simply

$$E = \frac{RC}{n}.$$

This is equivalent to the explanation in the previous paragraph. Using low birth weight babies among white mothers as an example,  $R/n$  is  $59/189$  which is 0.31217. Then we multiply this by the column total  $C = 96$  to get

$$\left(\frac{R}{n}\right)C = \frac{RC}{n} = \frac{59 \times 96}{189} = 29.96825.$$

Everything else works almost the same as it did for a chi-square goodness-of-fit test. We still compute  $\chi^2$  by adding up deviations across all cells:

$$\chi^2 = \sum \frac{(O - E)^2}{E}.$$

Even under the assumption of the null, there will still be some sampling variability. Like any hypothesis test, our job is to determine whether the deviations we see are possible due to pure chance alone. The random values of  $\chi^2$  that result from sampling variability will follow a chi-square model. But how many

degrees of freedom are there? This is a little different from the goodness-of-fit test. Instead of the number of cells minus one, we use the following formula:

$$df = (\#rows - 1)(\#columns - 1).$$

In our example we have 2 rows (“yes”, “no”) and 3 columns (“white”, “black”, “other”); therefore,

$$df = (2 - 1)(3 - 1) = 1 \times 2 = 2$$

and we have 2 degrees of freedom (even though there are 6 cells).

Let’s run through the rubric in its entirety.

## 18.5 Exploratory data analysis

### 18.5.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

You should type `?birthwt` at the Console to read the help file. We don’t have any information about how these mothers were selected. The “Source” at the end of the help file is a statistics textbook, so we’d have to track down that book to see where they got the data and if traced back to a primary source.

```
birthwt
```

```
##      low age lwt race smoke ptl ht ui ftv  bwt
## 85    0  19 182   2     0   0  0  1   0 2523
## 86    0  33 155   3     0   0  0  0   3 2551
## 87    0  20 105   1     1   0  0  0   1 2557
## 88    0  21 108   1     1   0  0  1   2 2594
## 89    0  18 107   1     1   0  0  1   0 2600
## 91    0  21 124   3     0   0  0  0   0 2622
## 92    0  22 118   1     0   0  0  0   1 2637
## 93    0  17 103   3     0   0  0  0   1 2637
## 94    0  29 123   1     1   0  0  0   1 2663
## 95    0  26 113   1     1   0  0  0   0 2665
## 96    0  19  95   3     0   0  0  0   0 2722
## 97    0  19 150   3     0   0  0  0   1 2733
## 98    0  22  95   3     0   0  1  0   0 2751
## 99    0  30 107   3     0   1  0  1   2 2750
## 100   0  18 100   1     1   0  0  0   0 2769
```

## 101	0	18	100	1	1	0	0	0	0	2769
## 102	0	15	98	2	0	0	0	0	0	2778
## 103	0	25	118	1	1	0	0	0	3	2782
## 104	0	20	120	3	0	0	0	1	0	2807
## 105	0	28	120	1	1	0	0	0	1	2821
## 106	0	32	121	3	0	0	0	0	2	2835
## 107	0	31	100	1	0	0	0	1	3	2835
## 108	0	36	202	1	0	0	0	0	1	2836
## 109	0	28	120	3	0	0	0	0	0	2863
## 111	0	25	120	3	0	0	0	1	2	2877
## 112	0	28	167	1	0	0	0	0	0	2877
## 113	0	17	122	1	1	0	0	0	0	2906
## 114	0	29	150	1	0	0	0	0	2	2920
## 115	0	26	168	2	1	0	0	0	0	2920
## 116	0	17	113	2	0	0	0	0	1	2920
## 117	0	17	113	2	0	0	0	0	1	2920
## 118	0	24	90	1	1	1	0	0	1	2948
## 119	0	35	121	2	1	1	0	0	1	2948
## 120	0	25	155	1	0	0	0	0	1	2977
## 121	0	25	125	2	0	0	0	0	0	2977
## 123	0	29	140	1	1	0	0	0	2	2977
## 124	0	19	138	1	1	0	0	0	2	2977
## 125	0	27	124	1	1	0	0	0	0	2922
## 126	0	31	215	1	1	0	0	0	2	3005
## 127	0	33	109	1	1	0	0	0	1	3033
## 128	0	21	185	2	1	0	0	0	2	3042
## 129	0	19	189	1	0	0	0	0	2	3062
## 130	0	23	130	2	0	0	0	0	1	3062
## 131	0	21	160	1	0	0	0	0	0	3062
## 132	0	18	90	1	1	0	0	1	0	3062
## 133	0	18	90	1	1	0	0	1	0	3062
## 134	0	32	132	1	0	0	0	0	4	3080
## 135	0	19	132	3	0	0	0	0	0	3090
## 136	0	24	115	1	0	0	0	0	2	3090
## 137	0	22	85	3	1	0	0	0	0	3090
## 138	0	22	120	1	0	0	1	0	1	3100
## 139	0	23	128	3	0	0	0	0	0	3104
## 140	0	22	130	1	1	0	0	0	0	3132
## 141	0	30	95	1	1	0	0	0	2	3147
## 142	0	19	115	3	0	0	0	0	0	3175
## 143	0	16	110	3	0	0	0	0	0	3175
## 144	0	21	110	3	1	0	0	1	0	3203
## 145	0	30	153	3	0	0	0	0	0	3203
## 146	0	20	103	3	0	0	0	0	0	3203
## 147	0	17	119	3	0	0	0	0	0	3225
## 148	0	17	119	3	0	0	0	0	0	3225

## 149	0	23	119	3	0	0	0	0	2	3232
## 150	0	24	110	3	0	0	0	0	0	3232
## 151	0	28	140	1	0	0	0	0	0	3234
## 154	0	26	133	3	1	2	0	0	0	3260
## 155	0	20	169	3	0	1	0	1	1	3274
## 156	0	24	115	3	0	0	0	0	2	3274
## 159	0	28	250	3	1	0	0	0	6	3303
## 160	0	20	141	1	0	2	0	1	1	3317
## 161	0	22	158	2	0	1	0	0	2	3317
## 162	0	22	112	1	1	2	0	0	0	3317
## 163	0	31	150	3	1	0	0	0	2	3321
## 164	0	23	115	3	1	0	0	0	1	3331
## 166	0	16	112	2	0	0	0	0	0	3374
## 167	0	16	135	1	1	0	0	0	0	3374
## 168	0	18	229	2	0	0	0	0	0	3402
## 169	0	25	140	1	0	0	0	0	1	3416
## 170	0	32	134	1	1	1	0	0	4	3430
## 172	0	20	121	2	1	0	0	0	0	3444
## 173	0	23	190	1	0	0	0	0	0	3459
## 174	0	22	131	1	0	0	0	0	1	3460
## 175	0	32	170	1	0	0	0	0	0	3473
## 176	0	30	110	3	0	0	0	0	0	3544
## 177	0	20	127	3	0	0	0	0	0	3487
## 179	0	23	123	3	0	0	0	0	0	3544
## 180	0	17	120	3	1	0	0	0	0	3572
## 181	0	19	105	3	0	0	0	0	0	3572
## 182	0	23	130	1	0	0	0	0	0	3586
## 183	0	36	175	1	0	0	0	0	0	3600
## 184	0	22	125	1	0	0	0	0	1	3614
## 185	0	24	133	1	0	0	0	0	0	3614
## 186	0	21	134	3	0	0	0	0	2	3629
## 187	0	19	235	1	1	0	1	0	0	3629
## 188	0	25	95	1	1	3	0	1	0	3637
## 189	0	16	135	1	1	0	0	0	0	3643
## 190	0	29	135	1	0	0	0	0	1	3651
## 191	0	29	154	1	0	0	0	0	1	3651
## 192	0	19	147	1	1	0	0	0	0	3651
## 193	0	19	147	1	1	0	0	0	0	3651
## 195	0	30	137	1	0	0	0	0	1	3699
## 196	0	24	110	1	0	0	0	0	1	3728
## 197	0	19	184	1	1	0	1	0	0	3756
## 199	0	24	110	3	0	1	0	0	0	3770
## 200	0	23	110	1	0	0	0	0	1	3770
## 201	0	20	120	3	0	0	0	0	0	3770
## 202	0	25	241	2	0	0	1	0	0	3790
## 203	0	30	112	1	0	0	0	0	1	3799



```

## 204  0 22 169  1  0  0  0  0  0 3827
## 205  0 18 120  1  1  0  0  0  2 3856
## 206  0 16 170  2  0  0  0  0  4 3860
## 207  0 32 186  1  0  0  0  0  2 3860
## 208  0 18 120  3  0  0  0  0  1 3884
## 209  0 29 130  1  1  0  0  0  2 3884
## 210  0 33 117  1  0  0  0  1  1 3912
## 211  0 20 170  1  1  0  0  0  0 3940
## 212  0 28 134  3  0  0  0  0  1 3941
## 213  0 14 135  1  0  0  0  0  0 3941
## 214  0 28 130  3  0  0  0  0  0 3969
## 215  0 25 120  1  0  0  0  0  2 3983
## 216  0 16  95  3  0  0  0  0  1 3997
## 217  0 20 158  1  0  0  0  0  1 3997
## 218  0 26 160  3  0  0  0  0  0 4054
## 219  0 21 115  1  0  0  0  0  1 4054
## 220  0 22 129  1  0  0  0  0  0 4111
## 221  0 25 130  1  0  0  0  0  2 4153
## 222  0 31 120  1  0  0  0  0  2 4167
## 223  0 35 170  1  0  1  0  0  1 4174
## 224  0 19 120  1  1  0  0  0  0 4238
## 225  0 24 116  1  0  0  0  0  1 4593
## 226  0 45 123  1  0  0  0  0  1 4990
##  4   1 28 120  3  1  1  0  1  0  709
## 10   1 29 130  1  0  0  0  1  2 1021
## 11   1 34 187  2  1  0  1  0  0 1135
## 13   1 25 105  3  0  1  1  0  0 1330
## 15   1 25  85  3  0  0  0  1  0 1474
## 16   1 27 150  3  0  0  0  0  0 1588
## 17   1 23  97  3  0  0  0  1  1 1588
## 18   1 24 128  2  0  1  0  0  1 1701
## 19   1 24 132  3  0  0  1  0  0 1729
## 20   1 21 165  1  1  0  1  0  1 1790
## 22   1 32 105  1  1  0  0  0  0 1818
## 23   1 19  91  1  1  2  0  1  0 1885
## 24   1 25 115  3  0  0  0  0  0 1893
## 25   1 16 130  3  0  0  0  0  1 1899
## 26   1 25  92  1  1  0  0  0  0 1928
## 27   1 20 150  1  1  0  0  0  2 1928
## 28   1 21 200  2  0  0  0  1  2 1928
## 29   1 24 155  1  1  1  0  0  0 1936
## 30   1 21 103  3  0  0  0  0  0 1970
## 31   1 20 125  3  0  0  0  1  0 2055
## 32   1 25  89  3  0  2  0  0  1 2055
## 33   1 19 102  1  0  0  0  0  2 2082
## 34   1 19 112  1  1  0  0  1  0 2084

```

```
## 35 1 26 117 1 1 1 0 0 0 2084
## 36 1 24 138 1 0 0 0 0 0 2100
## 37 1 17 130 3 1 1 0 1 0 2125
## 40 1 20 120 2 1 0 0 0 3 2126
## 42 1 22 130 1 1 1 0 1 1 2187
## 43 1 27 130 2 0 0 0 1 0 2187
## 44 1 20 80 3 1 0 0 1 0 2211
## 45 1 17 110 1 1 0 0 0 0 2225
## 46 1 25 105 3 0 1 0 0 1 2240
## 47 1 20 109 3 0 0 0 0 0 2240
## 49 1 18 148 3 0 0 0 0 0 2282
## 50 1 18 110 2 1 1 0 0 0 2296
## 51 1 20 121 1 1 1 0 1 0 2296
## 52 1 21 100 3 0 1 0 0 4 2301
## 54 1 26 96 3 0 0 0 0 0 2325
## 56 1 31 102 1 1 1 0 0 1 2353
## 57 1 15 110 1 0 0 0 0 0 2353
## 59 1 23 187 2 1 0 0 0 1 2367
## 60 1 20 122 2 1 0 0 0 0 2381
## 61 1 24 105 2 1 0 0 0 0 2381
## 62 1 15 115 3 0 0 0 1 0 2381
## 63 1 23 120 3 0 0 0 0 0 2410
## 65 1 30 142 1 1 1 0 0 0 2410
## 67 1 22 130 1 1 0 0 0 1 2410
## 68 1 17 120 1 1 0 0 0 3 2414
## 69 1 23 110 1 1 1 0 0 0 2424
## 71 1 17 120 2 0 0 0 0 2 2438
## 75 1 26 154 3 0 1 1 0 1 2442
## 76 1 20 105 3 0 0 0 0 3 2450
## 77 1 26 190 1 1 0 0 0 0 2466
## 78 1 14 101 3 1 1 0 0 0 2466
## 79 1 28 95 1 1 0 0 0 2 2466
## 81 1 14 100 3 0 0 0 0 2 2495
## 82 1 23 94 3 1 0 0 0 0 2495
## 83 1 17 142 2 0 0 1 0 0 2495
## 84 1 21 130 1 1 0 1 0 3 2495
```

```
glimpse(birthwt)
```

```
## Rows: 189
## Columns: 10
## $ low <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0~
## $ age <int> 19, 33, 20, 21, 18, 21, 22, 17, 29, 26, 19, 19, 22, 30, 18, 18, ~
## $ lwt <int> 182, 155, 105, 108, 107, 124, 118, 103, 123, 113, 95, 150, 95, 1~
## $ race <int> 2, 3, 1, 1, 1, 3, 1, 3, 1, 1, 3, 3, 3, 3, 1, 1, 2, 1, 3, 1, 3, 1~
```

```
## $ smoke <int> 0, 0, 1, 1, 1, 0, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 0, 1, 0, 0~
## $ ptl <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0~
## $ ht <int> 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0~
## $ ui <int> 1, 0, 0, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1~
## $ ftv <int> 0, 3, 1, 2, 0, 0, 1, 1, 1, 0, 0, 1, 0, 2, 0, 0, 0, 3, 0, 1, 2, 3~
## $ bwt <int> 2523, 2551, 2557, 2594, 2600, 2622, 2637, 2637, 2663, 2665, 2722~
```

### 18.5.2 Prepare the data for analysis.

```
# Although we've already done this above,
# we include it here again for completeness.
birthwt2 <- birthwt %>%
  mutate(low_fct = factor(low, levels = c(0, 1),
    labels = c("no", "yes")),
    race_fct = factor(race, levels = c(1, 2, 3),
    labels = c("white", "black", "other")))
birthwt2
```

##	low	age	lwt	race	smoke	ptl	ht	ui	ftv	bwt	low_fct	race_fct
## 85	0	19	182	2	0	0	0	1	0	2523	no	black
## 86	0	33	155	3	0	0	0	0	3	2551	no	other
## 87	0	20	105	1	1	0	0	0	1	2557	no	white
## 88	0	21	108	1	1	0	0	1	2	2594	no	white
## 89	0	18	107	1	1	0	0	1	0	2600	no	white
## 91	0	21	124	3	0	0	0	0	0	2622	no	other
## 92	0	22	118	1	0	0	0	0	1	2637	no	white
## 93	0	17	103	3	0	0	0	0	1	2637	no	other
## 94	0	29	123	1	1	0	0	0	1	2663	no	white
## 95	0	26	113	1	1	0	0	0	0	2665	no	white
## 96	0	19	95	3	0	0	0	0	0	2722	no	other
## 97	0	19	150	3	0	0	0	0	1	2733	no	other
## 98	0	22	95	3	0	0	1	0	0	2751	no	other
## 99	0	30	107	3	0	1	0	1	2	2750	no	other
## 100	0	18	100	1	1	0	0	0	0	2769	no	white
## 101	0	18	100	1	1	0	0	0	0	2769	no	white
## 102	0	15	98	2	0	0	0	0	0	2778	no	black
## 103	0	25	118	1	1	0	0	0	3	2782	no	white
## 104	0	20	120	3	0	0	0	1	0	2807	no	other
## 105	0	28	120	1	1	0	0	0	1	2821	no	white
## 106	0	32	121	3	0	0	0	0	2	2835	no	other
## 107	0	31	100	1	0	0	0	1	3	2835	no	white
## 108	0	36	202	1	0	0	0	0	1	2836	no	white
## 109	0	28	120	3	0	0	0	0	0	2863	no	other
## 111	0	25	120	3	0	0	0	1	2	2877	no	other

## 112	0	28	167	1	0	0	0	0	0	2877	no	white
## 113	0	17	122	1	1	0	0	0	0	2906	no	white
## 114	0	29	150	1	0	0	0	0	2	2920	no	white
## 115	0	26	168	2	1	0	0	0	0	2920	no	black
## 116	0	17	113	2	0	0	0	0	1	2920	no	black
## 117	0	17	113	2	0	0	0	0	1	2920	no	black
## 118	0	24	90	1	1	1	0	0	1	2948	no	white
## 119	0	35	121	2	1	1	0	0	1	2948	no	black
## 120	0	25	155	1	0	0	0	0	1	2977	no	white
## 121	0	25	125	2	0	0	0	0	0	2977	no	black
## 123	0	29	140	1	1	0	0	0	2	2977	no	white
## 124	0	19	138	1	1	0	0	0	2	2977	no	white
## 125	0	27	124	1	1	0	0	0	0	2922	no	white
## 126	0	31	215	1	1	0	0	0	2	3005	no	white
## 127	0	33	109	1	1	0	0	0	1	3033	no	white
## 128	0	21	185	2	1	0	0	0	2	3042	no	black
## 129	0	19	189	1	0	0	0	0	2	3062	no	white
## 130	0	23	130	2	0	0	0	0	1	3062	no	black
## 131	0	21	160	1	0	0	0	0	0	3062	no	white
## 132	0	18	90	1	1	0	0	1	0	3062	no	white
## 133	0	18	90	1	1	0	0	1	0	3062	no	white
## 134	0	32	132	1	0	0	0	0	4	3080	no	white
## 135	0	19	132	3	0	0	0	0	0	3090	no	other
## 136	0	24	115	1	0	0	0	0	2	3090	no	white
## 137	0	22	85	3	1	0	0	0	0	3090	no	other
## 138	0	22	120	1	0	0	1	0	1	3100	no	white
## 139	0	23	128	3	0	0	0	0	0	3104	no	other
## 140	0	22	130	1	1	0	0	0	0	3132	no	white
## 141	0	30	95	1	1	0	0	0	2	3147	no	white
## 142	0	19	115	3	0	0	0	0	0	3175	no	other
## 143	0	16	110	3	0	0	0	0	0	3175	no	other
## 144	0	21	110	3	1	0	0	1	0	3203	no	other
## 145	0	30	153	3	0	0	0	0	0	3203	no	other
## 146	0	20	103	3	0	0	0	0	0	3203	no	other
## 147	0	17	119	3	0	0	0	0	0	3225	no	other
## 148	0	17	119	3	0	0	0	0	0	3225	no	other
## 149	0	23	119	3	0	0	0	0	2	3232	no	other
## 150	0	24	110	3	0	0	0	0	0	3232	no	other
## 151	0	28	140	1	0	0	0	0	0	3234	no	white
## 154	0	26	133	3	1	2	0	0	0	3260	no	other
## 155	0	20	169	3	0	1	0	1	1	3274	no	other
## 156	0	24	115	3	0	0	0	0	2	3274	no	other
## 159	0	28	250	3	1	0	0	0	6	3303	no	other
## 160	0	20	141	1	0	2	0	1	1	3317	no	white
## 161	0	22	158	2	0	1	0	0	2	3317	no	black
## 162	0	22	112	1	1	2	0	0	0	3317	no	white

## 163	0	31	150	3	1	0	0	0	2	3321	no	other
## 164	0	23	115	3	1	0	0	0	1	3331	no	other
## 166	0	16	112	2	0	0	0	0	0	3374	no	black
## 167	0	16	135	1	1	0	0	0	0	3374	no	white
## 168	0	18	229	2	0	0	0	0	0	3402	no	black
## 169	0	25	140	1	0	0	0	0	1	3416	no	white
## 170	0	32	134	1	1	1	0	0	4	3430	no	white
## 172	0	20	121	2	1	0	0	0	0	3444	no	black
## 173	0	23	190	1	0	0	0	0	0	3459	no	white
## 174	0	22	131	1	0	0	0	0	1	3460	no	white
## 175	0	32	170	1	0	0	0	0	0	3473	no	white
## 176	0	30	110	3	0	0	0	0	0	3544	no	other
## 177	0	20	127	3	0	0	0	0	0	3487	no	other
## 179	0	23	123	3	0	0	0	0	0	3544	no	other
## 180	0	17	120	3	1	0	0	0	0	3572	no	other
## 181	0	19	105	3	0	0	0	0	0	3572	no	other
## 182	0	23	130	1	0	0	0	0	0	3586	no	white
## 183	0	36	175	1	0	0	0	0	0	3600	no	white
## 184	0	22	125	1	0	0	0	0	1	3614	no	white
## 185	0	24	133	1	0	0	0	0	0	3614	no	white
## 186	0	21	134	3	0	0	0	0	2	3629	no	other
## 187	0	19	235	1	1	0	1	0	0	3629	no	white
## 188	0	25	95	1	1	3	0	1	0	3637	no	white
## 189	0	16	135	1	1	0	0	0	0	3643	no	white
## 190	0	29	135	1	0	0	0	0	1	3651	no	white
## 191	0	29	154	1	0	0	0	0	1	3651	no	white
## 192	0	19	147	1	1	0	0	0	0	3651	no	white
## 193	0	19	147	1	1	0	0	0	0	3651	no	white
## 195	0	30	137	1	0	0	0	0	1	3699	no	white
## 196	0	24	110	1	0	0	0	0	1	3728	no	white
## 197	0	19	184	1	1	0	1	0	0	3756	no	white
## 199	0	24	110	3	0	1	0	0	0	3770	no	other
## 200	0	23	110	1	0	0	0	0	1	3770	no	white
## 201	0	20	120	3	0	0	0	0	0	3770	no	other
## 202	0	25	241	2	0	0	1	0	0	3790	no	black
## 203	0	30	112	1	0	0	0	0	1	3799	no	white
## 204	0	22	169	1	0	0	0	0	0	3827	no	white
## 205	0	18	120	1	1	0	0	0	2	3856	no	white
## 206	0	16	170	2	0	0	0	0	4	3860	no	black
## 207	0	32	186	1	0	0	0	0	2	3860	no	white
## 208	0	18	120	3	0	0	0	0	1	3884	no	other
## 209	0	29	130	1	1	0	0	0	2	3884	no	white
## 210	0	33	117	1	0	0	0	1	1	3912	no	white
## 211	0	20	170	1	1	0	0	0	0	3940	no	white
## 212	0	28	134	3	0	0	0	0	1	3941	no	other
## 213	0	14	135	1	0	0	0	0	0	3941	no	white

## 214	0	28	130	3	0	0	0	0	0	3969	no	other
## 215	0	25	120	1	0	0	0	0	2	3983	no	white
## 216	0	16	95	3	0	0	0	0	1	3997	no	other
## 217	0	20	158	1	0	0	0	0	1	3997	no	white
## 218	0	26	160	3	0	0	0	0	0	4054	no	other
## 219	0	21	115	1	0	0	0	0	1	4054	no	white
## 220	0	22	129	1	0	0	0	0	0	4111	no	white
## 221	0	25	130	1	0	0	0	0	2	4153	no	white
## 222	0	31	120	1	0	0	0	0	2	4167	no	white
## 223	0	35	170	1	0	1	0	0	1	4174	no	white
## 224	0	19	120	1	1	0	0	0	0	4238	no	white
## 225	0	24	116	1	0	0	0	0	1	4593	no	white
## 226	0	45	123	1	0	0	0	0	1	4990	no	white
## 4	1	28	120	3	1	1	0	1	0	709	yes	other
## 10	1	29	130	1	0	0	0	1	2	1021	yes	white
## 11	1	34	187	2	1	0	1	0	0	1135	yes	black
## 13	1	25	105	3	0	1	1	0	0	1330	yes	other
## 15	1	25	85	3	0	0	0	1	0	1474	yes	other
## 16	1	27	150	3	0	0	0	0	0	1588	yes	other
## 17	1	23	97	3	0	0	0	1	1	1588	yes	other
## 18	1	24	128	2	0	1	0	0	1	1701	yes	black
## 19	1	24	132	3	0	0	1	0	0	1729	yes	other
## 20	1	21	165	1	1	0	1	0	1	1790	yes	white
## 22	1	32	105	1	1	0	0	0	0	1818	yes	white
## 23	1	19	91	1	1	2	0	1	0	1885	yes	white
## 24	1	25	115	3	0	0	0	0	0	1893	yes	other
## 25	1	16	130	3	0	0	0	0	1	1899	yes	other
## 26	1	25	92	1	1	0	0	0	0	1928	yes	white
## 27	1	20	150	1	1	0	0	0	2	1928	yes	white
## 28	1	21	200	2	0	0	0	1	2	1928	yes	black
## 29	1	24	155	1	1	1	0	0	0	1936	yes	white
## 30	1	21	103	3	0	0	0	0	0	1970	yes	other
## 31	1	20	125	3	0	0	0	1	0	2055	yes	other
## 32	1	25	89	3	0	2	0	0	1	2055	yes	other
## 33	1	19	102	1	0	0	0	0	2	2082	yes	white
## 34	1	19	112	1	1	0	0	1	0	2084	yes	white
## 35	1	26	117	1	1	1	0	0	0	2084	yes	white
## 36	1	24	138	1	0	0	0	0	0	2100	yes	white
## 37	1	17	130	3	1	1	0	1	0	2125	yes	other
## 40	1	20	120	2	1	0	0	0	3	2126	yes	black
## 42	1	22	130	1	1	1	0	1	1	2187	yes	white
## 43	1	27	130	2	0	0	0	1	0	2187	yes	black
## 44	1	20	80	3	1	0	0	1	0	2211	yes	other
## 45	1	17	110	1	1	0	0	0	0	2225	yes	white
## 46	1	25	105	3	0	1	0	0	1	2240	yes	other
## 47	1	20	109	3	0	0	0	0	0	2240	yes	other

```
## 49  1  18 148  3  0  0  0  0  0 2282  yes  other
## 50  1  18 110  2  1  1  0  0  0 2296  yes  black
## 51  1  20 121  1  1  1  0  1  0 2296  yes  white
## 52  1  21 100  3  0  1  0  0  4 2301  yes  other
## 54  1  26  96  3  0  0  0  0  0 2325  yes  other
## 56  1  31 102  1  1  1  0  0  1 2353  yes  white
## 57  1  15 110  1  0  0  0  0  0 2353  yes  white
## 59  1  23 187  2  1  0  0  0  1 2367  yes  black
## 60  1  20 122  2  1  0  0  0  0 2381  yes  black
## 61  1  24 105  2  1  0  0  0  0 2381  yes  black
## 62  1  15 115  3  0  0  0  1  0 2381  yes  other
## 63  1  23 120  3  0  0  0  0  0 2410  yes  other
## 65  1  30 142  1  1  1  0  0  0 2410  yes  white
## 67  1  22 130  1  1  0  0  0  1 2410  yes  white
## 68  1  17 120  1  1  0  0  0  3 2414  yes  white
## 69  1  23 110  1  1  1  0  0  0 2424  yes  white
## 71  1  17 120  2  0  0  0  0  2 2438  yes  black
## 75  1  26 154  3  0  1  1  0  1 2442  yes  other
## 76  1  20 105  3  0  0  0  0  3 2450  yes  other
## 77  1  26 190  1  1  0  0  0  0 2466  yes  white
## 78  1  14 101  3  1  1  0  0  0 2466  yes  other
## 79  1  28  95  1  1  0  0  0  2 2466  yes  white
## 81  1  14 100  3  0  0  0  0  2 2495  yes  other
## 82  1  23  94  3  1  0  0  0  0 2495  yes  other
## 83  1  17 142  2  0  0  1  0  0 2495  yes  black
## 84  1  21 130  1  1  0  1  0  3 2495  yes  white
```

### 18.5.3 Make tables or plots to explore the data visually.

```
tabyl(birthwt2, low_fct, race_fct) %>%
  adorn_totals()
```

```
## low_fct white black other
##      no    73    15    42
##      yes    23    11    25
##      Total   96    26    67
```

```
tabyl(birthwt2, low_fct, race_fct) %>%
  adorn_totals() %>%
  adorn_percentages("col") %>%
  adorn_pct_formatting()
```

```
## low_fct white black other
```

##	no	76.0%	57.7%	62.7%
##	yes	24.0%	42.3%	37.3%
##	Total	100.0%	100.0%	100.0%

Commentary: Earlier we used row and column total to explain how expected cell counts arise. Here, however, we will revert back to our previous standard practice of generating one contingency table with counts and another with column percentages.

## 18.6 Hypotheses

### 18.6.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

The sample consists of 189 mothers who gave birth at the Baystate Medical Center in Springfield, Massachusetts in 1986. The population is presumably all mothers, although it's safest to conclude only about mothers who gave birth at this hospital.

### 18.6.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

$H_0$  : Low birth weight and race are independent.

$H_A$  : Low birth weight and race are associated.

### 18.6.3 Express the null and alternative hypotheses in symbols (when possible).

For a chi-square test for independence, this section is not applicable. With multiple categories in the response and predictor variables, there are no specific parameters of interest to express symbolically.

## 18.7 Model

### 18.7.1 Identify the sampling distribution model.

We will use a chi-square model with 2 degrees of freedom.



### 18.7.2 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - We hope that these 189 women are representative of all women who gave birth in this hospital (or, at best, in that region) around that time.
- 10%
  - We don't know how many women gave birth at this hospital, but perhaps over many years we might have more than 1890 women.
- Expected cell counts
  - You checked the cell counts as a part of Exercise 1. Note that all expected cell counts are larger than 5, so the condition is met.

## 18.8 Mechanics

### 18.8.1 Compute the test statistic.

```
obs_chisq <- birthwt2 %>%
  specify(response = low_fct, explanatory = race_fct) %>%
  hypothesize(null = "independence") %>%
  calculate(stat = "chisq")
obs_chisq

## Response: low_fct (factor)
## Explanatory: race_fct (factor)
## Null Hypothesis: independence
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1  5.00
```

### 18.8.2 Report the test statistic in context (when possible).

The value of  $\chi^2$  is 5.004813.

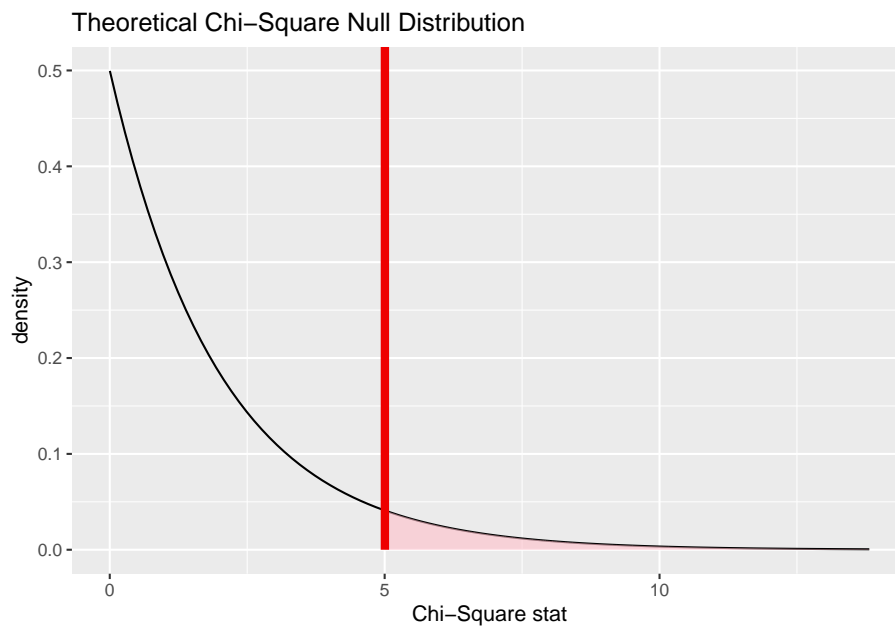
Commentary: As in the last chapter, there's not much context to report with a value of  $\chi^2$ , so the most we can do here is just report it in a full sentence.

### 18.8.3 Plot the null distribution.

```
low_race_test <- birthwt2 %>%  
  specify(response = low_fct, explanatory = race_fct) %>%  
  assume(distribution = "chisq")  
low_race_test
```

```
## A Chi-squared distribution with 2 degrees of freedom.
```

```
low_race_test %>%  
  visualize() +  
  shade_p_value(obs_chisq, direction = "greater")
```



### 18.8.4 Calculate the P-value.

```
low_race_test_p <- low_race_test %>%  
  get_p_value(obs_chisq, direction = "greater")  
low_race_test_p
```

```
## # A tibble: 1 x 1
```

```
##    p_value
##    <dbl>
## 1    0.0819
```

### 18.8.5 Interpret the P-value as a probability given the null.

The P-value is 0.0818877. If low birth weight and race were independent, there would be a 8.1887698% chance of seeing results at least as extreme as we saw in the data.

## 18.9 Conclusion

### 18.9.1 State the statistical conclusion.

We fail to reject the null hypothesis.

### 18.9.2 State (but do not overstate) a contextually meaningful conclusion.

There is insufficient evidence that low birth weight and race are associated.

### 18.9.3 Express reservations or uncertainty about the generalizability of the conclusion.

Given our uncertainty about how the data was collected, it's not clear what our conclusion means. Also, failing to reject the null is really a "non-conclusion" in that it leaves us basically knowing nothing. We don't have evidence of such an association (and there are good reasons to believe there may not be one), but failing to reject the null does not prove anything.

### 18.9.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.

It's possible that we have made a Type II error. It may be that low birth weight and race are associated, but our sample has not given enough evidence of such an association.

## 18.10 Confidence interval

There are no parameters of interest in a chi-square test, so there is no confidence interval to report.

## 18.11 Your turn

Use the `smoking` data set from the `openintro` package. Run a chi-square test for independence to determine if smoking status is associated with marital status.

The rubric outline is reproduced below. You may refer to the worked example above and modify it accordingly. Remember to strip out all the commentary. That is just exposition for your benefit in understanding the steps, but is not meant to form part of the formal inference process.

Another word of warning: the copy/paste process is not a substitute for your brain. You will often need to modify more than just the names of the data frames and variables to adapt the worked examples to your own work. Do not blindly copy and paste code without understanding what it does. And you should **never** copy and paste text. All the sentences and paragraphs you write are expressions of your own analysis. They must reflect your own understanding of the inferential process.

**Also, so that your answers here don't mess up the code chunks above, use new variable names everywhere.**

### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure. Please write up your answer here

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

**Identify the sample (or samples) and a reasonable population (or populations) of interest.** Please write up your answer here.

**Express the null and alternative hypotheses as contextually meaningful full sentences.**  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

**Express the null and alternative hypotheses in symbols (when possible).**  $H_0$  : *math*

$H_A$  : *math*

### Model

**Identify the sampling distribution model.** Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

### Mechanics

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

**Conclusion**

**State the statistical conclusion.** Please write up your answer here.

**State (but do not overstate) a contextually meaningful conclusion.** Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.** Please write up your answer here.

## 18.12 Conclusion

With two categorical variables, we can run a chi-square test for independence to test the null hypothesis that the two variables are independent. While technically we can run this test for any two categorical variables, if both variables have only two levels, we would usually choose to run a test for two proportions. The chi-square test for independence is useful when one or both of the response and predictor variables have three or more levels. The expected cell counts are derived from the data and then the chi-squared statistic is computed as usual. Using the correct degrees of freedom, we can test how much the observed cell counts deviate from the expected cell counts and derive a P-value.

### 18.12.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.





## Chapter 19

# Inference for one mean

2.0

### Functions introduced in this chapter

`rnorm`

## 19.1 Introduction

In this chapter, we'll learn about the Student  $t$  distribution and use it to perform a  $t$  test for a single mean.

### 19.1.1 Install new packages

There are no new packages used in this chapter.

### 19.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/17-inference\\_for\\_one\\_mean.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/17-inference_for_one_mean.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 19.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 19.2 Load packages

We load the standard `tidyverse` and `infer` packages as well as the `mosaic` package to run some simulation. The `openintro` package contains the `teacher` data and the `hsb2` data.

```
library(tidyverse)
library(infer)
library(mosaic)
library(openintro)
```

## 19.3 Simulating means

Systolic blood pressure (SBP) for women in the U.S. and Canada follows a normal distribution with a mean of 114 and a standard deviation of 14.

Suppose we gather a random sample of 4 women and measure their SBP. We can simulate doing that with the `rnorm` command:

```
set.seed(5151977)
SBP_sample <- rnorm(4, mean = 114, sd = 14)
SBP_sample
```

```
## [1] 99.75130 126.47739 99.53632 115.05247
```

We summarize our sample by taking the mean and standard deviation:

```
mean(SBP_sample)
```

```
## [1] 110.2044
```

```
sd(SBP_sample)
```

```
## [1] 13.05615
```

The sample mean  $\bar{y} = 110.2043696$  is somewhat close to the true population mean  $\mu = 114$  and the sample standard deviation  $s = 13.0561519$  is somewhat close to the true population standard deviation  $\sigma = 14$ . ( $\mu$  is the Greek letter “mu” and  $\sigma$  is the Greek letter “sigma”.)

Let’s simulate lots of samples of size 4. For each sample, we calculate the sample mean.

```
set.seed(5151977)
sims <- do(2000) * mean(rnorm(4, mean = 114, sd = 14))
sims
```

```
##          mean
## 1    110.95524
## 2    111.06853
## 3    109.91266
## 4    113.51487
## 5    114.84292
## 6    124.12671
## 7    110.52277
## 8    122.91483
## 9    113.79958
## 10   121.52306
## 11   119.45527
## 12   130.95196
## 13   106.25140
## 14   119.48189
## 15   122.95412
## 16   111.36293
## 17   115.26561
## 18   120.00887
## 19   111.12422
## 20   125.11449
## 21   112.54356
## 22   121.05007
## 23   111.92577
## 24   112.37685
## 25   108.60242
## 26   112.14135
## 27   121.03786
## 28   102.21504
## 29   131.42457
## 30   115.75208
## 31   118.57539
## 32   107.75367
## 33   113.73938
```

```
## 34 107.48598
## 35 104.02251
## 36 110.26283
## 37 114.03591
## 38 105.89310
## 39 112.81019
## 40 123.99549
## 41 102.07213
## 42 102.65507
## 43 119.93490
## 44 123.99603
## 45 119.72605
## 46 122.57296
## 47 112.79102
## 48 108.88674
## 49 109.46094
## 50 111.52494
## 51 106.51913
## 52 118.92374
## 53 122.65041
## 54 106.33611
## 55 114.84009
## 56 119.94925
## 57 87.48567
## 58 107.67256
## 59 112.29705
## 60 114.49032
## 61 106.00521
## 62 103.61574
## 63 114.44472
## 64 124.40115
## 65 107.25545
## 66 106.18013
## 67 107.38138
## 68 115.50453
## 69 118.83450
## 70 109.98443
## 71 133.63093
## 72 118.93599
## 73 112.55365
## 74 122.22781
## 75 119.94346
## 76 120.08051
## 77 115.73125
## 78 99.12175
## 79 110.20178
```

```
## 80    97.50553
## 81   126.13150
## 82   110.10237
## 83   116.45862
## 84   118.18392
## 85   120.15207
## 86   107.32720
## 87   117.33775
## 88    96.64247
## 89   109.86058
## 90   124.84727
## 91   109.67761
## 92   117.45921
## 93   110.36776
## 94   118.71447
## 95   122.94817
## 96   113.04646
## 97   116.69160
## 98   113.14800
## 99   117.60656
## 100  116.98939
## 101  113.87627
## 102  117.60049
## 103  119.06600
## 104  126.74302
## 105  116.53015
## 106  121.92932
## 107  107.90235
## 108  118.06294
## 109  116.88269
## 110  119.81950
## 111  127.56483
## 112  109.67205
## 113  113.93574
## 114  110.89664
## 115  115.59765
## 116   98.08517
## 117  108.69878
## 118  114.50676
## 119  109.82143
## 120  118.93792
## 121  121.50253
## 122  101.73570
## 123  117.77834
## 124  103.81164
## 125  101.48508
```

```
## 126 127.18256
## 127 119.56654
## 128 120.47221
## 129 123.70693
## 130 125.67436
## 131 124.50634
## 132 99.11626
## 133 113.36051
## 134 107.59688
## 135 119.69572
## 136 113.57789
## 137 114.00803
## 138 114.95061
## 139 117.94756
## 140 106.20955
## 141 112.69388
## 142 115.82052
## 143 124.41148
## 144 119.49821
## 145 114.44646
## 146 101.22920
## 147 109.58204
## 148 109.16187
## 149 105.36936
## 150 111.49145
## 151 118.48739
## 152 101.84622
## 153 115.05308
## 154 121.74454
## 155 115.84609
## 156 114.60402
## 157 121.84957
## 158 118.38499
## 159 117.98274
## 160 121.94268
## 161 112.60397
## 162 106.21758
## 163 121.90313
## 164 122.05917
## 165 128.85365
## 166 106.67919
## 167 120.88093
## 168 105.27210
## 169 133.73894
## 170 112.95960
## 171 114.62501
```

```
## 172 118.79292
## 173 114.05784
## 174 106.07207
## 175 122.25110
## 176 124.99923
## 177 111.32837
## 178 112.67882
## 179 118.10980
## 180 113.55150
## 181 109.94996
## 182 130.17665
## 183 117.41869
## 184 112.29039
## 185 115.18728
## 186 119.10711
## 187 121.18710
## 188 116.40250
## 189 123.58668
## 190 117.05543
## 191 114.30052
## 192 120.59040
## 193 108.93992
## 194 116.69512
## 195 123.65056
## 196 120.25289
## 197 119.10736
## 198 121.35013
## 199 108.22576
## 200 123.96013
## 201 120.50076
## 202 109.45569
## 203 124.60173
## 204 109.20374
## 205 109.14185
## 206 111.64284
## 207 127.80637
## 208 97.05353
## 209 104.42525
## 210 108.70502
## 211 123.53495
## 212 111.92085
## 213 103.79728
## 214 109.04242
## 215 101.15528
## 216 108.99493
## 217 115.66033
```

```
## 218 104.27866
## 219 127.74945
## 220 119.18990
## 221 99.37513
## 222 119.24557
## 223 107.03566
## 224 118.83983
## 225 118.84264
## 226 124.91099
## 227 103.66402
## 228 109.91857
## 229 116.49506
## 230 112.01135
## 231 110.40098
## 232 100.23115
## 233 115.89741
## 234 120.00895
## 235 110.26257
## 236 104.91429
## 237 121.20485
## 238 127.85001
## 239 121.99891
## 240 116.34753
## 241 113.57648
## 242 113.91281
## 243 117.83396
## 244 117.19323
## 245 123.04011
## 246 111.43295
## 247 108.88549
## 248 101.10892
## 249 108.54658
## 250 128.54127
## 251 132.02932
## 252 117.36163
## 253 100.19385
## 254 113.30224
## 255 120.65156
## 256 104.76686
## 257 118.55390
## 258 118.08333
## 259 118.85312
## 260 116.92587
## 261 125.34601
## 262 113.04661
## 263 127.01136
```



```
## 264 116.97079
## 265 115.09776
## 266 120.77965
## 267 112.78021
## 268 120.98030
## 269 96.97945
## 270 109.06035
## 271 113.31895
## 272 118.24567
## 273 128.56256
## 274 122.71663
## 275 122.79106
## 276 107.69711
## 277 122.51593
## 278 121.62137
## 279 115.44487
## 280 114.65932
## 281 100.87231
## 282 118.26446
## 283 108.46425
## 284 115.83714
## 285 121.39197
## 286 110.09557
## 287 113.85471
## 288 117.69545
## 289 116.22425
## 290 120.78184
## 291 126.43991
## 292 103.53681
## 293 116.32864
## 294 108.06495
## 295 106.65624
## 296 120.69772
## 297 119.37433
## 298 100.02332
## 299 118.59332
## 300 119.53438
## 301 107.11014
## 302 111.97493
## 303 103.47491
## 304 111.99805
## 305 118.71416
## 306 116.33954
## 307 125.49563
## 308 107.78016
## 309 102.12925
```

```
## 310 112.12212
## 311 117.51136
## 312 110.08975
## 313 114.72259
## 314 120.56031
## 315 122.04100
## 316 111.17129
## 317 116.39056
## 318 111.50435
## 319 104.30895
## 320 101.31131
## 321 114.53301
## 322 113.94972
## 323 116.04217
## 324 112.54460
## 325 113.52116
## 326 110.60055
## 327 117.48808
## 328 116.50048
## 329 119.46474
## 330 123.91257
## 331 111.94294
## 332 102.98073
## 333 109.80824
## 334 106.57737
## 335 113.14494
## 336 100.74728
## 337 100.16375
## 338 115.02875
## 339 110.51485
## 340 110.32509
## 341 120.91380
## 342 118.33534
## 343 111.63758
## 344 110.58353
## 345 118.32547
## 346 106.03945
## 347 114.78878
## 348 95.12731
## 349 115.50274
## 350 123.32999
## 351 104.88001
## 352 127.10250
## 353 127.14507
## 354 108.64777
## 355 112.02036
```

```
## 356 120.33362
## 357 120.23128
## 358 111.15694
## 359 123.51130
## 360 116.82204
## 361 104.68623
## 362 114.13924
## 363 111.40374
## 364 109.04713
## 365 118.19404
## 366 126.41994
## 367 119.38439
## 368 112.72901
## 369 106.14565
## 370 115.27480
## 371 112.79306
## 372 111.38774
## 373 115.34948
## 374 105.88397
## 375 127.93875
## 376 106.13218
## 377 103.12044
## 378 117.84138
## 379 117.41520
## 380 125.36306
## 381 105.82215
## 382 127.51360
## 383 103.99779
## 384 113.93482
## 385 104.04683
## 386 106.93355
## 387 107.05414
## 388 104.54855
## 389 125.37328
## 390 112.21401
## 391 113.13934
## 392 125.71206
## 393 105.71941
## 394 112.40308
## 395 108.61642
## 396 107.48780
## 397 118.09707
## 398 125.35679
## 399 97.45444
## 400 100.10943
## 401 116.58694
```

```
## 402 106.78057
## 403 111.91079
## 404 116.75726
## 405 108.17398
## 406 99.58362
## 407 114.57293
## 408 109.85168
## 409 121.84334
## 410 107.86493
## 411 127.12080
## 412 124.86587
## 413 99.53627
## 414 116.46358
## 415 124.81236
## 416 111.73796
## 417 108.87264
## 418 117.94757
## 419 115.56643
## 420 123.96318
## 421 113.77360
## 422 119.94670
## 423 108.32990
## 424 124.58518
## 425 114.06451
## 426 110.54113
## 427 114.85524
## 428 117.35423
## 429 125.28117
## 430 114.69364
## 431 106.83007
## 432 110.89630
## 433 115.50097
## 434 121.92301
## 435 118.78799
## 436 113.84525
## 437 120.64767
## 438 109.36883
## 439 121.13011
## 440 113.52213
## 441 115.16573
## 442 123.03323
## 443 111.16598
## 444 110.23718
## 445 121.01684
## 446 104.57516
## 447 114.23794
```

```
## 448 116.48334
## 449 112.93738
## 450 116.97262
## 451 123.01939
## 452 103.86612
## 453 108.16585
## 454 117.46619
## 455 102.80921
## 456 111.45025
## 457 113.71313
## 458 115.76154
## 459 107.26893
## 460 122.26012
## 461 136.24026
## 462 123.72361
## 463 110.92298
## 464 100.08531
## 465 112.24392
## 466 110.54597
## 467 111.99873
## 468 112.89430
## 469 112.26102
## 470 117.39683
## 471 117.50764
## 472 106.53525
## 473 105.80527
## 474 115.67630
## 475 100.35041
## 476 113.07986
## 477 114.39667
## 478 118.53729
## 479 125.13422
## 480 116.61993
## 481 113.62256
## 482 117.60229
## 483 121.42464
## 484 123.01585
## 485 110.59016
## 486 118.49153
## 487 116.60030
## 488 114.53784
## 489 126.91723
## 490 96.27709
## 491 103.54786
## 492 105.34090
## 493 113.60563
```

```
## 494 119.49589
## 495 120.85729
## 496 111.34998
## 497 108.19074
## 498 105.44374
## 499 111.48404
## 500 115.11209
## 501 113.31679
## 502 107.93316
## 503 121.78264
## 504 110.72774
## 505 108.02673
## 506 113.40761
## 507 121.72887
## 508 112.27018
## 509 105.09043
## 510 121.76014
## 511 116.73332
## 512 121.76908
## 513 117.22549
## 514 108.76471
## 515 107.87862
## 516 117.75028
## 517 110.29232
## 518 116.54346
## 519 109.25235
## 520 113.03419
## 521 117.64512
## 522 121.47154
## 523 114.08779
## 524 106.18617
## 525 119.07393
## 526 106.86533
## 527 115.46940
## 528 101.70763
## 529 97.54206
## 530 110.64455
## 531 107.06610
## 532 112.42027
## 533 118.73436
## 534 111.71727
## 535 104.05510
## 536 129.54831
## 537 117.09248
## 538 124.54404
## 539 116.33296
```

```
## 540 123.40549
## 541 102.44298
## 542 111.77241
## 543 120.33461
## 544 117.91417
## 545 108.52797
## 546 126.19165
## 547 113.31332
## 548 107.87823
## 549 119.66154
## 550 111.16370
## 551 109.34024
## 552 117.21184
## 553 119.04455
## 554 117.19549
## 555 107.11711
## 556 106.42320
## 557 121.77364
## 558 119.82572
## 559 113.46557
## 560 115.69528
## 561 110.62206
## 562 122.38456
## 563 122.98836
## 564 108.48447
## 565 106.09706
## 566 116.32697
## 567 116.35801
## 568 124.40857
## 569 116.82206
## 570 114.09462
## 571 115.40778
## 572 116.96016
## 573 98.85140
## 574 135.20693
## 575 119.75133
## 576 114.80116
## 577 108.22753
## 578 117.61364
## 579 116.36189
## 580 109.92622
## 581 111.91415
## 582 116.15821
## 583 105.58993
## 584 108.08802
## 585 117.67850
```

```
## 586 111.13633
## 587 132.24823
## 588 110.85715
## 589 87.83593
## 590 125.64538
## 591 109.59319
## 592 101.12824
## 593 113.94740
## 594 124.31554
## 595 118.67357
## 596 111.03314
## 597 121.03873
## 598 110.29637
## 599 112.24814
## 600 119.22314
## 601 124.26302
## 602 112.70908
## 603 97.54202
## 604 112.54098
## 605 117.30295
## 606 113.61166
## 607 126.07466
## 608 108.19994
## 609 117.06018
## 610 117.99884
## 611 124.48122
## 612 120.04676
## 613 120.79039
## 614 113.56916
## 615 106.28474
## 616 121.85101
## 617 121.80984
## 618 107.49041
## 619 110.51965
## 620 122.22094
## 621 112.96608
## 622 107.79417
## 623 109.04927
## 624 100.50307
## 625 117.33123
## 626 125.95204
## 627 122.03779
## 628 116.83302
## 629 110.13387
## 630 118.26938
## 631 123.07836
```



```
## 632 106.96144
## 633 119.32938
## 634 114.60838
## 635 104.26998
## 636 117.78356
## 637 112.10798
## 638 116.92210
## 639 122.20747
## 640 103.41158
## 641 104.35021
## 642 111.00875
## 643 126.15944
## 644 120.43646
## 645 103.26239
## 646 121.87818
## 647 109.79967
## 648 111.64820
## 649 116.67954
## 650 105.66557
## 651 112.75183
## 652 121.22979
## 653 114.24457
## 654 103.54787
## 655 101.95563
## 656 103.88058
## 657 124.59750
## 658 113.34938
## 659 104.30297
## 660 124.46201
## 661 114.08120
## 662 126.73495
## 663 117.66581
## 664 99.67641
## 665 107.33070
## 666 107.93766
## 667 113.07169
## 668 114.49677
## 669 109.61490
## 670 102.14626
## 671 118.50619
## 672 109.63734
## 673 125.07082
## 674 106.13135
## 675 120.89767
## 676 118.49616
## 677 121.94440
```

```
## 678 116.67561
## 679 110.53741
## 680 109.26362
## 681 121.35528
## 682 120.08566
## 683 106.30738
## 684 105.02832
## 685 116.33245
## 686 113.73313
## 687 121.30509
## 688 127.22500
## 689 115.56041
## 690 121.46557
## 691 118.54388
## 692 113.01171
## 693 130.12382
## 694 120.11217
## 695 105.06264
## 696 107.70540
## 697 116.29044
## 698 107.87553
## 699 99.27654
## 700 111.77306
## 701 112.65223
## 702 109.55930
## 703 116.77807
## 704 109.78229
## 705 119.13192
## 706 113.67539
## 707 118.85713
## 708 121.56431
## 709 116.28196
## 710 119.04540
## 711 109.45345
## 712 114.95872
## 713 115.29909
## 714 112.15066
## 715 116.73322
## 716 114.44525
## 717 111.12546
## 718 112.27558
## 719 113.56506
## 720 114.10238
## 721 100.49031
## 722 113.25783
## 723 111.85214
```

```
## 724 116.96490
## 725 108.83318
## 726 114.62116
## 727 106.61273
## 728 109.46670
## 729 123.27669
## 730 120.57396
## 731 103.87767
## 732 106.94421
## 733 108.34143
## 734 116.92814
## 735 110.42256
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## 1623 108.65630
## 1624 113.09310
## 1625 104.06154
## 1626 101.97438
## 1627 116.78661
## 1628 110.29435
## 1629 123.46893
## 1630 116.59034
## 1631 102.98659
## 1632 110.18205
## 1633 109.23006
## 1634 120.82451
## 1635 121.75575
## 1636 112.22303
## 1637 105.56302
## 1638 121.47504
## 1639 110.05367
## 1640 104.12317
## 1641 113.38777
## 1642 111.67334
## 1643 114.42323
```

```
## 1644 106.43060
## 1645 135.61538
## 1646 115.60082
## 1647 117.35365
## 1648 113.11384
## 1649 117.95815
## 1650 112.93363
## 1651 113.47685
## 1652 105.97033
## 1653 114.98154
## 1654 107.74614
## 1655 116.06752
## 1656 122.10031
## 1657 112.99138
## 1658 114.29996
## 1659 106.96841
## 1660 108.60291
## 1661 121.43145
## 1662 92.95222
## 1663 113.02822
## 1664 111.52353
## 1665 105.84010
## 1666 129.57170
## 1667 116.08035
## 1668 117.68832
## 1669 116.61900
## 1670 122.81734
## 1671 115.13204
## 1672 114.64356
## 1673 116.38536
## 1674 109.62207
## 1675 110.92888
## 1676 114.36763
## 1677 105.78363
## 1678 116.03262
## 1679 101.17360
## 1680 118.69656
## 1681 114.24099
## 1682 113.32326
## 1683 103.40502
## 1684 112.39048
## 1685 113.95558
## 1686 115.26227
## 1687 124.42804
## 1688 118.27150
## 1689 123.43397
```

```
## 1690 124.46988
## 1691 110.85936
## 1692 114.07836
## 1693 112.95339
## 1694 120.96305
## 1695 122.99706
## 1696 115.55654
## 1697 115.87079
## 1698 115.29694
## 1699 125.24085
## 1700 128.62481
## 1701 108.18660
## 1702 105.87661
## 1703 104.39514
## 1704 113.98015
## 1705 119.94162
## 1706 120.89582
## 1707 115.58448
## 1708 124.34802
## 1709 119.27451
## 1710 125.94165
## 1711 118.74313
## 1712 118.66193
## 1713 117.66496
## 1714 107.85271
## 1715 116.25929
## 1716 118.91040
## 1717 112.54798
## 1718 108.35888
## 1719 106.78994
## 1720 122.41502
## 1721 112.78309
## 1722 111.49801
## 1723 110.25142
## 1724 100.29204
## 1725 109.43258
## 1726 109.33677
## 1727 108.22743
## 1728 121.09013
## 1729 126.03049
## 1730 114.12910
## 1731 114.46407
## 1732 101.81167
## 1733 109.25527
## 1734 117.17836
## 1735 114.15739
```



```
## 1736 106.89497
## 1737 113.97003
## 1738 109.90265
## 1739 120.60851
## 1740 126.19142
## 1741 121.11972
## 1742 111.88430
## 1743 105.94220
## 1744 113.77476
## 1745 111.42531
## 1746 114.39007
## 1747 115.78462
## 1748 112.93819
## 1749 118.81692
## 1750 118.76391
## 1751 123.01901
## 1752 111.04410
## 1753 118.35484
## 1754 110.54607
## 1755 110.85959
## 1756 105.96548
## 1757 116.78229
## 1758 108.15793
## 1759 110.14765
## 1760 109.63972
## 1761 112.02199
## 1762 114.85539
## 1763 117.21206
## 1764 115.58728
## 1765 99.67584
## 1766 116.18988
## 1767 106.56255
## 1768 110.93185
## 1769 120.20929
## 1770 110.24173
## 1771 115.38537
## 1772 123.69769
## 1773 115.34699
## 1774 111.34985
## 1775 109.82229
## 1776 115.89685
## 1777 118.99048
## 1778 118.77597
## 1779 111.15591
## 1780 116.88276
## 1781 116.84949
```

```
## 1782 107.54415
## 1783 115.28064
## 1784 113.47038
## 1785 110.72918
## 1786 111.94738
## 1787 107.27141
## 1788 115.04275
## 1789 96.72293
## 1790 122.32240
## 1791 104.26958
## 1792 123.25807
## 1793 115.92358
## 1794 117.70162
## 1795 118.16755
## 1796 118.03596
## 1797 120.34519
## 1798 104.31188
## 1799 132.04806
## 1800 117.71137
## 1801 113.05951
## 1802 110.26341
## 1803 127.21428
## 1804 117.25141
## 1805 108.35096
## 1806 110.27506
## 1807 111.23149
## 1808 124.83066
## 1809 123.39050
## 1810 106.58225
## 1811 109.74921
## 1812 109.04106
## 1813 125.43409
## 1814 110.93092
## 1815 111.74767
## 1816 101.40743
## 1817 116.73829
## 1818 102.78626
## 1819 112.74032
## 1820 105.15150
## 1821 106.97115
## 1822 120.82963
## 1823 115.17882
## 1824 118.71154
## 1825 124.19609
## 1826 109.75987
## 1827 120.38832
```

```
## 1828 121.82306
## 1829 106.27523
## 1830 128.54055
## 1831 117.93971
## 1832 106.59459
## 1833 119.75123
## 1834 117.02807
## 1835 117.46441
## 1836 117.25068
## 1837 112.56719
## 1838 108.33113
## 1839 107.22700
## 1840 114.48208
## 1841 110.34761
## 1842 117.18823
## 1843 124.86804
## 1844 115.99743
## 1845 118.54041
## 1846 114.31177
## 1847 122.35911
## 1848 115.61515
## 1849 111.68315
## 1850 119.04893
## 1851 105.15279
## 1852 104.46286
## 1853 108.21831
## 1854 120.25840
## 1855 113.72293
## 1856 116.31275
## 1857 110.21878
## 1858 104.04796
## 1859 116.13271
## 1860 99.73447
## 1861 114.76161
## 1862 123.04099
## 1863 114.48397
## 1864 119.41272
## 1865 114.43066
## 1866 116.57754
## 1867 104.68885
## 1868 102.26670
## 1869 111.31379
## 1870 107.89620
## 1871 107.26937
## 1872 128.56182
## 1873 112.31984
```

```
## 1874 117.48175
## 1875 111.82601
## 1876 121.53766
## 1877 108.59204
## 1878 114.00073
## 1879 109.15453
## 1880 115.40349
## 1881 120.02438
## 1882 120.00529
## 1883 114.15522
## 1884 97.21296
## 1885 118.74600
## 1886 110.07800
## 1887 105.74195
## 1888 109.99513
## 1889 115.58094
## 1890 98.49195
## 1891 119.22469
## 1892 108.36079
## 1893 123.17149
## 1894 122.71776
## 1895 119.61528
## 1896 113.61297
## 1897 104.29065
## 1898 119.35944
## 1899 114.59634
## 1900 114.87640
## 1901 114.83493
## 1902 120.75232
## 1903 116.33686
## 1904 112.85593
## 1905 108.99668
## 1906 119.80091
## 1907 107.51762
## 1908 117.00237
## 1909 125.47799
## 1910 109.23858
## 1911 99.10170
## 1912 113.58951
## 1913 110.50543
## 1914 120.26970
## 1915 112.06393
## 1916 101.04741
## 1917 112.63951
## 1918 113.25368
## 1919 121.02941
```

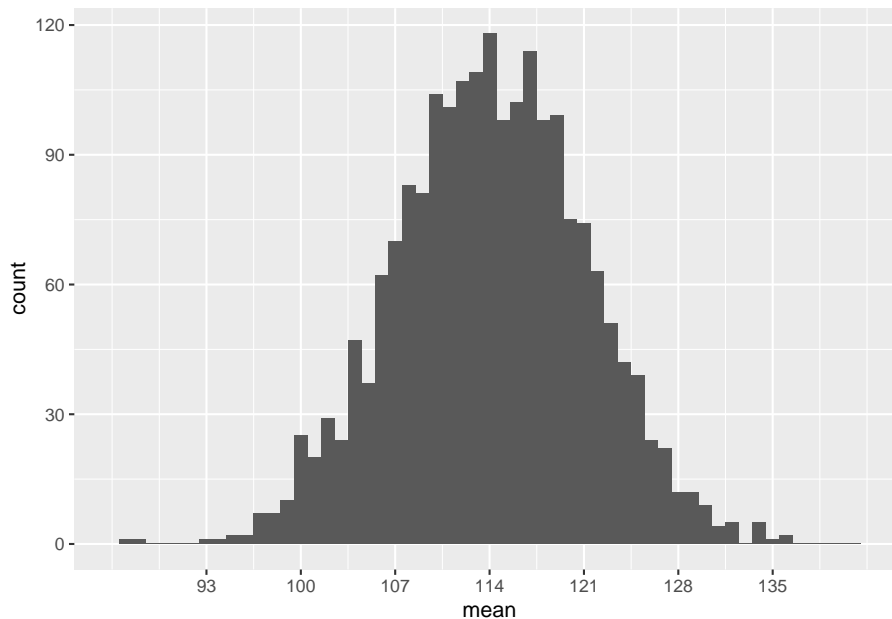
```
## 1920 120.40065
## 1921 102.51873
## 1922 122.20321
## 1923 121.08449
## 1924 119.55367
## 1925 115.73619
## 1926 108.47358
## 1927 113.91919
## 1928 115.65892
## 1929 117.53470
## 1930 113.44030
## 1931 112.06709
## 1932 106.90271
## 1933 113.75108
## 1934 118.57237
## 1935 115.23998
## 1936 108.66065
## 1937 108.24943
## 1938 112.21938
## 1939 124.59338
## 1940 113.36595
## 1941 107.43284
## 1942 115.07636
## 1943 116.41288
## 1944 114.93979
## 1945 112.58356
## 1946 118.89955
## 1947 113.45179
## 1948 109.08609
## 1949 122.58892
## 1950 101.93728
## 1951 106.47563
## 1952 120.28890
## 1953 109.49638
## 1954 104.30374
## 1955 112.77269
## 1956 124.76056
## 1957 118.72269
## 1958 123.78044
## 1959 110.63524
## 1960 109.31897
## 1961 107.73594
## 1962 116.17672
## 1963 105.96558
## 1964 119.74607
## 1965 118.69882
```

```
## 1966 115.85835
## 1967 104.62583
## 1968 113.57872
## 1969 128.22431
## 1970 115.12682
## 1971 114.34633
## 1972 106.33976
## 1973 112.85725
## 1974 109.54481
## 1975 126.89872
## 1976 106.20579
## 1977 114.33387
## 1978 118.06756
## 1979 120.88291
## 1980 112.68291
## 1981 126.43337
## 1982 110.43387
## 1983 114.83281
## 1984 116.18950
## 1985 105.62630
## 1986 122.38782
## 1987 118.93003
## 1988 113.00455
## 1989 121.08291
## 1990 124.71230
## 1991 111.14368
## 1992 111.19670
## 1993 114.69397
## 1994 113.91546
## 1995 111.82721
## 1996 112.65771
## 1997 118.70725
## 1998 109.79392
## 1999 114.41826
## 2000 114.76945
```

Again, we see that the sample means are close to 114, but there is some variability. Naturally, not every sample is going to have an average of exactly 114. So how much variability do we expect? Let's graph and find out. We're going to set the x-axis manually so that we can do some comparisons later.

```
ggplot(sims, aes(x = mean)) +
  geom_histogram(binwidth = 1) +
  scale_x_continuous(limits = c(86, 142),
                     breaks = c(93, 100, 107, 114, 121, 128, 135))
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



Most sample means are around 114, but there is a good range of possibilities from around 93 to 135. The population standard deviation  $\sigma$  is 14, but the standard deviation in this graph is clearly much smaller than that. (A large majority of the samples are within 14 of the mean!)

With some fancy mathematics, one can show that the standard deviation of this sampling distribution is not  $\sigma$ , but rather  $\sigma/\sqrt{n}$ . In other words, this sampling distribution of the mean has a standard error of

$$\frac{\sigma}{\sqrt{n}} = \frac{14}{\sqrt{4}} = 7.$$

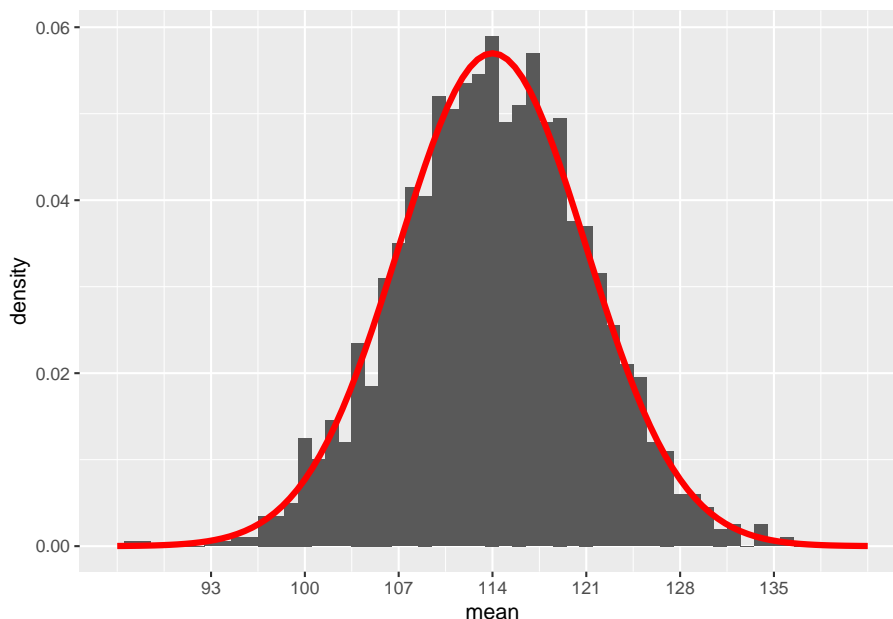
This makes sense: as the sample size increases, we expect the sample mean to be more and more accurate, so the standard error should shrink with large sample sizes.

Let's re-scale the y-axis to use percentages instead of counts. Then we should be able to superimpose the normal model  $N(114, 7)$  to check visually that it's the right fit.

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
ggplot(sims, aes(x = mean)) +
  geom_histogram(aes(y = ..density..), binwidth = 1) +
```

```
scale_x_continuous(limits = c(86, 142),
                   breaks = c(93, 100, 107, 114, 121, 128, 135)) +
stat_function(fun = dnorm, args = list(mean = 114, sd = 7),
             color = "red", size = 1.5)
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



Looks pretty good!

All we do now is convert everything to z scores. In other words, suppose we sample 4 individuals from a population distributed according to the normal model  $N(0, 1)$ . Now the standard error of the sampling distribution is

$$\frac{\sigma}{\sqrt{n}} = \frac{1}{\sqrt{4}} = 0.5.$$

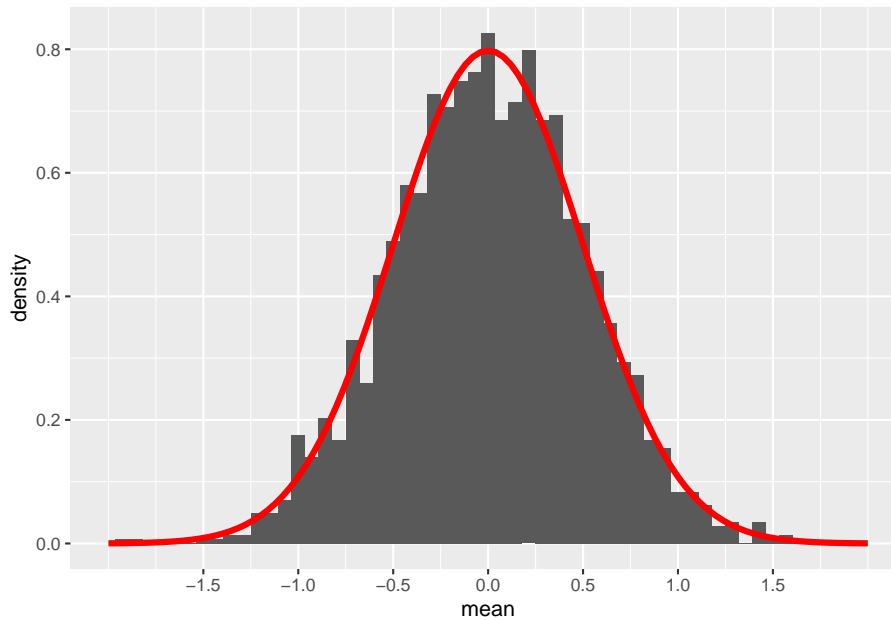
The following code will accomplish all of this. (Don't worry about the messy syntax. All I'm doing here is making sure that this graph looks exactly the same as the previous graph, except now centered at  $\mu = 0$  instead of  $\mu = 114$ .)

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
sims_z <- data.frame(mean = scale(sims$mean, center = 114, scale = 14))
ggplot(sims_z, aes(x = mean)) +
```



```
geom_histogram(aes(y = ..density..), binwidth = 1/14) +
  scale_x_continuous(limits = c(-2, 2),
                    breaks = c(-1.5, -1, -0.5, 0, 0.5, 1, 1.5)) +
  stat_function(fun = dnorm, args = list(mean = 0, sd = 0.5),
              color = "red", size = 1.5)
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



Remember that this is not the standard normal model  $N(0, 1)$ . The standard deviation in the graph above is not 1, but 0.5 because that is the standard error when using samples of size 4. ( $1/\sqrt{4} = 0.5$ .)

## 19.4 Unknown standard errors

If we want to run a hypothesis test, we will have a null hypothesis about the true value of the population mean  $\mu$ . For example,

$$H_0 : \mu = 114$$

Now we gather a sample and compute the sample mean, say 110.2043696. We would like to be able to compare the sample mean  $\bar{y}$  to the hypothesized value 114 using a z score:

$$z = \frac{(\bar{y} - \mu)}{\sigma/\sqrt{n}} = \frac{(110.2 - 114)}{\sigma/\sqrt{4}}.$$

However, we have a problem: we usually don't know the true value of  $\sigma$ . In our SBP example, we do happen to know it's 14, but we won't know this for a general research question.

The best we can do with a sample is calculate this z score replacing the unknown  $\sigma$  with the sample standard deviation  $s$ , 13.0561519. We'll call this a "t score" instead of a "z score":

$$t = \frac{(\bar{y} - \mu)}{s/\sqrt{n}} = \frac{(110.2 - 114)}{13.06/\sqrt{4}} = -0.58.$$

The problem is that  $s$  is not a perfect estimate of  $\sigma$ . We saw earlier that  $s$  is usually close to  $\sigma$ , but  $s$  has its own sampling variability. That means that our earlier simulation in which we assumed that  $\sigma$  was known and equal to 14 was wrong for the type of situation that will arise when we run a hypothesis test. How wrong was it?

## 19.5 Simulating t scores

Let's run the simulation again, but this time with the added uncertainty of using  $s$  to estimate  $\sigma$ .

The first step is to write a little function of our own to compute simulated t scores. This function will take a sample of size  $n$  from the true population  $N(\mu, \sigma)$ , calculate the sample mean and sample standard deviation, then compute the t score. Don't worry: you won't be required to do anything like this on your own.

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
sim_t <- function(n, mu, sigma) {
  sample_values <- rnorm(n, mean = mu, sd = sigma)
  y_bar <- mean(sample_values)
  s <- sd(sample_values)
  t <- (y_bar - mu)/(s / sqrt(n))
}
```

Now we can simulate doing this 2000 times.

```
set.seed(5151977)
sims_t <- do(2000) * sim_t(4, mu = 114, sigma = 14)
sims_t
```

```
##           sim_t
## 1      1.670726734
## 2     -0.975666678
## 3     -0.278839393
## 4      0.907808022
## 5     -1.527274531
## 6     -1.717671837
## 7     -0.610956296
## 8     -0.177107883
## 9     -0.081578742
## 10    -0.150764283
## 11     0.105561464
## 12     0.989851233
## 13     0.754578374
## 14    -0.221752375
## 15    -0.569806798
## 16     1.056144154
## 17     0.709520796
## 18     1.786249608
## 19    -0.022957371
## 20    -0.479076521
## 21     2.196497891
## 22    -0.057126903
## 23     0.723176732
## 24     0.462163070
## 25     2.305842397
## 26    -0.541132956
## 27    -1.155518891
## 28     1.893602331
## 29     3.587253178
## 30    -1.329845154
## 31     1.786070559
## 32     0.205368769
## 33    -0.617185683
## 34     1.408927566
## 35     0.174600728
## 36    -0.585585461
## 37     0.975358819
## 38     0.867186495
## 39    -0.509037457
## 40     0.463270308
```

```
## 41      2.961196587
## 42      0.250917786
## 43     -0.151400364
## 44      2.379911294
## 45      0.965542692
## 46     -1.639114331
## 47     -0.187393864
## 48      0.702999822
## 49     -1.649486008
## 50      0.642256403
## 51     -0.445978914
## 52     -0.870684799
## 53     -0.506327234
## 54      0.515425890
## 55      1.188525622
## 56      1.173749591
## 57     -3.089680034
## 58      1.479209494
## 59     10.039858675
## 60     -1.865677247
## 61      0.208720956
## 62      1.698415163
## 63      0.874459927
## 64     -0.414113539
## 65     -2.079229096
## 66     -0.641514036
## 67     -0.046016401
## 68     -2.051611648
## 69     -1.116638893
## 70     -2.568290582
## 71     -3.634987999
## 72      0.131241299
## 73     -0.317803823
## 74     -1.063949859
## 75      0.004811193
## 76      4.439627383
## 77     -1.364313839
## 78      1.645804106
## 79     -0.201914744
## 80      0.504043393
## 81      1.440774874
## 82     -3.291032994
## 83     -1.551130801
## 84     -0.802710562
## 85     -4.861382113
## 86      1.016265268
```

```
## 87      1.080518333
## 88      2.980709799
## 89      4.326429336
## 90      0.458414619
## 91      2.037994906
## 92     -1.820144738
## 93      1.040068322
## 94      2.555396424
## 95     -0.478768875
## 96     -0.929751963
## 97     -0.508981112
## 98     -0.569059363
## 99     -1.094024179
## 100     0.110893966
## 101    -0.923379631
## 102     0.408635917
## 103    -0.521992962
## 104     2.636311764
## 105     0.636091866
## 106     0.859720275
## 107     1.253116033
## 108     0.874350704
## 109    -0.867757352
## 110    -1.337827858
## 111     0.156515269
## 112    -2.023417372
## 113     0.789119890
## 114     0.664206505
## 115    -5.013338827
## 116     1.080852724
## 117    -0.468189050
## 118    -0.592941304
## 119    -0.224440854
## 120     1.566295593
## 121     0.104289555
## 122    -1.197675728
## 123    -1.007030300
## 124     0.407430926
## 125    -1.942399658
## 126    -3.000766684
## 127     0.061485310
## 128    -1.592080649
## 129     1.051971725
## 130     3.007244391
## 131    -0.926063447
## 132     0.360010372
```

```
## 133 -1.154431763
## 134  0.837885024
## 135 -0.865787271
## 136 -1.185354554
## 137  0.295746913
## 138 -0.396571358
## 139  0.887971205
## 140 -1.027778834
## 141 -1.056473957
## 142 -0.790085592
## 143  2.166777070
## 144  0.009600946
## 145  0.761096684
## 146 -0.445841081
## 147 -0.513983827
## 148  0.831912239
## 149  0.716585444
## 150 -0.341729523
## 151  1.959676409
## 152  0.501861848
## 153  1.419772119
## 154 -1.145028443
## 155  0.404685855
## 156  0.572805957
## 157 -1.261116341
## 158 -1.077860929
## 159 -0.340670950
## 160  3.191331484
## 161 -2.919014184
## 162  1.362479919
## 163  1.326437044
## 164 -0.619316503
## 165 -1.330164481
## 166  0.114571544
## 167  0.275918212
## 168 -1.609972483
## 169  0.746043178
## 170  0.571191844
## 171  1.155595866
## 172  0.134574629
## 173 -1.218916492
## 174 -1.492947751
## 175  1.012713541
## 176 -0.651309215
## 177 -2.690012483
## 178  0.381110576
```

```
## 179 -0.709852732
## 180  1.127924885
## 181  2.690832381
## 182  1.716396925
## 183 -0.697362354
## 184 -0.961945375
## 185  0.746108381
## 186 -1.524226171
## 187 -0.458618707
## 188 -0.055254402
## 189  1.020115666
## 190  0.018051809
## 191  0.979239006
## 192  0.785251827
## 193 -0.178483558
## 194 -1.244265037
## 195  0.744906482
## 196 -0.491305065
## 197 -0.345225608
## 198 -0.857919408
## 199  0.767931118
## 200  0.567650649
## 201  0.285171950
## 202 -0.912431467
## 203 -0.016306668
## 204 -0.018041076
## 205  0.864570995
## 206  1.856671982
## 207  0.481038270
## 208 -1.469329052
## 209  2.623871232
## 210 -0.712124175
## 211  0.392677868
## 212 -0.960771180
## 213  1.503009840
## 214 -1.308729342
## 215 -0.714134598
## 216  0.910092338
## 217  0.687880279
## 218 -0.706690653
## 219  1.039393080
## 220  1.285188816
## 221  2.082287808
## 222  0.065838057
## 223  1.905921689
## 224  1.228140674
```

```
## 225 -0.765591982
## 226  0.605332968
## 227 -0.017615429
## 228 -0.220003147
## 229 -0.921723662
## 230 -1.408301607
## 231  0.307375781
## 232 -0.384728667
## 233 -4.815204952
## 234  0.153630251
## 235 -0.544127519
## 236 -0.012780210
## 237  0.143751438
## 238  1.320877365
## 239 -1.291725993
## 240 -0.482246881
## 241  0.752661778
## 242  0.393190471
## 243  1.179327701
## 244  0.393345460
## 245 -3.793928233
## 246  5.181415482
## 247  0.564651863
## 248 -1.295222322
## 249 -1.416412176
## 250  0.491626455
## 251 -3.145790254
## 252  0.254944191
## 253  2.515832119
## 254  0.820769536
## 255  0.645464631
## 256 -0.270108112
## 257  1.810842034
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## 263  5.407605220
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## 267 -0.146152727
## 268  1.122665692
## 269 -0.757175673
## 270  0.183402023
```



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```



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## 905   -1.728999374
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## 908   -1.556073749
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## 914   -0.277136722
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## 1656 1.359927361
## 1657 -0.495495218
## 1658 -1.240287121
## 1659 -0.082211339
## 1660 -1.188018749
## 1661 -2.223184727
## 1662 0.705587014
## 1663 -0.848632473
## 1664 -2.613258924
## 1665 -0.863908222
## 1666 -2.107749753
## 1667 2.082153516
## 1668 1.496670703
## 1669 0.016416946
## 1670 1.014578005
## 1671 -0.361644011
## 1672 0.247235364
## 1673 1.144823453
## 1674 -0.047697451
## 1675 0.455343948
## 1676 0.994593364
## 1677 -0.822444222
## 1678 -0.244816328
## 1679 -1.082771869
## 1680 0.747409305
## 1681 -0.428650753
## 1682 -0.169425334
## 1683 1.605816199
## 1684 0.449971184
## 1685 0.730435284
## 1686 1.847506343
## 1687 -0.206396757
## 1688 0.380880583
## 1689 0.818313605
## 1690 -0.408848628
## 1691 -0.515786900
## 1692 0.974370595
## 1693 -0.133150873
## 1694 1.398333843
## 1695 -1.361151145
## 1696 0.433309662
```

```
## 1697 -0.946376931
## 1698 -0.670063632
## 1699 1.676048959
## 1700 -0.140611177
## 1701 0.053654636
## 1702 1.259689693
## 1703 -5.174206131
## 1704 0.788702296
## 1705 -1.993087093
## 1706 0.397864475
## 1707 2.134884681
## 1708 -0.710201299
## 1709 2.996060042
## 1710 0.510889890
## 1711 0.054068572
## 1712 0.605433933
## 1713 0.347134535
## 1714 1.103668504
## 1715 1.103076166
## 1716 1.332205225
## 1717 0.423082535
## 1718 -1.625596444
## 1719 -1.554203022
## 1720 0.006527303
## 1721 0.053696296
## 1722 -1.561823405
## 1723 0.207694829
## 1724 1.301721385
## 1725 0.603758316
## 1726 -2.775142964
## 1727 0.063536743
## 1728 2.740397766
## 1729 -2.752915518
## 1730 0.822732164
## 1731 -0.980567935
## 1732 3.973534763
## 1733 -0.740899772
## 1734 1.420636878
## 1735 -1.333517659
## 1736 -0.706797886
## 1737 -0.709147617
## 1738 3.371441854
## 1739 1.005492756
## 1740 -3.541571056
## 1741 -1.439834921
## 1742 3.286784985
```



```
## 1743 -0.122735530
## 1744 -0.437715190
## 1745  3.251385190
## 1746 -0.593354656
## 1747 -1.079917550
## 1748  0.606761232
## 1749 -1.127159142
## 1750  2.358211611
## 1751  0.763686667
## 1752  1.110251032
## 1753 -1.492509083
## 1754 -1.241463822
## 1755  4.439832289
## 1756  2.554971740
## 1757  0.660895643
## 1758  0.123687788
## 1759  1.333725257
## 1760  4.152832797
## 1761  1.217302777
## 1762  1.656895371
## 1763  0.353317077
## 1764 -0.657602012
## 1765 -0.381770876
## 1766  0.187400308
## 1767  1.939343087
## 1768  0.210374661
## 1769 -2.345500420
## 1770 -0.874157596
## 1771  0.540670356
## 1772  0.112802661
## 1773 -2.648256979
## 1774  0.597568786
## 1775 -0.137426550
## 1776  3.516064434
## 1777  0.102408252
## 1778  0.776033821
## 1779  0.930709076
## 1780  5.220574704
## 1781  0.736020501
## 1782 -0.990894962
## 1783 -0.274559644
## 1784 -1.016884505
## 1785 -0.221887192
## 1786 -0.445992770
## 1787  0.475688115
## 1788  0.785786694
```

```
## 1789 -0.130032635
## 1790 -0.394688042
## 1791 -2.323386527
## 1792  0.514375139
## 1793  1.492241939
## 1794  0.327791984
## 1795 -0.075720368
## 1796  0.514881334
## 1797 -1.119208961
## 1798 -0.180152878
## 1799  0.637308878
## 1800  3.964044307
## 1801 -0.144384160
## 1802  1.487932212
## 1803 -0.566635527
## 1804 -1.139370142
## 1805 -3.086612508
## 1806  0.862030400
## 1807  0.474449333
## 1808  0.961474292
## 1809 -0.538548656
## 1810  0.017726335
## 1811 -1.138437401
## 1812  0.121364311
## 1813 -0.978068660
## 1814  0.283660468
## 1815  0.242053638
## 1816 -0.117018330
## 1817  0.540950519
## 1818  1.580644887
## 1819  1.028931010
## 1820  1.015550193
## 1821  1.196996138
## 1822  0.230669296
## 1823  0.031274355
## 1824 -0.707303604
## 1825 -1.142676757
## 1826  1.804785405
## 1827  0.112926949
## 1828  0.477232896
## 1829 -0.476903681
## 1830 -0.692818107
## 1831  1.332466553
## 1832  2.318784256
## 1833  1.184052989
## 1834  1.141068630
```

```
## 1835    0.167916703
## 1836   -1.116243275
## 1837   -0.045689694
## 1838    0.596004263
## 1839   -0.748392267
## 1840   -0.060920315
## 1841   -1.444313228
## 1842   -0.044715427
## 1843   -0.056960004
## 1844    1.151901771
## 1845   -0.174865186
## 1846    0.545593634
## 1847   -0.692471122
## 1848   -0.734818390
## 1849    1.457787809
## 1850    0.875233226
## 1851    0.391506603
## 1852    1.740417860
## 1853   -0.388065238
## 1854   -0.877747675
## 1855    0.284135482
## 1856    0.111826285
## 1857    0.815318224
## 1858   -0.140032745
## 1859   -1.361405539
## 1860   -0.758963912
## 1861    0.360491065
## 1862   -0.205572385
## 1863   -0.363727621
## 1864    1.604171479
## 1865   -0.120997962
## 1866   -0.766683547
## 1867    0.468191113
## 1868   -1.837601301
## 1869    1.415300784
## 1870   -1.098654854
## 1871    0.035359762
## 1872    0.156320433
## 1873    1.539551984
## 1874    0.266961864
## 1875    1.352917387
## 1876   -0.404440536
## 1877    1.808759952
## 1878   -1.881284209
## 1879   -0.549492991
## 1880    2.526688917
```

```
## 1881 0.228924017
## 1882 0.513811303
## 1883 1.017006255
## 1884 -0.742499144
## 1885 -0.140586012
## 1886 -0.053718530
## 1887 0.803828055
## 1888 0.048360449
## 1889 -0.215828947
## 1890 -0.058291264
## 1891 0.864983841
## 1892 -1.356170107
## 1893 -0.617262864
## 1894 -1.402265309
## 1895 -0.523441459
## 1896 0.830853039
## 1897 0.317281478
## 1898 0.084830762
## 1899 2.121363127
## 1900 0.121462979
## 1901 0.834729191
## 1902 0.040652843
## 1903 0.722788277
## 1904 -0.747640271
## 1905 0.387297140
## 1906 -0.812770956
## 1907 1.454741416
## 1908 0.620606741
## 1909 0.833451137
## 1910 -1.683346033
## 1911 0.804701034
## 1912 -0.229263120
## 1913 0.046194911
## 1914 -0.166435185
## 1915 -1.112652661
## 1916 -1.073200728
## 1917 -0.046565310
## 1918 5.696117906
## 1919 -0.290236383
## 1920 1.207304711
## 1921 -0.762685782
## 1922 -1.497926637
## 1923 -0.822479149
## 1924 1.052504492
## 1925 1.198638323
## 1926 -0.126984105
```

```
## 1927 -2.196066627
## 1928 2.821882676
## 1929 0.888531536
## 1930 1.030936408
## 1931 0.557465035
## 1932 0.289026047
## 1933 -0.709589288
## 1934 1.018615649
## 1935 1.014718518
## 1936 0.118878497
## 1937 -2.353307515
## 1938 -0.463709158
## 1939 -3.089588325
## 1940 -2.124134818
## 1941 -3.397232314
## 1942 0.910430585
## 1943 -1.056790935
## 1944 -0.262030547
## 1945 0.607755870
## 1946 0.403856235
## 1947 1.412269952
## 1948 0.422769816
## 1949 -0.005290671
## 1950 -0.361825063
## 1951 2.228995584
## 1952 -0.093855139
## 1953 0.088769126
## 1954 2.985708776
## 1955 -1.616981175
## 1956 -0.814294262
## 1957 -0.579969780
## 1958 -0.532228413
## 1959 0.475891817
## 1960 -0.028348796
## 1961 -0.097690038
## 1962 -1.338162601
## 1963 -1.294586067
## 1964 0.687677162
## 1965 -0.201650989
## 1966 -0.658662267
## 1967 -0.364505858
## 1968 -0.822221317
## 1969 3.268173150
## 1970 -4.967636498
## 1971 -0.584376271
## 1972 -1.161526012
```

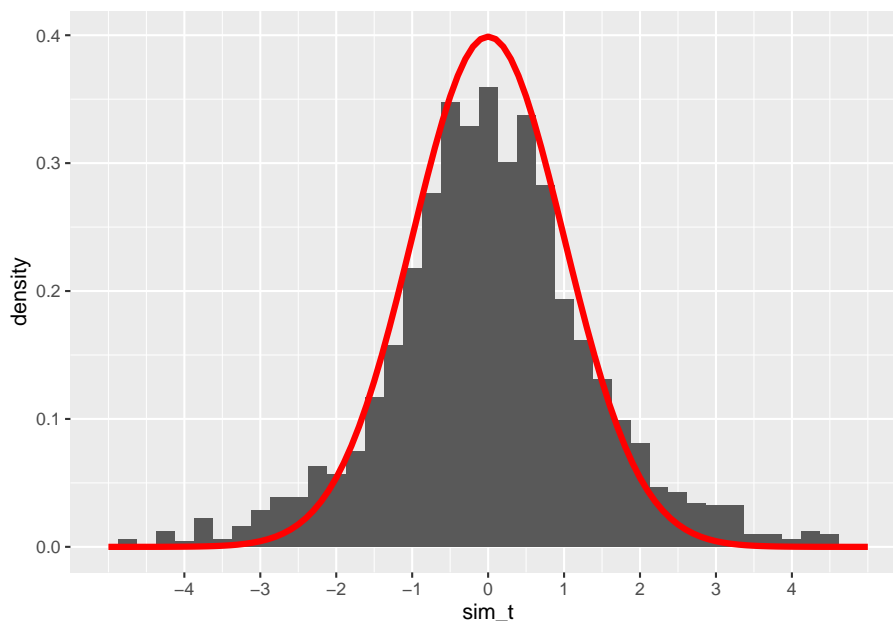
```
## 1973 -0.244878422
## 1974 3.032321344
## 1975 -1.812160139
## 1976 -1.261326720
## 1977 -2.309825696
## 1978 0.131785814
## 1979 -0.512137299
## 1980 -2.212688313
## 1981 -0.833872274
## 1982 0.185610652
## 1983 -0.141494928
## 1984 0.109487405
## 1985 0.089989645
## 1986 0.668121661
## 1987 -0.430441702
## 1988 0.792453656
## 1989 -1.400129839
## 1990 -0.215107105
## 1991 -0.085294745
## 1992 0.437635054
## 1993 1.414558604
## 1994 -1.470842044
## 1995 0.204152049
## 1996 -0.603812902
## 1997 0.788499060
## 1998 0.489937346
## 1999 -1.605398619
## 2000 0.409543307
```

Let's plot our simulated t scores alongside a normal distribution.

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
ggplot(sims_t, aes(x = sim_t)) +
  geom_histogram(aes(y = ..density..), binwidth = 0.25) +
  scale_x_continuous(limits = c(-5, 5),
                    breaks = c(-4, -3, -2, -1, 0, 1, 2, 3, 4)) +
  stat_function(fun = dnorm, args = list(mean = 0, sd = 1),
              color = "red", size = 1.5)
```

```
## Warning: Removed 19 rows containing non-finite values (stat_bin).
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```



These t scores are somewhat close to the normal model we had when we knew  $\sigma$ , but the fit doesn't look quite right. The peak of the simulated values isn't quite high enough, and the tails seem to spill out over the much thinner tails of the normal model.

William Gosset figured this all out in the early 20th century. While working for the Guinness brewery in Dublin, Ireland, he started noticing that his quality control tests (using very small sample sizes) didn't yield statistical results consistent with the normal models that were universally used at the time. At the encouragement of the company, which saw his work as a potential source of cost savings, he took some time off to study and consult with other statisticians. As a result, he found a new function that is similar to a normal distribution but is more spread out. This new function accounts for the extra variability one gets when using the sample standard deviation  $s$  as an estimate for the true population standard deviation  $\sigma$ . Guinness considered the result a "trade secret", so they wouldn't allow Gosset to publish under his own name. But they did permit him to publish his findings under the pseudonym "Student". He used data sets unrelated to brewing and submitted his work to the top statistical journal of the time.

The new function Gosset discovered became known as the *Student t distribution*. He realized that the spread of the t distribution depends on the sample size. This makes sense: the accuracy of  $s$  will be greater when we have a larger sample. In fact, for large enough samples, the t distribution is very close to a normal model.

Gosset used the term *degrees of freedom* to describe how the sample size influ-

ences the spread of the t distribution. It's somewhat mathematical and technical, so suffice it to say here that the number of degrees of freedom is simply the sample size minus 1:

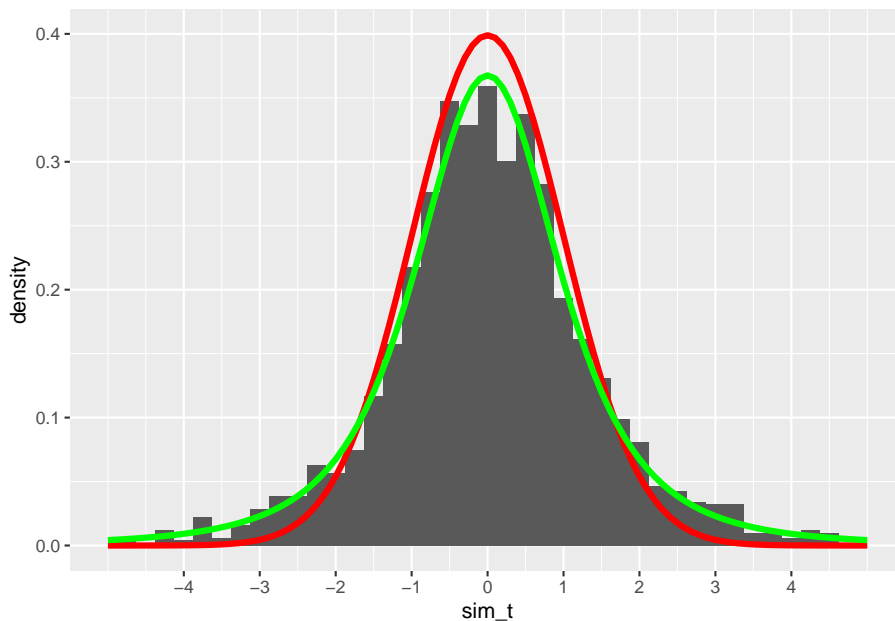
$$df = n - 1.$$

So is the t model correct for our simulated t scores? Our sample size was 4, so we should use a t model with 3 degrees of freedom. Let's plot it in green on top of our previous graph and see:

```
# Don't worry about the syntax here.
# You won't need to know how to do this on your own.
ggplot(sims_t, aes(x = sim_t)) +
  geom_histogram(aes(y = ..density..), binwidth = 0.25) +
  scale_x_continuous(limits = c(-5, 5),
                    breaks = c(-4, -3, -2, -1, 0, 1, 2, 3, 4)) +
  stat_function(fun = dnorm, args = list(mean = 0, sd = 1),
              color = "red", size = 1.5) +
  stat_function(fun = dt, args = list(df = 3),
              color = "green", size = 1.5)
```

```
## Warning: Removed 19 rows containing non-finite values (stat_bin).
```

```
## Warning: Removed 2 rows containing missing values (geom_bar).
```





The green curve fits the simulated values much better.

## 19.6 Inference for one mean

When we have a single numerical variable, we can ask if the sample mean is consistent or not with a null hypothesis. We will use a  $t$  model for our sampling distribution model as long as certain conditions are met.

One of the assumptions we made in the simulation above was that the true population was normally distributed. In general, we have no way of knowing if this is true. So instead we check the *nearly normal* condition: if a histogram or QQ plot of our data shows that the data is nearly normal, then there is a reasonable assumption that the whole population is shaped the same way.

If our sample size is large enough, the central limit theorem tells us that the sampling distribution gets closer and closer to a normal model. Therefore, we'll use a rule of thumb that says that if the sample size is greater than 30, we won't worry too much about any deviations from normality in the data.

The number 30 is somewhat arbitrary. If the sample size is 25 and a histogram shows only a little skewness, we're probably okay. But if the sample size is 10, we need for the data to be very normal to justify using the  $t$  model. The irony, of course, is that small sample sizes are the hardest to check for normality. We'll have to use our best judgment.

## 19.7 Outliers

We also need to be on the lookout for outliers. We've seen before that outliers can have a huge effect on means and standard deviations, especially when sample sizes are small. Whenever we find an outlier, we need to investigate.

Some outliers are mistakes. Perhaps someone entered data incorrectly into the computer. When it's clear that outliers are data entry errors, we are free to either correct them (if we know what error was made) or delete them from our data completely.

Some outliers are not necessarily mistakes, but should be excluded for other reasons. For example, if we are studying the weight of birds and we have sampled a bunch of hummingbirds and one emu, the emu's weight will appear as an outlier. It's not that its weight is "wrong", but it clearly doesn't belong in the analysis.

In general, though, outliers are real data that just happen to be unusual. It's not ethical simply to throw away such data points because they are inconvenient. (We only do so in very narrow and well-justified circumstances like the emu.) The best policy to follow when faced with such outliers is to run inference

twice—once with the outlier included, and once with the outlier excluded. If, when running a hypothesis test, the conclusion is the same either way, then the outlier wasn't all that influential, so we leave it in. If, when computing a confidence interval, the endpoints don't change a lot either way, then we leave the outlier in. However, when conclusions or intervals are dramatically different depending on whether the outlier was in or out, then we have no choice but to state that honestly.

## 19.8 Research question

The `teacher` data from the `openintro` package contains information on 71 teachers employed by the St. Louis Public School in Michigan. According to Google, the average teacher salary in Michigan was \$63,024 in 2010. So does this data suggest that the teachers in the St. Louis region of Michigan are paid differently than teachers in other parts of Michigan?

Let's walk through the rubric.

## 19.9 Exploratory data analysis

### 19.9.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

You should type `?teacher` at the Console to read the help file. Unfortunately, the help file does not give us a lot of information about how the data was collected. The only source listed is a website that no longer contains this data set. Besides, that website is just an open repository for data, so it's not clear that the site would have contained any additional information about the provenance of the data. We will have to assume that the data was collected accurately.

Here is the data set:

```
teacher
```

```
## # A tibble: 71 x 8
##   id   degree fte   years base  fica retirement total
## * <fct> <fct> <fct> <dbl> <int> <dbl>      <dbl> <dbl>
## 1 01    BA     1     5   45388 3472.    7689. 56549.
## 2 02    MA     1    15   60649 4640.   10274. 75563.
## 3 03    MA     1    16   60649 4640.   10274. 75563.
## 4 04    BA     1    10   54466 4167.    9227. 67859.
## 5 05    BA     1    26   65360 5000.   11072. 81432.
```

```
## 6 06 BA 1 28.5 65360 5000. 11072. 81432.
## 7 07 BA 1 12 58097 4444. 9842. 72383.
## 8 08 MA 1 32 68230 5220. 11558. 85008.
## 9 09 BA 1 25 65360 5000. 11072. 81432.
## 10 11 BA 1 12 58097 4444. 9842. 72383.
## # ... with 61 more rows
```

```
glimpse(teacher)
```

```
## Rows: 71
## Columns: 8
## $ id      <fct> 01, 02, 03, 04, 05, 06, 07, 08, 09, 11, 12, 13, 14, 15, 16, ~
## $ degree  <fct> BA, MA, MA, BA, BA, BA, BA, MA, BA, BA, BA, BA, BA, BA, MA, ~
## $ fte     <fct> 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ~
## $ years   <dbl> 5.0, 15.0, 16.0, 10.0, 26.0, 28.5, 12.0, 32.0, 25.0, 12.0, ~
## $ base    <int> 45388, 60649, 60649, 54466, 65360, 65360, 58097, 68230, 653~
## $ fica    <dbl> 3472.18, 4639.65, 4639.65, 4166.65, 5000.04, 5000.04, 4444.~
## $ retirement <dbl> 7688.73, 10273.94, 10273.94, 9226.54, 11071.98, 11071.98, 9~
## $ total    <dbl> 56548.91, 75562.59, 75562.59, 67859.19, 81432.02, 81432.02, ~
```

Since `total` is a numerical variable, we can use the `summary` function to produce the five-number summary. (The function also reports the mean.)

```
summary(teacher$total)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  24793   63758   74647   70289   81432   85008
```

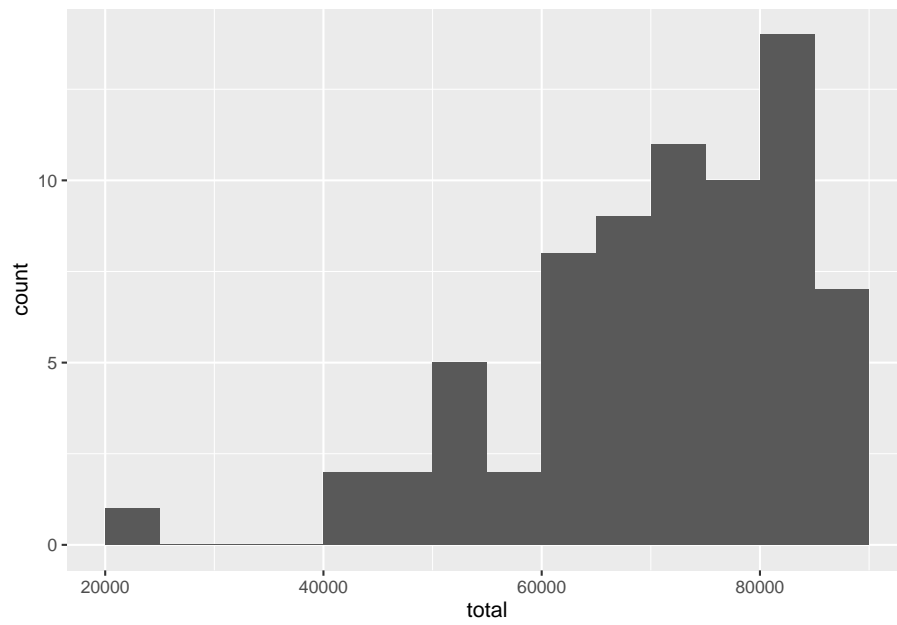
### 19.9.2 Prepare the data for analysis.

Not necessary here, but see the next section to find out what we do when we discover an outlier.

### 19.9.3 Make tables or plots to explore the data visually.

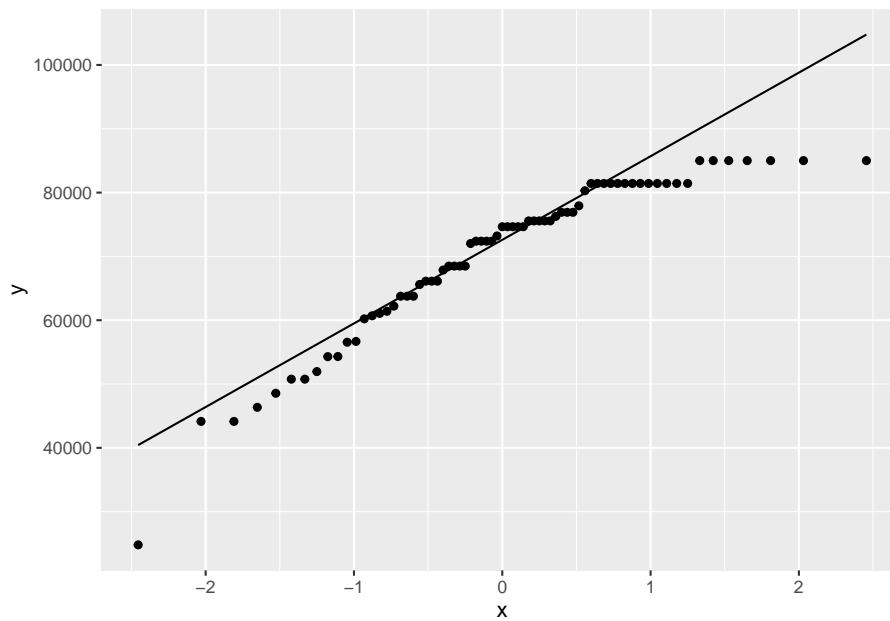
Here is a histogram.

```
ggplot(teacher, aes(x = total)) +
  geom_histogram(binwidth = 5000, boundary = 60000)
```



And here is a QQ plot.

```
ggplot(teacher, aes(sample = total)) +  
  geom_qq() +  
  geom_qq_line()
```



This distribution is quite skewed to the left. Of even more concern is the extreme outlier on the left.

With any outlier, we need to investigate.

**Exercise 1** Let's sort the data by `total` (ascending) using the `arrange` command.

```
teacher %>%
  arrange(total)
```

```
## # A tibble: 71 x 8
##   id  degree fte  years base  fica retirement total
##   <fct> <fct> <fct> <dbl> <int> <dbl>      <dbl> <dbl>
## 1 37    MA    0.5     1 19900 1522.    3371. 24793.
## 2 12    BA     1     0 35427 2710.    6001. 44138.
## 3 57    BA     1     0 35427 2710.    6001. 44138.
## 4 41    BA     1     1 37199 2846.    6302. 46346.
## 5 69    BA     1     2 38968 2981.    6601. 48550.
## 6 48    BA     1     3 40739 3117.    6901. 50757.
## 7 54    BA     1     3 40739 3117.    6901. 50757.
## 8 38    MA     1     2 41695 3190.    7063. 51948.
## 9 15    BA     1     4 43575 3333.    7382. 54290.
## 10 39    MA     1     3 43593 3335.    7385. 54313.
## # ... with 61 more rows
```

Can you figure out why the person with the lowest total salary is different from all the other teachers?

Please write up your answer here.

---

Based on your answer to the above exercise, hopefully it's clear that this is an outlier for which we can easily justify exclusion. We can use the `filter` command to get only the rows we want. There are lots of ways to do this, but it's easy enough to grab only salaries above \$30,000. (There's only one salary below \$30,000, so that outlier will be excluded.)

**CAUTION:** If you are copying and pasting from this example to use for another research question, the following code chunk is specific to this research question and not applicable in other contexts.

```
teacher2 <- teacher %>%  
  filter(total > 30000)
```

Check to make sure this had the desired effect:

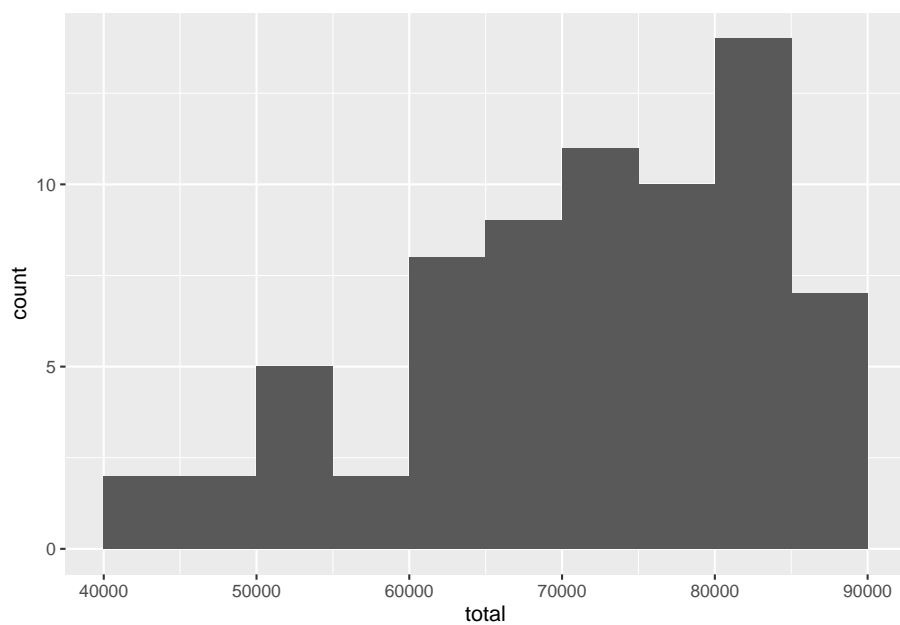
```
summary(teacher2$total)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.  
##  44139   63758   74647   70939   81432   85008
```

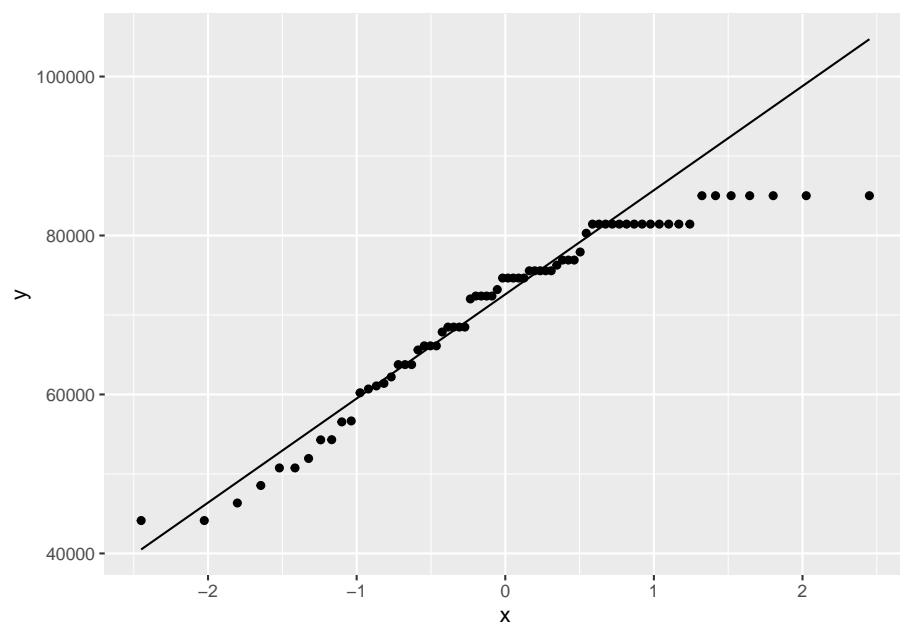
Notice how the min is no longer \$24,793.41.

Here are the new plots:

```
ggplot(teacher2, aes(x = total)) +  
  geom_histogram(binwidth = 5000, boundary = 60000)
```



```
ggplot(teacher2, aes(sample = total)) +  
  geom_qq() +  
  geom_qq_line()
```



The left skew is still present, but we have removed the outlier.

## 19.10 Hypotheses

### 19.10.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

The sample consists of 70 teachers employed by the St. Louis Public School in Michigan. We are using these 70 teachers as a hopefully representative sample of all teachers in that region of Michigan.

### 19.10.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

$H_0$  : Teachers in the St. Louis region earn \$63,024 on average. (In other words, these teachers are the same as the teachers anywhere else in Michigan.)

$H_A$  : Teachers in the St. Louis region do not earn \$63,024 on average. (In other words, these teachers are *not* the same as the teachers anywhere else in Michigan.)

### 19.10.3 Express the null and alternative hypotheses in symbols (when possible).

$$H_0 : \mu = 63024$$

$$H_A : \mu \neq 63024$$

## 19.11 Model

### 19.11.1 Identify the sampling distribution model.

We will use a  $t$  model with 69 degrees of freedom.

Commentary: The original `teacher` data had 71 observations. The `teacher2` data has only 70 observations because we removed an outlier. Therefore  $n = 70$  and thus  $df = n - 1 = 69$ .



### 19.11.2 Check the relevant conditions to ensure that model assumptions are met.

- Random
  - We know this isn't a random sample. We're not sure if this school is representative of other schools in the region, so we'll proceed with caution.
- 10%
  - This is also suspect, as it's not clear that there are 700 teachers in the region. One way to look at it is this: if there are 10 or more schools in the region, and all the school are about the size of the St. Louis Public School under consideration, then we should be okay.
- Nearly Normal
  - For this, we note that the sample size is much larger than 30, so we should be okay, even with the skewness in the data.

## 19.12 Mechanics

### 19.12.1 Compute the test statistic.

```
total_mean <- teacher2 %>%
  specify(response = total) %>%
  calculate(stat = "mean")
total_mean
```

```
## Response: total (numeric)
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1 70939.
```

```
total_t <- teacher2 %>%
  specify(response = total) %>%
  hypothesize(null = "point", mu = 63024) %>%
  calculate(stat = "t")
total_t
```

```
## Response: total (numeric)
## Null Hypothesis: point
```

```
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1  5.89
```

### 19.12.2 Report the test statistic in context (when possible).

The sample mean is \$70938.5725714.

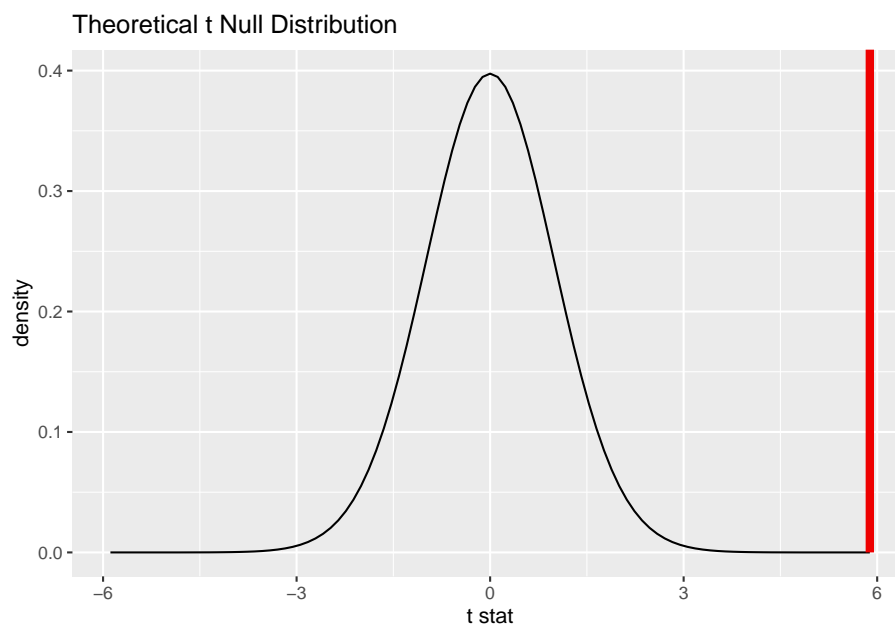
The t score is 5.886253. The mean teacher salary in our sample is almost 6 standard errors to the right of the null value.

### 19.12.3 Plot the null distribution.

```
total_test <- teacher2 %>%
  specify(response = total) %>%
  assume("t")
total_test
```

```
## A T distribution with 69 degrees of freedom.
```

```
total_test %>%
  visualize() +
  shade_p_value(obs_stat = total_t, direction = "two-sided")
```



Commentary: Although we are conducting a two-sided test, the area in the tails is so small that it can't really be seen in the picture above.

#### 19.12.4 Calculate the P-value.

```
total_test_p <- total_test %>%
  get_p_value(obs_stat = total_t, direction = "two-sided")
total_test_p
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1 0.000000129
```

#### 19.12.5 Interpret the P-value as a probability given the null.

$P < 0.001$ . If teachers in the St. Louis region truly earned \$63,024 on average, there would be only a 0.0000129% chance of seeing data at least as extreme as what we saw.

Commentary: When the P-value is this small, remember that it is traditional to report simply  $P < 0.001$ .

## 19.13 Conclusion

### 19.13.1 State the statistical conclusion.

We reject the null hypothesis.

### 19.13.2 State (but do not overstate) a contextually meaningful conclusion.

There is sufficient evidence that teachers in the St. Louis region do not earn \$63,024 on average.

### 19.13.3 Express reservations or uncertainty about the generalizability of the conclusion.

Because we do not know how this data was collected (was it every teacher in this region? was it a sample of some of the teachers? was it a representative sample?), we do not know if we can generalize it to all teachers in the region. Also, the data set was from 2010, so we know that this data cannot be applied to teachers in St. Louis, Michigan now.

### 19.13.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.

If we've made a Type I error, then the truth is that teachers in this region do make around \$63,024 on average, but our sample was way off.

## 19.14 Confidence interval

### 19.14.1 Check the relevant conditions to ensure that model assumptions are met.

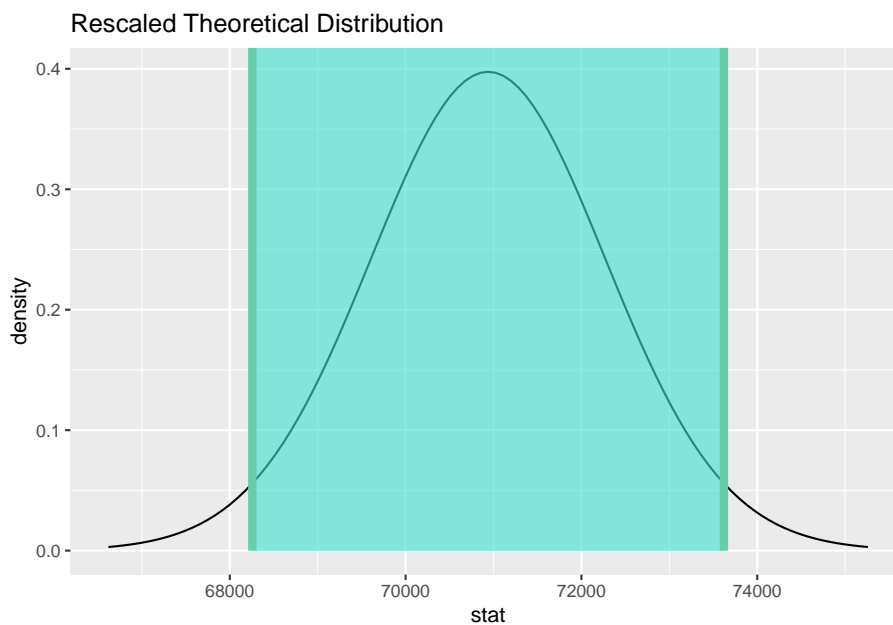
All the conditions have been checked already.

### 19.14.2 Calculate and graph the confidence interval.

```
total_ci <- total_test %>%
  get_confidence_interval(point_estimate = total_mean, level = 0.95)
total_ci
```

```
## # A tibble: 1 x 2
##   lower_ci upper_ci
##   <dbl>    <dbl>
## 1  68256.   73621.
```

```
total_test %>%
  visualize() +
  shade_confidence_interval(endpoints = total_ci)
```



### 19.14.3 State (but do not overstate) a contextually meaningful interpretation.

We are 95% confident that the true mean salary for teachers in the St. Louis region is captured in the interval (68256.2, 73620.95).

Commentary: As these are dollar amounts, it makes sense to round them to two decimal places. Even then, R is finicky and sometimes it will not respect your wishes.)

**19.14.4 If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test.**

Since \$63,024 is not contained in the confidence interval, it is not a plausible value for the mean teacher salary in the St Louis region of Michigan.

**19.14.5 When comparing two groups, comment on the effect size and the practical significance of the result.**

We are not comparing two groups.

## 19.15 Your turn

In the High School and Beyond survey (the `hsb2` data set from the `openintro` package), among the many scores that are recorded are standardized math scores. Suppose that these scores are normalized so that a score of 50 represents some kind of international average. (This is not really true. I had to make something up here to give you a baseline number with which to work.) The question is, then, are American students different from this international baseline?

The rubric outline is reproduced below. You may refer to the worked example above and modify it accordingly. Remember to strip out all the commentary. That is just exposition for your benefit in understanding the steps, but is not meant to form part of the formal inference process.

Another word of warning: the copy/paste process is not a substitute for your brain. You will often need to modify more than just the names of the data frames and variables to adapt the worked examples to your own work. Do not blindly copy and paste code without understanding what it does. And you should **never** copy and paste text. All the sentences and paragraphs you write are expressions of your own analysis. They must reflect your own understanding of the inferential process.

**Also, so that your answers here don't mess up the code chunks above, use new variable names everywhere.**

### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure. Please write up your answer here

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

Identify the sample (or samples) and a reasonable population (or populations) of interest. Please write up your answer here.

Express the null and alternative hypotheses as contextually meaningful full sentences.  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

Express the null and alternative hypotheses in symbols (when possible).  $H_0$  : *math*

$H_A$  : *math*

### Model

Identify the sampling distribution model. Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

### Mechanics

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
# IF CONDUCTING A SIMULATION...  
set.seed(1)  
# Add code here to simulate the null distribution.
```

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

### Conclusion

**State the statistical conclusion.** Please write up your answer here. {-}



**State (but do not overstate) a contextually meaningful conclusion.** Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.** Please write up your answer here.

### Confidence interval

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

```
# Add code here to calculate the confidence interval.
```

```
# Add code here to graph the confidence interval.
```

**Calculate and graph the confidence interval.**

**State (but do not overstate) a contextually meaningful interpretation.** Please write up your answer here.

**If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test. [Not always applicable.]** Please write up your answer here.

**When comparing two groups, comment on the effect size and the practical significance of the result. [Not always applicable.]** Please write up your answer here.

## 19.16 Additional exercises

After running inference above, answer the following questions:

**Exercise 2** Even though the result was *statistically* significant, do you think the result is *practically* significant? By this, I mean, are scores for American students so vastly different than 50? Do we have a lot of reason to brag about American scores based on your analysis?

Please write up your answer here.

**Exercise 3** What makes it possible for a small effect like this to be statistically significant even if it's not practically very different from 50? In other words, what has to be true of data to detect small but statistically significant effects?

Please write up your answer here.

## 19.17 Conclusion

When working with numerical data, we have to estimate a mean and a standard deviation. The extra variability in estimating both gives rise to a sampling distribution model with thicker tails called the Student t distribution. Using this distribution gives us a way to calculate P-values and confidence intervals that take this variation into account.

### 19.17.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.

## Chapter 20

# Inference for paired data

2.0

### Functions introduced in this chapter

No new R functions are introduced here.

## 20.1 Introduction

In this chapter we will learn how to run inference for two paired numerical variables.

### 20.1.1 Install new packages

There are no new packages used in this chapter.

### 20.1.2 Download the R notebook file

Check the upper-right corner in RStudio to make sure you're in your `intro_stats` project. Then click on the following link to download this chapter as an R notebook file (`.Rmd`).

[https://vectorposse.github.io/intro\\_stats/chapter\\_downloads/18-inference\\_for\\_paired\\_data.Rmd](https://vectorposse.github.io/intro_stats/chapter_downloads/18-inference_for_paired_data.Rmd)

Once the file is downloaded, move it to your project folder in RStudio and open it there.

### 20.1.3 Restart R and run all chunks

In RStudio, select “Restart R and Run All Chunks” from the “Run” menu.

## 20.2 Load packages

We load the standard `tidyverse` and `infer` packages. The `openintro` package will give access to the `textbooks` data and the `hsb2` data.

```
library(tidyverse)
library(infer)
library(openintro)
```

## 20.3 Paired data

Sometimes data sets have two numerical variables that are related to each other. For example, a diet study might include a pre-weight and a post-weight. The research question is not about either of these variables directly, but rather the difference between the variables, for example how much weight was lost during the diet.

When this is the case, we run inference for paired data. The procedure involves calculating a new variable `d` that represents the difference of the two paired variables. The null hypothesis is almost always that there is no difference between the paired variables, and that translates into the statement that the average value of `d` is zero.

## 20.4 Research question

The `textbooks` data frame (from the `openintro` package) has data on the price of books at the UCLA bookstore versus Amazon.com. The question of interest here is whether the campus bookstore charges more than Amazon.

## 20.5 Inference for paired data

The key idea is that we don’t actually care about the book prices themselves. All we care about is if there is a difference between the prices for each book. These are not two independent variables because each row represents a single book. Therefore, the two measurements are “paired” and should be treated

as a single numerical variable of interest, representing the difference between `ucla_new` and `amaz_new`.

Since we're only interested in analyzing the one numerical variable `d`, this process is nothing more than a one-sample *t* test. Therefore, there is really nothing new in this chapter.

Let's go through the rubric.

## 20.6 Exploratory data analysis

### 20.6.1 Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

You should type `textbooks` at the Console to read the help file. The data was collected by a person, David Diez. A quick Google search reveals that he is a statistician who graduated from UCLA. We presume he had access to accurate information about the prices of books at the UCLA bookstore and from Amazon.com at the time the data was collected.

Here is the data set:

```
textbooks
```

```
## # A tibble: 73 x 7
##   dept_abbr course isbn      ucla_new amaz_new more  diff
##   <fct>      <fct> <fct>      <dbl>      <dbl> <fct> <dbl>
## 1 Am Ind    " C170" 978-0803272620 27.7      28.0 Y   -0.28
## 2 Anthro    "9"     978-0030119194 40.6      31.1 Y    9.45
## 3 Anthro    "135T"  978-0300080643 31.7      32    Y   -0.32
## 4 Anthro    "191HB" 978-0226206813 16        11.5 Y    4.48
## 5 Art His   "M102K" 978-0892365999 19.0      14.2 Y    4.74
## 6 Art His   "118E"  978-0394723693 15.0      10.2 Y    4.78
## 7 Asia Am   "187B"  978-0822338437 24.7      20.1 Y    4.64
## 8 Asia Am   "191E"  978-0816646135 19.5      16.7 N    2.84
## 9 Ch Engr   "C125"  978-0195123401 124.      106.  N   17.6
## 10 Chicano  "M145B" 978-0896086265 17        13.3 Y    3.74
## # ... with 63 more rows
```

```
glimpse(textbooks)
```

```
## Rows: 73
## Columns: 7
```

```
## $ dept_abbr <fct> Am Ind, Anthro, Anthro, Anthro, Art His, Art His, Asia Am, A~
## $ course <fct> C170, 9, 135T, 191HB, M102K, 118E, 187B, 191E, C125, M145B,~
## $ isbn <fct> 978-0803272620, 978-0030119194, 978-0300080643, 978-02262068~
## $ ucla_new <dbl> 27.67, 40.59, 31.68, 16.00, 18.95, 14.95, 24.70, 19.50, 123.~
## $ amaz_new <dbl> 27.95, 31.14, 32.00, 11.52, 14.21, 10.17, 20.06, 16.66, 106.~
## $ more <fct> Y, Y, Y, Y, Y, Y, Y, N, N, Y, Y, N, Y, Y, N, N, N, N, N, ~
## $ diff <dbl> -0.28, 9.45, -0.32, 4.48, 4.74, 4.78, 4.64, 2.84, 17.59, 3.7~
```

The two paired variables are `ucla_new` and `amaz_new`.

### 20.6.2 Prepare the data for analysis.

Generally, we will need to create a new variable `d` that represents the difference between the two paired variables of interest. This uses the `mutate` command that adds an extra column to our data frame. The order of subtraction usually does not matter, but we will want to keep track of that order so that we can interpret our test statistic correctly. In the case of a one-sided test (which this is), it is especially important to keep track of the order of subtraction. Since we suspect the bookstore will charge more than Amazon, let's subtract in that order. Our hunch is that it will be a positive number, on average.

```
textbooks_d <- textbooks %>%
  mutate(d = ucla_new - amaz_new)
textbooks_d
```

```
## # A tibble: 73 x 8
##   dept_abbr course isbn      ucla_new amaz_new more   diff     d
##   <fct>      <fct> <fct>      <dbl>      <dbl> <fct> <dbl> <dbl>
## 1 Am Ind    "C170" 978-0803272620 27.7      28.0 Y    -0.28 -0.280
## 2 Anthro    "9"    978-0030119194 40.6      31.1 Y     9.45  9.45
## 3 Anthro    "135T" 978-0300080643 31.7      32   Y    -0.32 -0.320
## 4 Anthro    "191HB" 978-0226206813 16        11.5 Y     4.48  4.48
## 5 Art His   "M102K" 978-0892365999 19.0      14.2 Y     4.74  4.74
## 6 Art His   "118E" 978-0394723693 15.0      10.2 Y     4.78  4.78
## 7 Asia Am   "187B" 978-0822338437 24.7      20.1 Y     4.64  4.64
## 8 Asia Am   "191E" 978-0816646135 19.5      16.7 N     2.84  2.84
## 9 Ch Engr   "C125" 978-0195123401 124.      106. N    17.6  17.6
## 10 Chicano  "M145B" 978-0896086265 17        13.3 Y     3.74  3.74
## # ... with 63 more rows
```

If you look closely at the tibble above, you will see that there is a column already in our data called `diff`. It is the same as the column `d` we just created. So in this case, we didn't really need to create a new difference variable. However, since most data sets do not come pre-prepared with such a difference variable, it is good to know how to make one if needed.

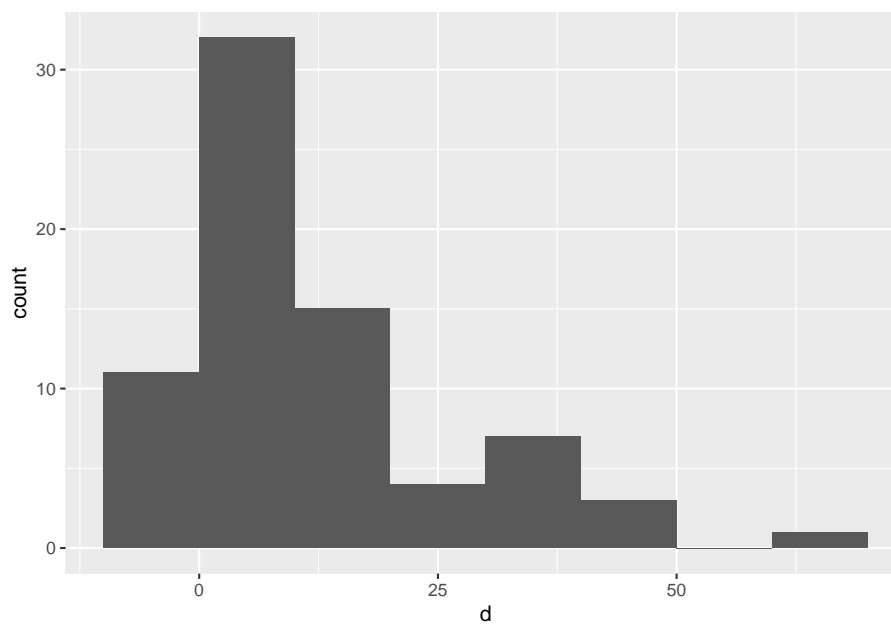
### 20.6.3 Make tables or plots to explore the data visually.

Here are summary statistics, a histogram, and a QQ plot for `d`.

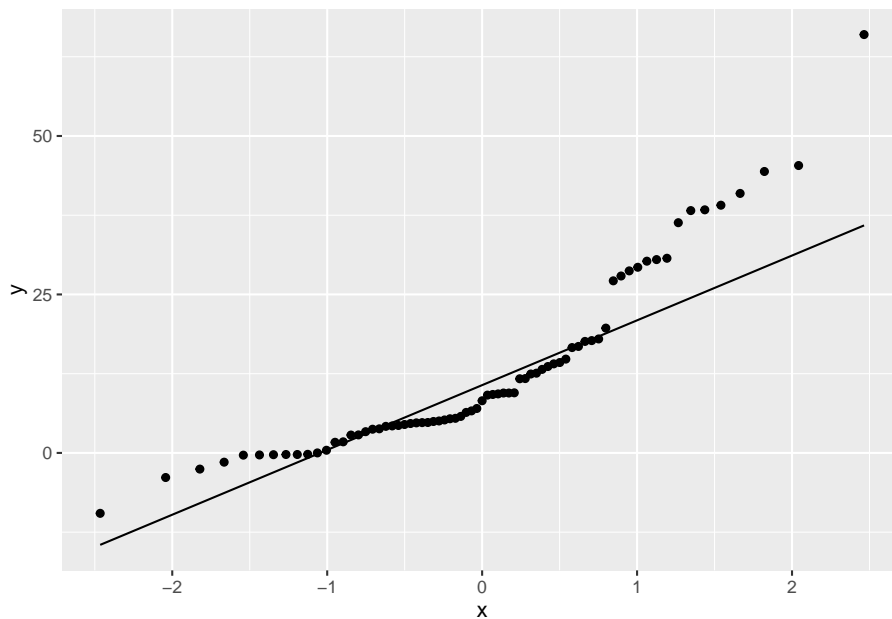
```
summary(textbooks_d$d)
```

```
##      Min. 1st Qu.  Median    Mean 3rd Qu.    Max.   \n##    -9.53   3.80    8.23   12.76   17.59   66.00
```

```
ggplot(textbooks_d, aes(x = d)) +  
  geom_histogram(binwidth = 10, boundary = 0)
```



```
ggplot(textbooks_d, aes(sample = d)) +  
  geom_qq() +  
  geom_qq_line()
```



The data is somewhat skewed to the right with one observation that might be a bit of an outlier. If the sample size were much smaller, we might be concerned about this point. However, it's not much higher than other points in that right tail, and it doesn't appear that its inclusion or exclusion will change the overall conclusion much. If you are concerned that the point might alter the conclusion, run the hypothesis test twice, once with and once without the outlier present to see if the main conclusion changes.

## 20.7 Hypotheses

### 20.7.1 Identify the sample (or samples) and a reasonable population (or populations) of interest.

The sample consists of 73 textbooks. The population is all textbooks that might be sold both at the UCLA bookstore and on Amazon.

### 20.7.2 Express the null and alternative hypotheses as contextually meaningful full sentences.

$H_0$  : There is no difference in textbooks prices between the UCLA bookstore and Amazon.



$H_A$  : Textbook prices at the UCLA bookstore are higher on average than on Amazon.

Commentary: Note we are performing a one-sided test. If we are conducting our own research with our own data, we can decide whether we want to run a two-sided or one-sided test. Remember that we only do the latter when we have a strong hypothesis in advance that the difference should be clearly in one direction and not the other. In this case, it's not up to us. We have to respect the research question as it was given to us: "The question of interest here is whether the campus bookstore charges more than Amazon."

**Exercise 1** What would the research question say if we were supposed to run a two-sided test instead? In other words, write down a slightly different research question about textbook prices that would prompt us to run a two-sided test.

Please write up your answer here.

### 20.7.3 Express the null and alternative hypotheses in symbols (when possible).

$$H_0 : \mu_d = 0$$

$$H_A : \mu_d > 0$$

Commentary: Since we're really just doing a one-sample t test, we could just call this parameter  $\mu$ , but the subscript  $d$  is a good reminder that it's the mean of the difference variable we care about (as opposed to the mean price of all the books at the UCLA bookstore or the mean price of all the same books on Amazon).

## 20.8 Model

### 20.8.1 Identify the sampling distribution model.

We use a t model with 72 degrees of freedom.

**Exercise 2** Explain how we got 72 degrees of freedom.

Please write up your answer here.

### 20.8.2 Check the relevant conditions to ensure that model assumptions are met.

- Random

- We do not know how exactly how David Diez obtained this sample, but the help file claims it is a random sample.
- 10%
  - We do not know how many total textbooks were available at the UCLA bookstore at the time the sample was taken, so we do not know if this condition is met. As long as there were at least 730 books, we are okay. We suspect that, based on the size of UCLA and the number of course offerings there, this is a reasonable assumption.
- Nearly normal
  - Although the sample distribution is skewed (with a possible mild outlier), the sample size is more than 30.

## 20.9 Mechanics

### 20.9.1 Compute the test statistic.

```
d_mean <- textbooks_d %>%
  specify(response = d) %>%
  calculate(stat = "mean")
d_mean
```

```
## Response: d (numeric)
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1  12.8
```

```
d_t <- textbooks_d %>%
  specify(response = d) %>%
  hypothesize(null = "point", mu = 0) %>%
  calculate(stat = "t")
d_t
```

```
## Response: d (numeric)
## Null Hypothesis: point
## # A tibble: 1 x 1
##   stat
##   <dbl>
## 1  7.65
```

### 20.9.2 Report the test statistic in context (when possible).

The mean difference in textbook prices is 12.7616438.

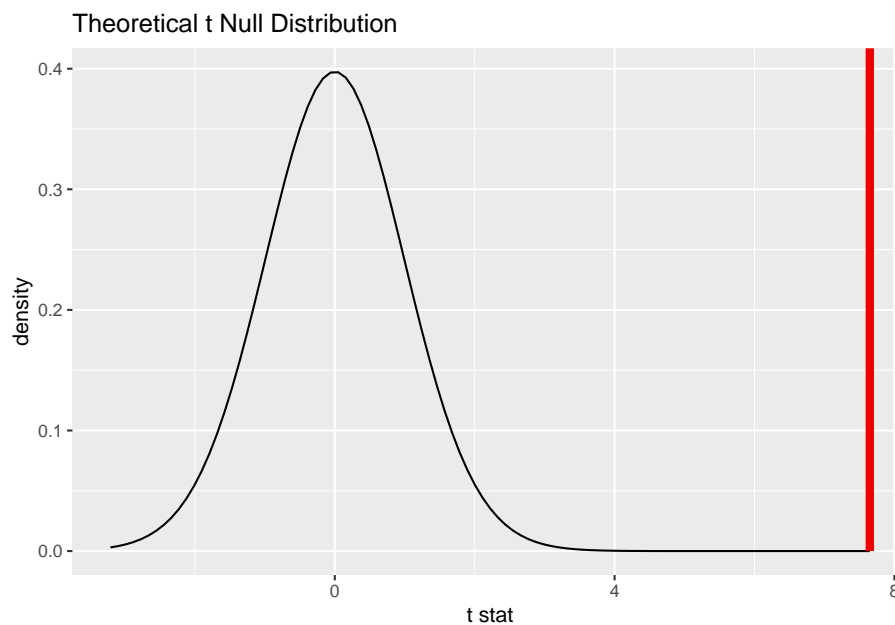
The value of  $t$  is 7.6487711. The mean difference in textbook prices is more than 7 standard errors above a difference of zero.

### 20.9.3 Plot the null distribution.

```
price_test <- textbooks_d %>%  
  specify(response = d) %>%  
  assume("t")  
price_test
```

```
## A T distribution with 72 degrees of freedom.
```

```
price_test %>%  
  visualize() +  
  shade_p_value(obs_stat = d_t, direction = "greater")
```



### 20.9.4 Calculate the P-value.

```
price_test_p <- price_test %>%
  get_p_value(obs_stat = d_t, direction = "greater")
price_test_p
```

```
## # A tibble: 1 x 1
##   p_value
##   <dbl>
## 1 3.46e-11
```

### 20.9.5 Interpret the P-value as a probability given the null.

$P < 0.001$ . If there were no difference in textbook prices between the UCLA bookstore and Amazon, there is only a 0% chance of seeing data at least as extreme as what we saw. (Note that the number is so small that it rounds to zero in the inline code above. That zero is technically incorrect. The P-value is never exactly zero. That's why we also are clear to state  $P < 0.001$ .)

## 20.10 Conclusion

### 20.10.1 State the statistical conclusion.

We reject the null hypothesis.

### 20.10.2 State (but do not overstate) a contextually meaningful conclusion.

We have sufficient evidence that UCLA prices are higher than Amazon prices.

Commentary: Note that because we performed a one-sided test, our conclusion is also one-sided in the hypothesized direction.

### 20.10.3 Express reservations or uncertainty about the generalizability of the conclusion.

We can be confident about the validity of this data, and therefore the conclusion drawn. We should be careful to limit our conclusion to the UCLA bookstore (and not extrapolate the findings, say, to other campus bookstores.) Depending

on when this data was collected, we may not be able to say anything about current prices at the UCLA bookstore either.

#### 20.10.4 Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.

If we made a Type I error, that would mean there was actually no difference in textbook prices, but that we got an unusual sample that detected a difference.

## 20.11 Confidence interval

### 20.11.1 Check the relevant conditions to ensure that model assumptions are met.

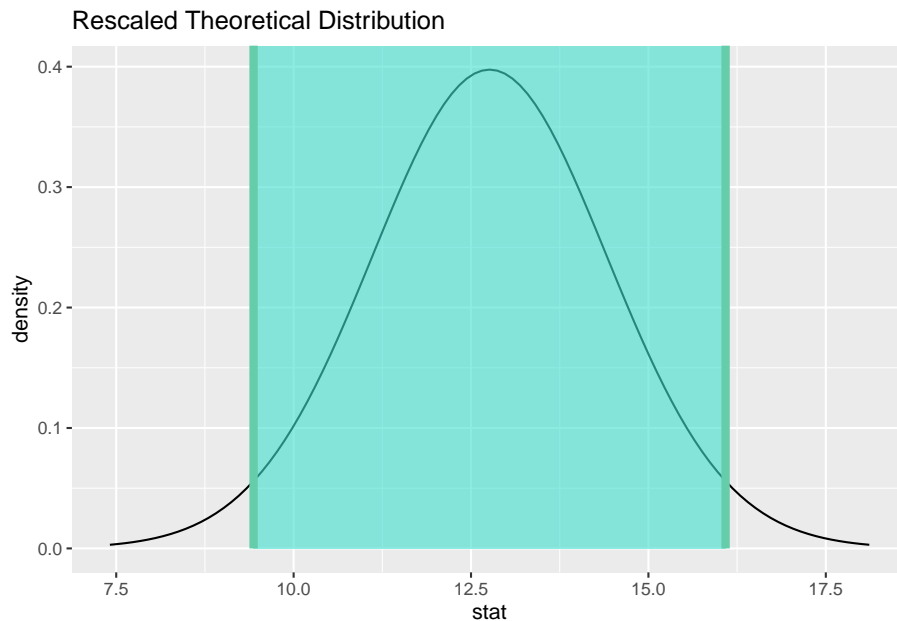
All necessary conditions have already been checked.

### 20.11.2 Calculate and graph the confidence interval.

```
price_ci <- price_test %>%
  get_confidence_interval(point_estimate = d_mean, level = 0.95)
price_ci
```

```
## # A tibble: 1 x 2
##   lower_ci upper_ci
##   <dbl>    <dbl>
## 1     9.44     16.1
```

```
price_test %>%
  visualize() +
  shade_confidence_interval(endpoints = price_ci)
```



### 20.11.3 State (but do not overstate) a contextually meaningful interpretation.

We are 95% confident that the true difference in textbook prices between the UCLA bookstore and Amazon is captured in the interval (9.4356361, 16.0876516). This was obtained by subtracting the Amazon price minus the UCLA bookstore. (In other words, since all differences in the confidence interval are positive, all plausible differences indicate that the UCLA prices are higher than the Amazon prices.)

Commentary: Don't forget that any time we find a number that represents a difference, we have to be clear in the conclusion about the direction of subtraction. Otherwise, we have no idea how to interpret positive and negative values.

### 20.11.4 If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test.

The confidence interval does not contain zero, which means that zero is not a plausible value for the difference textbook prices.

### 20.11.5 When comparing two groups, comment on the effect size and the practical significance of the result.

To think about the practical significance, imagine that you were a student at UCLA and that every textbook you needed was (on average) \$10 to \$15 more expensive in the bookstore than purchasing on Amazon. Multiplied across the number of textbooks you need, that could amount to a significant increase in expenses. In other words, that dollar figure is not likely a trivial amount of money for many students who require multiple textbooks each semester.

## 20.12 Your turn

The `hsb2` data set contains data from a random sample of 200 high school seniors from the “High School and Beyond” survey conducted by the National Center of Education Statistics. It contains, among other things, students’ scores on standardized tests in math, reading, writing, science, and social studies. We want to know if students do better on the math test or on the reading test.

Run inference to determine if there is a difference between math scores and reading scores.

The rubric outline is reproduced below. You may refer to the worked example above and modify it accordingly. Remember to strip out all the commentary. That is just exposition for your benefit in understanding the steps, but is not meant to form part of the formal inference process.

Another word of warning: the copy/paste process is not a substitute for your brain. You will often need to modify more than just the names of the data frames and variables to adapt the worked examples to your own work. Do not blindly copy and paste code without understanding what it does. And you should **never** copy and paste text. All the sentences and paragraphs you write are expressions of your own analysis. They must reflect your own understanding of the inferential process.

**Also, so that your answers here don’t mess up the code chunks above, use new variable names everywhere.**

### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure. Please write up your answer here

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

```
# Add code here to prepare the data for analysis.
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to make tables or plots.
```

Make tables or plots to explore the data visually.

### Hypotheses

Identify the sample (or samples) and a reasonable population (or populations) of interest. Please write up your answer here.

Express the null and alternative hypotheses as contextually meaningful full sentences.  $H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

Express the null and alternative hypotheses in symbols (when possible).  $H_0$  : *math*

$H_A$  : *math*

### Model

Identify the sampling distribution model. Please write up your answer here.

Check the relevant conditions to ensure that model assumptions are met. Please write up your answer here. (Some conditions may require R code as well.)



## Mechanics

```
# Add code here to compute the test statistic.
```

**Compute the test statistic.**

**Report the test statistic in context (when possible).** Please write up your answer here.

```
# IF CONDUCTING A SIMULATION...  
set.seed(1)  
# Add code here to simulate the null distribution.
```

```
# Add code here to plot the null distribution.
```

**Plot the null distribution.**

```
# Add code here to calculate the P-value.
```

**Calculate the P-value.**

**Interpret the P-value as a probability given the null.** Please write up your answer here.

## Conclusion

**State the statistical conclusion.** Please write up your answer here.

**State (but do not overstate) a contextually meaningful conclusion.** Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.** Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.** Please write up your answer here.

### Confidence interval

**Check the relevant conditions to ensure that model assumptions are met.** Please write up your answer here. (Some conditions may require R code as well.)

```
# Add code here to calculate the confidence interval.
```

```
# Add code here to graph the confidence interval.
```

**Calculate and graph the confidence interval.**

**State (but do not overstate) a contextually meaningful interpretation.** Please write up your answer here.

**If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test. [Not always applicable.]** Please write up your answer here.

**When comparing two groups, comment on the effect size and the practical significance of the result. [Not always applicable.]** Please write up your answer here.

## 20.13 Conclusion

Paired data occurs whenever we have two numerical measurements that are related to each other, whether because they come from the same observational

units or from closely related ones. When our data is structured as pairs of measurements in this way, we can subtract the two columns and obtain a difference. That difference variable is the object of our study, and now that it is represented as a single numerical variable, we can apply the one-sample  $t$  test from the last chapter.

### 20.13.1 Preparing and submitting your assignment

1. From the “Run” menu, select “Restart R and Run All Chunks”.
2. Deal with any code errors that crop up. Repeat steps 1—2 until there are no more code errors.
3. Spell check your document by clicking the icon with “ABC” and a check mark.
4. Hit the “Preview” button one last time to generate the final draft of the `.nb.html` file.
5. Proofread the HTML file carefully. If there are errors, go back and fix them, then repeat steps 1–5 again.

If you have completed this chapter as part of a statistics course, follow the directions you receive from your professor to submit your assignment.



## Appendix A

# Rubric for inference

2.0

This is the R Markdown outline for running inference, both a hypothesis test and a confidence interval.

### Exploratory data analysis

Use data documentation (help files, code books, Google, etc.) to determine as much as possible about the data provenance and structure.

Please write up your answer here

```
# Add code here to print the data
```

```
# Add code here to glimpse the variables
```

Prepare the data for analysis. [Not always necessary.]

```
# Add code here to prepare the data for analysis.
```

Make tables or plots to explore the data visually.

*# Add code here to make tables or plots.*

## Hypotheses

**Identify the sample (or samples) and a reasonable population (or populations) of interest.**

Please write up your answer here.

**Express the null and alternative hypotheses as contextually meaningful full sentences.**

$H_0$  : Null hypothesis goes here.

$H_A$  : Alternative hypothesis goes here.

**Express the null and alternative hypotheses in symbols (when possible).**

$H_0$  : *math*

$H_A$  : *math*

## Model

**Identify the sampling distribution model.**

Please write up your answer here.

**Check the relevant conditions to ensure that model assumptions are met.**

Please write up your answer here. (Some conditions may require R code as well.)

## Mechanics

Compute the test statistic.

```
# Add code here to compute the test statistic.
```

Report the test statistic in context (when possible).

Please write up your answer here.

Plot the null distribution.

```
# IF CONDUCTING A SIMULATION...  
set.seed(1)  
# Add code here to simulate the null distribution.
```

```
# Add code here to plot the null distribution.
```

Calculate the P-value.

```
# Add code here to calculate the P-value.
```

Interpret the P-value as a probability given the null.

Please write up your answer here.

## Conclusion

State the statistical conclusion.

Please write up your answer here.

State (but do not overstate) a contextually meaningful conclusion.

Please write up your answer here.

**Express reservations or uncertainty about the generalizability of the conclusion.**

Please write up your answer here.

**Identify the possibility of either a Type I or Type II error and state what making such an error means in the context of the hypotheses.**

Please write up your answer here.

## Confidence interval

**Check the relevant conditions to ensure that model assumptions are met.**

Please write up your answer here. (Some conditions may require R code as well.)

**Calculate and graph the confidence interval.**

```
# Add code here to calculate the confidence interval.
```

```
# Add code here to graph the confidence interval.
```

**State (but do not overstate) a contextually meaningful interpretation.**

Please write up your answer here.

**If running a two-sided test, explain how the confidence interval reinforces the conclusion of the hypothesis test. [Not always applicable.]**

Please write up your answer here.



**When comparing two groups, comment on the effect size and the practical significance of the result. [Not always applicable.]**

Please write up your answer here.