1.1 $x=1$: $\theta^x(1-\theta)^{1-x} = \theta^1(1-\theta)^0 = \theta = \mathbb{P}(X=1)$

$x=0$: $\theta^x(1-\theta)^{1-x} = \theta^0(1-\theta)^{1-0} = 1-\theta = \mathbb{P}(X=0)$

Therefore, $\mathbb{P}(X=x) = \theta^x(1-\theta)^{1-x}$, $x\in\{0,1\}$

1.2 $\mathbb{P}(x_i=x\,|\,\theta) = \theta^x(1-\theta)^{1-x}$, $x\in\{0,1\}$

$\Rightarrow \mathbb{P}(x_1,\cdots,x_n\,|\,\theta) = \theta^{\sum_i x_i}(1-\theta)^{n-\sum_i x_i} = \theta^{n_1}(1-\theta)^{n_0}$ ☐

1.3. $\mathcal{L}(\theta) = \mathbb{P}(x_1,\cdots,x_n|\theta) = \theta^{n_1}(1-\theta)^{n_0}$

$\ell(\theta) = \log\mathcal{L}(\theta) = n_1\log\theta + n_0\log(1-\theta)$

$$\frac{d\ell}{d\theta} = \frac{n_1}{\theta} - \frac{n_0}{1-\theta} \overset{\text{def}}{=} 0$$

$$n_1(1-\theta) - n_0\theta = 0$$

$$n_1 = \theta(n_1+n_0) = \theta n$$

$$\hat{\theta}_{ML} = \frac{n_1}{n}$$

1.4 By Bayes', $\mathbb{P}(\theta|x_1,\cdots,x_n) \propto \mathbb{P}(x_1,\cdots,x_n|\theta)\times\mathbb{P}(\theta)$

From above, $\mathbb{P}(x_1,\cdots,x_n|\theta) = \theta^{n_1}(1-\theta)^{n_0}$

$\theta\sim\mathrm{Beta}(a,b)$, $\mathbb{P}(\theta)\propto\theta^{a-1}(1-\theta)^{b-1}$

$$\begin{aligned}\mathbb{P}(\theta|x_1,\cdots,x_n) &\propto \theta^{n_1}(1-\theta)^{n_0}\times\theta^{a-1}(1-\theta)^{b-1}\\ &= \theta^{n_1+(a-1)}(1-\theta)^{n_0+(b-1)}\\ &\propto \theta^{n_1+a-1}\cdot(1-\theta)^{n_0+b-1}\end{aligned}$$