

COMP498G/691G COMPUTER VISION

LECTURE 12 EPIPOLAR GEOMETRY



Administrative

- Tonight's tutorial topics
 - Camera calibration (review)
 - Stereo

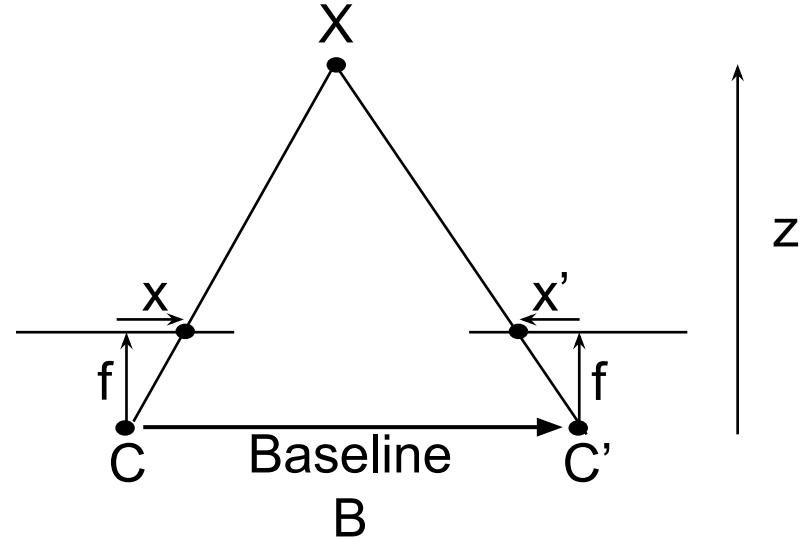
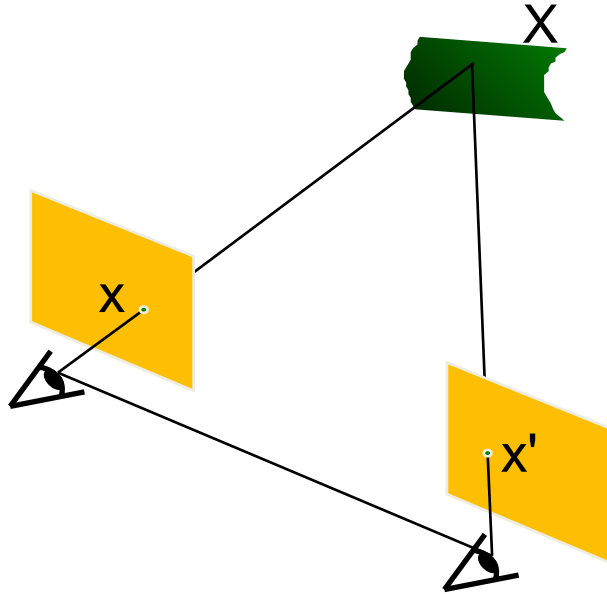
Today's Lecture

- Epipolar Geometry
 - General case with calibrated cameras
 - Slides acknowledgment: J. Hays, Derek Hoiem, Lana Lazebnik, Silvio Savarese, Steve Seitz.
 - Many figures from Hartley & Zisserman
- Questions

- Epipolar geometry
 - Relates cameras from two positions

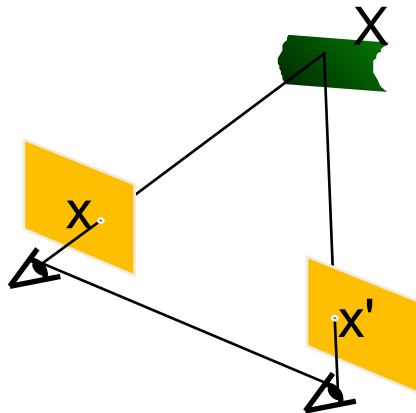
Depth from Stereo

- **Goal:** recover depth by finding image coordinate x' that corresponds to x



Depth from Stereo

- **Goal:** recover depth by finding image coordinate x' that corresponds to x
- Sub-Problems
 1. **Calibration:** How do we recover the relation of the cameras (if not already known)?
 2. **Correspondence:** How do we search for the matching point x' ?



Correspondence Problem

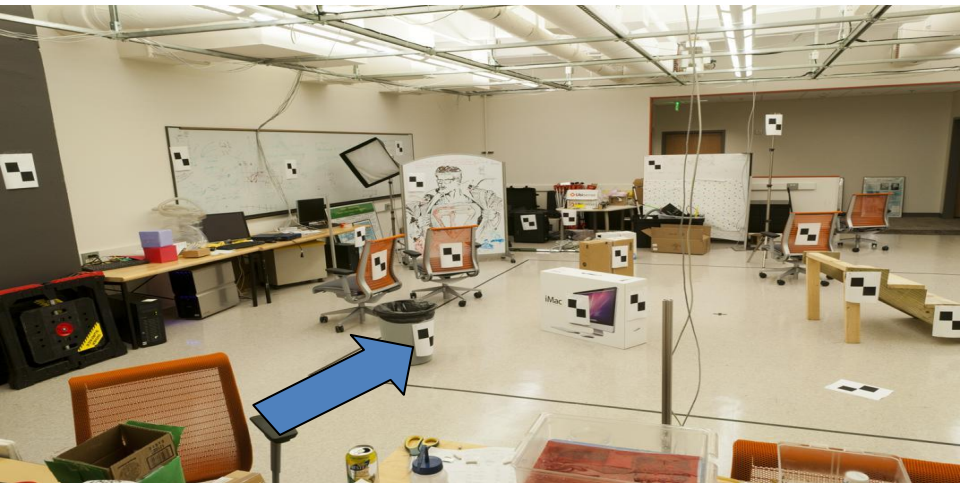


- We have two images taken from cameras with different intrinsic and extrinsic parameters
- How do we match a point in the first image to a point in the second? How can we constrain our search?

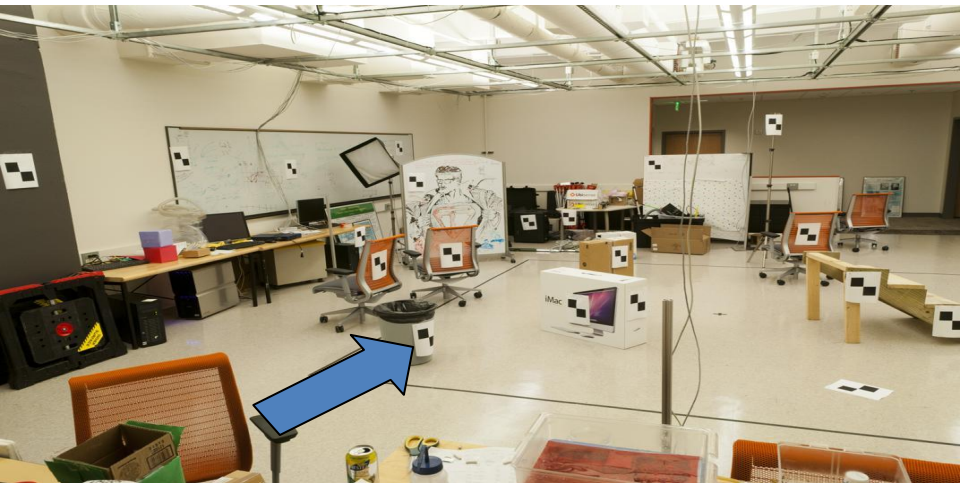
Where do we need to search?



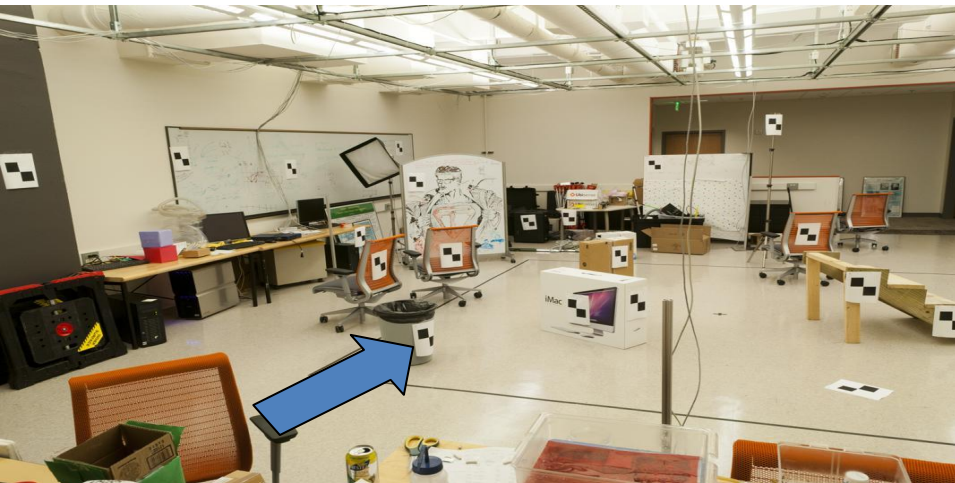
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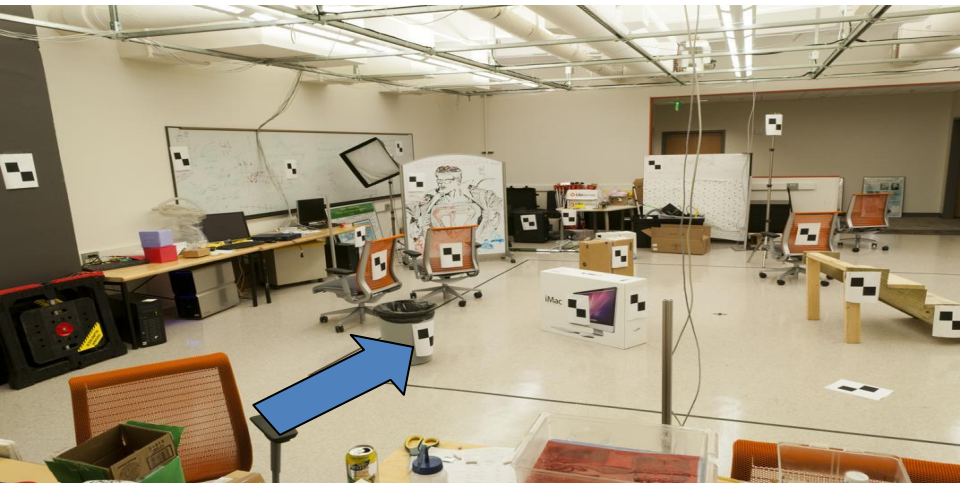
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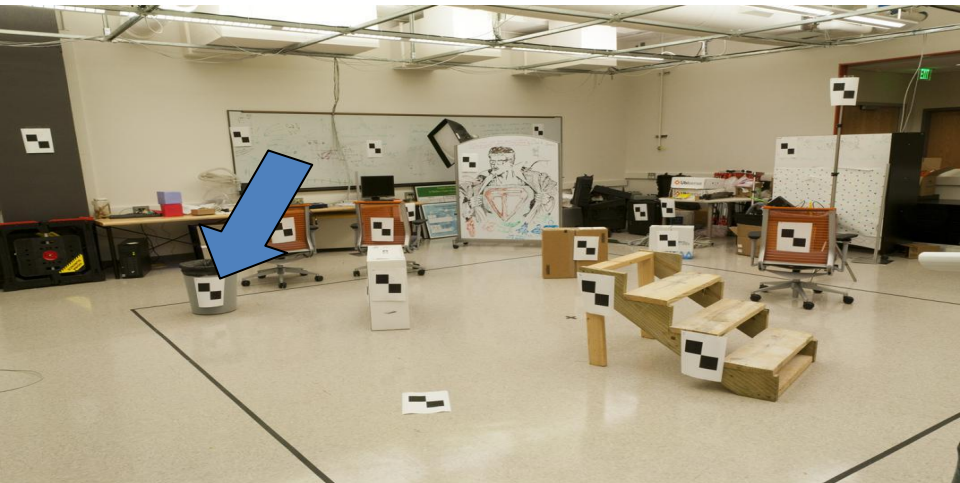
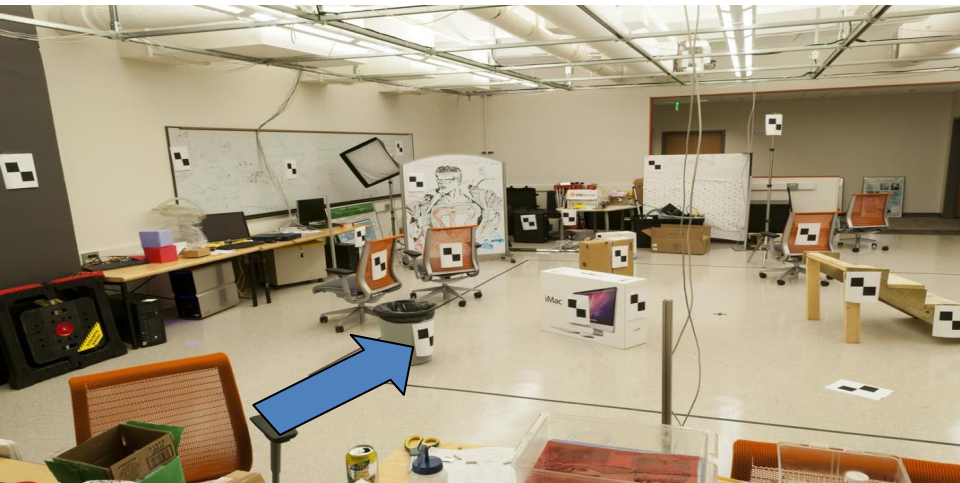
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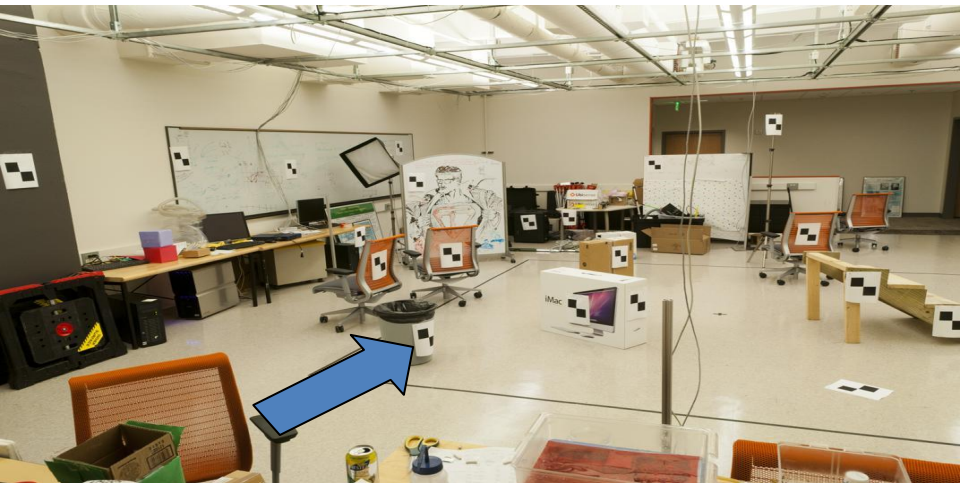
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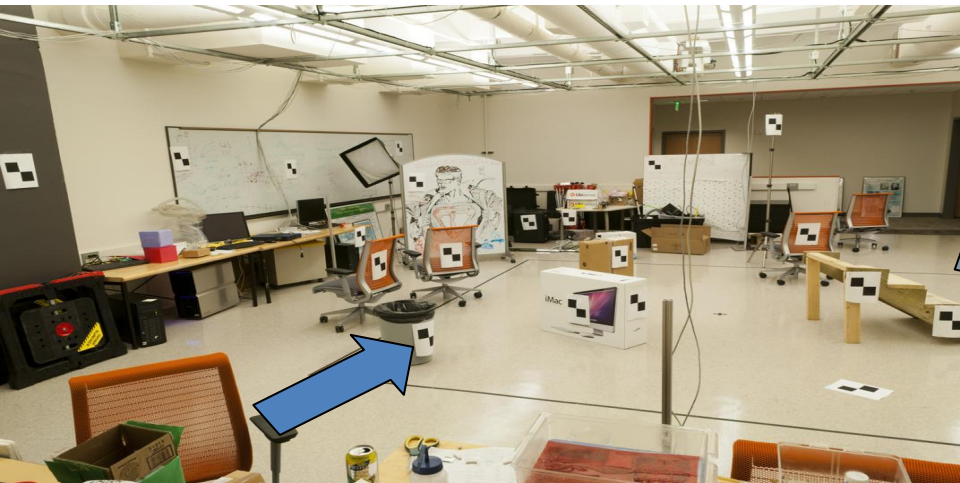
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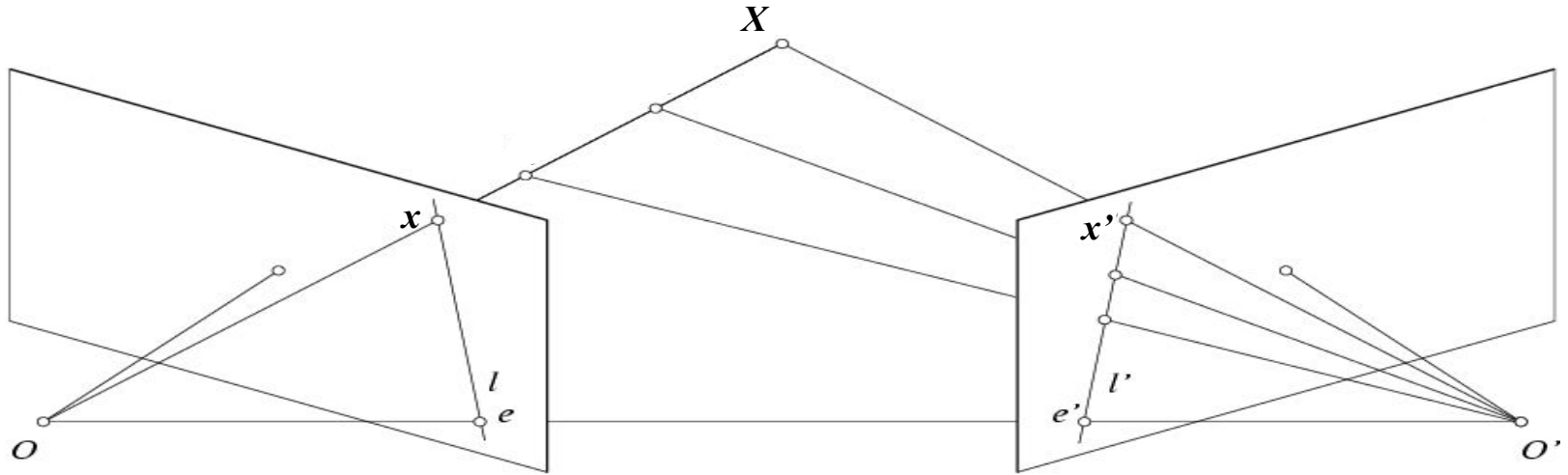


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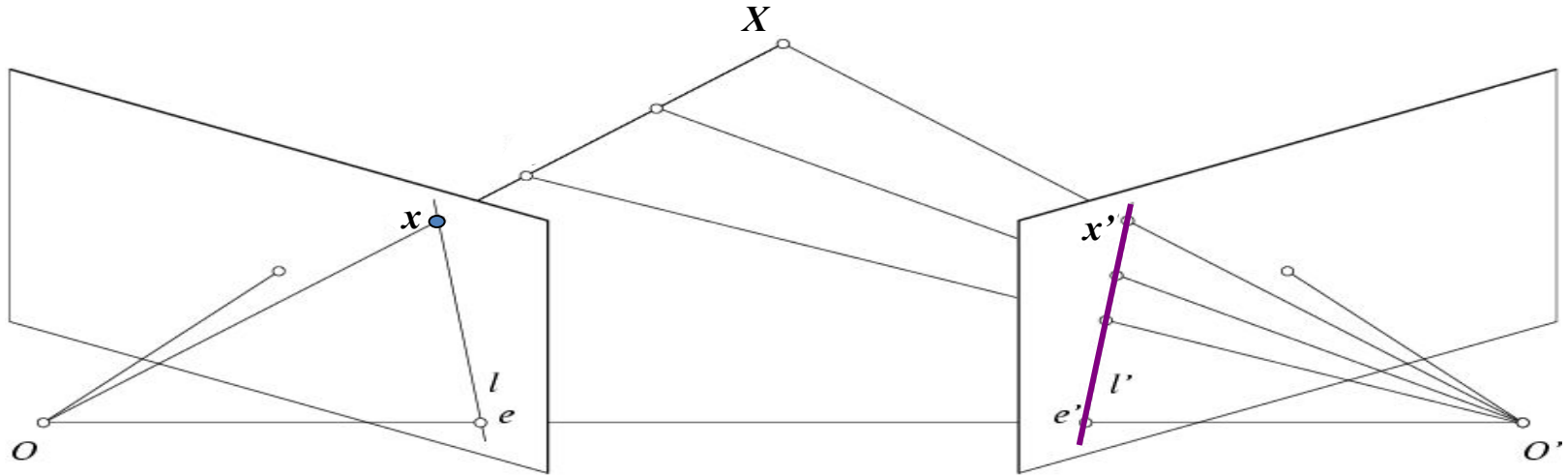


Key idea: Epipolar constraint

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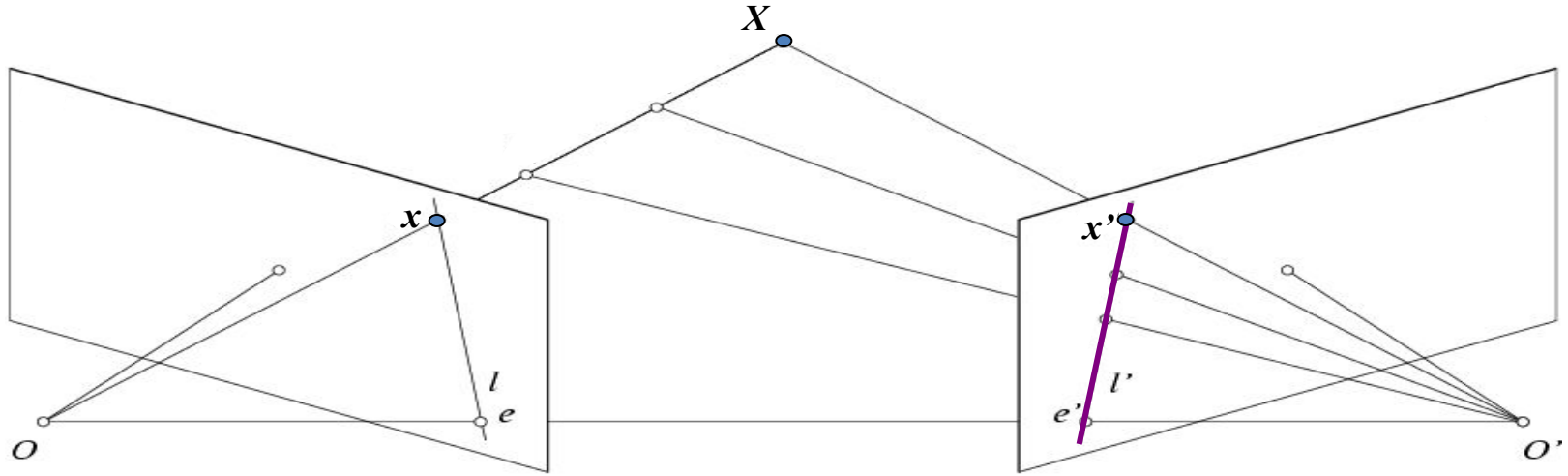


Key idea: Epipolar constraint



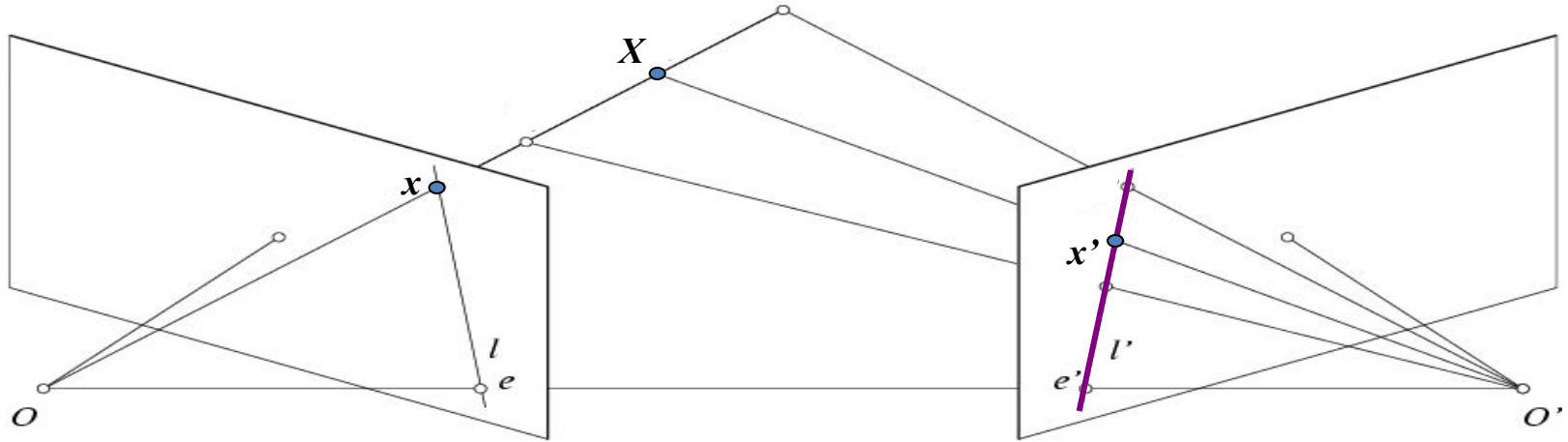
Potential matches for x have to lie on the corresponding line l' .

Key idea: Epipolar constraint



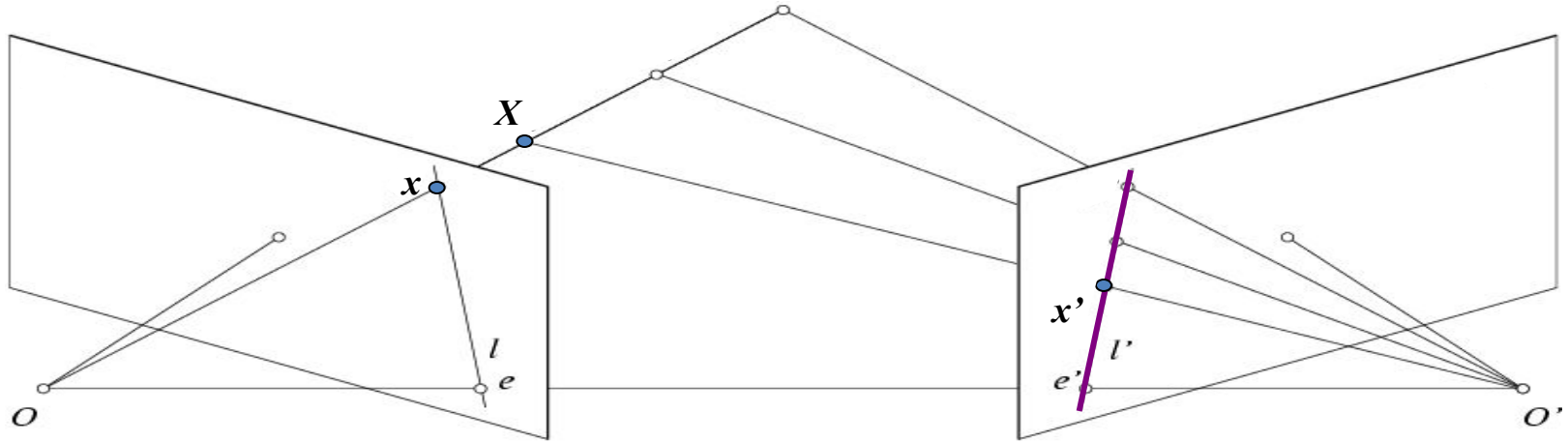
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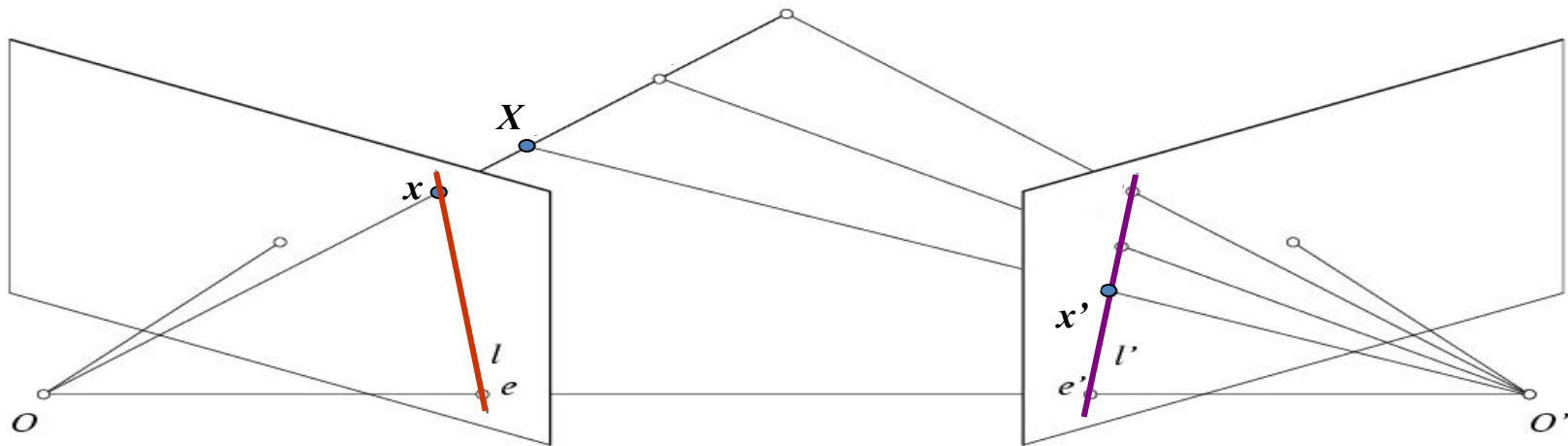
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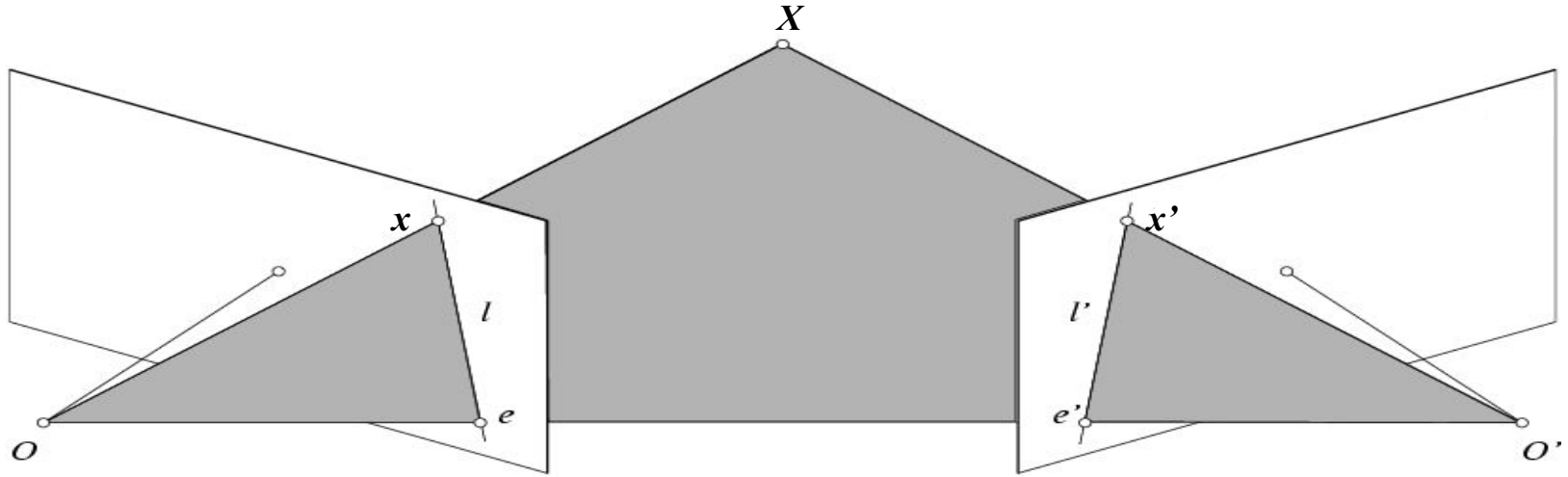
Potential matches for x' have to lie on the corresponding line l .

Wouldn't it be nice to know where matches can live? To constrain our 2d search to 1d.

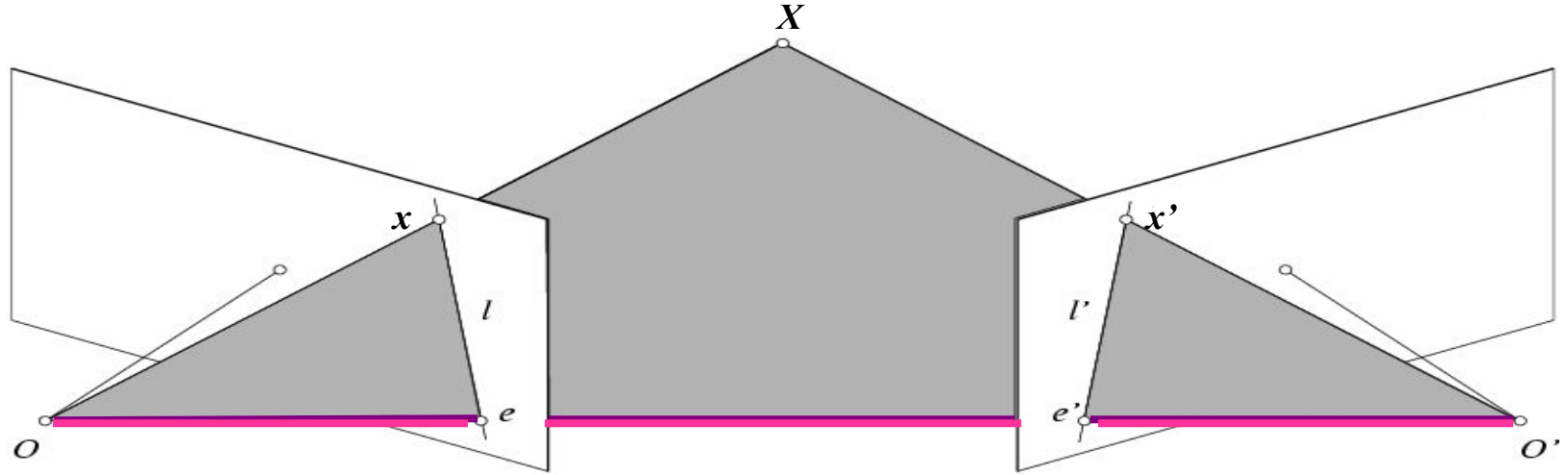
VLFeat's 800 most confident matches
among 10,000+ local features.



Epipolar geometry: notation

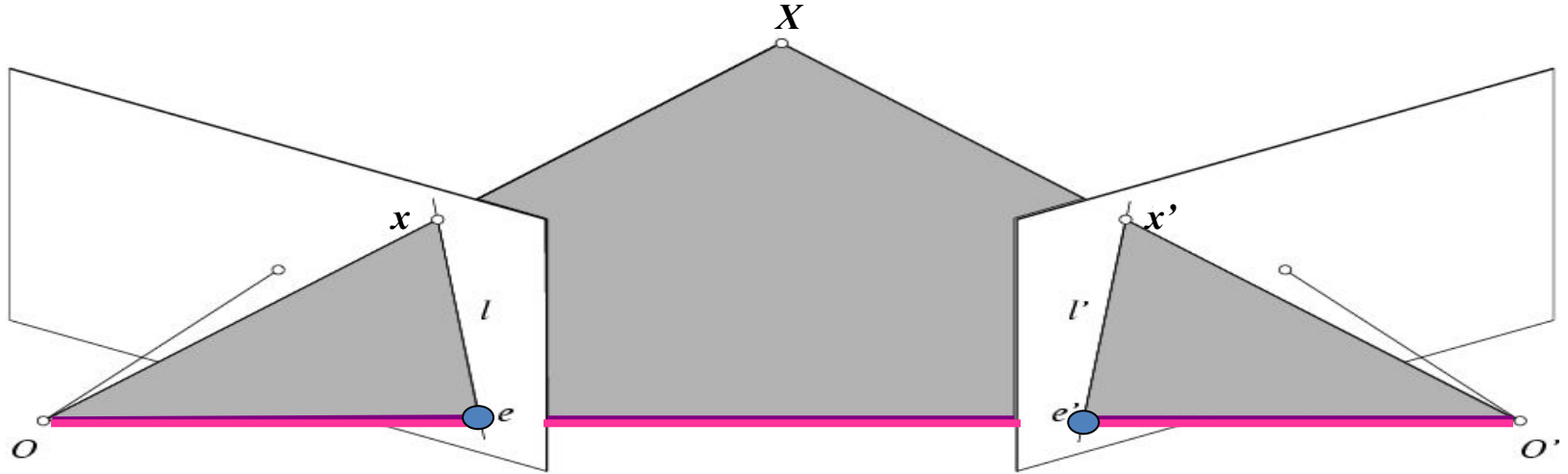


Epipolar geometry: notation



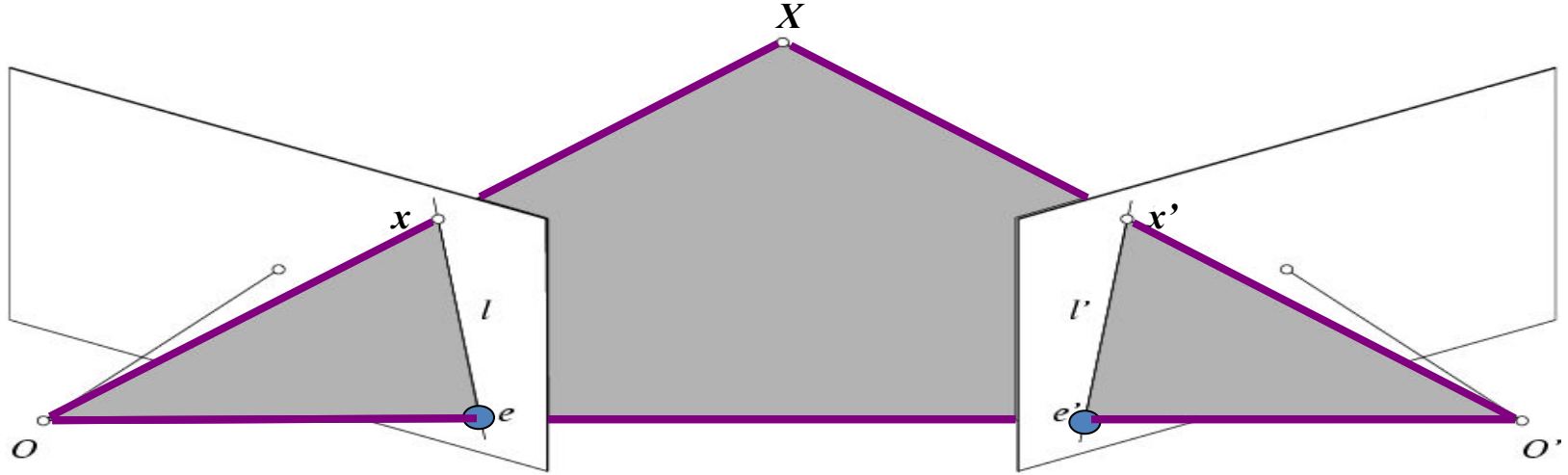
- **Baseline** – line connecting the two camera centers

Epipolar geometry: notation



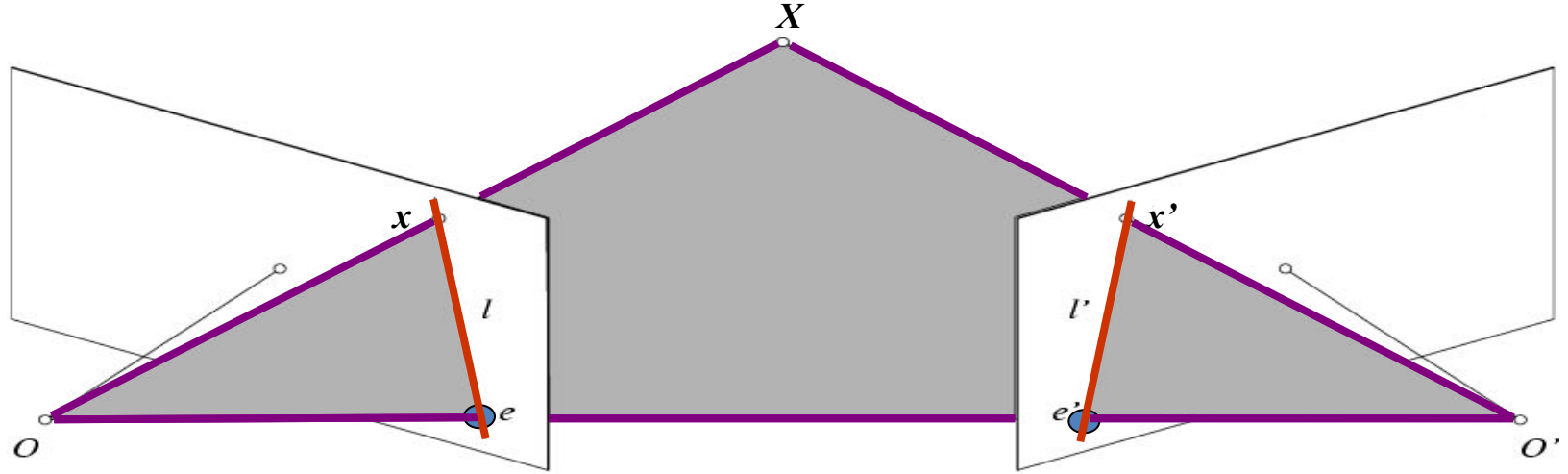
- **Baseline** – line connecting the two camera centers
- **Epipoles**
 - = intersections of baseline with image planes
 - = projections of the other camera center

Epipolar geometry: notation



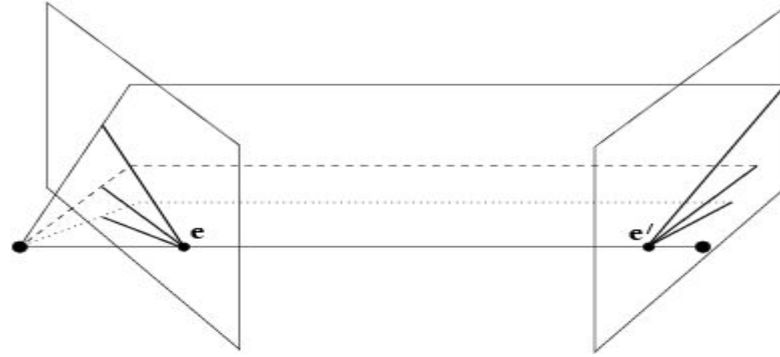
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Epipolar geometry: notation

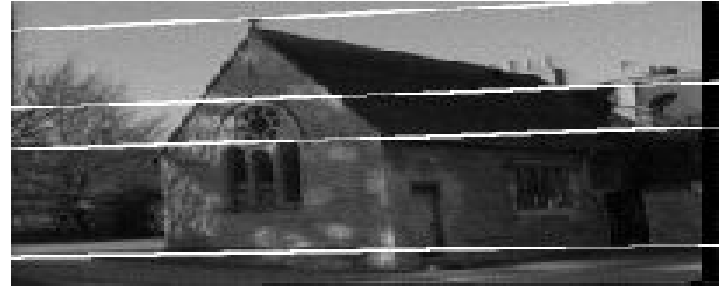


- **Baseline** – line connecting the two camera centers
- **Epipoles**
= intersections of baseline with image planes
= projections of the other camera center
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipolar Lines** - intersections of epipolar plane with image planes (always come in corresponding pairs)

Example: Converging cameras



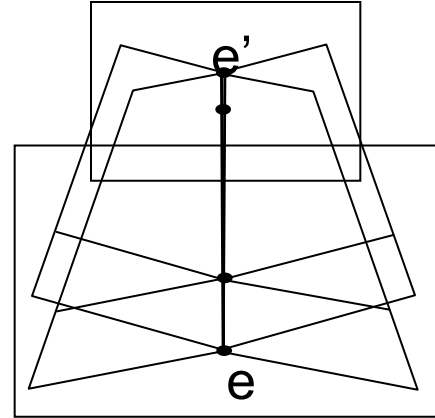
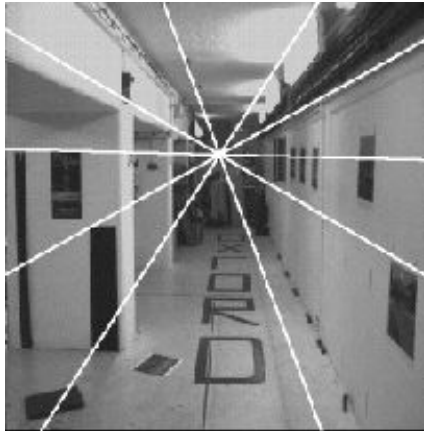
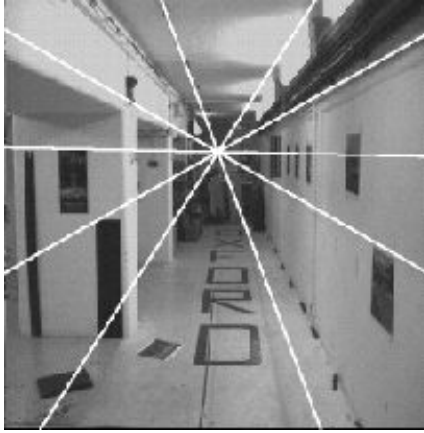
Example: Motion parallel to image plane



Example: Forward motion

What would the epipolar lines look like if the camera moves directly forward?

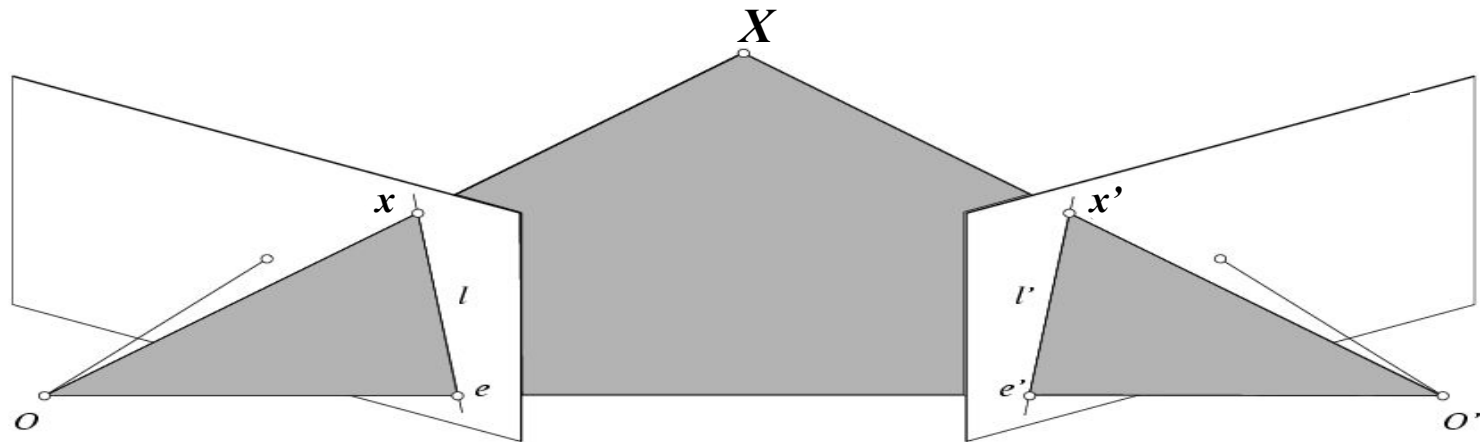
Example: Forward motion



Epipole has same coordinates in both images.

Points move along lines radiating from e:
“Focus of expansion”

Epipolar constraint: Calibrated case



Given the intrinsic parameters of the cameras:

1. Convert to normalized coordinates by pre-multiplying all points with the inverse of the calibration matrix; set first camera's coordinate system to world coordinates

$$\hat{x} = K^{-1} x = X \quad \leftarrow \text{3D scene point}$$

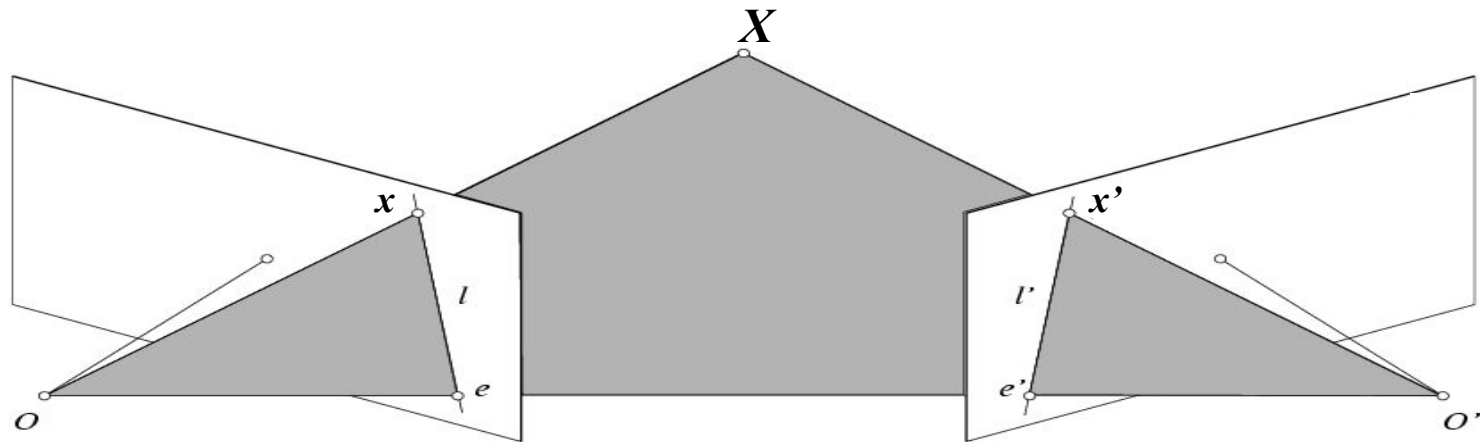
Homogeneous 2d point
(3D ray towards X)

2D pixel coordinate
(homogeneous)

$$\hat{x}' = K'^{-1} x' = X'$$

3D scene point in 2nd
camera's 3D coordinates

Epipolar constraint: Calibrated case



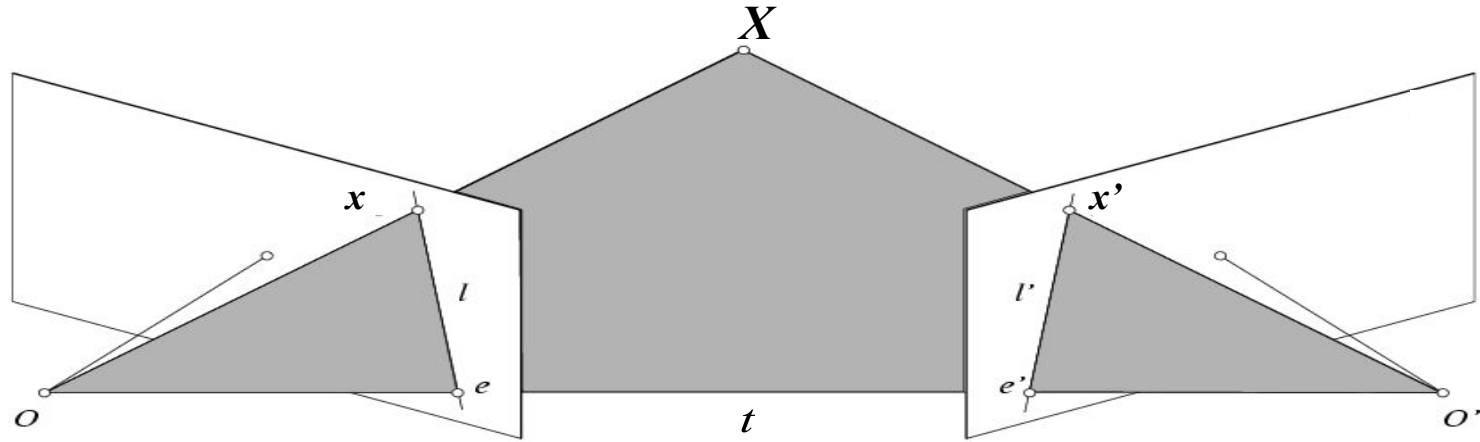
Given the intrinsic parameters of the cameras:

1. Convert to normalized coordinates by pre-multiplying all points with the inverse of the calibration matrix; set first camera's coordinate system to world coordinates
2. Define some R and t that relate X to X' as below

$$\hat{x} = K^{-1}x = \text{for some scale factor } X \quad \hat{x}' = K'^{-1}x' = X'$$

$$\hat{x} = R\hat{x}' + t$$

Epipolar constraint: Calibrated case



$$\hat{x} = K^{-l} x = X$$

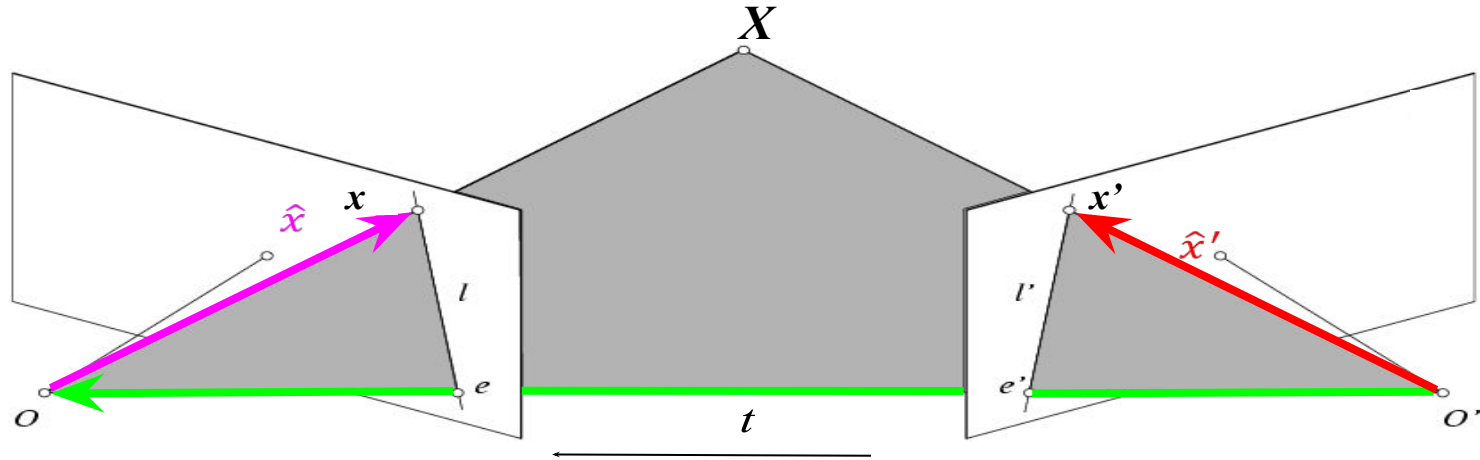
$$\hat{x}' = K'^{-l} x' = X'$$

$$\hat{x} = R\hat{x}' + t$$

$$\hat{x} \cdot [t \times (R\hat{x}')] = 0$$

(because \hat{x} , $R\hat{x}'$, and t are co-planar)

Epipolar constraint: Calibrated case



$$\hat{x} = K^{-l} x = X$$

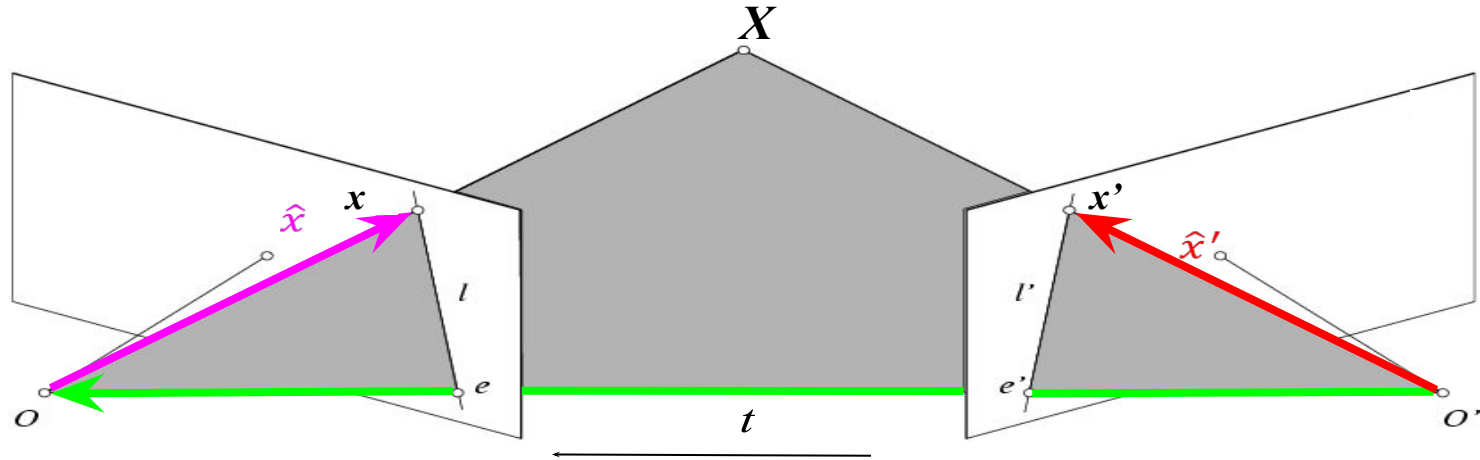
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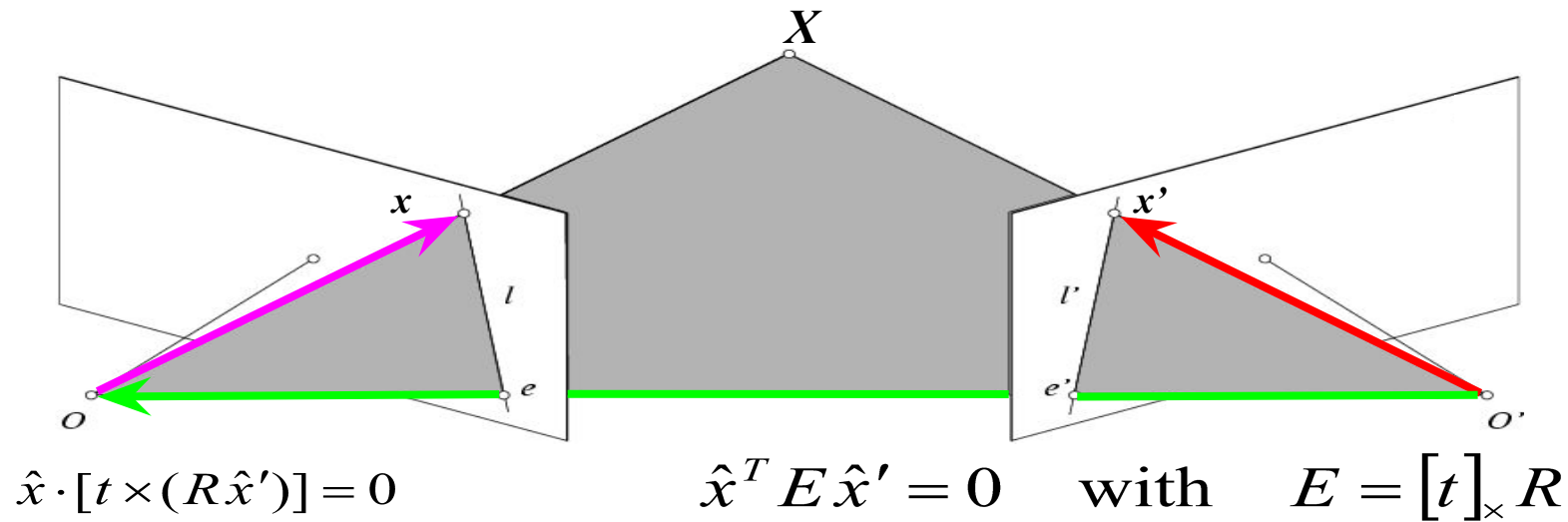
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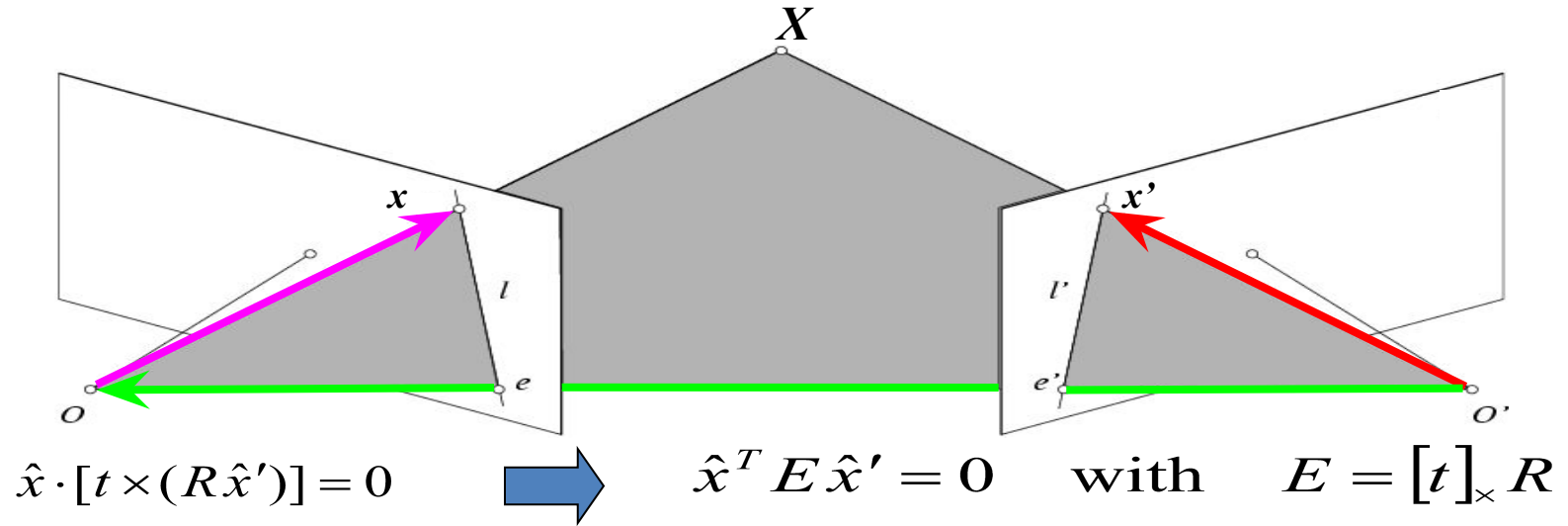
$$\hat{x} = R\hat{x}' + t \quad \Rightarrow \quad \hat{x} \cdot [t \times (R\hat{x}')] = 0$$

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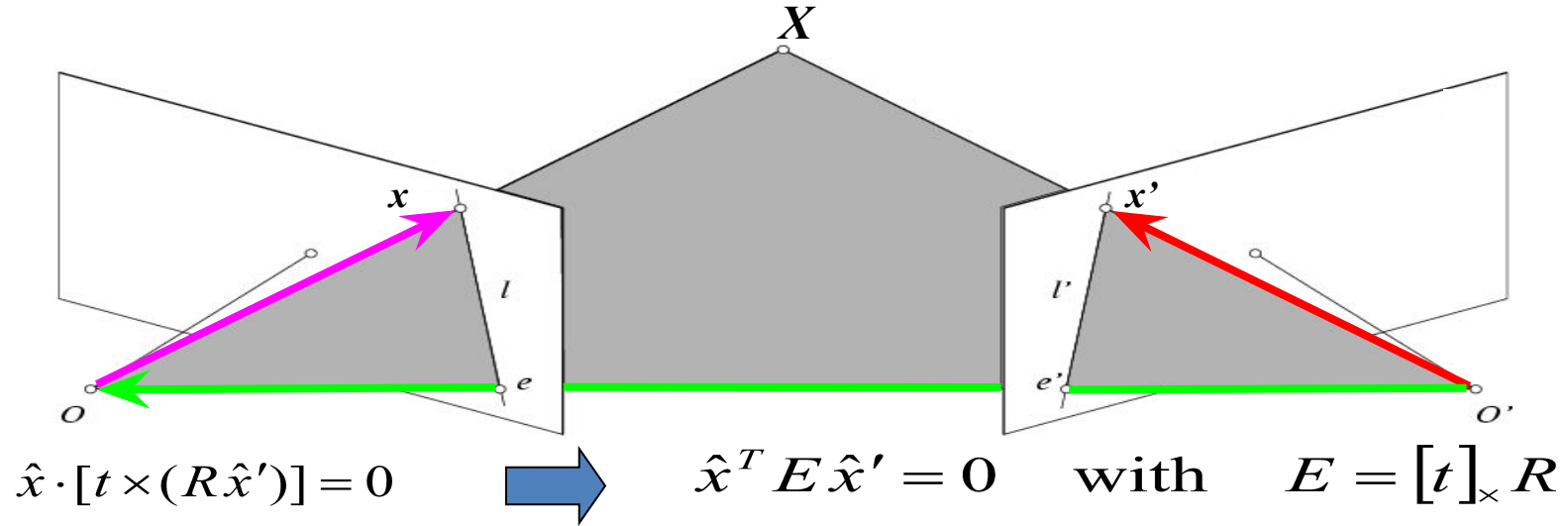
Essential matrix



Essential matrix

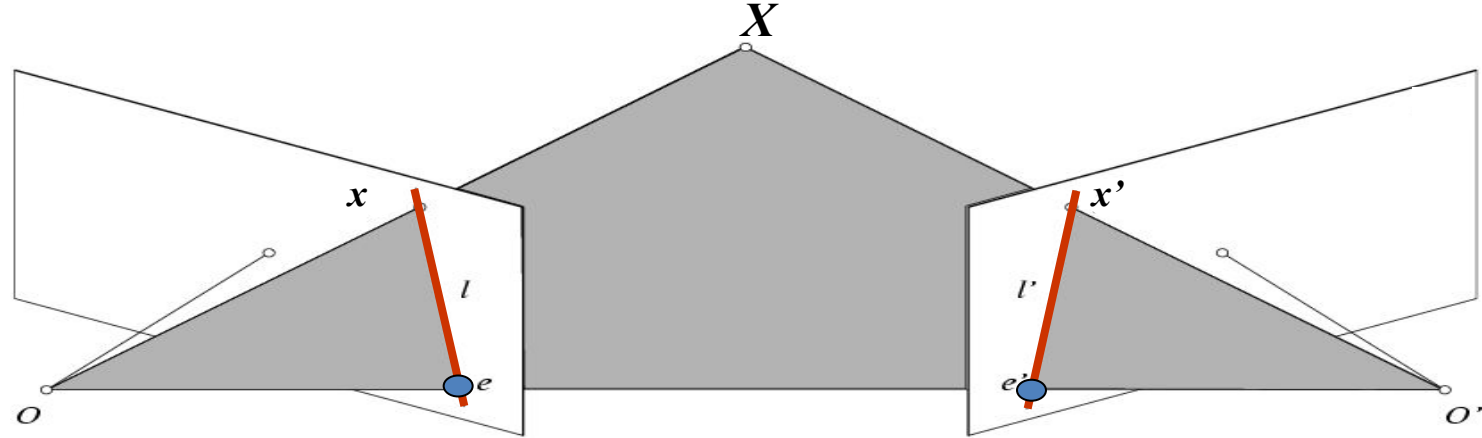


Essential matrix



Essential Matrix
(Longuet-Higgins, 1981)

Properties of the Essential matrix

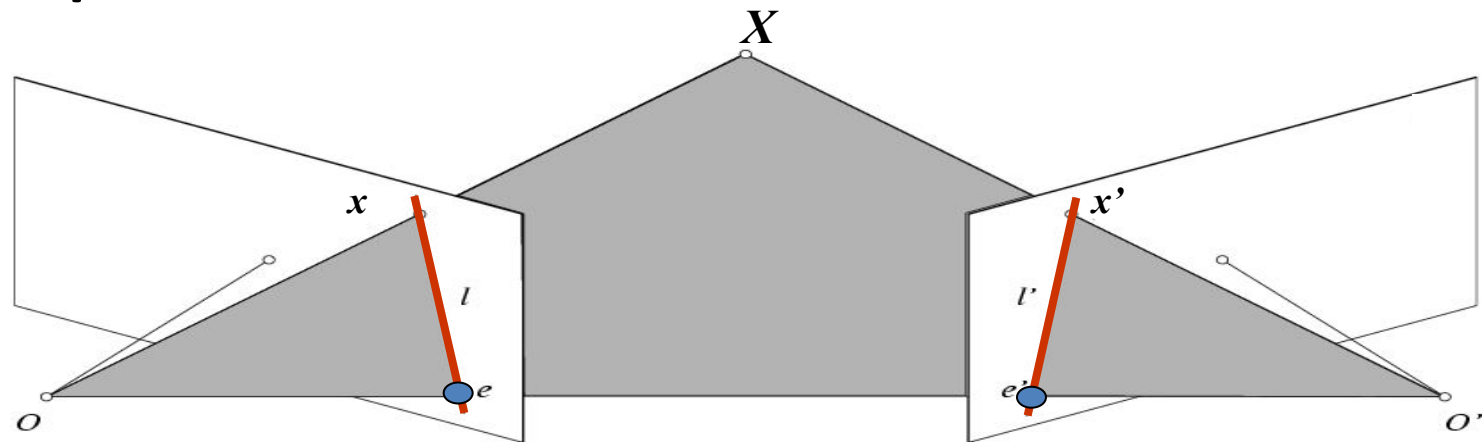


$$\hat{x} \cdot [t \times (R \hat{x}')] = 0 \quad \longrightarrow \quad \hat{x}^T E \hat{x}' = 0 \quad \text{with} \quad E = [t]_{\times} R$$

Drop \wedge below to simplify notation

Skew-sym
metric
matrix

Properties of the Essential matrix



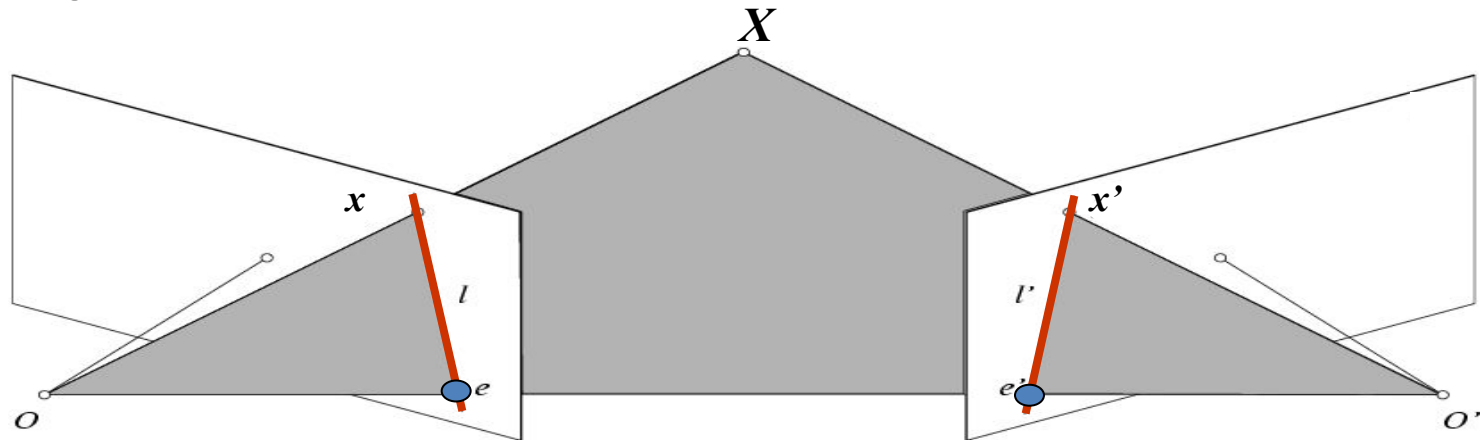
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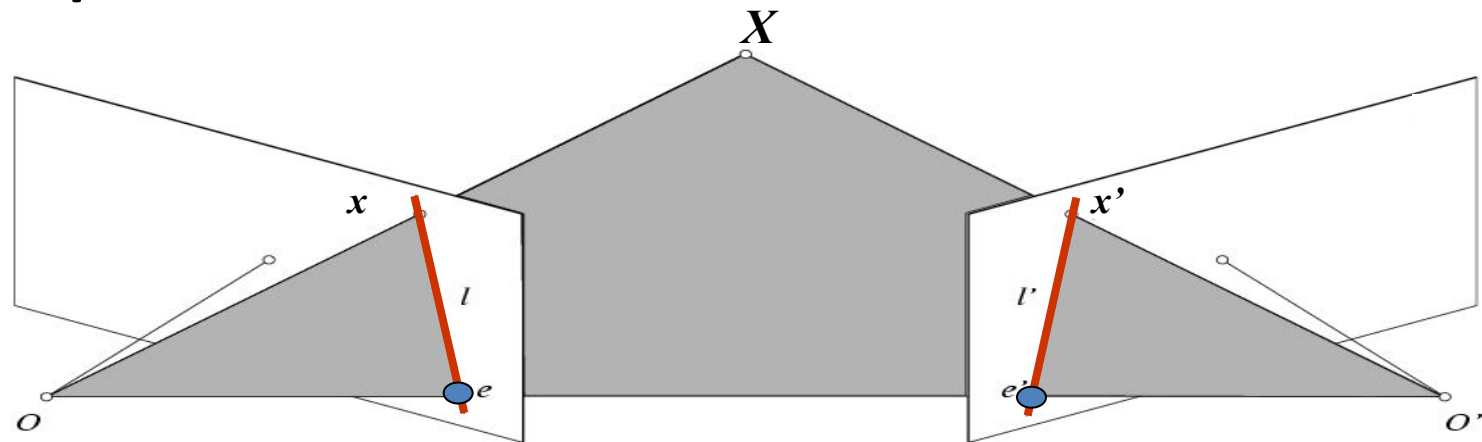
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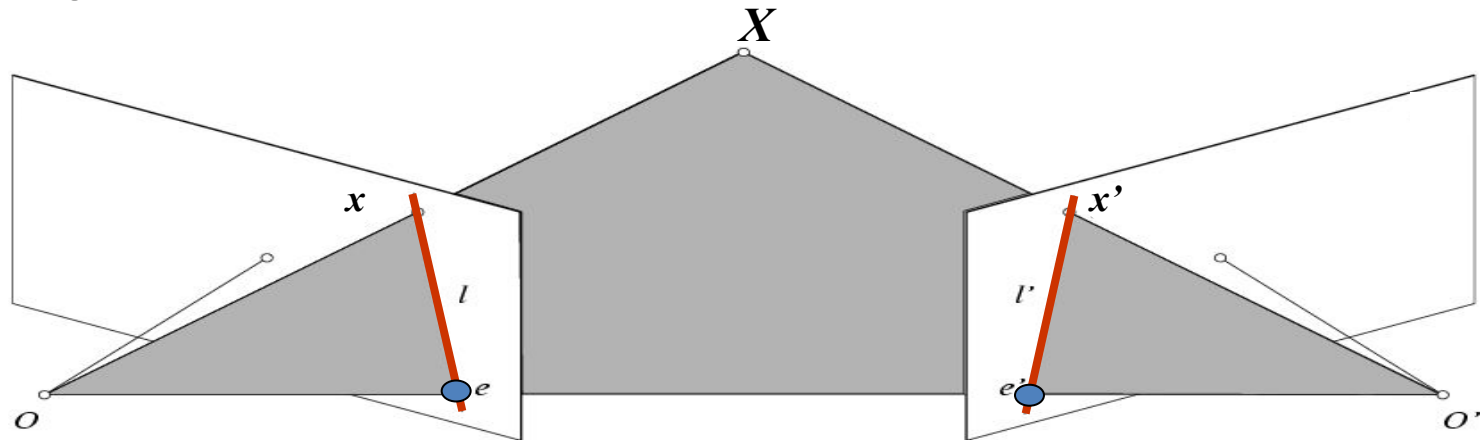
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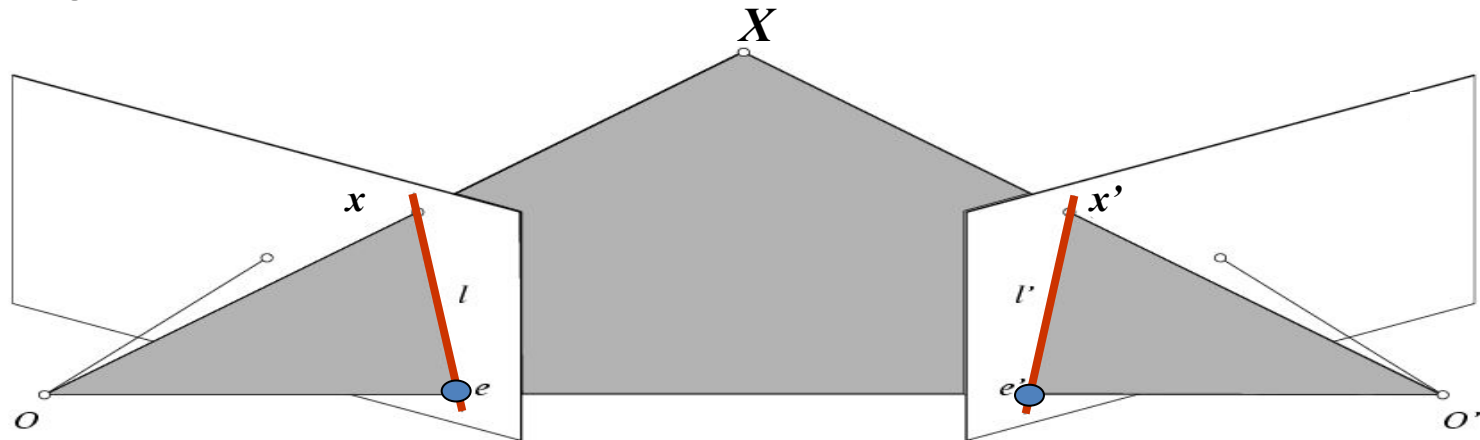
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- E is singular (rank two)
- E has five degrees of freedom
 - (3 for R , 2 for t because it's up to a scale)

Skew-sym
metric
matrix

The Fundamental Matrix

Without knowing K and K' , we can define a similar relation using *unknown* normalized coordinates

$$\hat{x}^T E \hat{x}' = 0$$

$$\hat{x} = K^{-1} x$$

$$\hat{x}' = K'^{-1} x'$$

$$x^T F x' = 0 \quad \text{with} \quad F = K^{-T} E K'^{-1}$$

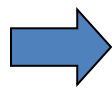
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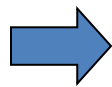
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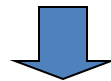
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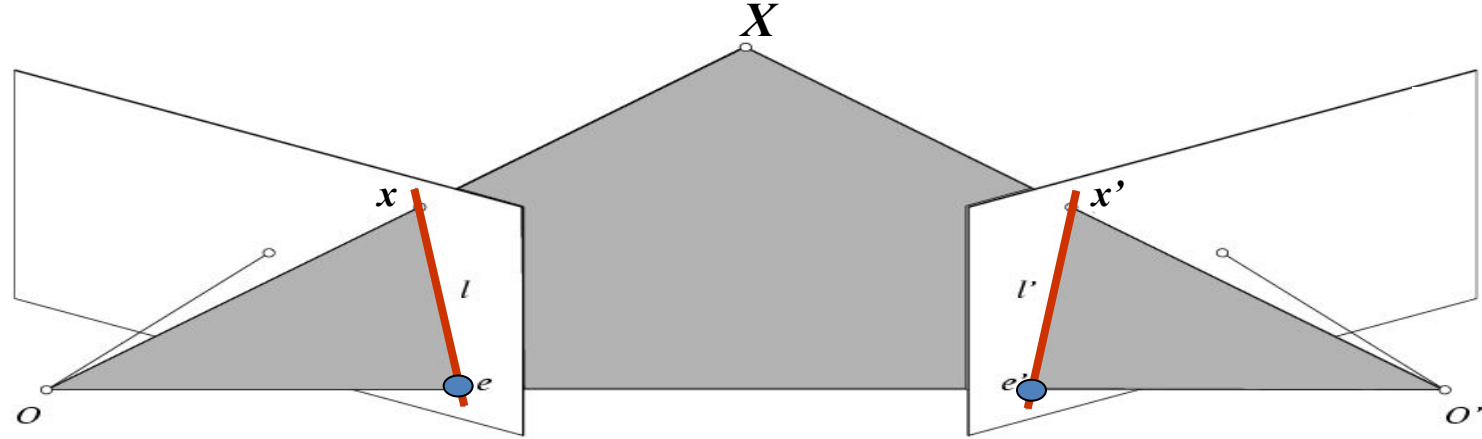


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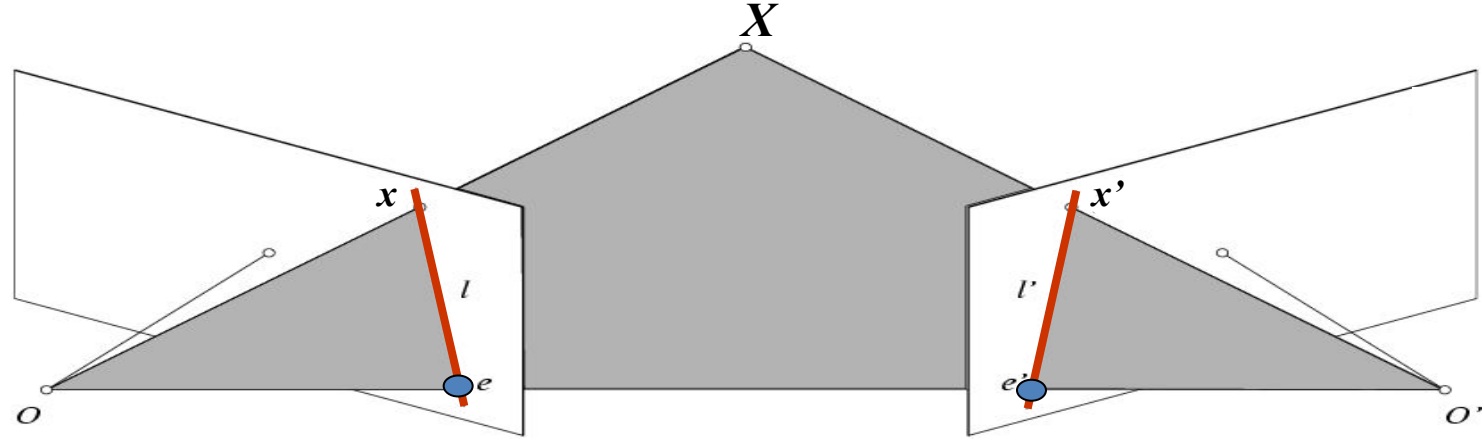
Fundamental Matrix
(Faugeras and Luong, 1992)

Properties of the Fundamental matrix



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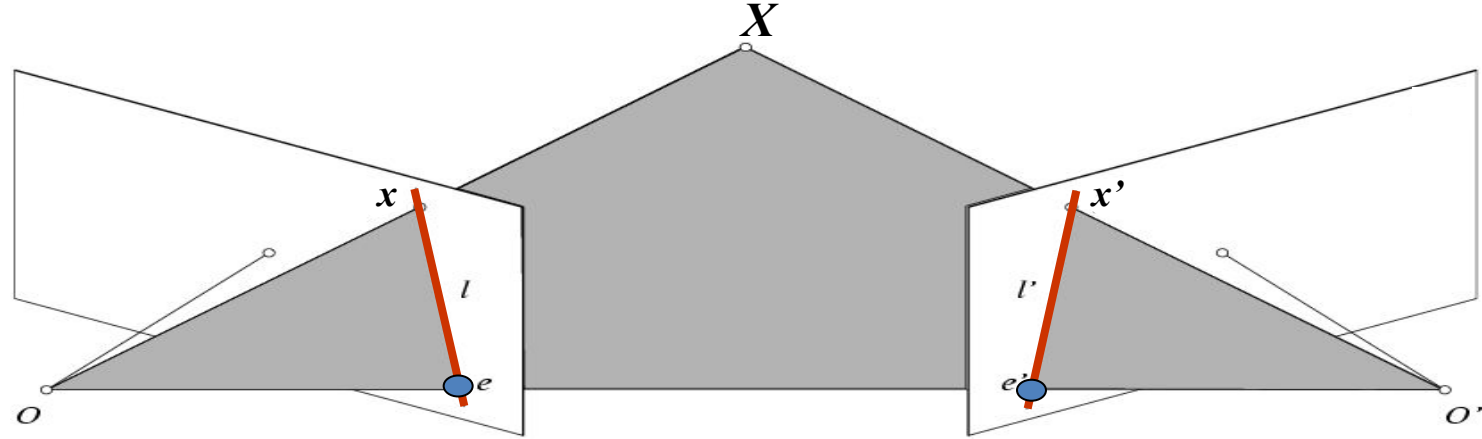
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- $F x'$ is the epipolar line associated with x' ($l = F x'$)

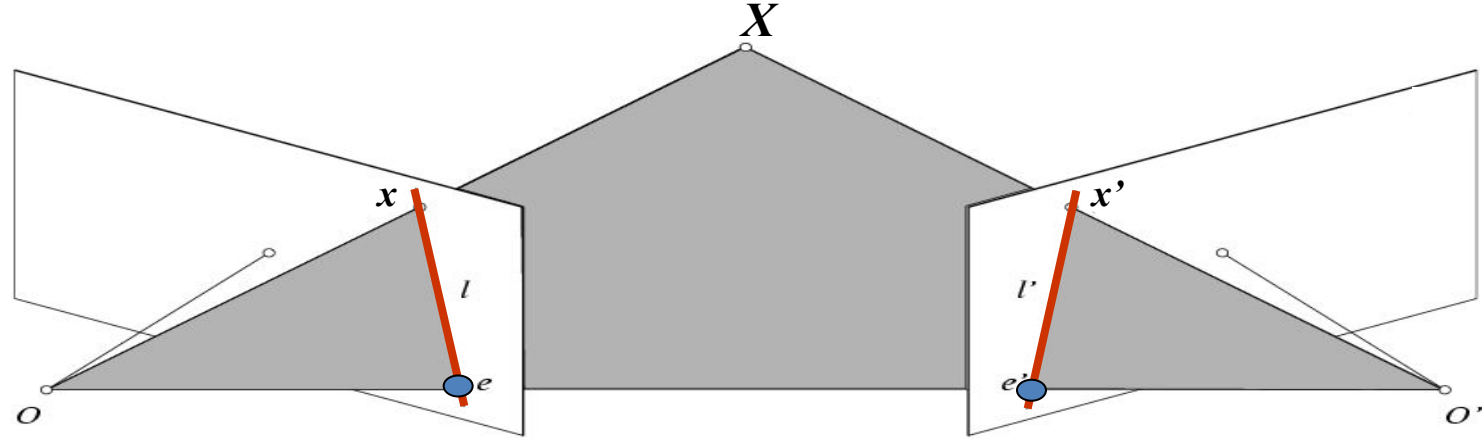
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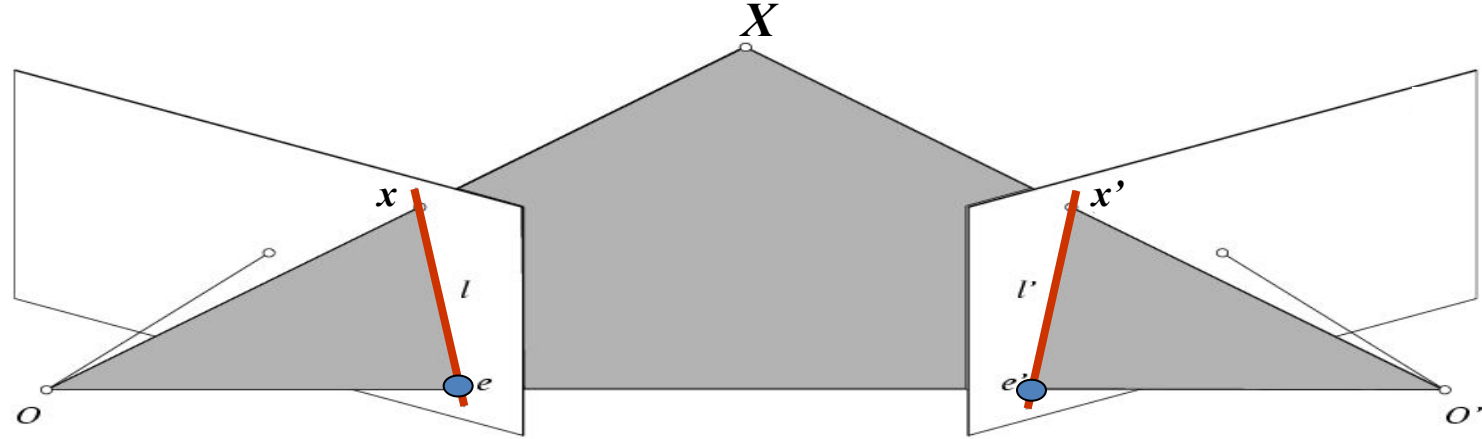
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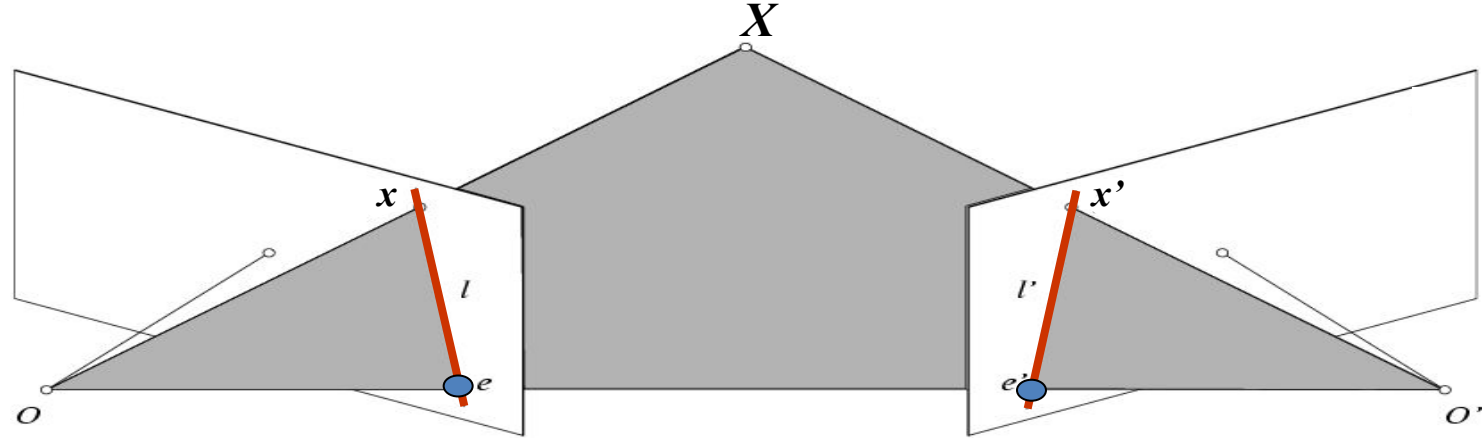
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- F is singular (rank two): $\det(F)=0$

Properties of the Fundamental matrix



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- $F e' = 0$ and $F^T e = 0$
- F is singular (rank two): $\det(F)=0$
- F has seven degrees of freedom: 9 entries but defined up to scale, $\det(F)=0$

Estimating the Fundamental Matrix

- 8-point algorithm
 - Least squares solution using SVD on equations from 8 pairs of correspondences
 - Enforce $\det(F)=0$ constraint using SVD on F
- 7-point algorithm
 - Use least squares to solve for null space (two vectors) using SVD and 7 pairs of correspondences
 - Solve for linear combination of null space vectors that satisfies $\det(F)=0$
- Minimize reprojection error
 - Non-linear least squares

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Note: estimation of F (or E) is degenerate for a planar scene.

8-point algorithm

1. Solve a system of homogeneous linear equations
 - a. Write down the system of equations

$$\mathbf{x}^T F \mathbf{x}' = 0$$

$$uu'f_{11} + uv'f_{12} + uf_{13} + vu'f_{21} + vv'f_{22} + vf_{23} + u'f_{31} + v'f_{32} + f_{33} = 0$$

$$A\mathbf{f} = \begin{bmatrix} u_1u_1' & u_1v_1' & u_1 & v_1u_1' & v_1v_1' & v_1 & u_1' & v_1' & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & u_nv_n' & u_n & v_nu_n' & v_nv_n' & v_n & u_n' & v_n' & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ \vdots \\ f_{33} \end{bmatrix} = \mathbf{0}$$

8-point algorithm

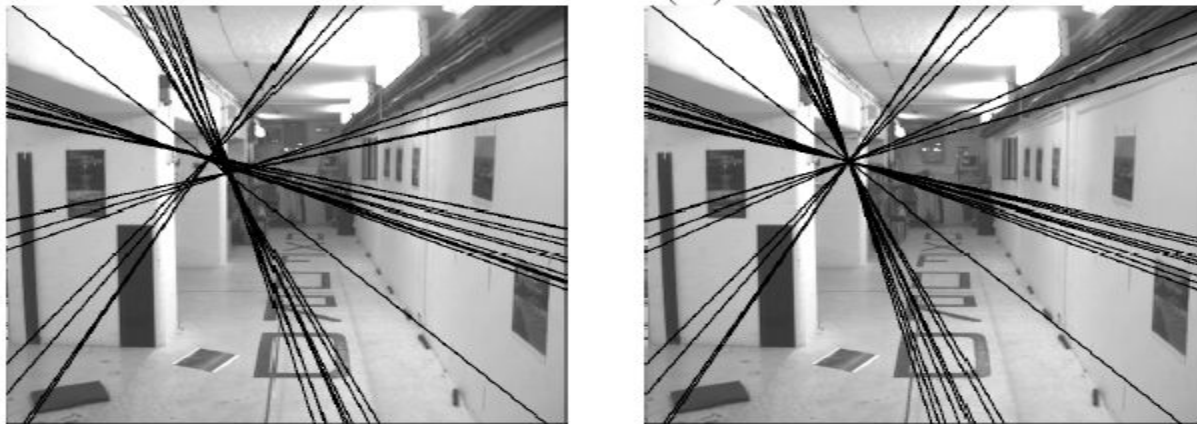
1. Solve a system of homogeneous linear equations
 - a. Write down the system of equations
 - b. Solve \mathbf{f} from $A\mathbf{f}=\mathbf{0}$ using SVD

Matlab:

```
[U, S, V] = svd(A);  
f = V(:, end);  
F = reshape(f, [3 3])';
```

Need to enforce singularity constraint

Fundamental matrix has rank 2 : $\det(\mathbf{F}) = 0$.



Left : Uncorrected \mathbf{F} – epipolar lines are not coincident.

Right : Epipolar lines from corrected \mathbf{F} .

8-point algorithm

1. Solve a system of homogeneous linear equations

a. Write down the system of equations

b. Solve \mathbf{f} from $\mathbf{A}\mathbf{f}=\mathbf{0}$ using SVD
Matlab:

```
[U, S, V] = svd(A);
```

```
f = V(:, end);
```

```
F = reshape(f, [3 3])';
```

2. Resolve $\det(\mathbf{F}) = 0$ constraint using SVD

Matlab:

```
[U, S, V] = svd(F);
```

```
S(3,3) = 0;
```

```
F = U*S*V';
```

8-point algorithm

1. Solve a system of homogeneous linear equations
 - a. Write down the system of equations
 - b. Solve \mathbf{f} from $A\mathbf{f}=\mathbf{0}$ using SVD
2. Resolve $\det(F) = 0$ constraint by SVD

Notes:

3. Use RANSAC to deal with outliers (sample 8 points)
 - How to test for outliers?

Problem with eight-point algorithm

$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -\mathbf{1}$$

Problem with eight-point algorithm

250906.36	183269.57	921.81	200931.10	146766.13	738.21	272.19	198.81
2692.28	131633.03	176.27	6196.73	302975.59	405.71	15.27	746.79
416374.23	871684.30	935.47	408110.89	854384.92	916.90	445.10	931.81
191183.60	171759.40	410.27	416435.62	374125.90	893.65	465.99	418.65
48988.86	30401.76	57.89	298604.57	185309.58	352.87	846.22	525.15
164786.04	546559.67	813.17	1998.37	6628.15	9.86	202.65	672.14
116407.01	2727.75	138.89	169941.27	3982.21	202.77	838.12	19.64
135384.58	75411.13	198.72	411350.03	229127.78	603.79	681.28	379.48

$$\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -\mathbf{1}$$

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Poor numerical conditioning

Problem with eight-point algorithm

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$$\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -\mathbf{1}$$

Poor numerical conditioning

Can be fixed by rescaling the data

The normalized eight-point algorithm

(Hartley, 1995)

The normalized eight-point algorithm

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- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels

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The normalized eight-point algorithm

(Hartley, 1995)

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute \mathbf{F} from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of \mathbf{F} and throw out the smallest singular value)

The normalized eight-point algorithm

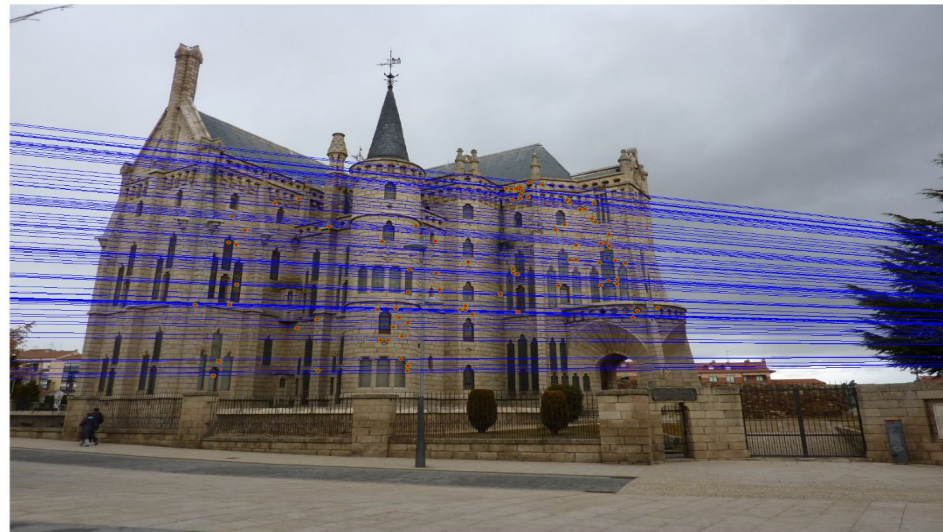
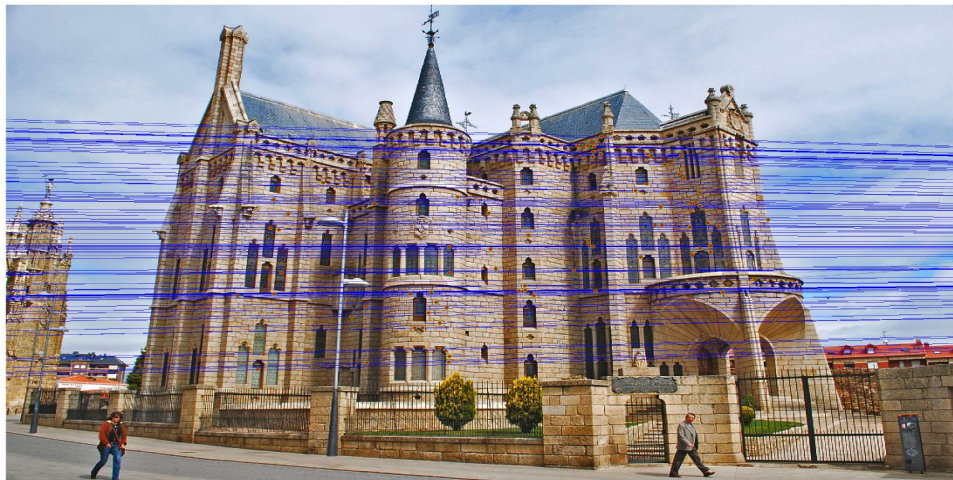
(Hartley, 1995)

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute \mathbf{F} from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of \mathbf{F} and throw out the smallest singular value)
- Transform fundamental matrix back to original units: if \mathbf{T} and \mathbf{T}' are the normalizing transformations in the two images, then the fundamental matrix in original coordinates is $\mathbf{T}'^T \mathbf{F} \mathbf{T}$

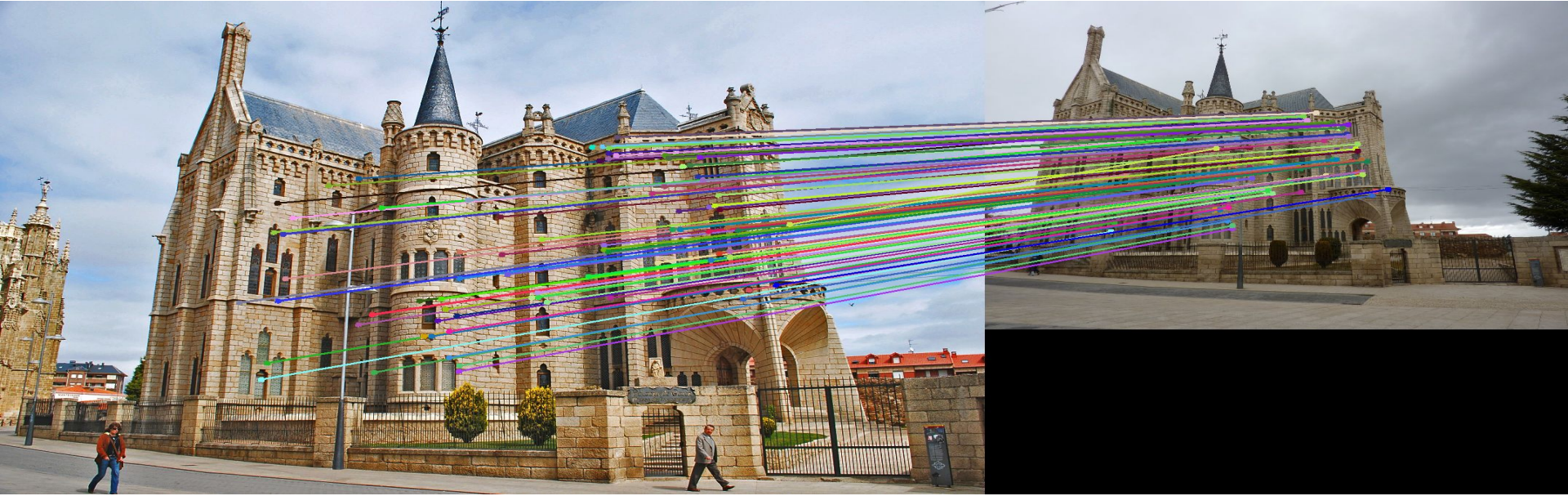
VLFeat's 800 most confident matches
among 10,000+ local features.



Epipolar lines



Keep only the matches that are “inliers” with respect to the “best” fundamental matrix



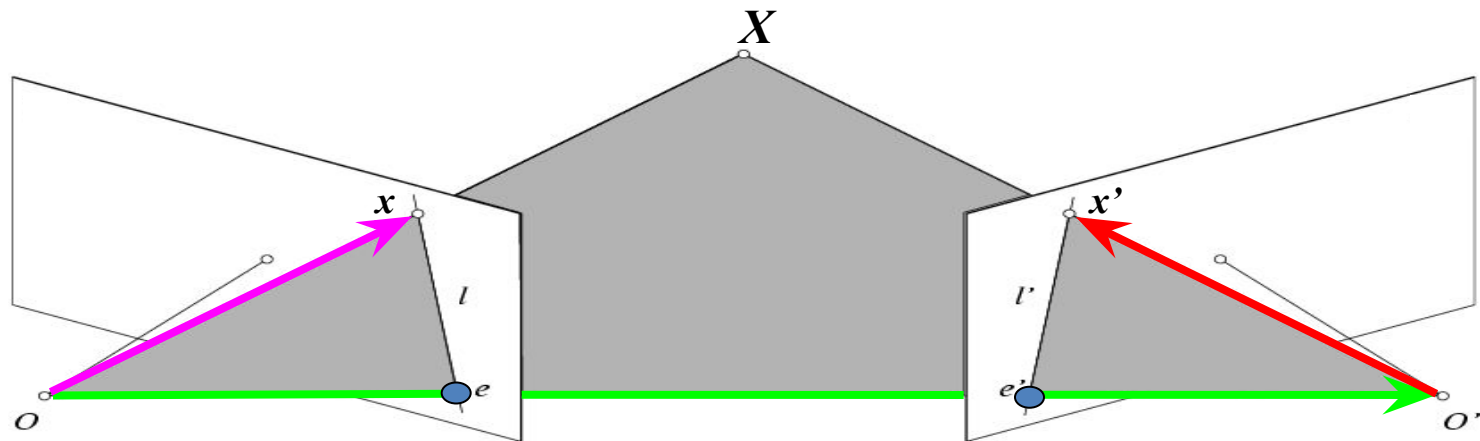
Review

- Depth from stereo: main idea is to triangulate from corresponding image points.
- Epipolar geometry defined by two cameras
 - We've assumed known extrinsic parameters relating their poses
- Epipolar constraint limits where points from one view will be imaged in the other
 - Makes search for correspondences quicker
- **Terms:** epipole, epipolar plane / lines, disparity, rectification, intrinsic/extrinsic parameters, essential matrix, baseline

A large, solid blue abstract shape that spans across the upper half of the slide. It has a jagged, angular appearance with several sharp points and a generally upward-sloping trend from left to right.

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Epipolar constraint: Uncalibrated case



- If we don't know K and K' , then we can write the epipolar constraint in terms of *unknown* normalized coordinates:

$$\hat{x}^T E \hat{x}' = 0$$

$$x = K \hat{x}, \quad x' = K' \hat{x}'$$

7-point algorithm

Computation of F from 7 point correspondences

- (i) Form the 7×9 set of equations $A\mathbf{f} = 0$.
- (ii) System has a 2-dimensional solution set.
- (iii) General solution (use SVD) has form

$$\mathbf{f} = \lambda \mathbf{f}_0 + \mu \mathbf{f}_1$$

- (iv) In matrix terms

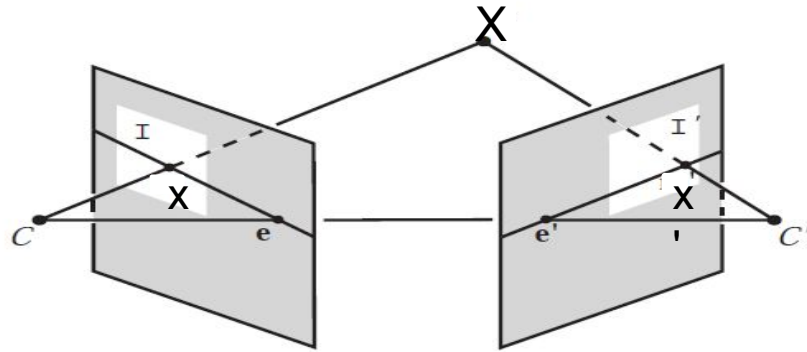
$$F = \lambda F_0 + \mu F_1$$

- (v) Condition $\det F = 0$ gives cubic equation in λ and μ .
- (vi) Either one or three real solutions for ratio $\lambda : \mu$.

Faster (need fewer points) and could be more robust (fewer points), but also need to check for degenerate cases

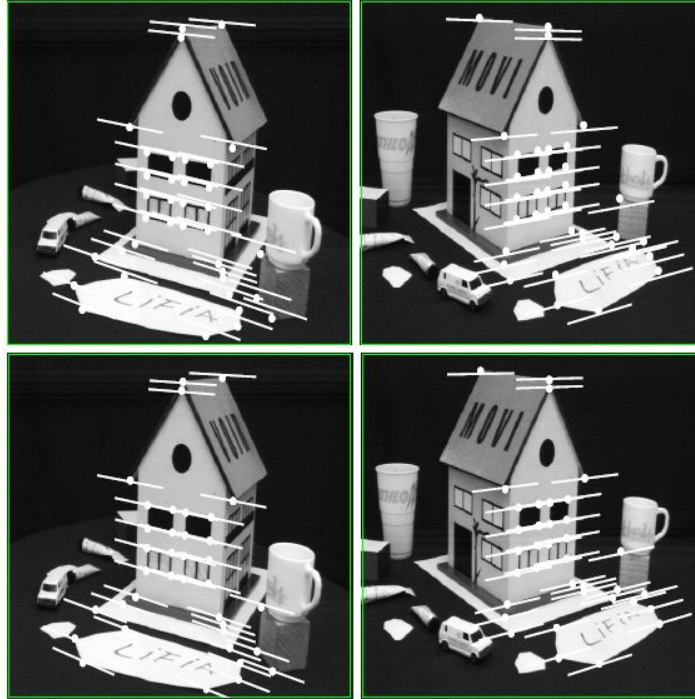
“Gold standard” algorithm

- Use 8-point algorithm to get initial value of F
- Use F to solve for P and P' (discussed later)
- Jointly solve for 3d points \mathbf{X} and \mathbf{F} that minimize error



See Algorithm 11.2 and Algorithm 11.3 in HZ (pages 284-285) for details

Comparison of estimation algorithms



	8-point	Normalized 8-point	Nonlinear least squares
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel

We can get projection matrices \mathbf{P} and \mathbf{P}' up to a projective ambiguity

$$\mathbf{P} = [\mathbf{I} \mid \mathbf{0}] \quad \mathbf{P}' = \begin{bmatrix} \overset{\text{K'*rotation}}{\downarrow} [\mathbf{e}']_{\times} \overset{\text{K'*translation}}{\swarrow} \mathbf{F} \mid \mathbf{e}' \end{bmatrix} \quad \mathbf{e}'^T \mathbf{F} = 0$$

Code: See HZ p. 255-256

```
function P = vgg_P_from_F(F)
[U,S,V] = svd(F);
e = U(:,3);
P = [-vgg_contreps(e)*F e];
```

If we know the intrinsic matrices (\mathbf{K} and \mathbf{K}'), we can resolve the ambiguity

From epipolar geometry to camera calibration

- Estimating the fundamental matrix is known as “weak calibration”
- We clf we know the calibration matrices of the two cameras, we can estimate the essential matrix: $E = K^T F K'$
- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters

Let's recap...

- [Fundamental matrix song](#)