Math/CS 4650 homework 4

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1. Compute the first two steps of the Jacobi and the Gauss–Seidel Methods with starting vector $[\mathbf{0}, \ldots, \mathbf{0}]$.

a) $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Jacobi:

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5 + v_0}{3} \\ \frac{4 + u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5 + v_1}{3} \\ \frac{4 + u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5 + 2}{3} \\ \frac{4 + \frac{5}{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{7}{3} \\ \frac{17}{6} \end{bmatrix} \approx \begin{bmatrix} 2.333 \\ 2.833 \end{bmatrix}$$

Gauss-Seidel:

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5 + 0}{3} \\ \frac{4 + \frac{5}{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{17}{6} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5 + \frac{17}{6}}{3} \\ \frac{4 + \frac{41}{18}}{2} \end{bmatrix} = \begin{bmatrix} \frac{47}{18} \\ \frac{119}{36} \end{bmatrix} \approx \begin{bmatrix} 2.61 \\ 3.306 \end{bmatrix}$$

b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Jacobi:

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \dfrac{0 + v_0 - 0w_0}{2} \\ \dfrac{2 + u_0 + w_0}{2} \\ \dfrac{0 - 0u_0 + v_0}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{0 + v_1 - 0w_1}{2} \\ \dfrac{2 + u_1 + w_1}{2} \\ \dfrac{0 - 0u_1 + v_1}{2} \end{bmatrix} = \begin{bmatrix} \dfrac{0 + 1 - 0}{2} \\ \dfrac{2 + 0 + 0}{2} \\ \dfrac{0 - 0 + 1}{2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} \\ 1 \\ \dfrac{1}{2} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 0.5 \end{bmatrix}$$

Gauss-Seidel:

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \dfrac{0}{2} \\ \dfrac{2}{2} \\ \dfrac{1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \dfrac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} \\ \dfrac{2 + \frac{1}{2} + \frac{1}{2}}{2} \\ \dfrac{3}{2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{2} \\ \dfrac{3}{2} \\ \dfrac{3}{4} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 0.75 \end{bmatrix}$$

c) $\begin{bmatrix}3 & 1 & 1\\1 & 3 & 1\\1 & 1 & 3\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}6\\3\\5\end{bmatrix}$

$$\begin{bmatrix}u_0\\v_0\\w_0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Jacobi:

$$\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}=\begin{bmatrix}\dfrac{6-v_0-w_0}{3}\\\dfrac{3-u_0-w_0}{3}\\\dfrac{5-u_0-v_0}{3}\end{bmatrix}=\begin{bmatrix}2\\1\\\dfrac{5}{3}\end{bmatrix}$$

$$\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}=\begin{bmatrix}\dfrac{6-v_1-w_1}{3}\\\dfrac{3-u_1-w_1}{3}\\\dfrac{5-u_1-v_1}{3}\end{bmatrix}=\begin{bmatrix}\dfrac{6-1-\frac{5}{3}}{3}\\\dfrac{3-2-\frac{5}{3}}{3}\\\dfrac{5-2-1}{3}\end{bmatrix}=\begin{bmatrix}\dfrac{10}{9}\\-\dfrac{2}{9}\\\dfrac{2}{3}\end{bmatrix}\approx\begin{bmatrix}1.111\\-0.222\\0.6667\end{bmatrix}$$

Gauss-Seidel:

$$\begin{bmatrix}u_1\\v_1\\w_1\end{bmatrix}=\begin{bmatrix}\dfrac{6}{3}\\\dfrac{3-2}{3}\\\dfrac{5-2-\frac{1}{3}}{3}\end{bmatrix}=\begin{bmatrix}2\\\dfrac{1}{3}\\\dfrac{8}{9}\end{bmatrix}$$

$$\begin{bmatrix}u_2\\v_2\\w_2\end{bmatrix}=\begin{bmatrix}\dfrac{6-\frac{1}{3}-\frac{8}{9}}{3}\\\dfrac{3-\frac{43}{27}-\frac{8}{9}}{3}\\\dfrac{5-\frac{43}{27}-\frac{14}{81}}{3}\end{bmatrix}=\begin{bmatrix}\dfrac{43}{27}\\\dfrac{14}{81}\\\dfrac{262}{243}\end{bmatrix}\approx\begin{bmatrix}2.0435\\0.1728\\1.0782\end{bmatrix}$$

2. **Rearrange the equations to form a strictly diagonally dominant system. Apply two steps of the Jacobi and Gauss–Seidel Methods from starting vector $[\mathbf{0}, \dots, \mathbf{0}]$.**

a) $5u + 4v = 6$

$u + 3v = -1$

$$\begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Jacobi:

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \dfrac{6 - 4v_0}{5} \\ \dfrac{-1 - u_0}{3} \end{bmatrix} = \begin{bmatrix} \dfrac{6}{5} \\ \dfrac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \dfrac{6 - 4v_1}{5} \\ \dfrac{-1 - u_1}{3} \end{bmatrix} = \begin{bmatrix} \dfrac{6 - \frac{4}{3}}{5} \\ \dfrac{-1 - \frac{6}{5}}{3} \end{bmatrix} = \begin{bmatrix} \dfrac{22}{15} \\ -\dfrac{11}{15} \end{bmatrix} \approx \begin{bmatrix} 1.467 \\ -0.73 \end{bmatrix}$$

Gauss-Seidel:

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \dfrac{6}{5} \\ \dfrac{-1 - \frac{6}{5}}{3} \end{bmatrix} = \begin{bmatrix} \dfrac{6}{5} \\ \dfrac{-11}{15} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \dfrac{6 + \frac{44}{15}}{5} \\ \dfrac{-1 - \frac{134}{75}}{3} \end{bmatrix} = \begin{bmatrix} \dfrac{134}{75} \\ -\dfrac{209}{225} \end{bmatrix} \approx \begin{bmatrix} 1.787 \\ -0.929 \end{bmatrix}$$

b) $3u - v + w = -2$

$u - 8v - 2w = 1$

$u + v + 5w = 4$

$$\begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Jacobi:

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \dfrac{-2 + v_0 - w_0}{3} \\ \dfrac{-1 + u_0 - 2w_0}{8} \\ \dfrac{4 - u_0 - v_0}{5} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{3} \\ -\dfrac{1}{8} \\ \dfrac{4}{5} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1 + v_1 - w_1}{3} \\ \dfrac{-1 + u_1 - 2w_1}{8} \\ \dfrac{4 - u_1 - v_1}{5} \end{bmatrix} = \begin{bmatrix} \dfrac{1 - \frac{1}{8} - \frac{4}{5}}{3} \\ \dfrac{4 - \frac{2}{3} - \frac{8}{5}}{1} \\ \dfrac{4 + \frac{2}{3} + \frac{1}{8}}{1} \end{bmatrix} = \begin{bmatrix} -\dfrac{39}{40} \\ -\dfrac{49}{120} \\ \dfrac{23}{24} \end{bmatrix} \approx \begin{bmatrix} -0.975 \\ -0.408 \\ 0.958 \end{bmatrix}$$

Gauss-Seidel:

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \dfrac{-2 +}{3} \\ \dfrac{-1 - \frac{2}{3}}{8} \\ \dfrac{4 + \frac{2}{3} + \frac{5}{24}}{5} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{3} \\ -\dfrac{5}{24} \\ \dfrac{39}{40} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{1 - \dfrac{5}{24} - \dfrac{39}{40}}{3} \\ \dfrac{-1 - \dfrac{191}{180} - 2 \times \dfrac{39}{40}}{8} \\ \dfrac{4 + \dfrac{191}{180} + \dfrac{361}{720}}{5} \end{bmatrix} = \begin{bmatrix} -\dfrac{191}{180} \\ -\dfrac{361}{720} \\ \dfrac{89}{80} \end{bmatrix} \approx \begin{bmatrix} -1.061 \\ -0.501 \\ 1.1125 \end{bmatrix}$$

c) $4u + 3w = 0$

$u + 4v = 5$

$v + 2w = 2$

$$\begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Jacobi:

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{4} w_0 \\ \dfrac{5 - u_0}{4} \\ \dfrac{2 - v_0}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{5}{4} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{4} w_1 \\ \dfrac{5 - u_1}{4} \\ \dfrac{2 - v_1}{2} \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{4} \\ \dfrac{5}{4} \\ \dfrac{2 + \dfrac{5}{4}}{3} \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{4} \\ \dfrac{5}{4} \\ \dfrac{3}{8} \end{bmatrix} = \begin{bmatrix} -0.75 \\ 1.25 \\ 0.375 \end{bmatrix}$$

Gauss-Seidel:

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4}w_0 \\ \frac{5}{4} \\ \frac{2-\frac{5}{4}}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5}{4} \\ \frac{3}{8} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\frac{3}{4}w_1 \\ \frac{5-u_1}{4} \\ \frac{2-v_1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{3}{4}\times\frac{3}{8} \\ \frac{5+\frac{9}{32}}{4} \\ \frac{2-\frac{169}{128}}{3} \end{bmatrix} = \begin{bmatrix} -\frac{9}{32} \\ \frac{169}{128} \\ \frac{87}{256} \end{bmatrix} \approx \begin{bmatrix} -0.281 \\ 1.320 \\ 0.34 \end{bmatrix}$$

3. Apply two steps of SOR to the systems in Exercise 1. Use starting vector $[\mathbf{0}, \ldots, \mathbf{0}]$ and $\boldsymbol{\omega} = \mathbf{1.5}$

a) $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$

$$u_{k+1} = -\frac{1}{2}u_k + \frac{3}{2}\times\frac{5+v_k}{3}$$
$$v_{k+1} = -\frac{1}{2}v_k + \frac{3}{2}\times\frac{4+u_{k+1}}{2}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}u_0 + \frac{3}{2}\times\frac{5+v_0}{3} \\ -\frac{1}{2}v_0 + \frac{3}{2}\times\frac{4+u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ \frac{39}{8} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}u_1 + \frac{1}{2}\times\frac{5+v_1}{3} \\ \frac{1}{2}v_1 + \frac{1}{2}\times\frac{4+u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\times\frac{5}{2}+\frac{1}{2}\times\frac{5+\frac{39}{8}}{3} \\ \frac{1}{2}\times\frac{39}{8}+\frac{1}{2}\times\frac{4+\frac{59}{16}}{2} \end{bmatrix} = \begin{bmatrix} \frac{59}{16} \\ \frac{213}{64} \end{bmatrix} \approx \begin{bmatrix} 3.6875 \\ 3.328 \end{bmatrix}$$

b) $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$$u_{k+1} = -\frac{1}{2}u_k + \frac{3}{2} \times \frac{v_k}{2}$$
$$v_{k+1} = -\frac{1}{2}v_k + \frac{3}{2} \times \frac{2 + u_{k+1} + w_k}{2}$$
$$w_{k+1} = -\frac{1}{2}w_k + \frac{3}{2} \times \frac{v_{k+1}}{2}$$
$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}u_0 + \frac{3}{2} \times \frac{v_0}{2} \\ -\frac{1}{2}v_0 + \frac{3}{2} \times \frac{2 + u_1 + w_0}{2} \\ -\frac{1}{2}w_0 + \frac{3}{2} \times \frac{v_1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ \frac{9}{8} \end{bmatrix}$$
$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}u_1 + \frac{3}{2} \times \frac{v_1}{2} \\ -\frac{1}{2}v_1 + \frac{3}{2} \times \frac{2 + u_2 + w_1}{2} \\ -\frac{1}{2}w_1 + \frac{3}{2} \times \frac{v_2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \times \frac{3}{2} + \frac{3}{2} \times \frac{\frac{3}{2}}{2} \\ \frac{1}{2} \times \frac{3}{2} + \frac{3}{2} \times \frac{2 + \frac{9}{8} + \frac{9}{8}}{2} \\ -\frac{1}{2} \times \frac{9}{8} + \frac{3}{2} \times \frac{\frac{39}{19}}{2} \end{bmatrix} = \begin{bmatrix} \frac{9}{8} \\ \frac{39}{19} \\ \frac{81}{64} \end{bmatrix} \approx \begin{bmatrix} 1.125 \\ 2.4375 \\ 1.2656 \end{bmatrix}$$

c) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$

$$u_{k+1} = -\frac{1}{2}u_k + \frac{3}{2} \times \frac{6 - v_k - w_k}{3}$$
$$v_{k+1} = -\frac{1}{2}v_k + \frac{3}{2} \times \frac{3 - u_{k+1} - w_k}{3}$$
$$w_{k+1} = -\frac{1}{2}w_k + \frac{3}{2} \times \frac{5 - u_{k+1} - v_{k+1}}{3}$$
$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}u_0 + \frac{3}{2} \times \frac{6}{3} \\ -\frac{1}{2}v_0 + \frac{3}{2} \times \frac{3 - 3}{3} \\ -\frac{1}{2}w_0 + \frac{3}{2} \times \frac{5 - 3}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}u_1 + \frac{3}{2} \times \frac{6 - v_1 - w_1}{3} \\ -\frac{1}{2}v_1 + \frac{3}{2} \times \frac{3 - u_2 - w_1}{3} \\ -\frac{1}{2}w_1 + \frac{3}{2} \times \frac{5 - u_2 - v_2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \times 3 + \frac{3}{2} \times \frac{6 - 0 - 1}{3} \\ -\frac{1}{2} \times 0 + \frac{3}{2} \times \frac{3 - 3 - \frac{5}{2}}{3} \\ -\frac{1}{2} \times 1 + \frac{3}{2} \times \frac{5 + 1 - \frac{1}{2}}{3} \end{bmatrix} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ \frac{5}{4} \end{bmatrix} = \begin{bmatrix} -1 \\ -0.5 \\ 1.25 \end{bmatrix}$$

4. Apply two steps of SOR to the systems in Exercise 2 after rearranging. Use starting vector $[0, \ldots, 0]$ and $\omega = 1$ and 1.2

a) $\omega = 1$

i. $\begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

$$u_{k+1} = \frac{6 - 4v_k}{5}$$

$$v_{k+1} = \frac{-1 - u_{k+1}}{3}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{6 - 4v_0}{5} \\ \frac{-1 - u_1}{3} \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{11}{15} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{-1 - 3v_1}{1} \\ \frac{6 - 5u_2}{4} \end{bmatrix} = \begin{bmatrix} \frac{134}{75} \\ -\frac{209}{225} \end{bmatrix} \approx \begin{bmatrix} 1.467 \\ -0.73 \end{bmatrix}$$

ii. $\begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

$$u_{k+1} = \frac{-2 + v_k - w_k}{3}$$

$$v_{k+1} = \frac{-1 + u_{k+1} - 2w_k}{8}$$

$$w_{k+1} = \frac{4 - u_{k+1} - v_{k+1}}{5}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} \dfrac{-2 + v_0 - w_0}{3} \\ \dfrac{-1 + u_1 - 2w_0}{8} \\ \dfrac{4 - u_1 - v_1}{5} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{3} \\ -\dfrac{5}{24} \\ -\dfrac{39}{40} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-2 + v_1 - w_1}{3} \\ \dfrac{-1 + u_2 - 2w_1}{8} \\ \dfrac{4 - u_2 - v_2}{5} \end{bmatrix} = \begin{bmatrix} -\dfrac{191}{180} \\ -\dfrac{361}{720} \\ \dfrac{89}{80} \end{bmatrix} \approx \begin{bmatrix} -1.061 \\ -0.501 \\ 1.1125 \end{bmatrix}$$

iii. $\begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

$$u_{k+1} = \frac{-3w_k}{4}$$
$$v_{k+1} = \frac{5 - u_{k+1}}{1}$$
$$w_{k+1} = \frac{2 - u_{k+1}}{3}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{4} w_0 \\ \dfrac{5 - u_1}{4} \\ \dfrac{2 - v_1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{4}{5} \\ \dfrac{3}{8} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\dfrac{3}{4} w_1 \\ \dfrac{5 - u_2}{4} \\ \dfrac{2 - v_2}{2} \end{bmatrix} = \begin{bmatrix} -\dfrac{9}{32} \\ \dfrac{169}{128} \\ \dfrac{87}{256} \end{bmatrix} \approx \begin{bmatrix} -0.281 \\ 1.320 \\ 0.34 \end{bmatrix}$$

b) $\omega = 1.2$

i. $\begin{bmatrix} 5 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

$$u_{k+1} = -\frac{1}{5}u_k + \frac{6}{5} \times \frac{6 - 4v_k}{5}$$

$$v_{k+1} = -\frac{1}{5}v_k + \frac{6}{5} \times \frac{-1 - u_{k+1}}{3}$$

$$\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \times \frac{6 - 4v_0}{5} \\ \frac{6}{5} \times \frac{-1 - u_1}{3} \end{bmatrix} = \begin{bmatrix} \frac{36}{25} \\ -\frac{122}{125} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \times u_1 + \frac{6}{5} \times \frac{6 - 4v_1}{5} \\ \frac{1}{5} \times v_1 + \frac{6}{5} \times \frac{-1 - u_2}{3} \end{bmatrix} \approx \begin{bmatrix} 2.089 \\ -1.040 \end{bmatrix}$$

ii. $\begin{bmatrix} 3 & -1 & 1 \\ 1 & -8 & -2 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$

$$u_{k+1} = -\frac{1}{5}u_k + \frac{6}{5} \times \frac{-2 + v_k - w_k}{3}$$

$$v_{k+1} = -\frac{1}{5}v_k + \frac{6}{5} \times \frac{-1 + u_{k+1} - 2w_k}{8}$$

$$w_{k+1} = -\frac{1}{5}w_k + \frac{6}{5} \times \frac{4 - u_{k+1} - v_{k+1}}{5}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5}u_0 + \frac{6}{5} \times \frac{-2 + v_0 - w_0}{3} \\ -\frac{1}{5}v_0 + \frac{6}{5} \times \frac{-1 + u_1 - 2w_0}{8} \\ -\frac{1}{5}w_0 + \frac{6}{5} \times \frac{4 - u_1 - v_1}{5} \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} \\ -\frac{27}{100} \\ \frac{1521}{1250} \end{bmatrix}$$

$$\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5}u_1 + \frac{6}{5} \times \frac{-2 + v_0 - w_0}{3} \\ -\frac{1}{5}v_1 + \frac{6}{5} \times \frac{-1 + u_1 - 2w_0}{8} \\ -\frac{1}{5}w_1 + \frac{6}{5} \times \frac{4 - u_1 - v_1}{5} \end{bmatrix} \approx \begin{bmatrix} -1.231 \\ -0.646 \\ 1.168 \end{bmatrix}$$

iii. $\begin{bmatrix} 4 & 0 & 3 \\ 1 & 4 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 2 \end{bmatrix}$

$$u_{k+1} = -\frac{1}{5}u_k + \frac{6}{5} \times (-\frac{3}{4}w_k)$$
$$v_{k+1} = -\frac{1}{5}v_k + \frac{6}{5} \times \frac{5 - u_{k+1}}{4}$$
$$w_{k+1} = -\frac{1}{5}w_k - \frac{6}{5} \times \frac{2 - v_{k+1}}{2}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5}u_0 + \frac{6}{5} \times (-\frac{3}{4}w_0) \\ -\frac{1}{5}v_0 + \frac{6}{5} \times \frac{5 - u_1}{4} \\ -\frac{1}{5}w_0 - \frac{6}{5} \times \frac{2 - v_1}{2} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{3}{2} \\ \frac{3}{10} \end{bmatrix}$$

$$\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5}u_1 + \frac{6}{5} \times (-\frac{3}{4}w_1) \\ -\frac{1}{5}v_1 + \frac{6}{5} \times \frac{5 - u_2}{4} \\ -\frac{1}{5}w_1 - \frac{6}{5} \times \frac{2 - v_2}{2} \end{bmatrix} = \begin{bmatrix} -\frac{27}{100} \\ \frac{1281}{1000} \\ -\frac{1857}{5000} \end{bmatrix} = \begin{bmatrix} -0.27 \\ 1.281 \\ 0.3714 \end{bmatrix}$$