Q1

1. let $Z = a + bX$.

the values of Z are $z_i = a + bx_i$ for $i = 1, \cdots, N$.

$$m(z) = \frac{1}{N}\sum_{i=1}^{N} z_i = \frac{1}{N}\sum_{i=1}^{N}(a+bx_i) = \frac{1}{N}\left(\sum_{i=1}^{N} a + \sum_{i=1}^{N} bx_i\right) = \frac{1}{N}\left(Na + b\sum_{i=1}^{N} x_i\right) = a + b\cdot\frac{1}{N}\sum_{i=1}^{N} x_i = a + b\cdot m(X)$$

therefore, $m(a+bX) = a + b\cdot m(X)$.

2. let $Z = a + bY$, $z_i = a + by_i$

$$m(Z) = a + b\cdot m(Y)$$

$$\mathrm{cov}(X, Z) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(z_i - m(Z)).$$

$$z_i - m(Z) = (a + by_i) - (a + bm(Y)) = b(y_i - m(Y)).$$

therefore, $\mathrm{cov}(X, Z) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))\cdot b(y_i - m(Y))$

$$= b\cdot\frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))(y_i - m(Y))$$

$$= b\cdot\mathrm{cov}(X, Y)$$

therefore, $\mathrm{cov}(X, a+bY) = b\cdot\mathrm{cov}(X, Y)$.

3. $\mathrm{cov}(X, X) = \frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))^2 = S^2 \rightarrow$ sample variance.

let $Z = a + bX$, $\mathrm{cov}(Z, Z) = \frac{1}{N}\sum_{i=1}^{N}(z_i - m(Z))^2$

since $m(Z) = a + b\cdot m(X)$,

$$z_i - m(Z) = (a + bx_i) - (a + bm(X)) = b(x_i - m(X)).$$

therefore, $(z_i - m(Z))^2 = [b(x_i - m(X))]^2 = b^2(x_i - m(X))^2$

$$\mathrm{cov}(Z, Z) = \frac{1}{N}\sum_{i=1}^{N} b^2(x_i - m(X))^2 = b^2\cdot\frac{1}{N}\sum_{i=1}^{N}(x_i - m(X))^2 = b^2\cdot S^2 = b^2\cdot\mathrm{cov}(X, X).$$

4. For a non-decreasing transformation g, the transformation of the median is the median of the transformed variable.

$$\mathrm{med}(g(X)) = g(\mathrm{med}(X)).$$

for p-th quantile $q_p(X)$, $P(X \le q_p(X)) = p$.

since g is non-decreasing, $P(g(X) \le g(q_p(X))) = P(X \le q_p(X)) = p$.

so $g(q_p(X))$ is the p-th quantile of $g(X)$.

IQR = $q_{0.75}(X) - q_{0.25}(X)$

after transformation, IQR = $g(q_{0.75}(X)) - g(q_{0.25}(X))$.

It is not necessarily equal to $g(IQR(X))$ unless g is linear.

Similarly, range (max - min) transforms as $g(\max(X)) - g(\min(X))$.

It is not necessarily $g(\text{range}(X))$ unless g is linear.

5. $m(g(X)) = g(m(X))$ is not always true. It is true only if g is linear.

The sample mean is a linear operator, but g may be nonlinear.

E.g. if X has values $\{1, 3\}$.

$m(X) = 2$

if $g(X) = X^2$, $g(m(X)) = 4$

$m(g(X)) = \frac{1}{2}(1^2 + 3^2) = 5$

therefore, $g(m(X)) \neq m(g(X))$.