Problem 1. Let A, B, C be three events. Find expressions for the events that of A, B, C:
(a) only B occurs,
(b) both A and B occur but not C,
(c) at least one event occurs,
(d) at least two events occur,
(e) all three occur,
(f) none occurs,
(g) at most one occurs,
(h) exactly one occurs.

Problem 2. A coin is tossed until either there are more heads than tails, or until the fifth toss, whichever comes first. Write down the sample space, and determine the probability of each outcome in the sample space.

Problem 3. An herpetologist wants to estimate the number of frogs in a pond. She captures 50 frogs in the pond, marks each with a dot of paint and releases them. A few days later she goes back, captures another sample of 60, finding 14 marked frogs and 46 unmarked.
(a) Assuming that the pond has n frogs, determine the probability $L(n)$ that a sample of 60 frogs contain 14 marked ones.
(b) Show that $L(n)$ is increasing for $n \le n_*$ and decreasing for $n \ge n_*$.
(c) Find the maximum likelihood estimate for n: that is the value of n which maximizes $L(n)$.
Hint: When does $L(n) \le L(n-1)$ hold?

Below, we use the notation $a_n \sim b_n$ to denote that $\lim_{n\to\infty} a_n/b_n = 1$.

Problem 4. A coin is tossed $2n$ times. Let p_n be the probability that exactly half the outcomes are heads.
(a) Find a formula for p_n.
(b) Calculate p_{n+1}/p_n, and show that p_n is decreasing in n (i.e., $p_{n+1} < p_n$).
(c) Show that p_n goes to 0 as $n \to \infty$ like the $\alpha/\sqrt{n}$ for some $\alpha > 0$. Using Stirling's formula

$$n! \sim \sqrt{2\pi n}(n/e)^n,$$

prove that there is some α so that $p_n \sim \alpha/\sqrt{n}$. Find the value of α.

Problem 5. A coin is tossed $3n$ times. Let q_n be the probability that exactly a third of the outcomes are heads.
(a) Find a formula for q_n.
(b) Show that

$$q_n \sim \frac{a}{\sqrt{n}} e^{-bn}$$

for some constants $a, b > 0$, and find their values.

Hint: Use Stirling's formula, and $x^n = e^{n \log x}$.

Remark. Please compare 4(c) with 5(b) and observe that q_n is much smaller than p_n for large values of n. Large deviations theory deals with estimating such exponentially small probabilities.

Problem 6. write a program in python that will do the following.

(a) Write a function `birthday(n)` that:

   (a) Generates a list containing n numbers uniformly distributed on $\{1, 2, \ldots, 365\}$ (think of this as the list of birthdays of n people.

   (b) Returns 1 (or `True`) if there is at least one pair of people with coinciding birthdays (a "match") and 0 (or `False`) otherwise.

(b) For each n, from 1 to 70, run the function `birthday(n)` 1000 times, and compute the proportion $X(n)$ of the 1000 times in which there was a match.

(c) Let $Y(n)$ be the actual probability of a match:

$$Y(n) = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}.$$

Generate a joint plot of $X(n)$ and $Y(n)$ for $n \in [1, 70]$.

(d) Repeat the steps above for Martians. (Hint: The Martian year has 669 Martian days.)