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# Cramér-type moderate deviation for general self-normalized non-linear statistics

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Let $\{(X_i, Y_i)\}_{i=1}^n$ be a sequence of independent bivariate random vectors. In this paper we establish a Cramér-type moderate deviation theorem for general self-normalized non-linear statistics $(\sum_{i=1}^n X_i + D_{1n})/(\sum_{i=1}^n Y_i^2(1 + D_{2n}))^{1/2}$

*Keywords:* Cramér-type moderate deviation; Stein's Method

## 1. Introduction

In this paper, we aim to prove a Cramér-type moderate deviation for general non-linear statistics . . .

## 2. Main Results

Let $(X_1, Y_1), (X_2, Y_2), \ldots, (X_n, Y_n)$ be independent bivariate random vectors, and for convenience presentation satisfying

$$\mathbb{E}X_i = 0 \text{ for } i \geq 1 \quad \text{and} \quad \sum_{i=1}^n \mathbb{E}X_i^2 = 1 = \sum_{i=1}^n \mathbb{E}Y_i^2$$

Let

$$S_n = \textstyle\sum_{i=1}^n X_i,\ V_n^2 = \sum_{i=1}^n Y_i^2 \text{ and } T_n = \frac{S_n + D_{1n}}{V_n(1+D_{2n})^{\frac{1}{2}}}$$

Where $D_{1n} = D_{1n}(X_1, \ldots, X_n, Y_1, \ldots, Y_n)$ is a measurable function of $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$ and $D_{2n}$ is also a measurable function of $\{X_i\}_{i=1}^n$ and $\{Y_i\}_{i=1}^n$. To ensure that $T_n$ is well defined, it is assumed that $1 + D_{2n} > 0$.

**Theorem 2.1.** *Suppose there exits some constants $c_0 > 0$ such that for $x > 0$ satisfying*

$$\mathbb{E}e^{min\{\frac{X_i^2}{Y_i^2 + c_0\mathbb{E}Y_i^2}, 2xX_i\}} < \infty$$

*and we assume* $\mathbb{E}|X_i|^3 < \infty$ *and* $\mathbb{E}|Y_i|^3 < \infty$ *for* $i \geq 1$. *Then there exist absolute positive constants* $0 < c_1 \leq \frac{1}{25}$ *and* $A > 0$ *such that*

$$\begin{aligned} P(T_n \geq x) \leq & [1-\Phi(x)]\Psi_x^* e^{O_1 R_x}(1 + O_2(1+x)L_{3,n} + O_3 Q_{n,x}) \\ & + P(|D_{1n}| > V_n/4x) + P(|D_{2n}| > 1/4x^2) \end{aligned} \tag{2.1}$$

$$P(T_n \geq x) \geq \Psi_x^* e^{-O_4 R_x}[1-\Phi(x)](1 - O_5(1+x)L_{3,n} - O_6 Q_{n,x}) \tag{2.2}$$

*where*

$$\Psi_x^* = \exp\{\frac{x^3}{6}\sum_{i=1}^n EX_i^3 - \frac{x^3}{2}\sum_{i=1}^n EX_iY_i^2\}$$

$$L_{3,n} = \sum_{i=1}^n (\mathbb{E}|X_i|^3 + \mathbb{E}|Y_i|^3)$$

$$\begin{aligned} \delta_{x,i} = & (1+x)^3 \left(\mathbb{E}[|X_i|^3 1\{|(1+x)X_i| > 1\}] + \mathbb{E}[|Y_i|^3 1\{|(1+x)Y_i| > 1\}]\right) \\ & + (1+x)^4 \left(\mathbb{E}[|X_i|^4 1\{|(1+x)X_i| \leq 1\}] + \mathbb{E}[|Y_i|^4 1\{|(1+x)Y_i| \leq 1\}]\right) \end{aligned} \tag{2.3}$$

$$r_{x,i} = \mathbb{E}[\exp\{\min(\frac{X_i^2}{Y_i^2 + c_0\mathbb{E}Y_i^2}, 2xX_i)\}1\{|(1+x)X_i| > 1\}]$$

$$R_{x,i} = \delta_{x,i} + r_{x,i},\ \delta_x = \sum_{i=1}^n \delta_{x,i},\ r_x = \sum_{i=1}^n r_{x,i},\ R_x = \delta_x + r_x$$

*for all* $x > 0$ *satisfying*

$$(1+x)L_{3,n} \leq c_1, \quad x^{-2}R_x \leq c_1, \tag{2.4}$$

$$x \leq \frac{\frac{1}{4} \wedge \frac{1}{2\sqrt{c_0}}}{[\max_i(\mathbb{E}|X_i|^3 + \mathbb{E}|Y_i|^3)]^{1/3}}, \quad \max_i r_{x,i} \leq c_1, \tag{2.5}$$

*where* $|O_i| \leq A, i = 1, \ldots, 6$

## 3. Proofs

**Proof of Theorem 2.1.** Since $(1 + D_{2n})^{\frac{1}{2}} \geq 1 + \min(D_{2n}, 0)$, and the elementary inequality:

$$1 + s/2 - s^2/2 \leq (1+s)^{\frac{1}{2}} \leq 1 + s/2, \quad s \geq -1 \tag{3.1}$$

Let $s = V_n^2 - 1$, then we have

$$(1 + V_n^2)/2 - (v_n^2 - 1)^2/2 \leq V_n \leq 1 + (V_n^2 - 1)/2 \tag{3.2}$$

Which leads to

$$\begin{aligned}V_n(1+D_{2n})^{\frac{1}{2}} &\geq V_n + V_n \min(D_{2n},0)\\ &\geq (1+V_n^2)/2 - (V_n^2-1)^2/2 + (1+(V_n^2-1)/2)\min(D_{2n},0)\\ &\geq (1+V_n^2)/2 - (V_n^2-1)^2 + \min(D_{2n},0)\end{aligned} \tag{3.3}$$

Using inequality $2ab \leq a^2+b^2$ yields the reverse inequality

$$V_n(1+D_{2n})^{\frac{1}{2}} \leq V_n^2/2 + (1+D_{2n})/2 \tag{3.4}$$

consequantly, for any $x>0$

$$\begin{aligned}\{T_n \geq x\} &= \{S_n + D_{1n} \geq xV_n(1+D_{2n})^{1/2}\}\\ &\subseteq \{S_n + D_{1n} \geq x((1+V_n^2)/2 - (V_n^2-1)^2 + \min(D_{2n},0))\}\\ &= \{xS_n - x^2V_n^2/2 \geq x^2/2 - x[x(V_n^2-1)^2 + D_{1n} - x\min(D_{2n},0)]\},\end{aligned} \tag{3.5}$$

and

$$\{T_n \geq x\} \supseteq \{xS_n - x^2V_n^2/2 \geq x^2/2 + x[xD_{2n}/2 - D_{1n}]\}. \tag{3.6}$$

We proof the theorem for two scenarios $0<x\leq 1$ and $x>1$, respectively. For $0<x\leq 1$, it is sufficient to prove a Berry-Esseen bound.

**Proposition 3.1.** *dd*

Then by (3.5), we have for $x \geq 1$

$$\begin{aligned}P(T_n \geq x) &= P(\frac{S_n + D_{1n}}{V_n(1+D_{2n})^{\frac{1}{2}}} \geq x)\\ &\leq P(S_n \geq xV_n(1+\min(D_{2n},0)) - D_{1n})\\ &\leq P(S_n \geq xV_n(1+\min(D_{2n},0)) - D_{1n}, |D_{1n}| \leq \frac{V_n}{4x}, |D_{2n}| \leq \frac{1}{4x^2})\\ &\quad + P(\frac{|D_{1n}|}{V_n} > \frac{1}{4x}) + P(|D_{2n}| > \frac{1}{4x^2})\\ &\leq P(S_n \geq xV_n(1+\min(D_{2n},0)) - D_{1n}, |D_{1n}| \leq \frac{V_n}{4x}, |D_{2n}| \leq \frac{1}{4x^2}, |V_n^2-1| \leq \frac{1}{2x})\\ &\quad + P(S_n \geq xV_n(1+\min(D_{2n},0)) - D_{1n}, |D_{1n}| \leq \frac{V_n}{4x}, |D_{2n}| \leq \frac{1}{4x^2}, |V_n^2-1| > \frac{1}{2x})\\ &\quad + P(\frac{|D_{1n}|}{V_n} > \frac{1}{4x}) + P(|D_{2n}| > \frac{1}{4x^2})\\ &\leq P(xS_n - x^2V_n^2/2 \geq x^2/2 - x\Delta_{1n}) + P(S_n \geq (x-1/2x)V_n, |V_n^2-1| > 1/2x)\\ &\quad + P(\frac{|D_{1n}|}{V_n} > \frac{1}{4x}) + P(|D_{2n}| > \frac{1}{4x^2}),\end{aligned} \tag{3.7}$$

where

$$\Delta_{1n} = \min\{x(V_n^2-1)^2 + \frac{1}{2x} + \frac{1}{16x^2}, \frac{1}{x}\} > 0, \tag{3.8}$$

by(3.6), we have

$$P(T_n \geq x) \geq P(xS_n - x^2V_n^2/2 \geq x^2/2 + x\Delta_{2n}), \tag{3.9}$$

where

$$\Delta_{2n} = xD_{2n}/2 - D_{1n}. \tag{3.10}$$

If we want to have a upper and lower bound of $P(T_n \geq x)$ then we need to proof the following three propositions.

**Proposition 3.2.** *There exits positive absolute constants*

$$P(xS_n - x^2V_n^2/2 \geq x^2/2 - x\Delta_{1n}) \leq \Psi_x^*[1-\Phi(x)]e^{A_1R_x}(1 + A_2(1+x)L_{3,n} + A_3Q_{n,x}). \tag{3.11}$$

*for $x > 1$ satisfying (2.4) and (2.5), where $|O_1| \leq A$ and $|O_1| \leq A$*

**Proposition 3.3.** *There exits positive absolute constants*

$$P(S_n \geq (x-1/2x)V_n, |V_n^2-1| > 1/2x) \leq AR_x[1-\Phi(x)]\Psi_x^* e^{AR_x}. \tag{3.12}$$

*for $x > 1$ satisfying (2.4) and (2.5), where $|O_1| \leq A$ and $|O_1| \leq A$*

**Proposition 3.4.** *There exists positive absolute constants*

$$P(xS_n - x^2V_n^2/2 \geq x^2/2 + x\Delta_{2n}) \geq \Psi_x^* e^{-A_2R_x}[1-\Phi(x)](1 - O_2(1+x)L_{3,n} - O_3Q_{n,x}). \tag{3.13}$$

*for $x > 1$ satisfying (2.4) and (2.5), where $|O_1| \leq A$ and $|O_1| \leq A$*

□

**Proof of Proposition 3.2.** We firstly give some notations related to conjugated method, which is the main tool to prove Proposition 3.2–3.4. For $1 \leq i \leq n$, let

$$W_i = 2xX_i - x^2Y_i^2,$$

and let $(\xi_i, \eta_i)$ be independent random vectors with distribution

$$V_i(x,y) = \frac{\mathbb{E}\{e^{\lambda W_i}1(X_i \leq x, Y_i \leq y)\}}{\mathbb{E}e^{\lambda W_i}}. \tag{3.14}$$

Denote

$$\tilde{W}_i = 2x\xi_i - x^2\eta_i^2,$$

then we have

$$\mathbb{E}\tilde{W}_i = \frac{\mathbb{E}W_i e^{\lambda W_i}}{\mathbb{E}e^{\lambda W_i}}$$
$$Var\tilde{W}_i = \frac{\mathbb{E}W_i^2 e^{\lambda W_i}}{\mathbb{E}e^{\lambda W_i}} - (\mathbb{E}\tilde{W}_i)^2$$
$$\mathbb{E}|\tilde{W}_i|^3 = \frac{\mathbb{E}|W_i|^3 e^{\lambda W_i}}{\mathbb{E}e^{\lambda W_i}},$$

then we give the expansion of the above moments. Next two lemmas are Lemma A.1 and A.2 of Gao et al. (2022))

**Lemma 3.1.** *For $\frac{1}{4} \le \lambda \le \frac{1}{4}$ and $x > 0$ satisfying (2.5), there exits an absolute constant $A$ such that*

$$\begin{aligned} Ee^{\lambda W_i} &= 1 + 2\lambda^2 x^2 EX_i^2 - \lambda x^2 EY_i^2 + \frac{4}{3}\lambda^3 x^3 EX_i^3 - 2\lambda^2 x^3 EX_i Y_i^2 + O_1 R_{x,i} \\ &= \exp\{2\lambda^2 x^2 EX_i^2 - \lambda x^2 EY_i^2 + \frac{4}{3}\lambda^3 x^3 EX_i^3 - 2\lambda^2 x^3 EX_i Y_i^2 + O_1 R_{x,i}\} \end{aligned} \tag{3.15}$$

$$EW_i e^{\lambda W_i} = 4\lambda x^2 EX_i^2 - x^2 EY_i^2 + 4\lambda^2 x^3 EX_i^3 - 4\lambda x^3 EX_i Y_i^2 + O_3 R_{x,i} \tag{3.16}$$

$$EW_i^2 e^{\lambda W_i} = 4x^2 EX_i^2 + 8\lambda x^3 EX_i^3 - 4x^3 EX_i Y_i^3 + O_4 R_{x,i} \tag{3.17}$$

$$E|W_i|^3 e^{\lambda W_i} = O_5 x^3 (E|X_i|^3 + E|Y_i|^3) + O_6 R_{x,i} \tag{3.18}$$

*Where $|O_i| \le A$ for $i = 1, \ldots, 6$.*

**Lemma 3.2.** *We have for $x$ satisfying (2.5) that*

$$(1+x)^4 (EX_i^2)^2 \le 2\delta_{x,i}$$
$$(1+x)^5 EX_i^2 E|X_i|^3 \le 2\delta_{x,i}$$
$$(1+x)^6 (E|X_i|^3)^2 \le \delta_{x,i},$$

*and similar results hod for $Y_i$. in addition if $x$ also satisfies (2.4), then*

$$(1+x)^4 L_{3,n}^2 \le 2\delta_x.$$

By lemmas 3.1 and 3.2, we can have under condition (2.5),

$$\mathbb{E}\tilde{W}_i = x^2\left(4\lambda\mathbb{E}X_i^2 - \mathbb{E}Y_i^2\right) + x^3\left(4\lambda\mathbb{E}X_i^3 - 4\lambda\mathbb{E}X_iY_i^2\right) + O_1 R_{x,i},$$
$$\mathrm{Var}\tilde{W}_i = 4x^2\mathbb{E}X_i^2 + x^3\left(8\lambda\mathbb{E}X_i^3 - 4\mathbb{E}X_iY_i^2\right) + O_2 R_{x,i},$$
$$\mathbb{E}|\tilde{W}_i|^3 = O_3 x^3\left(\mathbb{E}|X_i|^3 + \mathbb{E}|Y_i|^3\right) + O_4 R_{x,i}.$$

Let $m_n = \sum_{i=1}^n \mathbb{E}\tilde{W}_i$, $\sigma_n = \sqrt{\sum_{i=1}^n Var(\tilde{W}_i)}$, $\nu_n = \sum_{i=1}^n \mathbb{E}|\tilde{W}_i|^3$. Then we can have:

$$m_n = (4\lambda - 1)x^2 + x^3\left(4\lambda^2 \sum_{i=1}^n EX_i^3 - 4\lambda \sum_{i=1}^n EX_iY_i^2\right) + O_1 R_x, \tag{3.19}$$

$$\begin{aligned}
\sigma_n^2 =& 4x^2 + x^3\left(8\lambda\sum_{i=1}^n EX_i^3 - 4\sum_{i=1}^n EX_iY_i^2\right) + O_2R_x, && (3.20)\\
\nu_n =& O_3x^3L_{3,n} + O_4R_x. && (3.21)
\end{aligned}$$

Define $m(\lambda) = \sum_{i=1}^n \log \mathbb{E}e^{\lambda W_i}$, $\tilde{S}_n = \sum_{i=1}^n \tilde{W}_i$, $U_n = \left(\tilde{S}_n - m_n\right)/\sigma_n$, therefore $m_n = m'(\lambda)$, and $\sigma_n^2 = m''(\lambda)$. We can obtain a lemma, which can be directly obtained through Lemma A.3 in Gao et al. (2022).

**Lemma 3.3.** *For x satisfying (2.4) and (2.5), then the equation*

$$m'(\lambda) = x^2,$$

*has a unique solution $\lambda_1$, in addition, $\lambda_1$ satisfies $\frac{1}{4} < \lambda_1 < \frac{3}{4}$ and*

$$\left|\lambda_1 - \frac{1}{2} + x\left(\lambda_1^2\sum_{i=1}^n \mathbb{E}X_i^3 - \lambda_1\sum_{i=1}^n \mathbb{E}X_iY_i^2\right)\right| \leq Ax^{-2}R_x, \tag{3.22}$$

*and*

$$m(\lambda_1) = (2\lambda_1^2 - \lambda_1)x^2 + x^3\left(\frac{4}{3}\lambda_1^3\sum_{i=1}^n \mathbb{E}X_i^3 - 2\lambda_1^2\sum_{i=1}^n \mathbb{E}X_iY_i^2\right) + O_1R_x. \tag{3.23}$$

Now, we can proof the proposition 3.2, by definiton of $W_i$ we have

$$P(xS_n - x^2V_n^2/2 \geq x^2/2 - x\Delta_{1n}) = P\left(\sum_{i=1}^n (2xX_i - x^2Y_i^2) \geq x^2 - 2x\Delta_{1n}\right) = P\left(\sum_{i=1}^n W_i \geq x^2 - 2\Delta_{1n}\right).$$

Assume $\lambda_1$ is the solution to $m'(\lambda) = m_n = x^2$, by the conjugate method

$$\begin{aligned}
&P\left(\sum_{i=1}^n W_i \geq x^2 - 2x\Delta_{1n}\right)\\
=&\mathbb{E}\left(e^{\lambda_1\sum_{i=1}^n W_i}\right)\mathbb{E}\left(e^{-\lambda_1\sum_{i=1}^n \tilde{W}_i}\mathbf{1}\left\{\sum_{i=1}^n \tilde{W}_i \geq x^2 - 2x\tilde{\Delta}_{1n}\right\}\right)\\
=&\exp(m(\lambda_1))\mathbb{E}e^{-\lambda_1(\frac{\tilde{S}_n - m_n + m_n}{\sigma_n})\sigma_n}\mathbf{1}\left\{\frac{\tilde{S}_n - m_n}{\sigma_n} \geq -\frac{2x\tilde{\Delta}_{1n}}{\sigma_n}\right\}\\
=&\exp(m(\lambda_1))\mathbb{E}e^{-\lambda_1(\sigma_nU_n + m_n)}\mathbf{1}\left\{U_n \geq -\frac{2x\tilde{\Delta}_{1n}}{\sigma_n}\right\}\\
\leq&\exp(m(\lambda_1) - \lambda_1m_n)\mathbb{E}\left(e^{-\lambda_1\sigma_nU_n}\mathbf{1}\{U_n \geq 0\}\right)\\
&+\exp(m(\lambda_1) - \lambda_1m_n)\mathbb{E}\left(e^{-\lambda_1\sigma_nU_n}\mathbf{1}\{0 \geq U_n \geq -\frac{2x\tilde{\Delta}_{1n}}{\sigma_n}\}\right)\\
:=&H_1 + H_2 && (3.24)
\end{aligned}$$

where $\tilde{\Delta}_{1n} = \min\{x(\sum_{i=1}^{n}\eta_i^2 - 1)^2 + |\tilde{D}_{1n}| - x\min(\tilde{D}_{2n}, 0), 1/x\}$. Let $G_n(t)$ is the distribution of $U_n$, then we have

$$\begin{aligned} H_1 =& \exp(m(\lambda_1) - \lambda_1 x^2) \int_0^{+\infty} e^{-\lambda_1 \sigma_n t}\, dG_n(t) \\ =& \exp(m(\lambda_1) - \lambda_1 x^2) \int_0^{+\infty} e^{-\lambda_1 \sigma_n t}\, d(G_n(t) - \Phi(t)) \\ &+ \exp(m(\lambda_1) - \lambda_1 x^2) \int_0^{+\infty} e^{-\lambda_1 \sigma_n t}\, d\Phi(t) \\ :=& J_1 + J_2 \end{aligned} \tag{3.25}$$

by integral by parts, we have

$$\begin{aligned} J_1 =& \exp(m(\lambda_1) - \lambda_1 x^2) \int_0^{+\infty} e^{-\lambda_1 \sigma_n t}\, d(G_n(t) - \Phi(t)) \\ \le& 2\exp(m(\lambda_1) - \lambda_1 x^2) \sup_{z \in R} |P(U_n \le z) - \Phi(z)|. \end{aligned}$$

Applying the Berry-Esseen theorem to $\sup_{z\in R} |P(U_n \le z) - \Phi(z)|$, and by (3.20) and (3.21), we can find that for $x > 1$ satisfying (2.4),

$$\sup_{z \in R} |P(U_n \le z) - \Phi(z)| \le A\nu_n \sigma_n^{-3} \le A(L_{3,n} + x^{-3} R_x). \tag{3.26}$$

By applying (3.22) and (3.23) in Lemma 3.3 we have

$$|\lambda_1 - 1/2| \le A_1 x L_{3,n} + A_2 x^{-2} R_x,$$

and by Lemma 3.2 we have $x^4 L_{3,n}^2 \le 2\delta_x < 2R_x$, we obtain for x satisfying (2.4) and (2.5) that

$$(2\lambda_1^2 - \lambda_1)x^2 + \frac{1}{2}x^2 - \lambda_1 x^2 = 2x^2(\lambda_1 - \frac{1}{2})^2 \le O_1 R_x,$$

Then,by (3.23) we have

$$\begin{aligned} & m(\lambda_1) + \frac{1}{2}x^2 - \lambda_1 x^2 \\ =& (2\lambda_1^2 - \lambda_1)x^2 + \frac{1}{2}x^2 - \lambda_1 x^2 + x^3 \left( \frac{4}{3}\lambda_1^3 \sum_{i=1}^{n} \mathbb{E}X_i^3 - 2\lambda_1^2 \sum_{i=1}^{n} \mathbb{E}X_i Y_i^2 \right) + O_1 R_x \\ \le& x^3 \left( \frac{1}{6} \sum_{i=1}^{n} \mathbb{E}X_i^3 - \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}X_i Y_i^2 \right) + 8x^3 |\lambda_1 - 1/2| L_{3,n} + O_1 R_x \\ \le& x^3 \left( \frac{1}{6} \sum_{i=1}^{n} \mathbb{E}X_i^3 - \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}X_i Y_i^2 \right) + O_1 R_x, \end{aligned} \tag{3.27}$$

and since for $x > 1$ we have

$$e^{-\frac{x^2}{2}} \le 4x[1 - \Phi(x)], \tag{3.28}$$

we can obtain

$$\begin{aligned}J_1 \leq & 2A\exp(m(\lambda_1)-\lambda_1x^2)(L_{3,n}+x^{-3}R_x)\\ \leq & 2A\exp(m(\lambda_1)+\frac{1}{2}x^2-\lambda_1x^2)\exp(-x^2/2)(L_{3,n}+x^{-3}R_x)\\ \leq & [1-\Phi(x)]\Psi_*^x e^{O_1R_x}(xL_{3,n}+x^{-2}R_x).\end{aligned} \tag{3.29}$$

And for $J_2$, since,

$$\int_0^{+\infty} e^{-\lambda_1\sigma_n t}\, d\Phi(t) = \frac{e^{\lambda_1^2\sigma_n^2/2}}{\sqrt{2\pi}}\int_{\lambda_1\sigma_n}^{\infty} e^{-t^2/2}\, dt = \frac{1}{\sqrt{2\pi}}\psi(\lambda_1\sigma_n),$$

where $\psi(x)=\frac{1-\Phi(x)}{\Phi'(x)}=e^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt$. Following that for $x>1$,

$$\frac{x}{1+x^2}e^{-\frac{x^2}{2}} \leq \int_x^\infty e^{-\frac{t^2}{2}}\,dt \leq \frac{1}{x}e^{-\frac{x^2}{2}}, \tag{3.30}$$

we have,

$$\frac{x}{1+x^2}\leq\psi(x)\leq\frac{1}{x} \quad \text{and} \quad 0<-\psi'(x)=1-xe^{\frac{x^2}{2}}\int_x^\infty e^{-\frac{t^2}{2}}\,dt\leq\frac{1}{1+x^2}, \tag{3.31}$$

by (3.27), (3.28), (3.30) and (3.31) we can have,

$$\begin{aligned}J_2 = & \exp(m(\lambda_1)-\lambda_1x^2)\int_0^{+\infty}e^{-\lambda_1\sigma_n t}\,d\Phi(t)\\ \leq & \Psi_x^* e^{O_1R_x}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}\left(\psi(2\lambda_1x)+|\psi(\lambda_1\sigma_n)-\psi(2\lambda_1x)|\right)\\ \leq & \Psi_x^* e^{O_1R_x}e^{-\frac{x^2}{2}}\psi(2\lambda_1x)\left(1+\frac{\psi'(\theta)}{\psi(2\lambda_1x)}|\lambda_1\sigma_n-2\lambda_1x|\right)\\ \leq & [1-\Phi(x)]\Psi_x^* e^{O_1R_x}(1+3x^{-1}|\lambda_1\sigma_n-2\lambda_1x|),\end{aligned} \tag{3.32}$$

since (3.20), we have

$$\begin{aligned}& |\lambda_1\sigma_n-2\lambda_1x|\\ = & \frac{\lambda_1|\sigma_n^2-4x^2|}{\sigma_n+2x}\\ = & \frac{\lambda_1|x^3(8\lambda_1\sum_{i=1}^n EX_i^3-4\sum_{i=1}^n EX_iY_i^2)+O_2R_x|}{\sigma_n+2x}\\ \leq & A(x^2L_{3,n}+x^{-1}R_x),\end{aligned} \tag{3.33}$$

so we can have,

$$J_2 \leq [1-\Phi(x)]\Psi_x^* e^{O_1R_x}(1+xL^{3,n}+x^{-2}R_x). \tag{3.34}$$

Combine (3.29) and (3.34), we have

$$H_1 = J_1 + J_2 \le [1-\Phi(x)]\Psi_x^* e^{O_1 R_x}(1+O_2 x L_{3,n}). \tag{3.35}$$

Next we find an upper bound of $H_2$, by (3.27)

$$\begin{aligned} H_2 =& \exp(m(\lambda_1)-\lambda_1 m_n)\mathbb{E}\left(e^{-\lambda_1\sigma_n U_n}\mathbf{1}\{0 \ge U_n \ge -\frac{2x\tilde{\Delta}_{1n}}{\sigma_n}\}\right) \\ \le& \Psi_x^* e^{O_1 R_x} e^{-\frac{x^2}{2}}\mathbb{E}\left(e^{-\lambda_1\sigma_n U_n}\mathbf{1}\{0 \ge U_n \ge -\frac{2x\tilde{\Delta}_{1n}}{\sigma_n}\}\right) \end{aligned} \tag{3.36}$$

since $x\tilde{\Delta}_{1,n} \le 1$, We have

$$\begin{aligned} &\mathbb{E}\left(e^{-\lambda_1\sigma_n U_n}\mathbf{1}\{0 \ge U_n \ge -\frac{2x\tilde{\Delta}_{1n}}{\sigma_n}\}\right) \\ \le& \mathbb{E}\left(e^{\lambda_1 2x\tilde{\Delta}_{1n}}\mathbf{1}\{0 \ge U_n - \epsilon_n \ge -\frac{2x\tilde{\Delta}_{1n}}{\sigma_n}\}\right) \\ \le& e^2 P\left(-\frac{2x\tilde{\Delta}_{1n}}{\sigma_n} \le U_n \le 0\right) \end{aligned} \tag{3.37}$$

By applying the randomize concentration inequality by Shao and Zhou (2016),

$$P\left(-\frac{2x\tilde{\Delta}_{1n}}{\sigma_n} \le U_n \le 0\right) \le 17\nu_n\sigma_n^{-3} + 5x\sigma_n^{-1}\mathbb{E}|\tilde{\Delta}_{1n}| + 2x\sigma_n^{-2}\sum_{i=1}^n E|\tilde{W}_i(\tilde{\Delta}_{1n} - \tilde{\Delta}_{1n}^{(i)})| \tag{3.38}$$

where $\tilde{\Delta}_{1n}^{(i)}$ can be any random variable that is independent of $\tilde{W}_i$. By (3.26) we know that

$$\nu_n\sigma_n^{-3} \le A(L_{3,n} + x^{-3}R_x).$$

For the other two tems, by change of measure, the distribution of $(\xi_i, \eta_i)$ defined in (3.14), we have

$$\begin{aligned} E|\tilde{\Delta}_{1n}| =& \int\cdots\int |\Delta_{1n}(x_1,\ldots,x_n,y_1,\ldots,y_n)|dV_1(x_1,y_1)\ldots dV_n(x_n,y_n) \\ =& (\mathbb{E}e^{\lambda_1\sum_{i=1}^n W_i})^{-1}\int\cdots\int |\Delta_{1n}(x_1,\ldots,x_n,y_1,\ldots,y_n)|\prod_{i=1}^n\{e^{\lambda_1 W_i}\,dF_{(X_i,Y_i)}(x_i,y_i)\} \\ =& \exp\{-m(\lambda_1)\}\mathbb{E}\left(|\Delta_{1n}|e^{\lambda_1\sum_{i=1}^n W_i}\right), \end{aligned} \tag{3.39}$$

similarly we can also have,

$$E|\tilde{W}_i(\tilde{\Delta}_{1n} - \tilde{\Delta}_{1n}^{(i)})| = \exp\{-m(\lambda_1)\}E(|W_i(\Delta_{1n} - \Delta_{1n}^{(i)})|e^{\lambda_1\sum_{j=1}^n W_j}) \tag{3.40}$$

since $\Delta_{1n} = min\{x(V_n^2-1)^2 + |D_{1n}| - x\min(D_{2n},0), 1/x\}$, we have

$$dd \tag{3.41}$$

$$|\Delta_{1,n}-\Delta_{1,n}^{(i)}|\le x((V_n^2-1)^2-\{(V_n^2-1)^2\}^{(i)})+|D_{1,n}-D_{1,n}^{(i)}|+x|D_{2,n}-D_{2,n}^{(i)}|$$

$$|\Delta_{1,n}|\le x(V_n^2-1)^2+|D_{1,n}|+x|D_{2,n}|$$

and

$$V_n^2-1=\sum_{i=1}^n Y_i^2-\sum_{i=1}^n EY_i^2=\sum_{i=1}^n Z_i$$

$$(V_n^2-1)^2-((V_n^2-1)^2)^{(i)}=Z_i^2+2Z_i\sum_{j\ne i}Z_j,$$

by lemma3 we have

$$E((V_n^2-1)e^{\lambda\sum_{j=1}^n W_j})\le A_1\Psi_x^* e^{AR_x}x^{-2}R_x$$

$$\sum_{i=1}^n E(|W_i||Z_i^2+2Z_i\sum_{j\ne i}Z_j|e^{\lambda\sum_{j=1}^n W_j})\le A_1\Psi_x^* e^{AR_x}x^{-2}R_x$$

so

$$\begin{aligned}
H_2\le& A\Psi_x^* e^{AR_x}e^{-\frac{x^2}{2}}(L_{3,n}+x^{-3}R_x+\frac{x}{\sigma_n}(\Psi_*^x e^{AR_x})^{-1}\mathbb{E}\left(|\Delta_{1,n}|e^{\frac12\sum_{i=1}^n W_i}\right)\\
&+\frac{x}{\sigma_n^2}(\Psi_*^x e^{AR_x})^{-1}\sum_{i=1}^n E(|W_i(\Delta_{1,n}-\Delta_{1,n}^{(i)})|e^{\frac12\sum_{j=1}^n W_j}))\\
\le& A\Psi_x^* e^{AR_x}e^{-\frac{x^2}{2}}(L_{3,n}+x^{-3}R_x+\frac{x^2}{\sigma_n}(\Psi_x^* e^{AR_x})^{-1}\mathbb{E}\left((V_n^2-1)^2e^{\frac12\sum_{i=1}^n W_i}\right)\\
&+\frac{x}{\sigma_n}(\Psi_x^* e^{AR_x})^{-1}\mathbb{E}((|D_{1,n}|+x|D_{2,n}|)e^{\frac12\sum_{i=1}^n W_i})\\
&+\frac{x^2}{\sigma_n^2}(\Psi_x^* e^{AR_x})^{-1}\sum_{i=1}^n\mathbb{E}(|W_i||Z_i^2+2Z_i\sum_{j\ne i}|e^{\frac12\sum_{j=1}^n W_j})\\
&+\frac{x}{\sigma_n^2}(\Psi_x^* e^{AR_x})^{-1}\sum_{i=1}^n\mathbb{E}(|W_i(|D_{1,n}-D_{1,n}^{(i)}|+x|D_{2,n}-D_{2,n}^{(i)}|)|e^{\frac12\sum_{j=1}^n W_j}))\\
\le& A\Psi_x^* e^{AR_x}e^{-\frac{x^2}{2}}(L_{3,n}+x^{-3}R_x+x^{-1}R_x+x^{-2}R_x+\\
&+\frac{x}{\sigma_n}(\Psi_x^* e^{AR_x})^{-1}\mathbb{E}((|D_{1,n}|+x|D_{2,n}|)e^{\frac12\sum_{i=1}^n W_i})\\
&+\frac{x}{\sigma_n^2}(\Psi_x^* e^{AR_x})^{-1}\sum_{i=1}^n\mathbb{E}(|W_i(|D_{1,n}-D_{1,n}^{(i)}|+x|D_{2,n}-D_{2,n}^{(i)}|)|e^{\frac12\sum_{j=1}^n W_j}))
\end{aligned}\tag{3.42}$$

Since for $x>0$ we have $[1-\Phi(x)]\sim\frac{1}{x}e^{-\frac{x^2}{2}}$, we have $e^{\frac{x^2}{2}}[1-\Phi(x)]\ge Ax^{-1}$

$$\begin{aligned}
H_2\le& A\Psi_x^* e^{AR_x}e^{-\frac{x^2}{2}}[1-\Phi(x)]e^{\frac{x^2}{2}}(xL_{3,n}+x^{-2}R_x+R_x+x^{-1}R_x+Q_{n,x})\\
\le& A\Psi_x^* e^{AR_x}[1-\Phi(x)](xL_{3,n}+Q_{n,x})
\end{aligned}\tag{3.43}$$

Where

$$
\begin{aligned}
Q_{n,x} =&(\Psi_x^* e^{AR_x})^{-1}\mathbb{E}((x|D_{1,n}| + x^2|D_{2,n}|)e^{\frac{1}{2}\sum_{i=1}^n W_i}) \\
&+ (\Psi_x^* e^{AR_x})^{-1}\sum_{i=1}^n \mathbb{E}(|W_i(|D_{1,n} - D_{1,n}^{(i)}| + x|D_{2,n} - D_{2,n}^{(i)}|)|e^{\frac{1}{2}\sum_{j=1}^n W_j})
\end{aligned}
\tag{3.44}
$$

combine with

$$
H_1 \le A\Psi_x^* e^{AR_x}[1-\Phi(x)](1 + A_2 x L_{3,n} + A_3 x^{-2} R_x)
$$

so we have

$$
\begin{aligned}
&P(2xS_n - x^2 V_n^2 \ge x^2 - 2x\Delta_{1,n}) \\
=&H_1 + H_2 \\
\le&\Psi_x^*[1-\Phi(x)]e^{A_1 R_x}(1 + A_2(1+x)L_{3,n} + A_3 Q_{n,x})
\end{aligned}
\tag{3.45}
$$

so we finish the proof of proposition1.

□

**Proof.** Proof of proposition1

□

**Proof.** Proof of proposition2

$$
\begin{aligned}
&P(S_n \ge (x - 1/2x)V_n, |V_n^2 - 1| > 1/2x) \\
=&P(S_n/V_n \ge x - 1/2x, (1 + 1/2x)^{\frac{1}{2}} \le V_n \le B) \\
&+ P(S_n/V_n \ge x - 1/2x, V_n > B) \\
&+ P(S_n/V_n \ge x - 1/2x, V_n < (1 - 1/2x)^{\frac{1}{2}}) \\
=&\sum_{i=1}^3 P((W_n, V_n) \in \varepsilon_n)
\end{aligned}
\tag{3.46}
$$

Where

$$
\begin{aligned}
\varepsilon_1 &= \{(u,v) \in \mathbb{R} * \mathbb{R}^+ : \frac{u}{v} \ge x - \frac{1}{2x}, \sqrt{1 + \frac{1}{2x}} < v < B\} \\
\varepsilon_2 &= \{(u,v) \in \mathbb{R} * \mathbb{R}^+ : \frac{u}{v} \ge x - \frac{1}{2x}, v < \sqrt{1 - \frac{1}{2x}}\} \\
\varepsilon_3 &= \{(u,v) \in \mathbb{R} * \mathbb{R}^+ : \frac{u}{v} \ge x - \frac{1}{2x}, v > B\}
\end{aligned}
\tag{3.47}
$$

by chebyshev inequality

$$
P((S_n, V_n) \in \varepsilon_1) \le x^2 \exp\{-\inf_{(u,v)\in\varepsilon_1}(t_1 u - \lambda_1 v^2)\}\mathbb{E}((V_n^2 - 1)^2 \exp\{t_1 S_n - \lambda_1 V_n^2\})
$$

in lemma3 we have know that when $0 < r < r_0 < 1$ for a constant $r_0$ and for a number $w > r_0$, there exits a $A_1$ and $A_2$ depending on $w$ and $r_0$ such that

$$\begin{aligned}&\mathbb{E}\left((V_n^2-1)^2 e^{\sum_{i=1}^n (2rxX_i - wrx^2Y_i^2)}\right)\\ \le&\frac{A_1R_x}{x^2}\exp\{(2r^2-wr)x^2 - 2wr^2x^3\sum_{i=1}^n \mathbb{E}X_iY_i^2 + \frac{4}{3}r^3x^3\sum_{i=1}^n \mathbb{E}X_i^3 + A_2R_x\}\end{aligned} \tag{3.48}$$

in this case we have

$$\begin{aligned}&\mathbb{E}\left((V_n^2-1)^2 e^{\sum_{i=1}^n (t_1X_i - \lambda_1Y_i^2)}\right)\\ \le&\frac{A_2R_x}{x^2}\exp\{\frac{t_1^2}{2} - \lambda_1 + \frac{t_1^3}{6}\sum_{i=1}^n \mathbb{E}X_i^3 - \lambda_1t_1\sum_{i=1}^n \mathbb{E}X_iY_i^2 + A_2R_x\}\end{aligned} \tag{3.49}$$

so we have

$$\begin{aligned}&P((S_n,V_n)\in\varepsilon_1)\\ \le&A_1R_x\exp\{-\inf_{(u,v)\in\varepsilon_1}(t_1u-\lambda_1v^2)\}\exp\{\frac{t_1^2}{2} - \lambda_1 + \frac{t_1^3}{6}\sum_{i=1}^n \mathbb{E}X_i^3 - \lambda_1t_1\sum_{i=1}^n \mathbb{E}X_iY_i^2 + A_2R_x\}\end{aligned} \tag{3.50}$$

now we choose $t_1 = x\sqrt{1+\frac{1}{2x}}$ and $\lambda_1 = t_1(x-\frac{1}{2x})/8$, and since $B = \max\{50, 200c_0\}$, then we have

$$\inf_{(u,v)\in\varepsilon_1}(t_1u-\lambda_1v^2) = x^2 + \frac{x}{2} - \lambda_1(1+\frac{1}{2x}) - \frac{1}{2} - \frac{1}{4x}$$

so we have,

$$\begin{aligned}P((S_n,V_n)\in\varepsilon_1)\le&AR_x\exp\{-x^2-\frac{x}{2}+\frac{x\sqrt{1+\frac{1}{2x}}(x-\frac{1}{2x})(1+\frac{1}{2x})}{8}+\frac{1}{2}+\frac{1}{4x}\}\\ &*\exp\{\frac{x^2(1+\frac{1}{2x})}{2}-\frac{x\sqrt{1+\frac{1}{2x}}(x-\frac{1}{2x})}{8}+\frac{x^3(1+\frac{1}{2x})^{\frac{3}{2}}}{6}\sum_{i=1}^n\mathbb{E}X_i^3\\ &-\frac{x^2(1+\frac{1}{2x})(x-\frac{1}{2x})}{8}\sum_{i=1}^n\mathbb{E}X_iY_i^2+A_2R_x\}\\ \le&AR_xe^{A_2R_x}\exp\{-\frac{x^2}{2}\}\exp\{\frac{x^3}{6}\sum_{i=1}^n\mathbb{E}X_iY_i^3-\frac{x^3}{8}\sum_{i=1}^n\mathbb{E}X_iY_i^2\}\\ \lesssim&AR_xe^{A_2R_x}[1-\Phi(x)]\Psi_x^*\end{aligned} \tag{3.51}$$

as for

$$P((S_n,V_n)\in\varepsilon_2)\le x^2\exp\{-\inf_{(u,v)\in\varepsilon_2}(t_2u-\lambda_2v^2)\}\mathbb{E}\left((V_n^2-1)^2e^{t_2S_n-\lambda_2V_n^2}\right)$$

now we choose $t_2 = x\sqrt{1-\frac{1}{2x}}$ and $\lambda_2 = 2x^2 - 1$ and since $B = \max\{50, 200c_0\}$,

$$\inf_{(u,v)\in\varepsilon_2}(t_2 u - \lambda_2 v^2) = x^2 - \frac{x}{2} - \frac{1}{2} + \frac{1}{4x} - (2x^2-1)(1-\frac{1}{2x})$$

so

$$\begin{aligned} P((S_n, V_n)\in\varepsilon_2) \le & x^2\exp\{-x^2 + \frac{x}{2} + \frac{1}{2} - \frac{1}{4x} + (2x^2-1)(1-\frac{1}{2x})\} \\ & \frac{A_2 R_x}{x^2}\exp\{\frac{t_2^2}{2} - \lambda_2 + \frac{t_2^3}{6}\sum_{i=1}^n \mathbb{E}X_i^3 - \lambda_2 t_2\sum_{i=1}^n \mathbb{E}X_i Y_i^2 + A_2 R_x\} \\ \le & A R_x e^{A_2 R_x}\exp\{-\frac{x^2}{2}\}\Psi_x^* \\ \lesssim & A R_x e^{A_2 R_x}[1-\Phi(x)]\Psi_x^* \end{aligned} \tag{3.52}$$

as for

$$\begin{aligned} & P((S_n, V_n)\in\varepsilon_3) \\ = & P(\frac{S_n}{V_n} > x - \frac{1}{2x}, V_n > B) \\ \le & P(S_n^{in} > \frac{V_n}{10}(x-\frac{1}{2x}), V_n > B) + P(S_n^{out} > \frac{9V_n}{10}(x-\frac{1}{2x}), V_n > B) \\ \le & P(S_n^{in} > \frac{B}{10}(x-\frac{1}{2x}), V_n^{in} > \frac{B}{2}) + P(S_n^{in} > \frac{B}{10}(x-\frac{1}{2x}), V_n^{out} > \frac{B}{2}) \\ & P(S_n^{out} > \frac{9V_n}{10}(x-\frac{1}{2x}), V_n > B) \\ := & K_1 + K_2 + K_3 \end{aligned} \tag{3.53}$$

where

$$\begin{aligned} & X_i^{in} = X_i 1\{|(1+x)X_i| \le 1\} \quad X_i^{out} = X_i 1\{|(1+x)X_i| > 1\} \\ & Y_i^{in} = X_i 1\{|(1+x)X_i| \le 1\} \quad Y_i^{out} = X_i 1\{|(1+x)X_i| > 1\} \\ & S_n^{in} = \sum_{i=1}^n X_i^{in}, S_n^{out} = \sum_{i=1}^n X_i^{out} \\ & V_n^{in} = \sqrt{\sum_{i=1}^n (Y_i^{in})^2}, V_n^{out} = \sqrt{\sum_{i=1}^n (Y_i^{out})^2} \end{aligned} \tag{3.54}$$

$$\begin{aligned} K_1 = & P(S_n^{in} > \frac{B}{10}(x-\frac{1}{2x}), V_n^{in} > \frac{B}{2}) \\ \le & \frac{\mathbb{E}\{((V_n^{in})^2-1)^2 e^{\frac{xS_n^{in}}{2}}\}\exp\{-\frac{B}{20}x^2 + \frac{B}{40}\}}{(\frac{B^2}{4}-1)^2} \end{aligned} \tag{3.55}$$

we let $Z_i^{in} = (Y_i^{in})^2 = \mathbb{E}(Y_i^{in})^2$ then we have $(V_i^{in})^2 - 1 = \sum_{i=1}^n Z_i^{in} - \sum_{i=1}^n \mathbb{E}(Y_i^{out})^2$, so we have

$$\begin{aligned}\mathbb{E}\{((V_n^{in})^2-1)^2 e^{\frac{xS_n^{in}}{2}}\} \le& 2\sum_{i=1}^n \frac{\mathbb{E}(Z_i^{in})^2 e^{\frac{xX_i^{in}}{2}}}{\mathbb{E}e^{\frac{xX_i^{in}}{2}}} \prod_{j=1}^n \mathbb{E}e^{\frac{xX_j^{in}}{2}} + 2x^{-4}R_x^2 \prod_{j=1}^n \mathbb{E}e^{\frac{xX_j^{in}}{2}} \\ &+ 4\sum_{i\neq j} \frac{\mathbb{E}Z_i^{in} e^{\frac{xX_i^{in}}{2}}}{\mathbb{E}e^{\frac{xX_i^{in}}{2}}} \frac{\mathbb{E}Z_j^{in} e^{\frac{xX_j^{in}}{2}}}{\mathbb{E}e^{\frac{xX_j^{in}}{2}}} \prod_{j=1}^n \mathbb{E}e^{\frac{xX_j^{in}}{2}}\end{aligned} \tag{3.56}$$

by taylor expansion we have

$$\mathbb{E}e^{\frac{xX_i^{in}}{2}} = \exp\{\frac{1}{8}x^2\mathbb{E}X_i^2 + \frac{1}{48}x^3\mathbb{E}X_i^3 + O_1\delta_{x,i}\}$$

$$\mathbb{E}(Z_i^{in})^2 e^{\frac{xX_i^{in}}{2}} = O(x^{-4}\delta_{x,i})$$

$$\mathbb{E}Z_i^{in} e^{\frac{xX_i^{in}}{2}} = O(x(\mathbb{E}|X_i|^3 + \mathbb{E}|Y_i|^3)) + O(x^-2\delta_{x,i})$$

$$\mathbb{E}(Y_i^{out})^2 e^{\frac{xX_i^{in}}{2}} = O(x^{-2}\delta_{x,i})$$

combining the fact that $x^4L_{3,n}^2 \le 4\delta_x$, we have

$$\mathbb{E}\{((V_n^{in})^2-1)^2 e^{\frac{xS_n^{in}}{2}}\} \le \frac{AR_x}{x^2}\exp\{\frac{1}{8}x^2 + \frac{1}{48}x^3\sum_{i=1}^n \mathbb{E}X_i^3 + A_1R_x\}$$

so

$$\begin{aligned}K_1 \le& A(\frac{B^2}{4}-1)^{-2}x^{-2}R_x\exp\{(\frac{1}{8}-\frac{B}{20})x^2 + \frac{B}{40} + \frac{1}{48}\sum_{i=1}^n \mathbb{E}X_i^3 + A_1R_x\} \\ \le& C_1x^{-2}R_x\exp\{-\frac{19}{8}x^2 + \frac{B}{40} + \frac{1}{48}\sum_{i=1}^n \mathbb{E}X_i^3 + A_1R_x\} \\ \le& C_2R_x[1-\Phi(x)]\Psi_x^* e^{A_1R_x}\end{aligned} \tag{3.57}$$

last inequality holds because x satisfies condition (2.6) and

$$|x^3\sum_{i=1}^n \mathbb{E}X_i^3| \le x^3L_{x,n} \;\; and \;\; |x^3\sum_{i=1}^n \mathbb{E}X_iY_i^2| \le x^3L_{3,n}$$

similarly for $K_2$ er have

$$\begin{aligned}K_2 =& P(S_n^{in} > \frac{B}{10}(x-\frac{1}{2x}), V_n^{out} > \frac{B}{2}) \\ \le& \frac{4\mathbb{E}\{(V_n^{out})^2 e^{\frac{xS_n^{in}}{2}}\}\exp\{-\frac{B}{20}x^2 + \frac{B}{40}\}}{B^2}\end{aligned} \tag{3.58}$$

and

$$\begin{aligned}\mathbb{E}\{(V_n^{out})^2 e^{\frac{xS_n^{in}}{2}}\} &= \sum_{i=1}^{n} \frac{\mathbb{E}(Y_i^{out})^2 e^{\frac{xX_i^{in}}{2}}}{\mathbb{E}e^{\frac{xX_i^{in}}{2}}} \prod_{i=j}^{n} \mathbb{E}e^{\frac{xX_j^{in}}{2}} \\ &\leq \frac{AR_x}{x^2} \exp\{\frac{1}{8}x^2 + \frac{1}{48}x^3 \sum_{i=1}^{n} \mathbb{E}X_i^3 + A_1 R_x\}\end{aligned} \tag{3.59}$$

hence

$$K_2 \leq 4AB^{-2}x^{-2}R_x \exp\{(\frac{1}{8} - \frac{B}{20})x^2 + \frac{B}{40} + \frac{1}{48}\sum_{i=1}^{n}\mathbb{E}X_i^3 + A_1 R_x\} \tag{3.60}$$

in the same manner of proof of $K_1$, it follows that,

$$K_2 \leq AR_x[1 - \Phi(x)]\Psi_x^* e^{A_1 R_x}$$

as for $K_3$ we denote that

$$X_i^{out(1)} = X_i 1\{2xX_i \leq \frac{X_i^2}{Y_i^2 + c_0\mathbb{E}Y_i^2}\}, \quad X_i^{out(2)} = X_i^{out} - X_i^{out(1)}$$

$$S_n^{out(1)} = \sum_{i=1}^{n} X_i^{out(1)}, \quad S_n^{out(2)} = \sum_{i=1}^{n} X_i^{out(2)}$$

so we have

$$\begin{aligned}K_3 =& P(S_n^{out} > \frac{9V_n}{10}(x - \frac{1}{2x}), V_n > B) \\ \leq& P(S_n^{out(1)} > \frac{1V_n}{100}(x - \frac{1}{2x}), V_n > B) \\ &+ P(S_n^{out(2)} > \frac{89V_n}{100}(x - \frac{1}{2x}), V_n > B) \\ \leq& P(\sum_{i=1}^{n} 2xX_i^{out(1)} > \frac{V_n}{50}(x^2 - \frac{1}{2}), V_n > B) \\ &+ P(\sqrt{\sum_{i=1}^{n}\frac{(X_i^{out(2)})^2}{Y_i^2 + c_0\mathbb{E}Y_i^2}} > \frac{89V_n(x - \frac{1}{2x})}{100\sqrt{V_n^2 + c_0}}, V_n > B) \\ :=& K_4 + K_5\end{aligned} \tag{3.61}$$

since

$$\prod_{j \neq i} \mathbb{E}e^{2xX_j^{out(1)}} \leq \exp\{R_x\}$$

$$\prod_{j \neq i} \mathbb{E}e^{\frac{(X_j^{out(2)})^2}{Y_j^2 + c_0\mathbb{E}Y_j^2}} \leq \exp\{R_x\}$$

$$R_x \geq \sum_{i=1}^{n} \mathbb{E}[e^{2xX_i^{out(1)}}] + \sum_{i=1}^{n} \mathbb{E}[e^{\frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2}}]$$

then we choose $B = \max\{50, 200c_0\}$ then we have for $x > c > 2$

$$\begin{aligned}
&P(\sum_{i=1}^{n} 2xX_i^{out(1)} > \frac{V_n}{50}(x^2 - \frac{1}{2}), V_n > B) \\
\leq &P(\sum_{i=1}^{n} 2xX_i^{out(1)} > x^2 - 1/2) \\
\leq &(x^2 - 1/2)^{-1} \exp\{-0.99(x^2 - 1/2)\}\mathbb{E}[\sum_{I=1}^{n} 2xX_i^{out(1)} e^{0.99 \sum_{i=1}^{n} 2xX_i^{out(1)}}] \\
\leq &C_2 x^{-2} \exp\{-0.99x^2\} \sum_{i=1}^{n} \mathbb{E}[e^{2xX_i^{out(1)}}] \prod_{j \neq i} \mathbb{E}e^{2xX_j^{out(1)}} \\
\leq &AR_x[1 - \Phi(x)]\Psi_x^* \exp\{R_x\}
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
&P(\sqrt{\sum_{i=1}^{n} \frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2}} > \frac{89V_n(x - \frac{1}{2x})}{100\sqrt{V_n^2 + c_0}}, V_n > B) \\
\leq &P(\sum_{i=1}^{n} \frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2} > 0.792(x - \frac{1}{2x})^2) \\
\leq &P(\sum_{i=1}^{n} \frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2} > 0.6x^2) \\
\leq &C_1 x^{-2} \exp\{-0.54x^2\}\mathbb{E}[\sum_{i=1}^{n} \frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2} e^{0.99 \sum_{i=1}^{n} \frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2}}] \\
\leq &C_2 x^{-2} \exp\{-0.54x^2\} \sum_{i=1}^{n} \mathbb{E}[e^{\frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2}}] \prod_{j \neq i} \mathbb{E}e^{\frac{(X_i^{out(2)})^2}{Y_i^2 + c_0 \mathbb{E}Y_i^2}} \\
\leq &AR_x[1 - \Phi(x)]\Psi_x^* \exp\{R_x\}
\end{aligned} \tag{3.63}$$

combine that we have

$$P(S_n \geq (x - \frac{1}{2x})V_n, |V_n^2 - 1| > \frac{1}{2x}) \leq AR_x[1 - \Phi(x)]\Psi_x^* e^{AR_x} \tag{3.64}$$

□

**Proof.** Proof of proposition3

we also use change of measure

$$\begin{aligned}
&P(xS_n - x^2V_n^2/2 \geq x^2/2 + x\Delta_{2n}) \\
=&P(\sum_{i=1}^{n}(2xX_i - x^2Y_i^2) \geq x^2 + 2x\Delta_{2,n}) \\
=&\Psi_x^* e^{AR_x}\mathbb{E}[e^{-\frac{1}{2}\sigma_n U_n - \frac{1}{2}m_n} 1\{U_n \geq \epsilon_n + \frac{2x\tilde{\Delta}_{2,n}}{\sigma_n}\}] \\
\geq&\Psi_x^* e^{AR_x}\mathbb{E}[e^{-\frac{1}{2}\sigma_n U_n - \frac{1}{2}m_n} 1\{U_n \geq \epsilon_n\}] \\
&+ \Psi_x^* e^{AR_x}\mathbb{E}[e^{-\frac{1}{2}\sigma_n U_n - \frac{1}{2}m_n} 1\{\epsilon \leq U_n \leq \epsilon_n + \frac{2x\tilde{\Delta}_{2,n}}{\sigma_n}\}] \\
:=&H_1^{'} - H_2^{'}
\end{aligned} \tag{3.65}$$

similarly with the proof of proposition1 $H_1^{'} = J_1^{'} + J_2^{'}$, we drop $J_1^{'}$ and

$$\begin{aligned}
J_2^{'} &\geq \Psi_x^* e^{AR_x} e^{-\frac{1}{2}x^2}\frac{1}{\sqrt{2}}\Psi(x) \\
&\geq \Psi_x^* e^{AR_x}[1 - \Phi(x)]
\end{aligned} \tag{3.66}$$

$$H_2^{'} \leq [1 - \Phi(x)]\Psi_x^* e^{AR_x}(1 + O_1 x L_{3,n} + O_2 R_x + O_3 Q_{n,x}) \tag{3.67}$$

so we have

$$P(xS_n - x^2V_n^2/2 \geq x^2/2 + x\Delta_{2n}) \geq \Psi_x^* e^{-A_2 R_x}[1 - \Phi(x)](1 - O_2(1+x)L_{3,n} - O_3 Q_{n,x})$$

□

# References

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